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STRONG COMPACT PROPERTIES OF THE MAPPINGS AND K-RADON-NIKODYM PROPERTY

I. V. ORLOV AND F. S. STONYAKIN

Abstract. For mappings acting from an interval into a locally convex space, we study properties of strong compact variation and strong compact absolute continuity connected with an expansion of the space into subspaces generated by the compact sets. A description of strong K-absolutely continuous mappings in terms of indefinite Bochner integral is obtained. A special class of the spaces having K-Radon-Nikodym property is obtained. A relation between the K-Radon-Nikodym property and the classical Radon-Nikodym property is considered.

0. Introduction

It is well known that a mapping from an interval into a Banach space and, all the more, into a locally convex space (LCS) can be strongly absolutely continuous without being an indefinite Bochner integral. In this connection, a class of spaces having the Radon-Nikodym property (RNP) was singled out. By definition, for a space with (RNP), both of the above-cited properties of the mappings coincide. The theory of the spaces with (RNP) intensively develops and has numerous applications [1]–[8].

However, the class of spaces having (RNP) is sufficiently restricted [1, 2]. There is also a version, introduced in [9], of a Bochner integral for the spaces $L_2(\Sigma; \mathcal{H})$ for an operator–valued measure Σ over a Hilbert space $\mathcal{H}$. It appearse that there are the so-called Radon-Nikodym type theorems valid for spaces without (RNP) [10]–[15]. Among useful new notions that were introduced in this area lately, we consider here a suitable projective description of Banach spaces [16, 17], versions of a weak (RNP) [18], a use of special sequence spaces [19] and especially a use of compact sets with (RNP) [20].

In the paper [21] we used new convex compact properties of mappings into LCS, compact subdifferential and compact variation, to describe an indefinite Bochner integral. Note, in particular, a criterion in the case of Frechet spaces ([21], Theorem 3.2).

In this paper, developing the study of [21] in a new direction, we introduce strong compact properties of mappings into LCS, strong K-variation (V_K^s) and strong K-absolute continuity (AC_K^s). Here we are based on an expansion of the main space into an inductive scale of Banach spaces generated by compact sets. The properties of classes V_K^s and AC_K^s are studied in Sections 1–2. Note, in particular, the compact subdifferentiability a.e. (Theorems 1.1, 2.1).

On the basis of these properties, a description of AC_K^s as a certain subclass of the class $\mathcal{I}_B$ of the indefinite Bochner integrals (Theorem 3.2) is obtained. This made it possible to select a class of spaces having the K-Radon-Nikodym property $(RNP)_K$, where $AC_K^s = \mathcal{I}_B$. Theorem 3.3, establishing that the space c_0 has $(RNP)_K$ (together with ℓ_p, $1 \leqslant p < \infty$) is one of the main results of the paper. Examples in Section 3 show

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that in the chain $AC^s_K \subset \mathcal{I}_B \subset AC^s$ any combinations of strong inclusions and equalities are possible.

Throughout the paper, for arbitrary sets A, B in a LCS E, $\overline{co}A$ denotes the closed convex hull of A, and $B - B = \{x - x' \mid x \in B,\ x' \in B\}$.

1. Strong compact variation and its properties

In what follows, we consider a mapping $F : I \to E$ acting from a real segment $I = [a; b]$ into a real LCS E. Further $\mathcal{C}(E)$ is the system of all absolutely convex (a.c.) compacta $C \subset E$, $E_C = (\operatorname{span} C, \|\cdot\|_C)$ are Banach spaces generated by $C \in \mathcal{C}(E)$. Here the identical embeddings $E_C \hookrightarrow E$ are compact and $E = \lim_{\overrightarrow{C \in \mathcal{C}(E)}} E_C$ in the case of Banach E ([22], Theorem 3.3). Denote by $V^s(I, E)$ the class of mappings having a usual finite strong variation (respective to the all continuous seminorms on E).

Definition 1.1. We say that a mapping F possesses a *strong compact variation* on I ($F \in V^s_K(I, E)$) if there exists $C \in \mathcal{C}(E)$ such that $F : I \to F(a) + E_C$ and, in addition, $F \in V^s(I, E_C)$. Denote by $V^s_C(F)$ a strong total variation of F in E_C.

Let in the mention *the general properties* of mappings from the class $V^s_K(I, E)$.

Proposition 1.1. *The inclusion $V^s_K(I, E) \subset V^s(I, E)$ holds. In the case of* $\dim E < \infty$, *both classes coincide.*

Proof. Because $(F \in V^s_K(I, E)) \Rightarrow (F \in V^s(I, E_C)$ for some $C \in \mathcal{C}(E)$ and the embedding $E_C \hookrightarrow E$ is continuous), it is obvious that $F \in V^s(I, E)$. In the case of $\dim E < \infty$, the equality $E = E_C$ for a closed unit ball $C \subset E$ holds, whence $V^s_K(I, E) = V^s(I, E)$. $\square$

From the linearity of the classes $V^s(I, E_C)$ and the equality

$$V^s_K(I, E) = \bigcup_{C \in \mathcal{C}(E)} V^s(I, E_C),$$

we immediately get the following result.

Proposition 1.2. *Let E be a complete LCS. Then the class $V^s_K(I, E)$ is linear.*

Proposition 1.3. *If $F \in V^s_K(I, E_1)$, $A \in \mathcal{L}(E_1, E_2)$ then $A \circ F \in V^s_K(I, E_2)$.*

Proof. Let $V^s_C(F) < \infty$ for some $C \in \mathcal{C}(E_1)$. Then for each $x \in E_{1,C}$,

$$\begin{aligned} \|Ax\|_{A(C)} &= \inf\{\lambda > 0 \mid Ax \in \lambda \cdot A(C)\} = \inf\{\lambda > 0 \mid Ax \in A(\lambda \cdot C)\} \\ &\leqslant \inf\{\lambda > 0 \mid x \in \lambda C\} = \|x\|_C\ . \end{aligned} \tag{1.1}$$

Hence, for every partition $P : a = x_0 < x_1 < \cdots < x_n = b$, $\mathcal{P}(I) = \{P\}$, a use of (1.1) leads to an estimate of partial variation in $E_{2,A(C)}$,

$$\begin{aligned} V^s_{A(C)}(A \circ F, P) &= \sum_{k=1}^{n} \|A(F(x_k)) - A(F(x_{k-1}))\|_{A(C)} = \sum_{k=1}^{n} \|A(F(x_k) - F(x_{k-1}))\|_{A(C)} \\ &\leqslant \sum_{k=1}^{n} \|F(x_k) - F(x_{k-1})\|_C = V^s_C(F, P), \end{aligned}$$

whence the inequality $V^s_{A(C)}(A \circ F) \leqslant V^s_C(F) < \infty$ follows.

This means, in view of compactness of $A(C)$, that $A \circ F \in V^s_K(I, E_2)$. $\square$

Proposition 1.4. *Let E be a complete LCS. Then $F \in V^s_K(I_1 \bigcup I_2, E)$ if and only if $F|_{I_j} \in V^s_K(I_j, E)$, $j = 1, 2$.*

Proof. Indeed, $(F \in V^s(I_1 \bigcup I_2, E_C)) \Rightarrow \big(F|_{I_j} \in V^s_K(I_j, E), j = 1,2\big)$ according to the properties of strong variation in E_C. Conversely, if $F|_{I_j} \in V^s(I_j, E_{C_j}), j = 1,2$, set $C_3 = \overline{co}(C_1 \bigcup C_2)$. Then $C_3 \in \mathcal{C}(E)$ ([23], 8.13.4) and the required result follows from continuity of the embeddings $E_{C_j} \hookrightarrow E_C$ $(j = 1,2)$. □

Proposition 1.5. *$(F_1, F_2) \in V^s_K(I, E_1 \times E_2)$ if and only if $F_j \in V^s_K(I, E_j)$, $j = 1,2$. Moreover, if $V^s_{C_j}(F_j) < \infty$, then*

$$\frac{1}{2}\left[V^s_{C_1}(F_1) + V^s_{C_2}(F_2)\right] \leqslant V^s_{C_1 \times C_2}(F_1, F_2) \leqslant V^s_{C_1}(F_1) + V^s_{C_2}(F_2). \tag{1.2}$$

Proof. At first, calculate the norm in $E_{C_1 \times C_2}$,

$$\begin{aligned} \|(x_1, x_2)\|_{C_1 \times C_2} &= \inf\{\lambda > 0 \mid (x_1, x_2) \in \lambda(C_1 \times C_2)\} \\ &= \inf\{\lambda > 0 \mid (x_1, x_2) \in (\lambda C_1) \times (\lambda C_2)\} \\ &= \inf\{\lambda > 0 \mid x_1 \in \lambda C_1,\ x_2 \in \lambda C_2\} \\ &= \inf\Big[\{\lambda > 0 \mid x_1 \in \lambda C_1\} \bigcap \{\lambda > 0 \mid x_2 \in \lambda C_2\}\Big] \\ &= \max\left(\inf\{\lambda > 0 \mid x_1 \in \lambda C_1\}, \inf\{\lambda > 0 \mid x_2 \in \lambda C_2\}\right) \\ &= \max(\|x_1\|_{C_1}, \|x_2\|_{C_2}). \end{aligned} \tag{1.3}$$

Taking into account (1.3) and the elementary inequality $\max(\alpha, \beta) \leqslant \alpha + \beta$ $(\alpha \geqslant 0, \beta \geqslant 0)$, let's estimate now the partial variation of (F_1, F_2) in $E_{C_1 \times C_2}$ for a partition $P \in \mathcal{P}(I)$,

$$\begin{aligned} V^s_{C_1 \times C_2}((F_1, F_2), P) &= \sum_{k=1}^{n} \|(F_1(x_k), F_2(x_k)) - (F_1(x_{k-1}), F_2(x_{k-1}))\|_{C_1 \times C_2} \\ &= \sum_{k=1}^{n} \|(F_1(x_k) - F_1(x_{k-1}),\ F_2(x_k) - F_2(x_{k-1}))\|_{C_1 \times C_2} \\ &= \sum_{k=1}^{n} \max\left(\|F_1(x_k) - F_1(x_{k-1})\|_{C_1},\ \|F_2(x_k) - F_2(x_{k-1})\|_{C_2}\right) \\ &\leqslant \sum_{k=1}^{n} \left(\|F_1(x_k) - F_1(x_{k-1})\|_{C_1} + \|F_2(x_k) - F_2(x_{k-1})\|_{C_2}\right) \\ &= V^s_{C_1}(F_1, P) + V^s_{C_2}(F_2, P), \end{aligned}$$

whence the inequality in the right-hand side of (1.2) follows. Analogously, using the inequality $\max(\alpha, \beta) \geqslant \frac{\alpha+\beta}{2}$ in the preceding calculations we get the inequality in the left-hand side of (1.2). □

Next denote by $B(E_1 \times E_2; E_3)$ the class of bilinear continuous operators acting from $E_1 \times E_2$ into E_3.

Proposition 1.6. *If $F_j \in V^s_K(I, E_j)$, $j = 1,2$, $B \in B(E_1 \times E_2; E_3)$, then $B(F_1, F_2) \in V^s_K(I, E_3)$. If, in addition, $V^s_{C_j}(F_j) < \infty$, then*

$$V^s_{B(C_1 \times C_2)}(B(F_1, F_2)) \leqslant \sup_{x \in I} \|F_1(x)\|_{C_1} \cdot V^s_{C_2}(F_2) + \sup_{x \in I} \|F_2(x)\|_{C_2} \cdot V^s_{C_1}(F_1). \tag{1.4}$$

Proof. For a partition $P \in \mathcal{P}(I)$ it follows that

(1.5)
$$
\begin{aligned}
V^s_{B(C_1\times C_2)}(B(F_1,F_2),P) &= \sum_{k=1}^n \|B(F_1(x_k),F_2(x_k)) - B(F_1(x_{k-1}),F_2(x_{k-1}))\|_{B(C_1\times C_2)} \\
&= \sum_{k=1}^n \|B(F_1(x_k)-F_1(x_{k-1}),F_2(x_k)) + B(F_1(x_{k-1}),F_2(x_k)-F_2(x_{k-1}))\|_{B(C_1\times C_2)} \\
&\leqslant \sum_{k=1}^n \|B(F_1(x_k)-F_1(x_{k-1}),F_2(x_k))\|_{B(C_1\times C_2)} \\
&+ \sum_{k=1}^n \|B(F_1(x_{k-1}),F_2(x_k)-F_2(x_{k-1}))\|_{B(C_1\times C_2)}.
\end{aligned}
$$

Next, applying (1.1) to the linear operators $B(\cdot,F_2(x_k))$ and $B(F_1(x_{k-1}),\cdot)$, we find respectively that

$$
(1.6)\quad \begin{cases} \|B(F_1(x_k)-F_1(x_{k-1}),F_2(x_k))\|_{B(C_1,F_2(x_k))} \leqslant \|F_1(x_k)-F_1(x_{k-1})\|_{C_1}\ , \\ \|B(F_1(x_{k-1}),F_2(x_k)-F_2(x_{k-1}))\|_{B(F_1(x_{k-1}),C_2)} \leqslant \|F_2(x_k)-F_2(x_{k-1})\|_{C_2}\ . \end{cases}
$$

At last, setting, for simplicity, $F_j(a)=0$ and denoting $\lambda_j = \sup_{x\in I}\|F_j(x)\|_{C_j}$, $j=1,2$, we obtain

$$
\begin{aligned}
B(C_1,F_2(x_k)) \subset B(C_1\times\lambda_2 C_2) = \lambda_2\cdot B(C_1\times C_2);\ & B(F_1(x_{k-1}),C_2) \\
\subset B(\lambda_1 C_1\times C_2) = \lambda_1\cdot B(C_1\times C_2),&
\end{aligned}
$$

whence

$$
(1.7)\quad \begin{cases} \|B(F_1(x_k)-F_1(x_{k-1}),F_2(x_k))\|_{B(C_1\times C_2)} \\ \qquad \leqslant \lambda_2\cdot\|B(F_1(x_k)-F_1(x_{k-1}),F_2(x_k))\|_{B(C_1,\ F_2(x_k))}\ , \\ \|B(F_1(x_{k-1}),F_2(x_k)-F_2(x_{k-1}))\|_{B(C_1\times C_2)} \\ \qquad \leqslant \lambda_1\cdot\|B(F_1(x_{k-1}),F_2(x_k)-F_2(x_{k-1}))\|_{B(F_1(x_{k-1}),\ C_2)} \end{cases}
$$

follows. From (1.5), (1.6) and (1.7) we obtain

$$
V^s_{B(C_1\times C_2)}(B(F_1,F_2),P) \leqslant \lambda_1\cdot V^s_{C_2}(F_2,P) + \lambda_2\cdot V^s_{C_1}(F_1,P),
$$

which implies (1.4). □

Note further that $V^s_C<\infty$ and the continuous embedding $E_C \hookrightarrow E_{C'}$ implies that $V^s_{C'}<\infty$. More precisely, we have

Proposition 1.7. *If* $C_1,C_2\in\mathcal{C}(E)$, $C_1\subset\lambda\cdot C_2$ $(\lambda>0)$, *then*

$$
V^s_{C_2}(F) \leqslant \lambda\cdot V^s_{C_1}(F).
$$

Proof. This directly follows from the inequality $\|\cdot\|_{C_2}\leqslant\lambda\cdot\|\cdot\|_{C_1}$. □

Let's compare now the strong compact variation property V^s_K and the convex compact variation property V_K that was introduced by us earlier in ([21], Definition 1.3).

Definition 1.2. Given a partition $P\in\mathcal{P}(I)$ let's introduce a *partial convex variation*,

$$
V^{co}(F,P) = \sum_{k=1}^n w(F([x_{k-1};x_k]))\ ,
$$

where $w(A) := \overline{co}(A-A)$. We call the *total convex variation* of F on I the set

$$
V^{co}(F) = \overline{\bigcup_{P\in\mathcal{P}(I)} V^{co}(F,P)}\ .
$$

We call F a *compact variation mapping* ($F\in V_K(I,E)$) if $V^{co}(F)$ is a compact set.

Let's show that the property V_K^s is stronger than V_K.

Proposition 1.8. *If $F \in V_K^s(I,E)$, then $F \in V_K(I,E)$. Moreover, for all $C \in \mathcal{C}(E)$, the inclusion*

$$V_C^{co}(F) \subset V_C^s(F) \cdot C$$

holds.

Proof. It's easy to see that, for a partition $P \in \mathcal{P}(I)$,

$$\sum_{k=1}^{n} \sup \|wF([x_{k-1};x_k])\|_C = \sum_{k=1}^{n} \inf\{\lambda > 0 \mid wF([x_{k-1};x_k]) \subset \lambda C\} =: \sum_{k=1}^{n} \lambda_k \leqslant V_C^s(F).$$

From here, we get

$$V^{co}(F,P) = \sum_{k=1}^{n} wF([x_{k-1};x_k]) \subset \sum_{k=1}^{n} (\lambda_k \cdot C) = \Big(\sum_{k=1}^{n} \lambda_k\Big) \cdot C \subset V_C^s(F) \cdot C.$$

□

In ([21], Example 2.3) an example of a mapping from $V_K(I,E)$ which is nowhere K-subdifferentiable was constructed. The main result of this section states that any mapping from V_K^s is K-subdifferentiable almost everywhere. First, let us briefly recall the definition of a K-subdifferential ([21], Definition 2.2, [24], Definition 4.1).

Definition 1.3. Given $h > 0$, a *partial convex subdifferential* F at a point $x \in I$ is the set

$$\partial^{co} F(x,h) = \overline{co}\left\{ \frac{F(x+h') - F(x)}{h'} \;\middle|\; 0 < |h'| < h \right\}.$$

The set $\partial^{co}F(x)$ (namely, the intersection of all $\partial^{co}F(x,h)$) is called the *convex subdifferential* of F at x, if $\partial^{co}F(x,h) \subset \partial^{co}F(x) + U$ for each zero neighborhood $U \subset E$ and $|h| < \delta = \delta_U > 0$. Finally, the *$K$-subdifferential* is $\partial_K F(x) = \partial^{co}F(x)$ in case of compact $\partial^{co}F(x)$.

Theorem 1.1. *If $F \in V_K^s(I,E)$, then*

(i) *F is continuous everywhere on I, exept for at most a countable set of gap points;*
(ii) *F is K-subdifferentiable almost everywhere on I. In this case,*

$$\partial_K F(x) \in \varphi(x) \cdot C \tag{1.8}$$

for some summable $\varphi(x) \geqslant 0$, $F \in V^s(I, E_C)$ and a.e. $x \in I$.

Proof. Denote by $\Phi(x) = V_C^s(F|_{[a;x]})$, where $V_C^s(F) < \infty$, $C \in \mathcal{C}(E)$. Then, in view of Proposition 1.4, Φ increases on I. It follows that Φ is a.e. differentiable on I and $\varphi = \Phi'$ is nonnegative and summable over I. In addition, the obvious estimate

$$\|F(x+\triangle x) - F(x)\|_C \leqslant V_C^s(F|_{[x;x+\triangle x]}) = \Phi(x+\triangle x) - \Phi(x)$$

implies

(i) continuity exept for at most a countable set of gap points for F at the same points as Φ;
(ii) the inclusion

$$\frac{F(x+\triangle x) - F(x)}{\triangle x} \in \frac{\Phi(x+\triangle x) - \Phi(x)}{\triangle x} \cdot C\ .$$

The last estimate implies the inclusion

$$\partial^{co}F(x,h) \subset \partial^{co}\Phi(x,h) \cdot C\ ,$$

whence, taking into account differentiability a.e. of Φ and compactness of $\partial^{co}F(x,h)$, K-subdifferentiability a.e. of F and estimate (1.8) follow. □

Note, that (1.8) can be written in the following equivalent form:

$$\|\partial_K F(x)\|_C \leqslant \varphi(x)\ . \tag{1.9}$$

2. Strong compact absolute continuity and its properties

Consider now a more restricted class of mappings. First, denote by $AC^s(I,E)$ the class of mappings $F: I \to E$ having the usual strong absolute continuity property (with respect to each continuous seminorm on E).

Definition 2.1. We say that a mapping F is *strongly compactly absolutely continuous* on I ($F \in AC^s_K(I,E)$) if, for some $C \in \mathcal{C}(E)$, $F: I \to F(a) + E_C$ and in addition $F \in AC^s(I, E_C)$.

Now consider *general properties* of mappings from the class $AC^s_K(I,E)$, by analogy with the ones in the class $V^s_K(I,E)$.

Proposition 2.1. *The inclusion $AC^s_K(I,E) \subset AC^s(I,E)$ is valid. In the case of $\dim E < \infty$, the two the classes coincide.*

Proposition 2.2. *Let E be a complete LCS. Then the class $AC^s_K(I,E)$ is linear.*

Proposition 2.3. *Let E be a complete LCS. Then $F \in AC^s_K(I_1 \bigcup I_2, E)$ if and only if $F|_{I_j} \in AC^s_K(I_j,E)$, $j = 1,2$.*

The proofs of the propositions above are quite analogous with ones for propositions 1.1, 1.2 and 1.4, respectively.

Proposition 2.4. *If $F \in AC^s_K(I,E_1)$, $A \in \mathcal{L}(E_1,E_2)$ then $A \circ F \in AC^s_K(I,E_2)$.*

Proof. Using (1.1), for an arbitrary disjoint system $\bigcup_k (\alpha_k;\beta_k) \subset I$ and $F \in AC^s(I, E_{1,C})$, $C \in \mathcal{C}(E_1)$, we get

$$\begin{aligned}\sum_k \|A(F(\beta_k)) - A(F(\alpha_k))\|_{A(C)} &= \sum_k \|A(F(\beta_k) - F(\alpha_k))\|_{A(C)} \\ &\leqslant \sum_k \|F(\beta_k) - F(\alpha_k)\|_C \to 0\end{aligned}$$

as $\sum_k(\beta_k - \alpha_k) \to 0$, whence $A \circ F \in AC^s_K(I,E_2)$ follows. □

Proposition 2.5. *$(F_1,F_2) \in AC^s_K(I, E_1 \times E_2)$ if and only if $F_j \in AC^s_K(I,E_j)$, $j = 1,2$.*

Proof. Quite analogously with the proof of Proposition 1.5, for an arbitrary disjoint system $\bigcup_k (\alpha_k;\beta_k) \subset I$ we get

$$\begin{aligned}&\frac{1}{2}\Big(\sum_k \|F_1(\beta_k) - F_1(\alpha_k)\|_{C_1} + \sum_k \|F_2(\beta_k) - F_2(\alpha_k)\|_{C_2}\Big) \\ &\quad\leqslant \sum_k \|(F_1(\beta_k), F_2(\beta_k)) - (F_1(\alpha_k), F_2(\alpha_k))\|_{C_1 \times C_2} \\ &\quad\leqslant \sum_k \|F_1(\beta_k) - F_1(\alpha_k)\|_{C_1} + \sum_k \|F_2(\beta_k) - F_2(\alpha_k)\|_{C_2}\end{aligned}$$

for $F_j \in AC^s(I, E_{j,C_j})$, $j = 1,2$, whence

$$(F_1,F_2) \in AC^s(I, (E_1 \times E_2)_{C_1 \times C_2}) \Leftrightarrow \big(F_j \in AC^s(I, E_{j,C_j}),\ j = 1,2\big)\ .$$

□

Proposition 2.6. *If $F_j \in AC^s_K(I,E_j)$, $j = 1,2$, $B \in B(E_1 \times E_2; E_3)$ then $B(F_1,F_2) \in AC^s_K(I,E_3)$.*

Proof. Quite analogously with (1.5) – (1.6) – (1.7), for an arbitrary disjoint system $\bigcup_k(\alpha_k;\beta_k)\subset I$ and $F_j\in AC^s(I,E_{j,C_j})$ we get

$$\begin{aligned}\sum_k &\|B(F_1,F_2)(\beta_k)-B(F_1,F_2)(\alpha_k)\|_{B(C_1\times C_2)}\\ &\leqslant \sup_{x\in I}\|F_1(x)\|_{C_1}\cdot\sum_k\|F_2(\beta_k)-F_2(\alpha_k)\|_{C_2}\\ &+\sup_{x\in I}\|F_2(x)\|_{C_1}\cdot\sum_k\|F_1(\beta_k)-F_1(\alpha_k)\|_{C_1},\end{aligned}$$

whence $B(F_1,F_2)\in AC^s(I,E_{3,\ B(C_1\times C_2)})$ immediately follows. □

The following statement is analogous with Proposition 1.7.

Proposition 2.7. *If $C_1,C_2\in\mathcal{C}(E)$, $C_1\subset\lambda\cdot C_2$ and $F\in AC^s(I,E_{C_1})$ then $F\in AC^s(I,C_2)$.*

A partial inversion of Proposition 2.1 takes place.

Proposition 2.8. *Let E be Banach space, $E^*_\sigma=(E^*,\sigma(E^*,E))$. Then*

$$AC^s_K(I,E^*_\sigma)=AC^s(I,E^*)\ .$$

In particular, if E is a reflexive Banach space then

$$AC^s_K(I,E_\sigma)=AC^s(I,E)\ .$$

Proof. This directly follows from Banach-Alaoglu theorem on $*$-weak compactness of unit ball in E^* ([25], Theorem VII.8.1). □

Let's pass to the main results od this item. First, explain connection between strong K-variation and strong K-absolutely continuity.

Theorem 2.1. *If $F\in AC^s_K(I,E)$ then $F\in V^s_K(I,E)$.*

Proof. Let $F\in AC^s(I,E_C)$, $C\in\mathcal{C}(E)$. Following to the standard scheme ([26], Theorem IX.2.1), given $\varepsilon>0$ let $\delta>0$ be such that the inequality

$$\Big(\sum_k(\beta_k-\alpha_k)<\delta\Big)\Rightarrow\Big(\sum_k\|F(\beta_k)-F(\alpha_k)\|_C<\varepsilon\Big)$$

holds for an arbitrary disjoint system $\bigcup_k(\alpha_k;\beta_k)\subset I$. Given a partition $P\in\mathcal{P}(I)$, $\lambda(P)<\delta$ and fixed $j=\overline{1,n}$, let $P_j: x_j=y_0<y_1<\cdots<y_m=x_j$ be an arbitrary partition of $[x_{j-1};x_j]$. Then

$$\Big(\sum_{i=1}^m(y_i-y_{i-1})=\triangle x_j<\delta\Big)\Rightarrow\Big(\sum_{i=1}^m\|F(y_i)-F(y_{i-1})\|_C=V^s_C(F,P_j)<\varepsilon\Big)\ ,$$

whence $V^s_C(F|_{[x_{j-1};\ x_j]})\leqslant\varepsilon$ and hence, by Proposition 1.4, $V^s_C(F)\leqslant n\cdot\varepsilon$, i.e. $F\in V^s_K(I,E)$ follows. □

Theorems 2.1 and 1.1 immediately imply

Corollary 2.1. *If $F\in AC^s_K(I,E)$ then F is K-subdifferentiable a.e. on I.*

Next, for the case of Frechet space E in ([21], Theorem 3.2) equivalence of the conditions a.e. K-subdifferentiability and a.e. usual differentiability for $A\in AC^s(I,E)$ was proved. Then Proposition 2.1 and the last corollary imply

Corollary 2.2. *Let E be Frechet space. If $F\in AC^s_K(I,E)$ then F is differentiable a.e. on I.*

Note that F from AC^s_K can be nowhere differentiable if E isn't Frechet space (see [21], Example 2.2). Let's deduce now a criterion of the strong K-absolute continuity.

Theorem 2.2. *Let E be a separable LCS. Then F is strongly K-absolute continuous if and only if F possesses strong K-variation and F is weakly absolutely continuous.*

Proof. Necessity of the statement follows at once from Theorem 2.1 and Proposition 2.1. Conversely, if $F \in V^s_K(I,E)$ then F is K-subdifferentiable on $I \backslash e$, $mes(e) = 0$, by Corollary 2.1. In this case

$$(F \in V^s(I,E_C)) \Rightarrow (\partial_K F(x) \in \varphi(x) \cdot C, x \in I \backslash e)\,, \quad \text{where} \qquad (2.1)$$
$$\varphi(x) = \frac{d}{dx} V^s_C \left(F|_{[a;\; x]}\right) \geqslant 0$$

by virtue of (1.8). Next, since F is weakly absolutely continuous then F possesses weak Lusin N-property, whence weak null measure of $F(e)$ follows.

Let's check continuity of F. By theorem 1.1, F is continuous everywhere on I exept at most countable set of gaps. Assume that $F(x-0) \neq F(x+0)$, $x \in I$. Then, by corollary from Hahn-Banach theorem ([23], Corollary 2.1.4) such $\ell \in E^*$ exists that $\ell(F(x-0)) \neq \ell(F(x+0))$, $x \in I$. But that contradicts with weak continuity of F.

Thus, F satisfies all conditions of the generalized finite increments theorem for K-subdifferentials ([24], Theorem 6.2), namely: continuity on I, K-subdifferentiability on I exept $e \subset I$ with weak null measure of $F(e)$, estimation (2.1). Whence, applying the theorem on $[\alpha_k;\; \beta_k]$ we find

$$(2.2) \qquad F(\beta_k) - F(\alpha_k) \in \int_{\alpha_k}^{\beta_k} \varphi(t)\, dt \cdot C\ , \quad \text{i.e.} \quad \|F(\beta_k) - F(\alpha_k)\|_C \leqslant \int_{\alpha_k}^{\beta_k} \varphi(t)\, dt\ .$$

Summing inequalities (2.2) for an arbitrary disjoint system $\bigcup_k (\alpha_k;\; \beta_k)$ leads to

$$\sum_k \|F(\beta_k) - F(\alpha_k)\|_C \leqslant \int_{\bigcup_k (\alpha_k;\; \beta_k)} \varphi(t)\, dt\ ,$$

from here $F \in AC^s(I, E_C)$ directly follows. □

As consequence, a variant of Banach-Zaretsky theorem can be easily obtained.

Corollary 2.3. *Let E be a separable LCS. Then $F \in AC^s_K(I,E)$ if and only if F is continuous on I, F possesses strong K-variation and weak Lusin N-property on I.*

Let's select now a simple subclass of AC^s_K.

Definition 2.2. Say that F is *strongly compact Lipshitz* ($F \in \mathrm{Lip}^s_K(I,E)$) if $F : I \to F(a) + E_C$ for some $C \in \mathcal{C}(E)$ and moreover $F \in \mathrm{Lip}^s(I, E_C)$.

It's easy to check that the property Lip^s_K coincides with the convex compact Lipchitz property Lip_K ([21], Definition 1.5). The following results are immediately verified.

Theorem 2.3. *If $F \in \mathrm{Lip}^s_K(I,E)$ then $F \in AC^s_K(I,E)$.*

Corollary 2.4. *If $F \in C^1(I,E)$ then $F \in AC^s_K(I,E)$.*

3. A criterion of strong K-absolute continuity and K-Radon-Nikodym property

If $F \in AC^s_K(I,E)$ then, by virtue of Theorem 2.1, Corollary 2.1 and Proposition 1.8 F possesses convex K-variation and F is a.e. subdifferentiable on I. In case of a separable LCS E, these two conditions imply representability F in the form of indefinite Bochner integral ([21], Theorem 3.1). Thus, there is valid the following

Theorem 3.1. *Let E be a separable LCS. If $F \in AC^s_K(I,E)$ then F is indefinite Bochner integral, i.e.*

$$F(x) = F(a) + (B)\int_a^x f(t)\,dt \qquad (a \leqslant x \leqslant b) \tag{3.1}$$

where $f : I \to E$ is (B)-integrable on I.

In fact, the following criterion is valid.

Theorem 3.2. *Let E be a separable LCS. Then $F \in AC^s_K(I,E)$ if and only if*

(i) *F is an indefinite Bochner integral, i.e.* (3.1) *is fulfilled;*

(ii) *$\int_a^b \|f(t)\|_C dt < \infty$ for some $C \in \mathcal{C}(E)$.*

Proof. In case of $F \in AC^s_K(I,E)$, by virtue of Theorem 3.1, F is indefinite Bochner integral of the form (3.1). In addition ([21], Theorem 3.1) $f(x) \in \partial_K F(x)$ and therefore (1.9) implies $\|f(x)\|_C \leqslant \varphi(x)$, from which statement ($ii$) follows.

Conversely, let conditions (i)–(ii) be fulfilled. Since f is (B)-integrable then f is weakly integrable. From here, taking into account inclusion $f(t) \in \|f(t)\|_C \cdot C$, for arbitrary $x_1,\ x_2 \in I$ and $\ell \in E^*$ we obtain

$$\ell(F(x_2) - F(x_1)) = \int_{x_1}^{x_2} \ell(f(t))\,dt \leqslant \int_{x_1}^{x_2} \|f(t)\|_C dt \cdot \sup \ell(C)\ .$$

Hence, by corollary from Hahn-Banach theorem,

$$F(x_2) - F(x_1) \in \Big(\int_{x_1}^{x_2} \|f(t)\|_C dt\Big) \cdot C\ , \quad \text{i.e.} \quad \|F(x_2) - F(x_1)\|_C \leqslant \int_{x_1}^{x_2} \|f(t)\|_C dt\ .$$

We get precise analog of (2.2). It follows, just analogously with the proof of Theorem 2.2, that $F \in AC^s_K(I,E)$. □

Remark 3.1. However, it should not be supposed that condition (ii) of Theorem 3.2 means (B)-integrability f for some E_C, $C \in \mathcal{C}(E)$. Let, for example, E be Banach space. Then, according to Proposition 2.8, $AC^s_K(I,E^*_\sigma) = AC^s(I,E^*)$. At the same time, if E^* does not possess Radon-Nikodym property (see a simple criterion in [1, 2]), the class of indefinite Bochner integrals is strictly less than $AC^s(I,E^*)$.

Thus, denoting by $\mathcal{I}_B(I,E)$ the class of indefinite Bochner integrals (3.1), we obtain the relation

$$AC^s_K(I,E) \subset \mathcal{I}_B(I,E) \subset AC^s(I,E)\ . \tag{3.2}$$

As it is known, E possesses *Radon-Nikodym property* ($E \in RNP$) ([4]) if $\mathcal{I}_B(I,E) = AC^s(I,E)$. Let's introduce a strong compact analog of (RNP) by equating of two first terms in (3.2).

Definition 3.1. Say that LCS E possesses *K-Radon-Nikodym property* ($E \in (RNP)_K$) if each indefinite Bochner integral $F : I \to E$ belongs to class $AC^s_K(I,E)$, i.e. if

$$AC^s_K(I,E) = \mathcal{I}_B(I,E)\ .$$

Below it'll be shown that there exist Banach spaces having $(RNP)_K$ but not having (RNP). First, by analogy with a known case of ℓ_2 ([22], Definition 1.1) let's consider compact ellipsoids in the space c_0 of the tending to zero scalar sequences.

Definition 3.2. Call the *(nondegenerated) ellipsoid* in c_0 any set of the form

$$C_\varepsilon = \left\{ x = (x_k)_1^\infty \in c_0 \;\middle|\; \sup_{k \geqslant 1} (|x_k|/\varepsilon_k) \leqslant 1 \right\}\ , \tag{3.3}$$

where $\varepsilon = (\varepsilon_k > 0)_1^\infty$

By analogy with the case of ℓ_2, it is easy to check

Proposition 3.1. *An ellipsoid C_ε in c_0 is compact if and only if $\varepsilon_k \to 0$.*

Note that the same is true for ellipsoids in ℓ_p, $1 \leqslant p < \infty$. It can be proved by analogy with ([22], Theorem 1.2) that compact ellipsoids in c_0 (and also in ℓ_p, $1 < p < +\infty$) are universal compact sets, i.e. they absorb the all other compacta. Note also that norm generated by ellipsoid C_ε in $E_{C_\varepsilon} = \operatorname{span} C_\varepsilon$ has a form

$$\|x\|_{C_\varepsilon} = \sup_{k\geqslant 1} \left(|x_k|/\varepsilon_k\right) \ .$$

Let's formulate the main result.

Theorem 3.3. *The space c_0 possesses K-Radon-Nikodym property. More precisely, if $F : I \to c_0$ has a form (3.1) then*

$$\int_a^b \|f(t)\|_{C_\varepsilon} dt < \infty$$

for some compact ellipsoid $C_\varepsilon \subset c_0$.

Proof. 1) Since f is (B)-integrable on I then f, in particular ([27], Theorems 3.5.3 and 3.7.4), is weakly integrable on I, and therefore every coordinate function $f_k(t)$ of the mapping $f(t) = (f_1(t), f_2(t), \dots, f_k(t), \dots)$ is summable on I. Hence, for each compact ellipsoid $C_\varepsilon \subset c_0$ of the form (3.3) the function $\|f(t)\|_{C_\varepsilon}$ is supremum of a sequence of measurable functions and therefore is measurable, too. Denote further

$$K := \int_a^b \|f(t)\|_{c_0} dt = \int_a^b \Big(\sup_{k\geqslant 1} |f_k(t)|\Big) dt < \infty \ , \tag{3.4}$$

in view of (B)-integrability of f.

2) Let's construct now such sequence $(\varepsilon_k > 0)_1^\infty$, $\varepsilon_k \to 0$, that

$$K_\varepsilon := \int_a^b \|f(t)\|_{C_\varepsilon} dt = \int_a^b \Big[\sup_{k\geqslant 1}(|f_k(t)|/\varepsilon_k)\Big] dt < \infty \ .$$

First, let's construct by induction an auxiliary system of the sequences $\big\{\varepsilon^n = \big(\varepsilon^{nk} > 0\big)_{k=1}^\infty\big\}_{n=1}^\infty$. Set

$$\varepsilon^1 = (1, 1, \dots) \ ; \quad \varepsilon^{1k} = \left(1, 1, \dots, 1; \overbrace{\frac{1}{2}}^{k}, \frac{1}{2}, \dots\right) , \quad k \in \mathbb{N} \ .$$

Then

$$\|f(t)\|_{C_{\varepsilon^{11}}} \geqslant \|f(t)\|_{C_{\varepsilon^{12}}} \geqslant \dots \geqslant \|f(t)\|_{C_{\varepsilon^{1k}}} \geqslant \dots \geqslant \|f(t)\|_{c_0} \ ,$$

whence the integral sequence

$$I^{1k} := \int_a^b \|f(t)\|_{C_{\varepsilon^{1k}}} dt \quad (k \in \mathbb{N})$$

monotonically decreases and, taking into account (3.4), is situated between

$$I^1 := \inf_{k\geqslant 1} I^{1k} = \int_a^b \|f(t)\|_{c_0} dt = K \quad \text{and} \quad I^{11} := \sup_{k\geqslant 1} I^{1k} = \int_a^b \|f(t)\|_{C_{\varepsilon^{11}}} dt = 2K.$$

Hence, for every $t \in I$ the following limit

$$\varphi_1(t) = \lim_{k\to\infty} \|f(t)\|_{C_{\varepsilon^{1k}}} \tag{3.5}$$

exists. In addition, $\varphi_1(t) = \|f(t)\|_{c_0}$ because $\|f(t)\|_{c_0} = \|f(t)\|_{C_{\varepsilon^{1k}}} = |f_\ell(t)|$ for some $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$ large enough (here ℓ and k depend on t).

By theorem B. Levy ([25], Theorem III.6.3), it follows from (3.5) that $I^1 = \lim_{k\to\infty} I^{1k}$, whence $I^{1k_1} - I^1 < 1$ for some $k_1 \in \mathbb{N}$. Now set

$$\varepsilon^2 = \varepsilon^{1k_1} = \left(1, \dots, 1; \overbrace{\frac{1}{2}}^{k_1}, \frac{1}{2}, \dots\right).$$

By induction, suppose that the sequences

$$\varepsilon^p = \left(1, \dots, 1; \overbrace{\frac{1}{2}, \dots, \frac{1}{2}}^{k_1}; \overbrace{\frac{1}{3}, \dots, \frac{1}{3}}^{k_2}; \dots; \overbrace{\frac{1}{p-1}, \dots, \frac{1}{p-1}}^{k_{p-2}}; \overbrace{\frac{1}{p}}^{k_{p-1}}, \frac{1}{p}, \dots\right),$$

satisfying the condition

$$I^p - I^{p-1} < \frac{1}{2^{p-2}} \quad \left(I^p := \int_a^b \|f(t)\|_{C_{\varepsilon^p}}\, dt\right) \tag{3.6}$$

are constructed for $p = 2, \dots, n$. Describe construction of the sequence

$$\varepsilon^{n+1} = \left(1, \dots, 1; \overbrace{\frac{1}{2}, \dots, \frac{1}{2}}^{k_1}; \overbrace{\frac{1}{3}, \dots, \frac{1}{3}}^{k_2}; \dots; \overbrace{\frac{1}{n+1}, \dots, \frac{1}{n+1}}^{k_n}; \overbrace{\frac{1}{n+2}}^{k_{n+1}}, \frac{1}{n+2}, \dots\right),$$

satisfying the conditions

$$I^{n+1} - I^n < \frac{1}{2^{n-1}} \quad \left(I^{n+1} := \int_a^b \|f(t)\|_{C_{\varepsilon^{n+1}}}\, dt\right). \tag{3.7}$$

To this end, consider analogously with construction above collection of the sequences

$$\varepsilon^{nk} = \left(1, \dots, \frac{1}{n-1}; \overbrace{\frac{1}{n}, \dots, \frac{1}{n}}^{k_{n-1}}; \overbrace{\frac{1}{n+1}}^{k}, \frac{1}{n+1}, \dots\right), \quad (k \geqslant k_{n-1})$$

and choose such k_n that inequality

$$I^{nk_n} - I^n < \frac{1}{2^{n-1}} \quad \left(I^{nk_n} = \int_a^b \|f(t)\|_{C_{\varepsilon^{nk_n}}}\, dt\right) \tag{3.8}$$

holds. Setting $\varepsilon^{n+1} := \varepsilon^{nk_n}$ we obtain the required result. Thus by induction, the collection of the sequences $\{\varepsilon^n\}_1^\infty$ is constructed.

3) Denote by ε the coordinate-wise limit of ε^n as $n \to \infty$. Hence,

$$\varepsilon = \left(1, \dots, 1; \overbrace{\frac{1}{2}, \dots, \frac{1}{2}}^{k_1}; \overbrace{\frac{1}{3}, \dots, \frac{1}{3}}^{k_2}; \dots; \overbrace{\frac{1}{n}, \dots, \frac{1}{n}}^{k_{n-1}}; \overbrace{\frac{1}{n+1}, \dots \frac{1}{n+1}}^{k_n}; \dots\right).$$

Denoting by $\varepsilon = (\varepsilon_k)_1^\infty$ we obtain $\varepsilon_k \to 0$, whence by Proposition 3.1 ellipsoid C_ε is compact in c_0. Moreover, for every $x \in c_0$:

$$\|x\|_{C_{\varepsilon^1}} \leqslant \|x\|_{C_{\varepsilon^2}} \leqslant \dots \leqslant \|x\|_{C_{\varepsilon^n}} \leqslant \dots \leqslant \|x\|_{C_\varepsilon}. \tag{3.9}$$

In particular, sequence of the integrals $\left\{I^n = \int_a^b \|f(t)\|_{C_{\varepsilon^n}}\, dt\right\}_1^\infty$ monotonically increases. It is easily follows from (3.6)–(3.8) that $\{I^n\}_1^\infty$ is Cauchy sequence. Hence, by B. Levy theorem the limit

$$I^{(\varepsilon)} := \lim_{n\to\infty} I^n = \int_a^b \varphi^\varepsilon(t)\, dt, \quad \varphi^\varepsilon(t) = \lim_{n\to\infty} \|f(t)\|_{C_{\varepsilon^n}},$$

exists. In addition, φ^ε is summable on I in view of $I^{(\varepsilon)} < \infty$.

Let's show now that

$$\varphi^{\varepsilon}(t) = \|f(t)\|_{C_\varepsilon} \quad (\forall t \in I) \ . \tag{3.10}$$

Fix $t \in T$ and consider both admissible cases.

a) The equality

$$\|f(t)\|_{C_\varepsilon} = \sup_{k\geqslant 1} (|f_k(t)|/\varepsilon_k) = |f_{k_0}(t)|/\varepsilon_{k_0}$$

holds for some $k_0 \in \mathbb{N}$. Then, in view of (3.9), the equality

$$\|f(t)\|_{C_{\varepsilon k}} = |f_{k_0}(t)|/\varepsilon_{k_0} = \|f(t)\|_{C_\varepsilon}$$

holds, whence (3.10) follows.

b) The equality

$$\|f(t)\|_{C_\varepsilon} = \sup_{k\geqslant 1} (|f_k(t)|/\varepsilon_k) = \lim_{\ell\to\infty} (|f_{k_\ell}(t)|/\varepsilon_{k_\ell})$$

holds for some increasing sequence $(|f_{k_\ell}(t)|/\varepsilon_{k_\ell})_{\ell=1}^{\infty}$. Then for any $\delta > 0$ there is such $\ell_0 \in \mathbb{N}$ that

$$(|f_{k_\ell}(t)|/\varepsilon_k) > \|f(t)\|_{C_\varepsilon} - \delta$$

for all $\ell \geqslant \ell_0$, i.e. $\|f(t)\|_{C_{\varepsilon_{k_\ell}}} \geqslant (|f_{k_\ell}(t)|/\varepsilon_{k_\ell}) > \|f(t)\|_{C_\varepsilon} - \delta$, whence $\varphi^{\varepsilon}(t) \geqslant \|f(t)\|_{C_\varepsilon}$. The inverse inequality follows from (3.9) and therefore the equality (3.10) is true.

So, the function $\|f(t)\|_{C_\varepsilon} = \varphi^{\varepsilon}(t)$ is summable on I and hence, the mapping

$$F(x) = F(a) + (B)\int_a^x f(t)\,dt \ ,$$

belongs to the class AC_K^s by virtue of Theorem 3.2. □

Remind that the space c_0 doesn't possess the classical (RNP), that is in this case

$$AC_K^s(I, c_0) = \mathcal{I}_B(I, c_0) \subsetneqq AC^s(I, c_0) \ . \tag{\textbf{A}}$$

Note that modifying *mutatus mutandis* proof of Theorem 3.3 respectively compact ellipsoids in ℓ_p $(1 \leqslant p < \infty)$:

$$C_\varepsilon = \left\{ x = (x_k)_1^{\infty} \in \ell_p \ \middle|\ \sum_{k=1}^{\infty} (|x_k|^p/(\varepsilon_k)^p) \leqslant 1 \right\} , \quad \varepsilon_k \to \infty \ ,$$

it can be proved K-Radon-Nikodym property for the spaces ℓ_p $(1 \leqslant p < \infty)$. So, there is valid the following

Theorem 3.4. *The spaces ℓ_p $(1 \leqslant p < +\infty)$ possess K-Radon-Nikodym property. In this case,*

$$AC_K^s(I, \ell_p) = \mathcal{I}_B(I, \ell_p) = AC^s(I, \ell_p) \ . \tag{\textbf{B}}$$

in view of $\ell_p \in (RNP)$ for $1 \leqslant p < +\infty$.

Note that in the work [19] the special sequence spaces, Banach lattices with (RNP), were investigated.

Finally, let's show now that the case $AC_K^s \neq \mathcal{I}_B$ is possible, too.

Example 3.1. Let E_T be space of the all real functions $\xi : T = [0;1] \to \mathbb{R}$ equipped with pointwise convergence topology, $\{\|\cdot\|_t\}_{t\in T}$ is corresponding defining system of seminorms, $\|\xi(\cdot)\|_t = |\xi(t)|$. Denote by E_T^t the subspaces of E_T generated by $\|\cdot\|_t$, by $\widehat{E}_T^t$ their completions respective to factor norms and by φ_t the canonical embeddings of E_T into $\widehat{E}_T^t$. Here

$$\varphi_t(\xi(\cdot)) = \xi(t) \ . \tag{3.11}$$

Remind that Bochner integrability of $f : I = [0;1] \to E_T$ means the same for the all factor mappings from I into $\widehat{E}_T^t$, $t \in T$, i.e., in connection with the case, summability of the all functions $f(\cdot)(t) = f(\cdot, t)$, $t \in T$.

Set $f(s)(t) = f(s,t) = 0$ for $s \leqslant t \leqslant 1$, $f(s,t) = 1/2\sqrt{s-t}$ for $0 < t < s$. Then $\|f(s)\|_t = |f(s,t)| = f(s,t)$, $t \in T$, $f : I \to E_T$.

Next, set $F(s)(t) = F(s,t) = 0$ for $s \leqslant t \leqslant 1$, $F(s,t) = \sqrt{s-t}$ for $0 < t < s$. Let's show that

$$F(s) = (B)\int_0^s f(u)\,du \quad (s \in I)\ . \tag{3.12}$$

a) It's easy to see that

$$F'(s)(t) = f(s,t) \quad as \quad s \in I\backslash\{t\}\ . \tag{3.13}$$

b) Let's check summability of $f(\cdot,t)$ over I. Consider the cut-off functions

$$f^N(s,t) = f(s,t) \quad \text{for} \quad f(s,t) \leqslant N, \quad f^N(s,t) = 0 \quad \text{for} \quad f(s,t) > N.$$

Then $f^N(\cdot,t)$, $t \in T$, are summable over I and, denoting by $t_N : f(t_N, t) = N$ and taking into account (3.13), we obtain

$$(R)\int_0^1 f^N(s,t)\,ds = (R)\int_{t_N}^1 f(s,t)\,ds = \sqrt{1-t} - \sqrt{t_N - t} \quad \text{as} \quad N \to \infty\ .$$

Hence, by B. Levy theorem $f(\cdot,t)$ is summable over I. By virtue of (3.11), now the equality (3.12) follows from

$$\varphi_t(F(s)) = F(s,t) = \int_0^s f(u,t)\,du = \int_0^s \varphi_t(f(u))\,du \quad (\forall s \in I)\ .$$

Let's check, at last, nowhere differentiability of F over I. Direct calculation shows

$$\left\| \frac{F(s+\triangle s) - F(s)}{\triangle s} \right\|_{t=s} = \left| \frac{F(s+\triangle s) - F(s)}{\triangle s} \right| = \frac{1}{2\sqrt{\triangle s}} \to \infty \quad \text{as} \quad \triangle s \to +\,0\ .$$

Therefore, by Corollary 2.2, $F \notin AC^s_K(I, E_T)$.

Since in the analyzed case $F \in AC^s(I, E_T)$ means that $F(\cdot, t)$ are absolutely continuous for $t \in T$ then

$$AC^s_K(I, E_T) \subsetneqq \mathcal{I}_B(I, E_T) = AC^s(I, E_T)\ . \tag{\textbf{C}}$$

Combining the relations (A) and (C) for direct sum of the corresponding spaces, it can be obtained also the relation

$$AC^s_K(I, E) \subsetneqq \mathcal{I}_B(I, E) \subsetneqq AC^s(I, E)\ . \tag{\textbf{D}}$$

So, all the possible relations (A)–(D) are realized.

Final remarks

Comparing Remark 3.1 with Theorem 3.2 leads to the following natural hypotheses for the case of Banach (and, possible, even Fréchet) space E:

1. $F \in AC^s_K(I, E)$ if and only if $F \in \mathcal{I}_B(I, E)$.
2. $F \in AC^s_K(I, E)$ if and only if $F \in \mathcal{I}_B(I, E_C)$ for some $C \in \mathcal{C}(E)$ (namely, for $F \in AC^s(I, E_{C'})$ and $E_{C'} \hookrightarrow\hookrightarrow E_C$).

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Taurida National V. Vernadsky University, 4, Vernadsky ave., Simpheropol, 95007, Ukraine
E-mail address: old@crimea.edu

Taurida National V. Vernadsky University, 4, Vernadsky ave., Simpheropol, 95007, Ukraine
E-mail address: fedyor@mail.ru

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