

## BOUNDARY PROBLEMS FOR THE WAVE EQUATION WITH THE LÉVY LAPLACIAN IN SHILOV'S CLASS

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ABSTRACT. We present solutions to some boundary value and initial-boundary value problems for the "wave" equation with the infinite dimensional Lévy Laplacian  $\Delta_L$

$$\frac{\partial^2 U(t, x)}{\partial t^2} = \Delta_L U(t, x)$$

in the Shilov class of functions.

### 1. INTRODUCTION

The theory of linear elliptic and parabolic equations with the Lévy Laplacian is now well developed (see for example [1]).

This paper is devoted to the construction of solution of the boundary value and initial-boundary value problems for the equation

$$\frac{\partial^2 U(t, x)}{\partial t^2} = \Delta_L U(t, x)$$

with the Lévy Laplacian  $\Delta_L$  ("wave" equation) in fundamental domains of the Shilov functional class.

It should be noted that in the Schilov functional class the Lévy Laplacian is a "derivative" (see (4)). As a result, in this class the equation  $\frac{\partial^2 U(t, x)}{\partial t^2} = \Delta_L U(t, x)$  is reduced to the equation

$$\frac{\partial^2 u(t, (a_1, x)_H, \dots, (a_m, x)_H, \zeta)}{\partial t^2} = \frac{\partial u(t, (a_1, x)_H, \dots, (a_m, x)_H, \zeta)}{\partial \zeta} \Big|_{\zeta = \frac{\|x\|_H^2}{2}}.$$

### 2. PRELIMINARIES

Let  $H$  be a real separable Hilbert space with inner product  $(\cdot, \cdot)_H$  and the norm  $\|\cdot\|_H$ , and let  $F(x)$  be a scalar function defined on  $H$ .

The infinite dimensional Laplacian was introduced by P. Lévy [2]. If  $F$  is twice strongly differentiable at a point  $x_0$  then the Lévy Laplacian of  $F$  in this point is defined (if it exists) by the formula

$$(1) \quad \Delta_L F(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (F''(x_0) f_k, f_k)_H,$$

where  $F''(x)$  is the Hessian of the function  $F(x)$  and  $\{f_k\}_1^\infty$  is a chosen orthonormal basis in  $H$ .

Let  $\Omega$  be a bounded domain in  $H$  (that is, a bounded open set in  $H$ ),  $\bar{\Omega} = \Omega \cup \Gamma$ , where  $\Gamma$  is the boundary of  $\Omega$ . We suppose that

$$(2) \quad \Omega = \{x \in H : 0 \leq Q(x) < R^2\}, \quad \Gamma = \{x \in H : Q(x) = R^2\},$$

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with some  $R \geq 0$ , where  $Q(x)$  is a twice differentiable function such that  $\Delta_L Q(x) = \gamma$  for a positive nonzero constant  $\gamma$ . Such kind of domains are called fundamental.

For example, the domains

$$1) \bar{\Omega} = \{x \in H : \|x\|_H^2 \leq R^2\} \text{ (a ball)}$$

and

$$2) \bar{\Omega} = \{x \in H : (Bx, x)_H \leq R^2\}, \text{ where } B = \gamma E + A, \text{ } E \text{ is the unit operator,}$$

$A$  is a compact linear operator in  $H$  (ellipsoid),  
are fundamental.

We put

$$T(x) = \frac{R^2 - Q(x)}{\gamma}.$$

Obviously, the real valued function  $T(x), x \in H$ , possesses the properties

$$0 < T(x) \leq \frac{R^2}{\gamma} \quad \text{for } x \in \Omega; \quad T(x) = 0 \quad \text{for } x \in \Gamma; \quad \Delta_L T(x) = -1.$$

### 3. THE SHILOV CLASS OF FUNCTIONS

Let  $\mathfrak{C}$  denote the Shilov class of functions [3], that is a set of functions of the form

$$(3) \quad F(x) = f\left((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\right),$$

where the elements  $a_1, \dots, a_m$  belong to  $H$ ,  $f(\xi_1, \dots, \xi_m, \zeta)$  is a real-valued bounded continuous function of  $m+1$  variables, defined and continuous in the domain  $G \subseteq R^{m+1}$ , and

$$x \in \bar{\Omega} \implies \left((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\right) \in G.$$

Denote by  $\mathfrak{C}^*$  the subset of functions from  $\mathfrak{C}$  which are continuously differentiable in  $\frac{\|x\|_H^2}{2}$ . For any  $F \in \mathfrak{C}^*$ , we have [3]

$$(4) \quad \Delta_L F(x) = \frac{\partial f((a_1, x)_H, \dots, (a_m, x)_H, \zeta)}{\partial \zeta} \Big|_{\zeta = \frac{\|x\|_H^2}{2}}, \quad x \in H.$$

Note that in the Shilov class functions the Lévy Laplacian does not depend on the choice of a basis.

**3.1. Initial problem with homogeneous boundary condition.** 1. First we consider the problem

$$(5) \quad \frac{\partial^2 V_1(t, x)}{\partial t^2} = \Delta_L V_1(t, x) \quad (t > 0),$$

$$(6) \quad V_1(0, x) = F(x),$$

$$(7) \quad V_1(t, 0) = 0,$$

where  $F(x)$  is a given function, and  $V_1(t, x) \in C([0, \infty), H) \cap C^{2,1}((0, \infty), H)$ .

**Theorem 1.** *Let  $F \in \mathfrak{C}^*$ ,*

$$F(x) = f\left((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\right),$$

where  $f(\xi_1, \dots, \xi_m, \zeta)$  is a bounded, continuous, twice differentiable in  $\zeta$  function on  $R^{m+1}$ ,  $a_k \in H, k = 1, \dots, m$ . Assume also  $f(\xi_1, \dots, \xi_m, 0) = 0$ .

Then

$$(8) \quad V_1(t, x) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f\left((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) e^{-z^2} dz$$

is a solution of problem (5)–(7).

*Proof.* Let us rewrite the function  $F(x)$  in the form  $F(x) = f\left(P_a x, \frac{\|x\|_H^2}{2}\right)$ , where  $P_a$  is a projection to an  $m$ -dimensional subspace spanned on vectors  $a_1, \dots, a_m$ .

It follows from (8) that

$$\begin{aligned}
 \frac{\partial^2 V_1(t, x)}{\partial t^2} &= \frac{2f'_\zeta(P_a x, 0)\sqrt{\frac{\|x\|_H^2}{2}}}{\sqrt{\pi} t} \exp\left(\frac{-t^2}{2\|x\|_H^2}\right) \\
 &- \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^\infty f'_\zeta\left(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) \frac{1}{2z^2} e^{-z^2} dz \\
 &+ \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^\infty t^2 f''_{\zeta\zeta}\left(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) \frac{1}{4z^4} e^{-z^2} dz \quad (t > 0).
 \end{aligned}
 \tag{9}$$

Taking into account (4), we deduce from (8) that

$$\Delta_L V_1(t, x) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^\infty f'_\zeta\left(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) e^{-z^2} dz,$$

because  $\Delta_L \frac{\|x\|_H^2}{2} = 1$ ,  $\Delta_L \left(\frac{\|x\|_H^2}{2}\right)^{-1/2} = -\frac{1}{2} \left(\frac{\|x\|_H^2}{2}\right)^{-3/2}$ .

Applying the integration by parts formula, we derive

$$\begin{aligned}
 \Delta_L V_1(t, x) &= -\frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^\infty f'_\zeta\left(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) \frac{1}{2z} d\left(e^{-z^2}\right) \\
 &= \frac{2f'_\zeta(P_a x, 0)\sqrt{\frac{\|x\|_H^2}{2}}}{\sqrt{\pi} t} \exp\left(\frac{-t^2}{2\|x\|_H^2}\right) \\
 &- \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^\infty f'_\zeta\left(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) \frac{1}{2z^2} e^{-z^2} dz \\
 &+ \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^\infty t^2 f''_{\zeta\zeta}\left(P_a x, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) \frac{1}{4z^4} e^{-z^2} dz.
 \end{aligned}
 \tag{10}$$

Substituting (9) and (10) into (5), we obtain an identity.

Setting  $t = 0$  in (8), we obtain  $V_1(0, x) = F(x)$ , and setting  $x = 0$  in (8) we obtain  $V_1(t, 0) = 0$ . □

2. Now consider the auxiliary problem

$$\frac{\partial^2 V_2(t, x)}{\partial t^2} = \Delta_L V_2(t, x) \quad (t > 0, x \in \Omega),
 \tag{11}$$

$$V_2(0, x) = 0,
 \tag{12}$$

$$V_2(t, x)\Big|_\Gamma = h(t),
 \tag{13}$$

where  $h(t) = V_1(t, x)|_\Gamma$ ,  $V_1(t, x)$  is the solution of problem (5)–(7);  $V_2(t, x) \in C([0, \infty), \bar{\Omega}) \cap C^{2,1}((0, \infty), \Omega)$ .

**Theorem 2.** *Let the conditions of Theorem 1 are satisfied. Let  $f\left(\xi_1, \dots, \xi_m, \frac{R^2}{2}\right) = 0$ . Suppose, in addition, that the Fourier sine-transformation  $\hat{h}(\beta) = \sqrt{\frac{2}{\pi}} \int_0^\infty h(\tau) \sin \beta \tau d\tau$*

of the function

$$h(t) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{R^2}{2}}}}^{\infty} f\left((a_1, x)_H, \dots, (a_m, x)_H, \frac{R^2}{2} - \frac{t^2}{4z^2}\right) e^{-z^2} dz$$

exists, and  $\beta^2 \hat{h}(\beta) \exp(\frac{R^2 \beta^2}{2}) \in L_1(0, \infty)$ .

We put also  $\bar{\Omega} = \{x \in H : \|x\|_H^2 \leq R^2\}$ .

Then

$$(14) \quad V_2(t, x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{h}(\beta) e^{T(x)\beta^2} \operatorname{sint} \beta d\beta$$

is a solution of problem (11)–(13).

*Proof.* Since, by the assumptions of the theorem,  $f(P_a x, \frac{R^2}{2}) = 0$ , we have

$$h(0) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} f\left(P_a x, \frac{R^2}{2}\right) e^{-z^2} dz = 0.$$

From (14), by direct computation, we deduce

$$(15) \quad \frac{\partial^2 V_2(t, x)}{\partial t^2} = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \beta^2 \hat{h}(\beta) e^{T(x)\beta^2} \operatorname{sint} \beta d\beta.$$

Taking into account that

$$\Delta_L e^{T(x)\beta^2} = \beta^2 e^{T(x)\beta^2} \Delta_L T(x) = -\beta^2 e^{T(x)\beta^2},$$

we obtain

$$(16) \quad \Delta_L V_2(t, x) = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} \beta^2 \hat{h}(\beta) e^{T(x)\beta^2} \operatorname{sint} \beta d\beta.$$

The substitution of (15) and (16) into (11) gives an identity.

Setting  $t = 0$  in (14), we obtain  $V_2(0, x) = 0$ .

At the boundary  $\Gamma$  we have  $T(x) = 0$  that yields

$$V_2(t, x)|_{\Gamma} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{h}(\beta) \operatorname{sint} \beta d\beta = h(t),$$

where  $h(t) = \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{R^2}{2}}}}^{\infty} f\left(P_a x, \frac{R^2}{2} - \frac{t^2}{4z^2}\right) e^{-z^2} dz$ . □

3. At the end, we consider the initial-boundary value problem with homogeneous boundary condition for the wave equation with the Lévy Laplacian, namely

$$(1) \quad \frac{\partial^2 V(t, x)}{\partial t^2} = \Delta_L V(t, x) \quad (t > 0, x \in \Omega),$$

$$(2) \quad V(0, x) = F(x),$$

$$(3) \quad V(t, x) = 0 \quad \text{on } \Gamma,$$

where  $F(x)$  is a given function, and  $V(t, x) \in C([0, \infty), \bar{\Omega}) \cap C^{2,1}((0, \infty), \Omega)$ .

**Theorem 3.** Suppose, that  $F \in \mathfrak{C}^*$ ,

$$F(x) = f\left((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2}\right)$$

and the conditions of Theorems 1 and 2 are satisfied.

Then

$$(20) \quad \begin{aligned} V(t, x) &= \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f\left((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) e^{-z^2} dz \\ &\quad - \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{h}(\beta) e^{T(x)\beta^2} \operatorname{sint} \beta d\beta, \end{aligned}$$

is a solution of problem (17)–(19).

*Proof.* It follows from (20) and (8), (14) that

$$V(t, x) = V_1(t, x) - V_2(t, x).$$

The function  $V(t, x)$  solves (17) (since  $\frac{\partial^2 V_1(t, x)}{\partial t^2} = \Delta_L V_1(t, x)$ ,  $\frac{\partial^2 V_2(t, x)}{\partial t^2} = \Delta_L V_2(t, x)$ ). It satisfies the initial condition (18) as  $V(0, x) = F(x) - 0 = F(x)$ , and the boundary condition (19), because of  $V(t, x)|_{\Gamma} = h(t) - h(t) = 0$ . So the function  $V(t, x)$  given by (20) is a solution of problem (17)–(19).  $\square$

**3.2. Boundary problem with homogeneous initial condition.** Now we shall deal with the boundary problem

$$(4) \quad \frac{\partial^2 W(t, x)}{\partial t^2} = \Delta_L W(t, x) \quad (t > 0, x \in \Omega),$$

$$(5) \quad W(0, x) = 0,$$

$$(6) \quad W(t, x) = G(t, x) \quad \text{on } \Gamma,$$

with homogeneous boundary condition, where  $G(t, x)$  is a given function, and  $W(t, x) \in C([0, \infty), \bar{\Omega}) \cap C^{2,1}((0, \infty), \Omega)$ .

**Theorem 4.** *Let  $\bar{\Omega}$  be a fundamental domain. Suppose that the function  $G(t, x)$  is twice differentiable with respect to  $t$  and*

$$G(t, x) = g\left(t, (b_1, x)_H, \dots, (b_n, x)_H, \frac{\|x\|_H^2}{2}\right),$$

belongs to  $\mathfrak{C}^*$  for every  $t \in [0, \infty)$ . Here  $g(t, \xi_1, \dots, \xi_n, \zeta)$  is a function on  $R^{n+2}$  such that  $g(0, \xi_1, \dots, \xi_n, \zeta) = 0$ ,  $b_k \in H$ ,  $k = 1, \dots, n$ . It is assumed also that the functions  $t^2 g(t, \xi_1, \dots, \xi_n, \zeta)$  and  $t^2 \exp\left(\frac{R^2 t^2}{\gamma}\right) \hat{g}(t, \xi_1, \dots, \xi_n, \zeta)$  are absolutely Lebesgue integrable in  $t$  over  $[0, \infty)$ .

Then the formula

$$(24) \quad W(t, x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{g}\left(\gamma, (b_1, x)_H, \dots, (b_n, x)_H, T(x) + \frac{\|x\|_H^2}{2}\right) e^{T(x)\gamma^2} \sin t\gamma \, d\gamma,$$

yields a solution of problem (21)–(23).

$$\text{Here } \hat{g}(\gamma, \xi_1, \dots, \xi_n, \zeta) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\tau, \xi_1, \dots, \xi_n, \zeta) \sin \gamma \tau \, d\tau.$$

*Proof.* Let us rewrite the function  $G(t, x)$  in the form  $G(t, x) = g(t, P_b x, \frac{\|x\|_H^2}{2})$ , where  $P_b$  is a projection into the  $n$ -dimensional space spanned over vectors  $b_1, \dots, b_n$ . Since the functions  $t^k g(t, \xi_1, \dots, \xi_n, \zeta)$ ,  $k = 0, 1, 2$ , are absolutely integrable, their Fourier sine-transformation in  $t$  exists.

We deduce from (24)

$$(25) \quad \frac{\partial^2 W(t, x)}{\partial t^2} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \gamma^2 \hat{g}\left(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\right) e^{T(x)\gamma^2} \sin t\gamma \, d\gamma,$$

$$(26) \quad \begin{aligned} \Delta_L W(t, x) &= -\sqrt{\frac{2}{\pi}} \int_0^\infty \gamma^2 \hat{g}\left(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\right) e^{T(x)\gamma^2} \sin t\gamma \, d\gamma \\ &+ \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{g}'_\zeta\left(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\right) e^{T(x)\gamma^2} \sin t\gamma \, d\gamma \left[ \Delta_L T(x) + \Delta_L \frac{\|x\|_H^2}{2} \right] \\ &= -\sqrt{\frac{2}{\pi}} \int_0^\infty \gamma^2 \hat{g}\left(\gamma, P_b x, T(x) + \frac{\|x\|_H^2}{2}\right) e^{T(x)\gamma^2} \sin t\gamma \, d\gamma, \end{aligned}$$

because

$$\Delta_L e^{T(x)\gamma^2} = \gamma^2 e^{T(x)\gamma^2} \Delta_L T(x) = -\gamma^2 e^{T(x)\gamma^2}, \quad \Delta_L T(x) = -1, \quad \Delta_L \frac{\|x\|_H^2}{2} = 1.$$

Substituting (25) and (26) into (21) we derive an identity.

Setting  $t = 0$  in (24), we obtain  $W(0, x) = 0$ .

At the boundary  $\Gamma$  we have  $T(x) = 0$  and  $\|x\|_H^2 = R^2$ , hence (24) yields

$$W(t, x)\Big|_{\Gamma} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{g}\left(\gamma, P_b x, \frac{\|x\|_H^2}{2}\right) \operatorname{sint} \gamma \, d\gamma = g\left(t, P_b x, \frac{\|x\|_H^2}{2}\right) = G(t, x).$$

□

**3.3. Initial-boundary value problem.** Consider the initial-boundary value problem for the wave equation with the Lévy Laplacian

$$(27) \quad \frac{\partial^2 U(t, x)}{\partial t^2} = \Delta_L U(t, x) \quad (t > 0, x \in \Omega),$$

$$(28) \quad U(0, x) = F(x),$$

$$(29) \quad U(t, x) = G(t, x) \quad \text{on } \Gamma,$$

where  $F(x)$ ,  $G(t, x)$  are given functions, and  $U(t, x) \in C([0, \infty), \bar{\Omega}) \cap C^{2,1}((0, \infty), \Omega)$ .

Assume that the conditions of Theorems 3 and 4 are satisfied. The Theorem 3, 4 imply the following assertion

**Corollary.** *The function*

$$\begin{aligned} U(t, x) &= \frac{2}{\sqrt{\pi}} \int_{\frac{t}{2\sqrt{\frac{\|x\|_H^2}{2}}}}^{\infty} f\left((a_1, x)_H, \dots, (a_m, x)_H, \frac{\|x\|_H^2}{2} - \frac{t^2}{4z^2}\right) e^{-z^2} \, dz \\ &\quad - \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{h}(\beta) e^{T(x)\beta^2} \operatorname{sint} \beta \, d\beta \\ &\quad + \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{g}\left(\gamma, (b_1, x)_H, \dots, (b_n, x)_H, \frac{R^2}{2}\right) e^{T(x)\gamma^2} \operatorname{sint} \gamma \, d\gamma, \end{aligned}$$

gives a solution of problem (27)–(29).

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