

ON THE H -RING STRUCTURE OF INFINITE GRASSMANNIANS

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ABSTRACT. The H -ring structure of certain infinite dimensional Grassmannians is discussed using various algebraic and analytical methods but avoiding cellular arguments. These methods allow us to treat these Grassmannians in a greater generality.

INTRODUCTION

Infinite dimensional Grassmannians are often used as realizations of classifying spaces like $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$. The two spaces mentioned above classify KO_0 and K_0 respectively; consequently they possess H -ring structures induced from the direct sum, switch, and tensor product constructions of virtual vector bundles. In standard textbook constructions, the homotopy additive structure of these Grassmannians is often described in explicit terms, see e. g. [4]. However, the existence of the homotopy product map is rather inferred from principles of homotopy theory instead of constructed explicitly. The main tools are approximation, weak equivalences, and universality. One might get an impression that cellular arguments are unavoidable in that respect. But the truth is that the H -ring structure of the infinite Grassmannians is much more an algebraic, or perhaps analytic, matter than a combinatorial one. Our objective here is to work out this structure without obscuring it with cellular topology. First, we discuss the algebraic H -ring structure. We take a ring \mathfrak{A} endowed by a polymetric structure. Then we consider the virtual Grassmannian $\mathcal{G}^{(2)}(\mathfrak{A})$ and the ordinary infinite Grassmannian $\mathcal{G}(\mathfrak{A})$ associated to \mathfrak{A} . We will show that they possess commutative unital H -ring structures. Strictly speaking, this holds if \mathfrak{A} is a commutative ring, but one can formulate this phenomenon in terms of tensor products such that it applies more generally. Second, if \mathfrak{A} is a locally convex algebra, then the algebraic H -ring structure implies a topological, in fact, smooth, H -ring structure. Third, if \mathfrak{A} satisfies somewhat stronger conditions, then the smooth H -structure implies a smooth algebraic H -structure without stabilization. This all applies to $\mathfrak{A} = \mathbb{R}$ and $\mathfrak{A} = \mathbb{C}$, corresponding to the classical cases mentioned above. We conclude the paper with a notion of dimension, and the discussion of Fredholm operators.

1. POLYMETRIC RINGS AND ASSOCIATED MATRIX SPACES

We say that the set Ω is an infinite set of polynomial growth (spg) if it is endowed by a set of real valued functions $\mathcal{S}_\Omega^{-\infty}$ on Ω such that there is a bijection $q : \Omega \rightarrow \mathbb{N}$ so that $\mathcal{S}_\Omega^{-\infty} = q^* \mathcal{S}_\mathbb{N}^{-\infty}$, where $\mathcal{S}_\mathbb{N}^{-\infty}$ is the set of real valued functions of at most polynomial growth on \mathbb{N} . We say that Ω is a finite set of polynomial growth if it is finite, and it is endowed with the set of arbitrary real valued functions $\mathcal{S}_\Omega^{-\infty}$. If Ω_1 and Ω_2 are spg's, then one can naturally construct the spg's $\Omega_1 \dot{\cup} \Omega_2$ and $\Omega_1 \times \Omega_1$. If $\Omega = \Omega_1 \dot{\cup} \Omega_2$ as spg's, then we say that Ω decomposes to Ω_1 and Ω_2 . (An arbitrary set-theoretical decomposition is not sufficient in general.)

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We say that a topological ring \mathfrak{A} is a polymetric ring if

- a) its topology is induced by a family of “seminorms” $p : \mathfrak{A} \rightarrow [0, +\infty)$ such that $p(0) = 0, p(-X) = p(X), p(X + Y) \leq p(X) + p(Y)$;
- b) for each “seminorm” p there exists a “seminorm” \tilde{p} such that $p(XY) \leq \tilde{p}(X)\tilde{p}(Y)$ holds.

This is a large class of topological rings: it includes locally convex algebras just as discrete rings with the convention $p(X) = 1$ for $X \neq 0$. In what follows, \mathfrak{A} is assumed to be a separated, sequentially complete polymetric ring.

Suppose that \mathfrak{A} is a polymetric ring, Ω is an spg. Then we can define the algebra of rapidly decreasing matrices $\mathcal{K}_\Omega(\mathfrak{A})$ and the algebra of matrices of pseudodifferential size $\Psi_\Omega(\mathfrak{A})$; essentially as in [3]. (In the special case when \mathfrak{A} is the discrete ring, the space $\mathcal{K}_\Omega(\mathfrak{A})$ is the space of matrices with finitely many non-zero elements, and $\Psi_\Omega(\mathfrak{A})$ is the space of matrices such that every column and row has only finitely many non-zero elements.) More generally, we can take the spaces of $\Omega' \times \Omega$ matrices $\Psi_{\Omega',\Omega}(\mathfrak{A})$ and $\mathcal{K}_{\Omega',\Omega}(\mathfrak{A})$. (This bold \times is reserved for matrix shape.) If \mathfrak{A} is a polymetric ring, then we may take its unital extension \mathfrak{A}^+ . We will be a bit vague about this construction: In the general case, it may be the group $\mathbb{Z} \oplus \mathfrak{A}$ endowed with the naturally extended structure, but if \mathfrak{A} is a locally convex algebra over \mathbb{K} , then there is no danger in taking $\mathbb{K} \oplus \mathfrak{A}$ with the naturally extended algebra structure. Let, in general, $1_\Omega = \sum_{\omega \in \Omega} \mathbf{e}_{\omega,\omega} \in \Psi_\Omega(\mathfrak{A}^+)$; and let, in general, 0_Ω denote the nullmatrix over Ω . The matrix $A \in \Psi_{\Omega',\Omega}(\mathfrak{A}^+)$ is invertible if there is an element $B \in \Psi_{\Omega,\Omega'}(\mathfrak{A}^+)$ such that $AB = 1_{\Omega'}$ and $BA = 1_\Omega$. We denote the set of those by $\Psi_{\Omega',\Omega}(\mathfrak{A}^+)^*$, and call the them units. The unit group $\mathcal{K}_\Omega(\mathfrak{A})^*$ is the group of invertible elements $1_\Omega + A \in \Psi_\Omega(\mathfrak{A}^+)$, where $A \in \mathcal{K}_\Omega(\mathfrak{A})$, but with topology induced from $\mathcal{K}_\Omega(\mathfrak{A})$. Furthermore, let $\Psi_{\Omega',\Omega}^{(2)}(\mathfrak{A}^+)$ be the space of pairs $\langle B, A \rangle$, where $A, B \in \Psi_{\Omega',\Omega}(\mathfrak{A}^+)$, but $B - A \in \mathcal{K}_{\Omega',\Omega}(\mathfrak{A})$. Its topology is induced jointly from A, B and $B - A$ with respect to the appropriate spaces, respectively. Then the unit group $\Psi_{\Omega',\Omega}^{(2)}(\mathfrak{A}^+)^*$ can be taken. Conjugation induces a continuous map $\text{Ad} : \Psi_{\Omega',\Omega}(\mathfrak{A}^+)^* \times \mathcal{K}_\Omega(\mathfrak{A})^* \rightarrow \mathcal{K}_{\Omega'}(\mathfrak{A})^*$, etc.

Some simple invertible matrices in $\Psi_{\Omega',\Omega}(\mathfrak{A}^+)$ are as follows. Let $r : \Omega \rightarrow \Omega'$ be an isomorphism of spg's. Then we take $\hat{r} = \sum_{\omega \in \Omega} \mathbf{e}_{r(\omega),\omega} \in \Psi_{\Omega',\Omega}(\mathfrak{A}^+)^*$. For $A \in \Psi_\Omega(\mathfrak{A}^+)$ the map $r_* : A \mapsto \hat{r}A\hat{r}^\top$ has the effect that $r_* \left(\sum_{n,m \in \Omega} a_{n,m} \mathbf{e}_{n,m} \right) = \sum_{n,m \in \Omega} a_{n,m} \mathbf{e}_{r(n),r(m)}$. Hence we call such r_* isomorphic relabeling maps. More generally, if $r : \Omega \rightarrow \Omega'$ is only a map of spg's such that $r(\Omega)$ and $\Omega' \setminus r(\Omega)$ decomposes Ω' as spg's, and r induces an isomorphism between the spg structure of Ω and the one of Ω restricted to $r(\Omega)$, then \hat{r} and r_* can be taken. We still call these r_* (not necessarily isomorphic) relabeling maps. Another natural operation is the direct sum of matrices. For example, if $A \in \Psi_\Omega(\mathfrak{A}), B \in \Psi_{\Omega'}(\mathfrak{A})$, then we can consider $A \oplus B \in \Psi_{\Omega \cup \Omega'}(\mathfrak{A})$ which is a colloquial notation for the block matrix $\begin{bmatrix} A & \\ & B \end{bmatrix} \in \Psi_{\Omega \cup \Omega'}(\mathfrak{A})$. Indexing direct sums might be confusing, especially if the construction is iterated. We take disjoint union for the index set, but if we take direct sum of matrices with the same index set Ω , then we might use $\Omega, \Omega', \Omega'', \dots$ for the components, or $\{0\} \times \Omega, \{1\} \times \Omega, \{2\} \times \Omega, \dots$, depending on the situation. Later, when we consider block matrices of (matrices indexed by Ω) indexed by Γ then we consider those matrices as matrices indexed by $\Gamma \times \Omega$. A $\Gamma_2 \times \Gamma_1$ block matrix of $\Omega_2 \times \Omega_1$ matrices will be considered as a $\Gamma_2 \times \Omega_2 \times \Gamma_1 \times \Omega_1$ matrix.

We apply the following notational conventions in unital polymetric rings:

- (a) We write \bar{a} for $1 - a$. (For $a \in \Psi_\Omega(\mathfrak{A}^+)$ it is, of course, $\bar{a} = 1_\Omega - a$.)
- (b) For $\Xi \times \Xi$ matrices over $\Psi_\Omega(\mathfrak{A}^+)$, we use the notation

$$s_{n,m}(a) = 1_{(\Xi \setminus \{n,m\}) \times \Omega} + \bar{a} \mathbf{e}_{n,n} + a \mathbf{e}_{n,m} + a \mathbf{e}_{m,n} + \bar{a} \mathbf{e}_{m,m}.$$

We see that $s_{n,m}(a)$ is an involution if a is an idempotent. It means that we have a partial switch between the n, m positions.

(c) Sometimes we write A^g for gAg^{-1} . We use the abbreviation $A \xrightarrow{g} B$ for $gAg^{-1} = B$. In fact, sometimes we say that g is a morphism between A and B .

(d) If $b - a \in \mathcal{K}_\Omega(\mathfrak{A})$, then we use the notation $a \approx b$.

(e) $\mathfrak{Invol}(\mathfrak{A})$ denotes the subspace of involutions in \mathfrak{A} .

2. VIRTUAL GRASSMANNIANS AND NATURAL OPERATIONS ON THEM

2.1. We define the virtual Grassmannian $\mathcal{G}_\Omega^{(2)}(\mathfrak{A})$ as $\mathfrak{Invol}(\Psi_\Omega^{(2)}(\mathfrak{A}^+))$, i. e. the space of pairs $\langle b, a \rangle$ such that $a, b \in \Psi_\Omega(\mathfrak{A}^+)$, $b - a \in \mathcal{K}_\Omega(\mathfrak{A})$, and a, b are idempotents. For us, such pairs are virtual idempotents $b \text{“} - \text{”} a$; we will call them as pairs of idempotents. We refer to the first term as the leading term, and we refer to the second term as the base term.

(a) The sum of pairs operation is defined for $\langle b, a \rangle \in \mathcal{G}_\Omega^{(2)}(\mathfrak{A})$ and $\langle d, c \rangle \in \mathcal{G}_\Xi^{(2)}(\mathfrak{A})$ as

$$\langle b, a \rangle \oplus \langle d, c \rangle := \langle b \oplus d, a \oplus c \rangle \in \mathcal{G}_{\Omega \dot{\cup} \Xi}^{(2)}(\mathfrak{A}).$$

(b) We define the inverse pair for $\langle b, a \rangle \in \mathcal{G}_\Omega^{(2)}(\mathfrak{A})$ as

$$\langle b, a \rangle^{\text{inv}} := \langle \bar{b}, \bar{a} \rangle \in \mathcal{G}_\Omega^{(2)}(\mathfrak{A}).$$

(c) A very special element is the pair $\mathbf{0} := \langle *, * \rangle$ of 0×0 , i. e. empty, matrices. We call this as the additive neutral element.

These operations satisfy the natural additive associative, commutative, and neutral element identities

$$\begin{aligned} (\langle b, a \rangle \oplus \langle d, c \rangle) \oplus \langle f, e \rangle &\simeq \langle b, a \rangle \oplus (\langle d, c \rangle \oplus \langle f, e \rangle), \\ \langle b, a \rangle \oplus \langle d, c \rangle &\simeq \langle d, c \rangle \oplus \langle b, a \rangle, \\ \langle b, a \rangle &\simeq \langle b, a \rangle \oplus \mathbf{0} \simeq \mathbf{0} \oplus \langle b, a \rangle, \end{aligned}$$

where “ \simeq ” means that we have equality after we make natural identifications in the index sets, i. e. after a particularly simple isomorphic relabeling. What is apparently lacked is a natural additive inverse element identity.

2.2. Let \otimes be a suitable tensor product operation of rings. Again, we will be somewhat vague about the meaning of this term: In general, we may mean a projective tensor product of polymetric rings, but, if \mathfrak{A} is commutative, then we may also consider the ordinary product as a tensor product operation, i. e. tensor product over itself.

(d) For $\langle b, a \rangle \in \mathcal{G}_\Omega^{(2)}(\mathfrak{A})$, $\langle d, c \rangle \in \mathcal{G}_\Xi^{(2)}(\mathfrak{B})$, we define the products of pairs of involutions as

$$\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle := \langle b \otimes d + \bar{b} \otimes c, a \otimes d + \bar{a} \otimes c \rangle \in \mathcal{G}_{\Omega \times \Xi}^{(2)}(\mathfrak{A} \otimes \mathfrak{B}),$$

and

$$\langle b, a \rangle \overrightarrow{\otimes} \langle d, c \rangle := \langle b \otimes d + a \otimes \bar{d}, b \otimes c + a \otimes \bar{c} \rangle \in \mathcal{G}_{\Omega \times \Xi}^{(2)}(\mathfrak{A} \otimes \mathfrak{B}).$$

So, we have two natural product operations, which may be somewhat strange.

(e) Another special element is $\mathbf{1} := \langle 1, 0 \rangle$, where the elements are 1×1 matrices, or rather “scalars”. Again, we will be vague about the ring it is over, we may mean \mathbb{Z} or the base field \mathbb{K} of an algebra.

The natural multiplicative associativity, distributive, and neutral element rules hold

$$\begin{aligned} (\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle) \overleftarrow{\otimes} \langle f, e \rangle &\simeq \langle b, a \rangle \overleftarrow{\otimes} (\langle d, c \rangle \overleftarrow{\otimes} \langle f, e \rangle), \\ \langle b, a \rangle \overleftarrow{\otimes} (\langle d, c \rangle \oplus \langle f, e \rangle) &\simeq (\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle) \oplus (\langle b, a \rangle \overleftarrow{\otimes} \langle f, e \rangle), \\ (\langle b, a \rangle \oplus \langle d, c \rangle) \overleftarrow{\otimes} \langle f, e \rangle &\simeq (\langle b, a \rangle \overleftarrow{\otimes} \langle f, e \rangle) \oplus (\langle d, c \rangle \overleftarrow{\otimes} \langle f, e \rangle), \\ \langle b, a \rangle &\simeq \langle b, a \rangle \overleftarrow{\otimes} \mathbf{1} \simeq \mathbf{1} \overleftarrow{\otimes} \langle b, a \rangle; \end{aligned}$$

and similarly for the other product. We also see that

$$\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle \simeq \langle d, c \rangle \overrightarrow{\otimes} \langle b, a \rangle.$$

What we clearly miss is the equivalence of the two product operations, which is equivalent to the problem of multiplicative commutativity.

2.3. We must look for a weaker equivalence relation in order to get the missing identities. That will be a notion of algebraic homotopy. Let us define $\mathbf{0}_\Xi = \langle 0_\Xi, 0_\Xi \rangle$ and $\mathbf{0}'_\Xi = \langle 1_\Xi, 1_\Xi \rangle$, in general. We will also use the notation $\mathbf{I}_\Xi = \langle 1_\Xi, 1_\Xi \rangle$ but in different context. We say that the maps $f_1 : X \rightarrow \mathcal{G}_{\Omega_1}^{(2)}(\mathfrak{A})$ and $f_2 : X \rightarrow \mathcal{G}_{\Omega_2}^{(2)}(\mathfrak{A})$ are algebraically homotopic if there are index sets $\Omega_{10}, \Omega_{11}, \Omega_{20}, \Omega_{21}$, and a map

$$g : X \rightarrow \Psi_{\Omega_2 \dot{\cup} \Omega_{20} \dot{\cup} \Omega_{21}, \Omega_1 \dot{\cup} \Omega_{10} \dot{\cup} \Omega_{11}}^{(2)}(\mathfrak{A})$$

which allows a multiplicative inverse g^{-1} such that

$$f_1 \oplus \mathbf{0}_{\Omega_{10}} \oplus \mathbf{0}'_{\Omega_{11}} \xrightarrow{g} f_2 \oplus \mathbf{0}_{\Omega_{20}} \oplus \mathbf{0}'_{\Omega_{21}};$$

i. e. if after stabilization the values are conjugate. We denote this as $f_1 \simeq_{\text{alg}} f_2$. This is an equivalence relation: if $f_2 \simeq_{\text{alg}} f_3$ is realized by a similar map h , then

$$f_1 \oplus \mathbf{0}_{\Omega_{10} \dot{\cup} \Omega_{20}} \oplus \mathbf{0}'_{\Omega_{11} \dot{\cup} \Omega_{21}} \xrightarrow{(h \oplus \mathbf{I}_{\Omega_{20} \dot{\cup} \Omega_{21}})(g \oplus \mathbf{I}_{\Omega_{20} \dot{\cup} \Omega_{21}})} f_3 \oplus \mathbf{0}_{\Omega_{20} \dot{\cup} \Omega_{30}} \oplus \mathbf{0}'_{\Omega_{21} \dot{\cup} \Omega_{31}}$$

indeed. So, algebraic homotopy is a combination of equivalence by stabilization and equivalence by conjugation. In what follows, dependence on X will often be suppressed.

Using this notion of equivalence, which generalizes natural equivalence, we may set out to demonstrate the missing identities of additive inverse and equality of products. However, weakening the equivalence relation introduces further problems. When we consider the generalization of classical structures up to homotopy, we have to worry not only about the classical identities

$$\text{Expr}_1(f_1, \dots, f_n) \simeq_{\text{alg}} \text{Expr}_2(f_1, \dots, f_n),$$

but also about the compatibility of the operations with algebraic homotopy, i. e.

$$\forall i f_i \simeq_{\text{alg}} f'_i \Rightarrow \text{Op}(f_1, \dots, f_n) \simeq_{\text{alg}} \text{Op}(f'_1, \dots, f'_n).$$

Homotopy compatibility is rather trivial in the topological setting but it is less trivial in the algebraic setting. Nevertheless, we can reduce this problem:

- (i) We can check compatibility in the variables separately.
- (ii) Even there, it is sufficient to check it in two special cases: First we must check invariance for stabilization, i. e. the case $f'_i = f \oplus \mathbf{0}_{\Xi_0} \oplus \mathbf{0}'_{\Xi_1}$, and, second, to check conjugation invariance when $f_i \xrightarrow{g} f'_i$.

Taking direct sum of operators, we can see easily that the sum operation is compatible with algebraic homotopy. We do not have to worry about the homotopy compatibility of the operations $\mathbf{0}$ and $\mathbf{1}$. The homotopy compatibility of the additive inverse and the product are left to be demonstrated, although these problems are not equally hard. We summarize what are the identities we want to prove:

- (1) Additive inverse.
- (2) Homotopy compatibility of the inverse.
- (3) Equality of products (or multiplicative commutativity).
- (4) Homotopy compatibility of the products.

3. TOOLS: REGULARIZATION AND TAMING

In this section, we introduce some tools to deal with algebraic homotopy effectively. Regarding the definition, one might believe that we allow excessively large classes of objects and morphisms in our Grassmannians. We will show that this is not the case. By “regularization”, one can reduce the variety of base objects, and by “taming”, one can replace the morphisms by smooth ones.

Lemma 3.1. (Virtual cancellation). *For $a \in \Psi_\Omega(\mathfrak{A}^+)$, we have $\langle a, a \rangle \simeq_{\text{alg}} \mathbf{0}$.*

Remark. The exact meaning of this statement is the existence of an algebraic homotopy of functions $\langle \text{id}_{\mathfrak{Invol}(\Psi_\Omega^{(2)}(\mathfrak{A}^+))}, \text{id}_{\mathfrak{Invol}(\Psi_\Omega^{(2)}(\mathfrak{A}^+))} \rangle \simeq_{\text{alg}} \mathbf{0}$ with domain $\mathfrak{Invol}(\Psi_\Omega^{(2)}(\mathfrak{A}^+))$, but we allow the colloquially of using variables instead of functions here, and in the future.

Proof. In terms of $(\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z}$ and $\mathbb{Z} \times (\mathbb{Z} + \frac{1}{2})$ block matrices, let us consider

$$\mathbf{B}(a) = \sum_{n \in \mathbb{Z}} a \mathbf{e}_{n-\frac{1}{2}, n} + \bar{a} \mathbf{e}_{n+\frac{1}{2}, n}, \quad \text{and} \quad \mathbf{B}(a)^{-1} = \sum_{n \in \mathbb{Z}} a \mathbf{e}_{n, n-\frac{1}{2}} + \bar{a} \mathbf{e}_{n, n+\frac{1}{2}}.$$

If the index set $\{0\} \times \Omega$ is replaced with Ω , then it yields

$$1_{\mathbb{Z} \times \Omega} \oplus a \oplus 0_{\mathbb{Z} \times \Omega} \xrightarrow{\mathbf{B}(a)} 1_{(-\frac{1}{2}-\mathbb{N}) \times \Omega} \oplus 0_{(\frac{1}{2}+\mathbb{N}) \times \Omega}.$$

After doubling the terms, it provides an algebraic homotopy as required. \square

3.2. Regularization. For $\langle b, a \rangle \in \mathcal{G}_\Omega^{(2)}(\mathfrak{A})$, we define its regularized matrix as the $\{0, 1\} \times \Omega$ block matrix

$$\mathbf{R}\langle b, a \rangle := s_{01}(a)(b \oplus \bar{a})s_{01}(a) = \begin{bmatrix} \bar{a}(b-a)\bar{a} & \bar{a}(b-a)a \\ a(b-a)\bar{a} & 1_\Omega + a(b-a)a \end{bmatrix}.$$

Then $\mathbf{R}\langle b, a \rangle \approx \mathbf{R}\langle a, a \rangle = 0_\Omega \oplus 1_\Omega = \begin{bmatrix} 0_\Omega & \\ & 1_\Omega \end{bmatrix}$, so we can define the regularized pair

$$\mathbf{R}\langle b, a \rangle := \langle \mathbf{R}\langle b, a \rangle, 0_\Omega \oplus 1_\Omega \rangle.$$

The trivial pair is

$$\mathbf{R}\mathbf{0}_\Omega = \langle 0_\Omega \oplus 1_\Omega, 0_\Omega \oplus 1_\Omega \rangle.$$

3.3. Regularization of morphisms. Let $\langle \psi, \phi \rangle \in \Psi_{\Omega', \Omega}^{(2)}(\mathfrak{A}^+)^*$, and let $a \in \Psi_\Omega(\mathfrak{A}^+)$ be an idempotent. The regularized morphism is the $\{0, 1\} \times \Omega' \times \{0, 1\} \times \Omega$ block matrix

$$\mathbf{R}\langle \psi, \phi \rangle, a := s_{01}(a^\phi)(\psi \oplus \phi)s_{01}(a).$$

Then $\mathbf{R}\langle \langle \psi, \phi \rangle, a \rangle \approx \mathbf{R}\langle \langle \phi, \phi \rangle, a \rangle = \phi \oplus \phi$, so we can define

$$\mathbf{R}\langle \langle \psi, \phi \rangle, a \rangle := \langle \mathbf{R}\langle \langle \psi, \phi \rangle, a \rangle, \phi \oplus \phi \rangle.$$

It yields

$$\mathbf{R}\langle b, a \rangle \xrightarrow{\mathbf{R}\langle \langle \psi, \phi \rangle, a \rangle} \mathbf{R}\langle b^\psi, a^\phi \rangle.$$

Based upon this, it is also reasonable to write $\mathbf{R}\langle \langle \psi, \phi \rangle_{\langle b, a \rangle} \rangle$ instead of $\mathbf{R}\langle \langle \psi, \phi \rangle, a \rangle$, etc. even if there is no dependence on b .

Lemma 3.4. *\mathbf{R} is an operation algebraically homotopic to the identity*

$$\langle b, a \rangle \simeq_{\text{alg}} \mathbf{R}\langle b, a \rangle.$$

Proof. By cancellation and conjugation $\langle b, a \rangle \simeq_{\text{alg}} \langle b, a \rangle \oplus \langle \bar{a}, \bar{a} \rangle \xrightarrow{\langle s_{12}(a), s_{12}(a) \rangle} \mathbf{R}\langle b, a \rangle$. \square

The transitivity of algebraic homotopy automatically ensures that \mathbf{R} is compatible with algebraic homotopy. Nevertheless, one can also show this directly, using the straightforward compatibility with direct sums and regularized morphisms. Ultimately, by regularization, we can bring the base terms into simple form. Another natural expectation is that morphisms should not deviate much from identity. This can be achieved as follows.

3.5. Translation. Using $\{0, 0', 0''\} \times \Omega$ block matrices we set

$$\mathbf{H}\langle b, a \rangle := s_{0'0''}(a) s_{0'0''}(b) s_{00'}(b) s_{00'}(a).$$

Then $\mathbf{H}\langle b, a \rangle \approx \mathbf{H}\langle a, a \rangle = 1_{\{0,0',0''\} \times \Omega}$, so we can define

$$\mathbf{H}\langle b, a \rangle := \langle \mathbf{H}\langle b, a \rangle, 1_{\{0,0',0''\} \times \Omega} \rangle.$$

It yields

$$\langle b \oplus \bar{a} \oplus a, a \oplus \bar{a} \oplus a \rangle \xleftarrow{\mathbf{H}\langle b, a \rangle} \langle a \oplus \bar{a} \oplus b, a \oplus \bar{a} \oplus a \rangle.$$

3.6. Regularized translation. The regularized version is given by $\{0, 1, 0', 1', 0'', 1''\} \times \Omega$ block matrices as follows: Let

$$\mathbf{HR}\langle b, a \rangle := s_{01}(a) s_{0'1'}(\bar{a}) s_{0''1''}(a) s_{0'0''}(a) s_{0'0''}(b) s_{00'}(b) s_{00'}(a) s_{01}(a) s_{0'1'}(\bar{a}) s_{0''1''}(a).$$

Then $\mathbf{HR}\langle b, a \rangle \approx \mathbf{HR}\langle a, a \rangle = 1_{\{0,1,0',1',0'',1''\} \times \Omega}$, which allows us to define

$$\mathbf{HR}\langle b, a \rangle := \langle \mathbf{HR}\langle b, a \rangle, 1_{\{0,1,0',1',0'',1''\} \times \Omega} \rangle.$$

This yields

$$\mathbf{R}\langle b, a \rangle \oplus \mathbf{R}0_\Omega \oplus \mathbf{R}0_\Omega \xleftarrow{\mathbf{HR}\langle b, a \rangle} \mathbf{R}0_\Omega \oplus \mathbf{R}0_\Omega \oplus \mathbf{R}\langle b, a \rangle.$$

3.7. Taming. If $\langle b, a \rangle \in \mathcal{G}_\Omega^{(2)}(\mathfrak{A})$, $\langle \psi, \phi \rangle \in \Psi_\Omega^{(2)}(\mathfrak{A}^+)^*$, then let

$$\mathbf{T}(\langle \psi, \phi \rangle_{\langle b, a \rangle}) := (\psi \oplus 1_\Omega \oplus 1_\Omega) \mathbf{H}\langle b, a \rangle (\phi^{-1} \oplus 1_\Omega \oplus 1_\Omega) (\mathbf{H}\langle b, a \rangle)^{-1}.$$

One can see that $\mathbf{T}(\langle \psi, \phi \rangle_{\langle b, a \rangle}) \approx \mathbf{T}(\langle \phi, \phi \rangle_{\langle a, a \rangle}) = 1_{\{0,0',0''\} \times \Omega}$, so we can define

$$\mathbf{T}(\langle \psi, \phi \rangle_{\langle b, a \rangle}) := \langle \mathbf{T}(\langle \psi, \phi \rangle_{\langle b, a \rangle}), 1_{\{0,0',0''\} \times \Omega} \rangle.$$

If ϕ commutes with a , then

$$\langle b \oplus \bar{a} \oplus a, a \oplus \bar{a} \oplus a \rangle \xrightarrow{\mathbf{T}(\langle \psi, \phi \rangle_{\langle b, a \rangle})} \langle b^\psi \oplus \bar{a} \oplus a, a \oplus \bar{a} \oplus a \rangle.$$

In particular, using the variant

$$\mathbf{T}'(\langle \psi, \phi \rangle_{\langle b, a \rangle}) = \langle s_{0'0''}(a) \mathbf{T}(\langle \psi, \phi \rangle_{\langle b, a \rangle}) s_{0'0''}(a), 1_{\{0,0',0''\} \times \Omega} \rangle$$

we obtain the following

Corollary 3.8. (Stable taming). $\langle b, a \rangle \xrightarrow{\langle \psi, \phi \rangle} \langle \tilde{b}, a \rangle$ implies

$$\langle b \oplus 1_\Omega \oplus 0_\Omega, a \oplus 1_\Omega \oplus 0_\Omega \rangle \xrightarrow{\mathbf{T}'(\langle \psi, \phi \rangle_{\langle b, a \rangle})} \langle \tilde{b} \oplus 1_\Omega \oplus 0_\Omega, a \oplus 1_\Omega \oplus 0_\Omega \rangle,$$

where the conjugating base term is $1_{\{0,0',0''\} \times \Omega}$.

This amounts to the statement that, up to stabilization, all algebraic homotopies can be realized by smooth morphisms whenever they have a chance.

3.9. Regularized taming. The commutation assumption is satisfied automatically if we apply taming after regularization. In terms of $\{0, 1, 0', 1', 0'', 1''\} \times \Omega$ block matrices, let

$$\begin{aligned} \mathbf{TR}(\langle \psi, \phi \rangle_{\langle b, a \rangle}) &:= (\mathbf{R}(\langle \psi, \phi \rangle, a) \oplus 1_{\{0,1\} \times \Omega} \oplus 1_{\{0,1\} \times \Omega}) \mathbf{HR}\langle b, a \rangle \\ &\quad \times (\mathbf{R}(\langle \phi, \phi \rangle, a) \oplus 1_{\{0,1\} \times \Omega} \oplus 1_{\{0,1\} \times \Omega}) (\mathbf{HR}\langle b, a \rangle)^{-1}. \end{aligned}$$

Then $\mathbf{TR}(\langle \psi, \phi \rangle_{\langle b, a \rangle}) \approx \mathbf{TR}(\langle \phi, \phi \rangle_{\langle a, a \rangle}) = 1_{\{0,1,0',1',0'',1''\} \times \Omega}$, so we can set

$$\mathbf{TR}(\langle \psi, \phi \rangle_{\langle b, a \rangle}) := \langle \mathbf{TR}(\langle \psi, \phi \rangle_{\langle b, a \rangle}), 1_{\{0,1,0',1',0'',1''\} \times \Omega} \rangle.$$

It yields

$$\mathbf{R}\langle b, a \rangle \oplus \mathbf{R}\mathbf{0}_\Omega \oplus \mathbf{R}\mathbf{0}_\Omega \xrightarrow{\mathbf{TR}(\langle \psi, \phi \rangle_{\langle b, a \rangle})} \mathbf{R}\langle b^\psi, a^\phi \rangle \oplus \mathbf{R}\mathbf{0}_\Omega \oplus \mathbf{R}\mathbf{0}_\Omega.$$

3.10. Regular relabeling. Let us mention one last operation homotopic to the identity, which, however, depends on the index set. Suppose that $\theta : \Omega \rightarrow \mathbb{N}$ is a relabeling map. Then consider the operation $\mathbf{R}\theta_*$, which is the composition of relabeling in both components followed by regularization. From the preceding discussion, it is clear that the map

$$\begin{aligned} \mathbf{R}\theta_* : \mathcal{G}_\Omega^{(2)}(\mathfrak{A}) &\rightarrow \mathcal{G}_{\{0,1\} \times \mathbb{N}}^{(2)}(\mathfrak{A}) \\ \langle b, a \rangle &\mapsto \langle \mathbf{R}\langle \theta_* b, \theta_* a \rangle, \mathbf{R}\mathbf{0}_\mathbb{N} \rangle \end{aligned}$$

is homotopic to the identity. In fact, the target can be considered to be the smaller space

$$\mathcal{G}_\mathbb{N}(\mathfrak{A}) := \{ \langle b, \mathbf{R}\mathbf{0}_\mathbb{N} \rangle : b \in \Psi_{\{0,1\} \times \mathbb{N}}(\mathfrak{A}^+), b^2 = b, b \approx \mathbf{R}\mathbf{0}_\mathbb{N} \},$$

the standard infinite single-space Grassmannian.

3.11. On polynomial constructions. We may observe that our constructions are all finite matrix polynomials in terms of the initial data, except the conjugating matrices $\mathbf{B}(a)$ in the proof of Lemma 3.1. But even that term was a matrix of finite Toeplitz type. However, even conjugating matrices of finitely multiple Toeplitz type will become essentially finite matrices if we apply the taming construction, because the finitely many components of diagonal type will cancel out, leaving an infinite but inert part of identity on stabilization terms. As a consequence, if $\langle b, a \rangle, \langle \tilde{b}, \tilde{a} \rangle \in \mathcal{G}_\Omega^{(2)}(\mathfrak{A})$ and

$$\mathbf{R}\langle b, a \rangle \simeq_{\text{alg}} \mathbf{R}\langle \tilde{b}, \tilde{a} \rangle$$

such that \simeq_{alg} is realized by a of finitely multiple Toeplitz type block matrix, then due to the universally applicable regularized taming construction

$$\mathbf{R}\langle b, a \rangle \simeq_{\text{alg}} \mathbf{R}\mathbf{R}\langle b, a \rangle \simeq_{\text{alg}} \mathbf{R}\mathbf{R}\langle \tilde{b}, \tilde{a} \rangle \simeq_{\text{alg}} \mathbf{R}\langle \tilde{b}, \tilde{a} \rangle,$$

where all \simeq_{alg} are realized by finite block matrices. That amounts to the fact, which will extend to our later experience, and which can also be checked case by case, that the regularized operations and their related algebraic homotopies can be realized by finite block matrices in terms of the initial data.

4. ESTABLISHING THE ALGEBRAIC H -RING STRUCTURE

4.1. Rule of the additive inverse. In terms of $\{0, 0'\} \times \Omega$ block matrices, we have

$$\langle b, a \rangle_\Omega \oplus \langle b, a \rangle_\Omega^{\text{inv}} \xrightarrow{\langle s_{00'}(b), s_{00'}(a) \rangle} \langle 0_\Omega \oplus 1_\Omega, 0_\Omega \oplus 1_\Omega \rangle \simeq_{\text{alg}} \mathbf{0}.$$

4.2. Compatibility of the additive inverse. The stabilization part follows from

$$(\langle b, a \rangle \oplus \mathbf{0}_{\Omega_0} \oplus \mathbf{0}'_{\Omega_1})^{\text{inv}} = \langle \bar{b}, \bar{a} \rangle \oplus \mathbf{0}'_{\Omega_0} \oplus \mathbf{0}_{\Omega_1} \simeq_{\text{alg}} \langle \bar{b}, \bar{a} \rangle = \langle b, a \rangle^{\text{inv}}.$$

The conjugation part is obvious from $\langle b, a \rangle^{\text{inv}} \xrightarrow{\langle \psi, \phi \rangle} \langle b^\psi, a^\phi \rangle^{\text{inv}}$.

4.3. Commutativity of the product. Using $\{0, 1, 0', 1'\} \times \Omega \times \Xi$ matrices, we set

$$\begin{aligned} \overleftarrow{\mathbf{C}}(\langle b, a \rangle, \langle d, c \rangle) \\ := s_{10}(b \otimes c + a \otimes \bar{c}) s_{10'}(\bar{a} \otimes \bar{c}) s_{10}(\overline{b \otimes d}) s_{10'}(\bar{a} \otimes \bar{c}) s_{10'}(\bar{a} \otimes \bar{c}) s_{10}(a \otimes d + \bar{a} \otimes c). \end{aligned}$$

One can see that

$$\overleftarrow{\mathbf{C}}(\langle b, a \rangle, \langle d, c \rangle) \approx \overleftarrow{\mathbf{C}}(\langle a, a \rangle, \langle c, c \rangle) = s_{00'}(\bar{a} \otimes \bar{c}) s_{11'}(\bar{a} \otimes \bar{c}),$$

hence one can define

$$\overleftarrow{\mathbf{C}}(\langle b, a \rangle, \langle d, c \rangle) := \overleftarrow{\mathbf{C}}(\langle b, a \rangle, \langle d, c \rangle), s_{00'}(\bar{a} \otimes \bar{c}) s_{11'}(\bar{a} \otimes \bar{c}).$$

Then, computation yields that

$$\mathbf{R}(\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle) \oplus \mathbf{R}\mathbf{0}_{\Omega \times \Xi} \xrightarrow{\overleftarrow{\mathbf{C}}(\langle b, a \rangle, \langle d, c \rangle)} \mathbf{R}(\langle b, a \rangle \overrightarrow{\otimes} \langle d, c \rangle) \oplus \mathbf{R}\mathbf{0}_{\Omega \times \Xi},$$

which demonstrates commutativity.

4.4. Compatibility of the product. First, we prove the compatibility of $\overleftarrow{\otimes}$ in the second variable. The stabilization part follows from distributivity and

$$\langle b, a \rangle \overleftarrow{\otimes} \langle 1_{\Xi_1} \oplus 0_{\Xi_0}, 1_{\Xi_1} \oplus 0_{\Xi_0} \rangle = \langle 1_{\Omega \times \Xi_1} \oplus 0_{\Omega \times \Xi_0}, 1_{\Omega \times \Xi_1} \oplus 0_{\Omega \times \Xi_0} \rangle.$$

The conjugation part follows from

$$\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle \xrightarrow{\langle b \otimes \theta + \bar{b} \otimes \chi, a \otimes \theta + \bar{a} \otimes \chi \rangle} \langle b, a \rangle \overleftarrow{\otimes} \langle d^\theta, c^\chi \rangle.$$

One can do the compatibility of $\overrightarrow{\otimes}$ in the first variable in the same way. But then, the equivalence of $\overleftarrow{\otimes}$ and $\overrightarrow{\otimes}$, from the previous point, implies compatibility in each variable.

Remark 4.5. One might prefer the variant additive inverse, the classical switch operation

$$\langle b, a \rangle^{\text{inv}'} = \langle a, b \rangle.$$

In order to show equivalence to the usual additive inverse, it is sufficient to show that the variant operation

$$\langle b, a \rangle' = \langle \bar{a}, \bar{b} \rangle$$

is algebraically homotopic to the identity operation. But this follows from

$$\mathbf{R}\langle a, b \rangle \xrightarrow{\langle s_{01}(b) s_{01}(a), 1_{\{0,1\} \times \Omega} \rangle} \mathbf{R}\langle a, b \rangle'.$$

Corollary 4.6. For idempotents $b \approx \tilde{b} \approx a$, we have $\langle b, \tilde{b} \rangle \simeq_{\text{alg}} \langle b, a \rangle \oplus \langle \tilde{b}, a \rangle^{\text{inv}}$.

Proof. Indeed, we see $\langle b, \tilde{b} \rangle \simeq_{\text{alg}} \langle a \oplus \bar{a} \oplus b, a \oplus \bar{a} \oplus \tilde{b} \rangle \xrightarrow{\mathbf{H}(b, a)} \langle b \oplus \bar{a} \oplus a, a \oplus \bar{a} \oplus \tilde{b} \rangle \simeq_{\text{alg}} \langle b, a \rangle \oplus \langle a, \tilde{b} \rangle \simeq_{\text{alg}} \langle b, a \rangle \oplus \langle \tilde{b}, a \rangle^{\text{inv}}$. \square

Remark 4.7. In classical algebraic K -theory, the class of $\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle$ is $[b \otimes d + \bar{b} \otimes c] - [a \otimes d + \bar{a} \otimes c] = [b][d] + ([1_\Omega] - [b])[c] - [a][d] - ([1_\Omega] - [a])[c] = ([b] - [a])([d] - [c])$, i. e. the product of the classes of $\langle b, a \rangle$ and $\langle d, c \rangle$.

4.8. Summary. We have proved that the functor

$$\Omega, \mathfrak{A} \mapsto \mathcal{G}_\Omega^{(2)}(\mathfrak{A})$$

is a commutative unital algebraic homotopy ring. This is a very flexible construction, but it has some strange features:

- Addition changes the index sets $\Omega_1, \Omega_2 \mapsto \Omega_1 \dot{\cup} \Omega_2$.
- Multiplication also changes the index sets $\Omega_1, \Omega_2 \mapsto \Omega_1 \times \Omega_2$, but it also changes the rings $\mathfrak{A}, \mathfrak{B} \rightarrow \mathfrak{A} \otimes \mathfrak{B}$. We have certain flexibility in what kind of tensor products we consider.
- The multiplicative unit element calls for a base ring like \mathbb{Z} or \mathbb{R} , which may not be embedded into \mathfrak{A} .

Our ultimate objective is, however, to study single-space Grassmannians.

4.9. Single-space Grassmannian. Suppose that \mathfrak{A} is a commutative ring. (The multiplication can be considered as tensor product.) Consider the space $\mathcal{G}_\mathbb{N}(\mathfrak{A})$. Then we can consider the alternative operations

$$a \tilde{+} b := \mathbf{R}\theta_{1*}(a \oplus b), \quad \tilde{-} a := \mathbf{R}\theta_{2*}(a^{\text{inv}}), \quad \tilde{0} := \mathbf{R}\mathbf{0}_\mathbb{N}, \quad a \tilde{\otimes} b = \mathbf{R}\theta_{3*}(a \otimes b),$$

and, if \mathfrak{A} is unital

$$\tilde{1} = \mathbf{R}\theta_{4*}\mathbf{1},$$

where the maps $\theta_1 : \{0, 1\} \times \mathbb{N} \times \{0, 1\} \times \mathbb{N} \rightarrow \mathbb{N}$, $\theta_2 : \{0, 1\} \times \mathbb{N} \rightarrow \mathbb{N}$, $\theta_3 : \{0, 1\} \times \mathbb{N} \times \{0, 1\} \times \mathbb{N} \rightarrow \mathbb{N}$, $\theta_4 : \{0\} \rightarrow \mathbb{N}$ are some fixed, not necessarily isomorphic relabeling maps. These new operations are algebraic operations in strict sense. From the algebraic homotopy ring structure of the functorial Grassmannian, it is immediately clear that the new operations yield an algebraic homotopy ring by restriction.

Moreover, due to the taming construction described earlier, we can use algebraic homotopy in strong sense: we say that $f_1 : X \rightarrow \mathcal{G}_{\mathbb{N}}(\mathfrak{A})$ and $f_2 : X \rightarrow \mathcal{G}_{\mathbb{N}}(\mathfrak{A})$ are smoothly algebraically homotopic

$$f_1 \simeq_{\text{sm alg}} f_2$$

if there is a map $g : X \rightarrow \mathcal{K}_{\{0,1\} \times \{0,1\} \times \mathbb{N}}(\mathfrak{A})^*$ such that

$$f_2 \oplus \mathbf{R}0_{\mathbb{N}} \xrightarrow{g} f_1 \oplus \mathbf{R}0_{\mathbb{N}}.$$

It is an equivalence relation. Transitivity, for example, follows as in the general case, except one needs a final relabeling in the stabilizing indices.

Furthermore, as the operations are sufficiently regularized, and as it was explained 3.11, the operations and the realizing algebraic homotopies can be chosen to be matrix polynomials. Here we take the extended sense that tensor products and relabeling matrices which are constant are still in the notion of polynomiality. Hence, we obtain:

Theorem 4.10. *If \mathfrak{A} is a (unital) commutative polymetric ring, then $\mathcal{G}_{\mathbb{N}}(\mathfrak{A})$ with the operations above is a (unital) commutative “strong polynomial” algebraic H -ring. By “strong polynomial” we mean that the algebraic homotopies (for the identities and compatibilities) are induced by maps $\mathcal{G}_{\mathbb{N}}(\mathfrak{A}) \times \dots \times \mathcal{G}_{\mathbb{N}}(\mathfrak{A}) \rightarrow \mathcal{K}_{\{0,1\} \times \{0,1\} \times \mathbb{N}}(\mathfrak{A})^*$ assembled polynomially.*

4.11. It must be clear that the operations on the Grassmannian allow several variants, but they are OK as long as one can deduce the existence of algebraic homotopies from the functorial Grassmannian. It must be also clear that several intermediate constructions are allowed, like a product $\mathcal{G}_{\mathbb{N}}(\mathfrak{A}) \times \mathcal{G}_{\mathbb{N}}(\mathfrak{B}) \rightarrow \mathcal{G}_{\mathbb{N}}(\mathfrak{A} \otimes \mathfrak{B})$. This comment also applies for later constructions, but we will not emphasize it further.

5. ESTABLISHING THE SMOOTH TOPOLOGICAL H -RING STRUCTURE

This section applies to locally convex algebras \mathfrak{A} . The key point is that we can deal with stabilization internally, without adjoining extra variables. In short terms, we can make extra space using homotopies.

5.1. First, we discuss the stabilization of the algebra $\mathcal{K}_{\mathbb{N}}(\mathfrak{A})$, which will be extended to the larger idempotents. Stabilization can be organized as in [3]. For $\theta \in [0, \frac{\pi}{2}]$ we consider

$$C(\theta) = \begin{bmatrix} s & ts & t^2s & t^3s & t^4s & \dots \\ -t & s^2 & ts^2 & t^2s^2 & t^3s^2 & \ddots \\ & -t & s^2 & ts^2 & t^2s^2 & \ddots \\ & & -t & s^2 & ts^2 & \ddots \\ & & & -t & s^2 & \ddots \\ & & & & -t & \ddots \\ & & & & & \ddots \end{bmatrix},$$

where $t = \sin \theta$ and $s = \cos \theta$. Then a stabilizing homotopy is given by

$$T_{\mathcal{K}} : \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) \times \left[0, \frac{\pi}{2}\right] \rightarrow \mathcal{K}_{\mathbb{N}}(\mathfrak{A})$$

$$A, \theta \mapsto T_{\mathcal{K}}(A, \theta) = C(\theta)AC(\theta)^{\top}.$$

It yields a smooth homotopy between the identity $T_{\mathcal{K}}(\cdot, 0) = \text{id}_{\mathcal{K}_{\mathbb{N}}(\mathfrak{A})}$ and the relabeling map $T_{\mathcal{K}}(\cdot, \frac{\pi}{2}) = r_*$, where $r(n) = n+1$. It stabilizes by one extra matrix entry, which may seem insufficient. However, using it as an $\mathbb{N} \times \Omega$ block matrix construction, it achieves stabilization by infinitely many entries.

5.2. There is a slightly more complicated, yet elegant way to achieve stabilization: We can consider $C(\theta)$ as stabilization of the space $\mathcal{H}_{\mathbb{N}}(\mathfrak{A})$, and then we consider the odd or even “quantized” situation. Without going into details, the topology of $\mathcal{H}_{\mathbb{N}}(\mathfrak{A})$ is induced by the seminorms

$$p_{\alpha} \left(\sum_{n \in \mathbb{N}} a_n \mathbf{e}_n \right) = \sum_{n \in \mathbb{N}} p(a_n) \alpha^n \quad (\alpha > 0).$$

The odd quantized space $\text{Qu}^1 \mathcal{H}_{\mathbb{N}}(\mathfrak{A})$ is the space with basis $\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k}$, where $i_1 < i_2 < \cdots < i_k$. The seminorms induced ultimately are

$$p \left(\sum a_{i_1, \dots, i_k} \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k} \right) = \sum (\alpha^{i_1} + \cdots + \alpha^{i_k}) p(a_{i_1, \dots, i_k}).$$

This quantized space is isomorphic to the space of rapidly decreasing sequences $\mathcal{S}_{\mathbb{N}}(\mathfrak{A})$ by

$$V : \quad \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k} \mapsto \mathbf{e}_{2^{i_1} + \dots + 2^{i_k}}.$$

The even quantized space $\text{Qu}^0 \mathcal{H}_{\mathbb{N}}(\mathfrak{A})$ is the space with basis $\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_k}$, where $i_1 \leq i_2 \leq \cdots \leq i_k$. The induced seminorms are, similarly,

$$p \left(\sum a_{i_1, \dots, i_k} \mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_k} \right) = \sum (\alpha^{i_1} + \cdots + \alpha^{i_k}) p(a_{i_1, \dots, i_k}).$$

This is also isomorphic to $\mathcal{S}_{\mathbb{N}}(\mathfrak{A})$, although this is less transparent. For this reason, it is more practical to use the odd quantization.

We can consider the quantized matrices $\text{Qu}^0 C(\theta)$ and $\text{Qu}^1 C(\theta)$. For example,

$$\text{Qu}^1 C(\theta)(\mathbf{e}_0 \wedge \mathbf{e}_1) = (s\mathbf{e}_0 - t\mathbf{e}_1) \wedge (ste_0 + s^2\mathbf{e}_1 - t\mathbf{e}_2) = s\mathbf{e}_0 \wedge \mathbf{e}_1 - ste_0 \wedge \mathbf{e}_2 + t^2\mathbf{e}_1 \wedge \mathbf{e}_2,$$

or, in other terms,

$$V\text{Qu}^1 C(\theta)\mathbf{e}_3 = s\mathbf{e}_3 - ste_5 + t^2\mathbf{e}_6.$$

This yields a smooth homomorphic homotopy $V\text{Qu}^1 T_{\mathcal{K}}$ between $V\text{Qu}^1 T_{\mathcal{K}}(\cdot, 0) = \text{id}_{\mathcal{K}_{\mathbb{N}}(\mathfrak{A})}$ and the “halving” relabeling map $V\text{Qu}^1 T_{\mathcal{K}}(\cdot, \frac{\pi}{2}) = hv_*$ where $hv(n) = 2n$. It has the nice property that it achieves infinite stabilization immediately, and it allows plenty of direct sum decompositions. In fact, we can consider any homomorphic homotopy Hv instead of $V\text{Qu}^1 T_{\mathcal{K}}$ as long as it yields a smooth homotopy as above.

5.3. It is natural to make an identification

$$\mathcal{K}_{\mathbb{N}}(\mathfrak{A}) \equiv \left[\begin{array}{cc} \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) & \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) \\ \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) & \mathcal{K}_{\mathbb{N}}(\mathfrak{A}) \end{array} \right]^{\sim}.$$

In what follows, \sim indicates that we consider a decomposition of the index set \mathbb{N} into two copies of \mathbb{N} through the indexing convention $(0, n) \leftrightarrow 2n, (1, n) \leftrightarrow 1 + 2n$. This looks especially simple if we use binary numbers with digits written in reverse order.

Hv yields a homotopy between

$$\text{id}_{\mathcal{K}_{\mathbb{N}}(\mathfrak{A})} \quad \text{and} \quad \text{id}_{\mathcal{K}_{\mathbb{N}}(\mathfrak{A})} \tilde{\oplus} 0_{\mathbb{N}}$$

through algebra homomorphisms. But we want to stabilize idempotents. This can be done by setting

$$Hv \left(\begin{bmatrix} b_{00} & b_{01} \\ b_{10} & 1_{\mathbb{N}} + b_{11} \end{bmatrix} \right) := \begin{bmatrix} Hv(b_{00}) & Hv(b_{01}) \\ Hv(b_{10}) & 1_{\mathbb{N}} + Hv(b_{11}) \end{bmatrix}.$$

This yields a smooth homotopy between

$$b \equiv \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & 1_{\mathbb{N}} + b_{11} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & 1_{\mathbb{N}} + b_{11} \end{bmatrix} \tilde{\oplus} \begin{bmatrix} 0_{\mathbb{N}} & \\ & 1_{\mathbb{N}} \end{bmatrix} \equiv \begin{bmatrix} b & \\ & \mathbf{R}0_{\mathbb{N}} \end{bmatrix} \sim.$$

In simple terms, this means that by using a smooth homotopy, we can always make free space, i. e. we can stabilize. It also means that we can conjugate freely by smooth terms. Indeed, using the conjugating matrices

$$\begin{bmatrix} \cos \alpha & -\hat{h}\nu^{\top} \sin \alpha \\ \hat{h}\nu \sin \alpha & 1_{2\mathbb{N}} \cos \alpha + 1_{2\mathbb{N}+1} \end{bmatrix} \begin{bmatrix} \phi \\ 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & \hat{h}\nu^{\top} \sin \alpha \\ -\hat{h}\nu \sin \alpha & 1_{2\mathbb{N}} \cos \alpha + 1_{2\mathbb{N}+1} \end{bmatrix},$$

we quickly obtain a smooth homotopy between

$$\begin{bmatrix} b & \\ & \mathbf{R}0_{\mathbb{N}} \end{bmatrix} \sim \quad \text{and} \quad \begin{bmatrix} b^{\phi} & \\ & \mathbf{R}0_{\mathbb{N}} \end{bmatrix} \sim.$$

As the homotopies used are smooth, and the conjugating terms used are smooth (in fact polynomials) in terms of the initial data, we obtain

Theorem 5.4. *If \mathfrak{A} is a (unital) commutative locally convex algebra, then $\mathcal{G}_{\mathbb{N}}(\mathfrak{A})$ is a (unital) commutative “smooth” topological H -ring. By “smooth” we mean that the topological homotopies are induced by smooth maps $\mathcal{G}_{\mathbb{N}}(\mathfrak{A}) \times \dots \times \mathcal{G}_{\mathbb{N}}(\mathfrak{A}) \times [0, 1] \rightarrow \mathcal{G}_{\mathbb{N}}(\mathfrak{A})$.*

Now, the exact meaning of “smooth” in the setting of infinite-dimensional spaces remains somewhat unclear, but if one tries his favorite definition, then he will most likely agree.

6. ESTABLISHING THE SMOOTH TOPOLOGICAL ALGEBRAIC H -RING STRUCTURE

6.1. We say that the polymetric ring \mathfrak{A} is strong if for every seminorm p there is a seminorm \tilde{p} such that $p(a_1 \dots a_n) \leq \tilde{p}(a_1) \dots \tilde{p}(a_n)$ for any $n \in \mathbb{N}$. These algebras behave well with respect to forming \mathfrak{A}^+ , and $\mathcal{K}_{\Omega}(\mathfrak{A})$, etc.

Suppose that \mathfrak{A} is a strong locally convex algebra. Let $[a, b] \subset \mathbb{R}$ be a closed interval. If $A : [a, b] \rightarrow \mathfrak{A}$ is a, say, continuous function, then $C(t) = A(t) dt$ is a continuous ordered measure on $[a, b]$. For such a continuous measure, the time-ordered exponential

$$\exp_t C(t) = 1 + \int_{t_1} C(t_1) + \dots + \int_{t_1 \leq \dots \leq t_n} C(t_n) \dots C(t_1) + \dots$$

can be considered. One can check

Lemma 6.2. *If \mathfrak{A} is a strong locally convex algebra, and $P : [a, b] \rightarrow \mathfrak{A}$ is a smooth idempotent-valued map, then the map*

$$A_p : \{(t_1, t_2) : t_1, t_2 \in [a, b], t_1 \leq t_2\} \rightarrow \mathfrak{A}$$

$$(t_1, t_2) \mapsto A_P(t_1, t_2) = \exp_t \dot{P}(t)P(t) - P(t)\dot{P}(t) dt|_{[t_1, t_2]}$$

is also smooth, and

$$P(t_1) \xrightarrow{A_P(t_1, t_2)} P(t_2).$$

Remark 6.3. What happens above corresponds to parallel transport along a connection, which can be written in local form as $\nabla = d - dP P + P dP = P.d.P + (1 - P).d.(1 - P)$.

Consequently, smooth homotopies can be lifted to conjugation. Hence we obtain

Theorem 6.4. *If \mathfrak{A} is a (unital) commutative strong locally convex algebra, then $\mathcal{G}_{\mathbb{N}}(\mathfrak{A})$ is a (unital) commutative “smooth topological algebraic” H -ring. By “smooth topological algebraic” we mean that the topological homotopies are induced by conjugating by smooth maps $\mathcal{G}_{\mathbb{N}}(\mathfrak{A}) \times \dots \times \mathcal{G}_{\mathbb{N}}(\mathfrak{A}) \times [0, 1] \rightarrow \mathcal{K}_{\{0,1\} \times \mathbb{N}}(\mathfrak{A})^*$.*

7. THE CORE INFORMATION OF VIRTUAL IDEMPOTENTS

7.1. Let us use the abbreviations $h_{ij} := a^{[i]}(b-a)a^{[j]}$, where $a^{[0]} = a$ and $a^{[1]} = \bar{a}$. Then

$$R(\langle b, a \rangle) = \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & 1_\Omega + h_{11} \end{bmatrix}.$$

In other terms, the h_{ij} 's contain the information what is left after regularization. It turns out that the regularization of various operations can be expressed in terms of that data:

$$R(\langle b, a \rangle^{\text{inv}}) = \begin{bmatrix} h_{11} & h_{10} \\ h_{01} & 1_\Omega + h_{00} \end{bmatrix};$$

and similarly, for

$$R(\langle b, a \rangle') = \begin{bmatrix} h'_{00} & h'_{01} \\ h'_{10} & 1_\Omega + h'_{11} \end{bmatrix},$$

it yields

$$\begin{aligned} &= s_{01}(h_{00} - h_{01} + h_{10} - h_{11}) \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & 1_\Omega + h_{11} \end{bmatrix} s_{01}(h_{00} + h_{01} - h_{10} - h_{11}) \\ &= \begin{bmatrix} h_{00} & h_{10} \\ h_{01} & 1_\Omega + h_{11} \end{bmatrix} - \bar{s}_{01}(h_{01}h_{10} + h_{01}h_{11} + h_{11}h_{10} - h_{10}h_{01}). \end{aligned}$$

Furthermore, if we use the abbreviations

$$R(\langle d, c \rangle) = \begin{bmatrix} k_{00} & k_{01} \\ k_{10} & 1_\Omega + k_{11} \end{bmatrix}, \quad R(\langle d, c \rangle') = \begin{bmatrix} k'_{00} & k'_{01} \\ k'_{10} & 1_\Omega + k'_{11} \end{bmatrix},$$

then it yields

$$\begin{aligned} R(\langle b, a \rangle \overleftarrow{\otimes} \langle d, c \rangle) &= \begin{bmatrix} 0_{\Omega \times \Xi} & \\ & 1_{\Omega \times \Xi} \end{bmatrix} + \begin{bmatrix} h_{11} \otimes k'_{00} & h_{11} \otimes k'_{01} \\ h_{11} \otimes k'_{10} & h_{11} \otimes k'_{11} \end{bmatrix} + \begin{bmatrix} h_{00} \otimes k_{00} & h_{00} \otimes k_{01} \\ h_{00} \otimes k_{10} & h_{00} \otimes k_{11} \end{bmatrix} \\ &+ \begin{bmatrix} h_{10} \otimes k_{11} + h_{10} \otimes k_{01} + h_{01} \otimes k_{01} + h_{01} \otimes k_{00} \\ h_{01} \otimes k_{11} + h_{01} \otimes k_{10} + h_{10} \otimes k_{10} + h_{10} \otimes k_{00} \end{bmatrix}; \end{aligned}$$

and we obtain a similar formula for $R(\langle b, a \rangle \overrightarrow{\otimes} \langle d, c \rangle)$ except the terms $h_{ij} \otimes k'_{lm}$ should be replaced by $h'_{lm} \otimes k_{ij}$. The additive neutral element corresponds to $\mathbf{R0} = \mathbf{0}$. The multiplicative neutral idempotent corresponds to the “scalar” matrix $\mathbf{R1} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$.

7.2. More generally, we can talk about regular idempotents. The general matrix

$$H = \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & 1_\Omega + h_{11} \end{bmatrix}$$

is an idempotent if and only if

$$\begin{aligned} h_{00}h_{00} + h_{01}h_{10} &= h_{00}, & h_{00}h_{01} + h_{01}h_{11} &= 0, \\ h_{10}h_{00} + h_{11}h_{10} &= 0, & h_{10}h_{01} + h_{11}h_{11} &= -h_{11}. \end{aligned}$$

We say that such an idempotent is regular if the identities

$$\begin{aligned} h_{00}h_{10} = h_{01}h_{00} &= 0, & h_{00}h_{11} = h_{01}h_{01} &= 0, \\ h_{10}h_{10} = h_{11}h_{00} &= 0, & h_{10}h_{11} = h_{11}h_{01} &= 0. \end{aligned}$$

also hold. Inspired by the formulas of the previous point, one can define “regular” operations $\oplus, \text{inv}, ', \overleftarrow{\otimes}, \overrightarrow{\otimes}$ for regular idempotents. They satisfy the same identities as their ordinary counterparts; although checking that they yield new regular idempotents is quite tedious already. Using the standard machinery, one can show that the “regular” operations are homotopic to the “ordinary” operations. In those terms, the regularization operation \mathbf{R} is a natural trivial homomorphism from pairs of idempotents to regular idempotents.

7.3. Regular dimension. Based upon the observations above, we propose a notion of dimension of virtual idempotents which behaves well with respect to non-unital rings and products: For $\langle b, a \rangle$ let $\dim \langle b, a \rangle$ be the infimum of the cardinality of those Ξ such that

$$\langle b, a \rangle \simeq_{\text{alg}} \langle U, \mathbf{R0}_\Xi \rangle,$$

where U is a regular idempotent. Then we obtain

Theorem 7.4. *The dimension defined above has the following properties:*

- (a) $\mathbf{p}_1 \simeq_{\text{alg}} \mathbf{p}_2$ implies $\dim \mathbf{p}_1 = \dim \mathbf{p}_2$.
- (b) $\dim \mathbf{p} = 0$ holds if and only if $\mathbf{p} \simeq \mathbf{0}$.
- (c) $\dim \mathbf{p}_1 \oplus \mathbf{p}_2 \leq \dim \mathbf{p}_1 + \dim \mathbf{p}_2$.
- (d) $\dim \mathbf{p}^{\text{inv}} = \dim \mathbf{p}$.
- (e) $\dim \mathbf{p}_1 \otimes \mathbf{p}_2 \leq \dim \mathbf{p}_1 \dim \mathbf{p}_2$.
- (f) $\dim \mathbf{1} = 1$, if it applies.

8. FINITE DIMENSIONALITY

From the previous discussion, it is clear that a Grassmannian element is finite dimensional if it can be represented by a pair of finite matrices. If \mathfrak{A} is a discrete ring, then finite dimensionality holds automatically. More generally, every Grassmannian element is finite dimensional if we can approximate in \mathfrak{A} in traditional sense. We demonstrate this statement in order to connect to the classical viewpoint.

8.1. We say that a polymetric \mathfrak{A} is norm-strong if there is a distinguished seminorm q such that for each seminorm p there is an other seminorm \tilde{p} and $C, i, j \in \mathbb{N}$ such that for all X_1, X_2, \dots, X_n ($n \geq 1$), we have

$$p(X_1 \dots X_n) \leq Cn^i \sum_{h: \{1, \dots, n\} \rightarrow \{q, \tilde{p}\}, \#\{r: h_r \neq q\} \leq j} h_1(X_1) \dots h_n(X_n).$$

This class of polymetric rings also behaves well with respect to taking \mathfrak{A}^+ and $\mathcal{K}_\Omega(\mathfrak{A})$.

8.2. Suppose now that \mathfrak{A} is a norm-strong polymetric ring. If $P \approx \mathbf{R0}_\Omega$ is an idempotent, then we can take ε such that

$$\varepsilon = P - \mathbf{R0}_\Omega \quad \text{except in finitely many entries,}$$

yet so that in a distinguished seminorm $\|\varepsilon\|$ can be arbitrarily small. Then

$$\tilde{P}_\varepsilon = P - \varepsilon$$

differs from $\mathbf{R0}_\Omega$ only in finitely many entries; however, it is not necessarily an idempotent.

In general, if \tilde{P} is not an idempotent but quite close to being one, then we may try the idempotent operation

$$\text{idem } \tilde{P} = \int \frac{\tilde{P}z}{(1 - \tilde{P}) + \tilde{P}z} \frac{|dz|}{2\pi},$$

where we apply formal integration for rapidly decreasing Laurent series. For the sake of brevity, we will write 1 for $1_{\{0,1\} \times \Omega}$ in the rest of the section.

In the present case, due to the norm-strong structure, this machinery works using classical Neumann series. Indeed, if ε is so small that $\|P\varepsilon\| + \|(1 - P)\varepsilon\| < \frac{1}{2}$ then

$$\begin{aligned} P_\varepsilon &:= \text{idem } \tilde{P}_\varepsilon = \int (P - \varepsilon)z((1 - P) + Pz^{-1}) \left(1 - \varepsilon(z - 1)((1 - P) + Pz^{-1})\right)^{-1} \frac{|dz|}{2\pi} \\ &= P + (2P\varepsilon P - \varepsilon P - P\varepsilon) + (P\varepsilon\varepsilon + \varepsilon P\varepsilon + \varepsilon\varepsilon P - 3P\varepsilon P\varepsilon - 3P\varepsilon\varepsilon P - 3\varepsilon P\varepsilon P + 6P\varepsilon P\varepsilon P) + \dots \end{aligned}$$

is convergent. According to its definition, idem \tilde{P}_ε is an idempotent which differs from $\mathbf{R0}_\Omega$ in finitely many terms. It is easily to quantify that if ε is very small, then idem \tilde{P}_ε is very close to P .

Now, if an idempotent Q is sufficiently close to the idempotent P , then $1 - P - Q = (1 - 2Q)(1 - P - Q + 2QP)$ becomes invertible. Indeed, the first term in the product is an involution, and the second term is invertible by Neumann series arguments. But then

$$P \xrightarrow{1-P-Q} Q, \quad \text{and} \quad P \xrightarrow{1-P-Q+2QP} Q.$$

According to this, if $\|\varepsilon\|$ is sufficiently small, then

$$\langle P, \mathbf{R0}_\Omega \rangle \xrightarrow{\langle 1-P-P_\varepsilon+2P_\varepsilon P, 1 \rangle} \langle P_\varepsilon, \mathbf{R0}_\Omega \rangle.$$

After eliminating the unnecessarily stabilization terms in the latter expression, we see finite dimensionality.

Remark 8.3. When we send P into Q we can choose between $1 - P - Q$ and $(1 - P - Q)^{-1}$. If \mathfrak{A} allows a continuous division by 2, then we have more symmetrical choice: We can take the geometric mean between $1 - P - Q$ and $(1 - P - Q)^{-1}$. This yields the sign operator

$$\text{sgn}(1 - P - Q) = \int \frac{\frac{(1-P)+(1-Q)}{2} - \frac{P+Q}{2}z}{\frac{(1-P)+(1-Q)}{2} + \frac{P+Q}{2}z} \frac{|dz|}{2\pi}.$$

Here it was understood that $\frac{P+Q}{2} = \mathbf{R0}_\Omega + \frac{(P-\mathbf{R0}_\Omega)+(Q-\mathbf{R0}_\Omega)}{2}$, and similarly for $\frac{(1-P)+(1-Q)}{2}$. This yields

$$P \xrightarrow{\text{sgn}(1-P-Q)} Q, \quad \text{and} \quad P \xrightarrow{(1-2Q)\text{sgn}(1-P-Q)=\text{sgn}(1-P-Q)(1-2P)} Q,$$

which are much more symmetrical choices. This construction fits to the case of $*$ -algebras.

8.4. Using the same techniques, one can prove the following version of Corollary 3.8: If $\langle b, a \rangle \simeq_{\text{alg}} \langle \tilde{b}, a \rangle$, where b, \tilde{b}, a are finite matrices, $b \approx \tilde{b} \approx a$, then there is a finite index set Ω' and $\psi, \phi \approx 1_{\Omega \cup \Omega'}$ such that $\langle b \oplus \mathbf{R0}_{\Omega'}, a \oplus \mathbf{R0}_{\Omega'} \rangle \xrightarrow{\langle \psi, \phi \rangle} \langle \tilde{b} \oplus \mathbf{R0}_{\Omega'}, a \oplus \mathbf{R0}_{\Omega'} \rangle$. As a consequence, the discussion of individual Grassmannian elements can be reduced to finite matrices. This is, however, of little help in terms of the global H -ring structure.

9. FREDHOLM OPERATORS

Here we discuss how Fredholm operators can be placed in this context. It is instructive to see how standard elements of Fredholm module theory come up in the context of virtual Grassmannians. See e. g. [2] for comparison.

9.1. Connectors. Suppose that $\xi, a \in \Psi_\Omega(\mathfrak{A}^+)$, a is an idempotent, and ξ is an invertible element such that $\bar{a}^\xi \approx a$. In such a pair $[\xi, a]$, we call the first term as the connector element, and the second term as the base idempotent. We define the index

$$\text{Ind} [\xi, a] = \langle \bar{a}^\xi, a \rangle.$$

The nicest case is when ξ is an involution. We can define the sum $[\xi_1, a_1] \oplus [\xi_2, a_2] = [\xi_1 \oplus \xi_2, a_1 \oplus a_2]$ and the additive inverse $[\xi, a]^{\text{inv}} = [\xi, \bar{a}]$.

Suppose now that c is another idempotent, and σ is invertible such that $\bar{c}^\sigma \approx c$ but in $\Psi_\Xi(\mathfrak{A})$. Let us consider the product of indices

$$\text{Ind} [\xi, a] \overleftarrow{\otimes} \text{Ind} [\sigma, c] = \langle \bar{a}^\xi, a \rangle \overleftarrow{\otimes} \langle \bar{c}^\sigma, c \rangle = \langle \bar{a}^\xi \otimes \bar{c}^\sigma + a^\xi \otimes c, a \otimes \bar{c}^\sigma + \bar{a} \otimes c \rangle.$$

One can see that

$$a \otimes \bar{c} + \bar{a} \otimes c \xrightarrow{a \otimes \sigma + \bar{a} \otimes 1_\Xi} a \otimes \bar{c}^\sigma + \bar{a} \otimes c$$

and

$$a \otimes c + \bar{a} \otimes \bar{c} \xrightarrow{\xi a \otimes 1_{\Xi} + \xi \bar{a} \otimes \sigma} \bar{a}^{\xi} \otimes \bar{c}^{\sigma} + a^{\xi} \otimes c.$$

Hence, it is reasonable to define

$$[\xi, a] \overleftarrow{\otimes} [\sigma, c] := [(a \otimes \sigma + \bar{a} \otimes 1_{\Xi})^{-1}(\xi a \otimes 1_{\Xi} + \xi \bar{a} \otimes \sigma), a \otimes \bar{c} + \bar{a} \otimes c].$$

Then the product of indices is equivalent to the index of the product. Indeed

$$\text{Ind} [\xi, a] \overleftarrow{\otimes} [\sigma, b] \xrightarrow{\langle a \otimes \sigma + \bar{a} \otimes 1_{\Xi}, a \otimes \sigma + \bar{a} \otimes 1_{\Xi} \rangle} \text{Ind} [\xi, a] \overleftarrow{\otimes} \text{Ind} [\sigma, b].$$

A bonus is that the connector term of $[\xi, a] \overleftarrow{\otimes} [\sigma, c]$ is an involution if ξ, σ are involutions.

Lemma 9.2. *The following hold:*

- (a) $[\xi_1, a_1] \approx [\xi_2, a_2] \Rightarrow \text{Ind} [\xi_1, a_1] \simeq_{\text{alg}} \text{Ind} [\xi_2, a_2]$;
- (b) $\xi \bar{a} \xi^{-1} = a \Rightarrow \text{Ind} [\xi, a] \simeq_{\text{alg}} \mathbf{0}$;
- (c) $\text{Ind} [\xi, a] \simeq_{\text{alg}} \text{Ind} [\xi^{-1}, a], \langle \xi^2 a \xi^{-2}, a \rangle \simeq_{\text{alg}} \mathbf{0}$;
- (d) $\text{Ind} [\xi_1, a_1] \oplus [\xi_2, a_2] \simeq \text{Ind} [\xi_1, a_1] \oplus \text{Ind} [\xi_2, a_2]$;
- (e) $\text{Ind} [\xi, a]^{\text{inv}} \simeq (\text{Ind} [\xi, a])^{\text{inv}}$;
- (f) $\text{Ind} [\xi, a] \overleftarrow{\otimes} [\sigma, b] \simeq_{\text{alg}} \text{Ind} [\xi, a] \overleftarrow{\otimes} \text{Ind} [\sigma, b]$.

Proof. The points (b), (d), (e), (f) are immediate.

- (a₁) $\xi_1 \approx \xi_2 \Rightarrow \text{Ind} [\xi_1, a] \simeq_{\text{alg}} \text{Ind} [\xi_2, a]$ follows from $\text{Ind} [\xi_1, a] \xrightarrow{\langle \xi_2 \xi_1^{-1}, 1_{\Omega} \rangle} \text{Ind} [\xi_2, a]$.
- (a₂) $a_1 \approx a_2 \Rightarrow \text{Ind} [\xi, a_1] \simeq_{\text{alg}} \text{Ind} [\xi, a_2]$ comes as follows: The algebraic homotopies

$$\text{Ind} [\xi, a_1] \oplus [\xi, \bar{a}_2] \xrightarrow{\langle s_{12}(a_2), s_{12}(a_2) \rangle} \text{Ind} [s_{12}(a_2)(\xi \oplus \xi) s_{12}(a_2), \mathbf{R}\langle a_1, a_2 \rangle] \simeq_{\text{alg}}$$

(by (a₁))

$$\simeq_{\text{alg}} \text{Ind} \left[\left[\begin{array}{c} \xi \\ \xi \end{array} \right], \mathbf{R}\langle a_1, a_2 \rangle \right] = \langle \mathbf{R}\langle a_1^{\xi}, a_2^{\xi} \rangle, \mathbf{R}\langle a_1, a_2 \rangle \rangle \simeq_{\text{alg}}$$

(by Corollary 4.6)

$$\simeq_{\text{alg}} \mathbf{R}\langle a_1^{\xi}, a_2^{\xi} \rangle \oplus \mathbf{R}\langle a_1, a_2 \rangle^{\text{inv}} \simeq_{\text{alg}} \mathbf{R}\langle a_1, a_2 \rangle \oplus \mathbf{R}\langle a_1, a_2 \rangle^{\text{inv}} \simeq_{\text{alg}} \mathbf{0}$$

imply the statement.

(c) According to (a₂), we see that $\text{Ind} [\xi, a] \simeq_{\text{alg}} \text{Ind} [\xi, \xi^{-1} \bar{a} \xi] \simeq_{\text{alg}} \text{Ind} [\xi, \xi^{-1} \bar{a} \xi]' = \text{Ind} [\xi^{-1}, a]$. But then, according to Corollary 4.6, we also see that $\langle \xi a \xi^{-1}, \xi^{-1} a \xi \rangle \simeq_{\text{alg}} \mathbf{0}$. Consequently, $\langle \xi^2 a \xi^{-2}, a \rangle \simeq_{\text{alg}} \mathbf{0}$. \square

9.3. Fredholm connectors. In general, ξ can always be replaced by an involution. Indeed, suppose that a is an idempotent, $\xi a = \bar{a} \xi$, and ξ allows a parametrix η such that $\xi \eta \approx \eta \xi \approx 1$. Then it is not hard to see that, for example,

$$\tilde{\xi} = (a + a \xi \bar{a} - \bar{a})(a - a \eta \bar{a} - \bar{a})(a + a \xi \bar{a} - \bar{a})$$

will be an involution. This is a reasonable substitute.

Lemma 9.4. *If ξ is invertible, then $\text{Ind} [\xi, a] \simeq_{\text{alg}} \text{Ind} [\tilde{\xi}, a]$.*

Proof. Apply the following lemma with $\psi = \tilde{\xi} \xi^{-1}$. \square

Lemma 9.5. *Suppose that ψ is invertible such that $\psi - \bar{a} \psi \bar{a} \approx a$ or $\psi - a \psi a \approx \bar{a}$. Then*

$$\langle \psi a \psi^{-1}, a \rangle \simeq_{\text{alg}} \mathbf{0}.$$

Proof. Consider the first case. Then

$$\langle \psi a \psi^{-1}, a \rangle \simeq_{\text{alg}} \left\langle \left[\begin{array}{c} \psi \\ 1 \end{array} \right] \left[\begin{array}{c} a \\ 0 \end{array} \right] \left[\begin{array}{c} \psi^{-1} \\ 1 \end{array} \right], \left[\begin{array}{c} a \\ 0 \end{array} \right] \right\rangle \rightarrow \left\langle \left[\begin{array}{c} a \\ 0 \end{array} \right], \left[\begin{array}{c} a \\ 0 \end{array} \right] \right\rangle \simeq_{\text{alg}} \mathbf{0},$$

where the conjugation in the middle is induced by the replacement in the conjugating term according to the observation

$$\begin{bmatrix} \psi & \\ & 1 \end{bmatrix} \approx \begin{bmatrix} \bar{a}\psi\bar{a} + a & \bar{a}\psi a \\ a\psi\bar{a} & \bar{a} + a\psi a \end{bmatrix} = \begin{bmatrix} \bar{a} & a \\ a & \bar{a} \end{bmatrix} \begin{bmatrix} \psi & \\ & 1 \end{bmatrix} \begin{bmatrix} \bar{a} & a \\ a & \bar{a} \end{bmatrix}.$$

The second case is similar. \square

The price we paid for having $\tilde{\xi}$ is that we have departed from the case of conjugating matrices which are manifestly unitary in the $*$ -algebraic case. However, it is this setting which is applied in the discussion of the index of Fredholm operators as follows.

9.6. Fredholm operators. We say that $\psi \in \Psi_{\Omega_1, \Omega_0}(\mathfrak{A}^+)$ is a Fredholm operator if there is a parametrix $\phi \in \Psi_{\Omega_0, \Omega_1}(\mathfrak{A}^+)$ such that $\phi\psi \approx 1_{\Omega_0}$ and $\psi\phi \approx 1_{\Omega_1}$. We say that $\xi \in \Psi_{\Omega_0 \cup \Omega_1}(\mathfrak{A}^+)$ is a connector element for the Fredholm operator ψ , if it is invertible and

$$\xi \approx \begin{bmatrix} & \phi \\ \psi & \end{bmatrix}.$$

One can easily obtain such an element by taking the involution

$$F(\psi, \phi) = \begin{bmatrix} 1_{\Omega_0} - \phi\psi & 2\phi - \phi\psi\phi \\ \psi & \psi\phi - 1_{\Omega_1} \end{bmatrix} = \begin{bmatrix} 1_{\Omega_0} & \phi \\ & -1_{\Omega_1} \end{bmatrix} \begin{bmatrix} 1_{\Omega_0} & \\ -\psi & -1_{\Omega_1} \end{bmatrix} \begin{bmatrix} 1_{\Omega_0} & \phi \\ & -1_{\Omega_1} \end{bmatrix}.$$

9.7. To a connector element ξ as above, we can associate the index

$$\text{Ind}[\xi, 0_{\Omega_0} \oplus 1_{\Omega_1}] = \left\langle \xi \begin{bmatrix} 1_{\Omega_0} & \\ & 0_{\Omega_1} \end{bmatrix} \xi^{-1}, \begin{bmatrix} 0_{\Omega_0} & \\ & 1_{\Omega_1} \end{bmatrix} \right\rangle.$$

It is easy to see that, up to smooth conjugation, this Grassmannian element depends only on ψ . Nevertheless, in order to get a well-defined index element, we define the index of the Fredholm pair (ψ, ϕ) as

$$\text{Ind}_F(\psi, \phi) := \text{Ind}[F(\psi, \phi), 0_{\Omega_0} \oplus 1_{\Omega_1}].$$

9.8. We can define the sum of Fredholm pairs as $(\psi_1, \phi_1) \oplus (\psi_2, \phi_2) := (\psi_1 \oplus \psi_2, \phi_1 \oplus \phi_2)$, and the inverse pair as $(\psi, \phi)^{\text{inv}} = (\phi, \psi)$. If (ψ', ϕ') is a Fredholm pair such that $\psi' \in \Psi_{\Omega_2, \Omega_1}$, $\phi' \in \Psi_{\Omega_1, \Omega_2}$, then we define the composition as $(\psi', \phi') \circ (\psi, \phi) = (\psi'\psi, \phi\phi')$.

Lemma 9.9. *The following hold:*

- (a) $(\psi_1, \phi_1) \approx (\psi_2, \phi_2) \Rightarrow \text{Ind}_F(\psi_1, \phi_1) \simeq_{\text{alg}} \text{Ind}_F(\psi_2, \phi_2)$;
- (b) if ψ or ϕ is invertible, then $\text{Ind}_F(\psi, \phi) \simeq_{\text{alg}} \mathbf{0}$;
- (c) $\text{Ind}_F(\psi_1, \phi_1) \oplus (\psi_2, \phi_2) \simeq \text{Ind}(\psi_1, \phi_1) \oplus \text{Ind}(\psi_2, \phi_2)$;
- (d) $\text{Ind}_F(\psi, \phi)^{\text{inv}} \simeq (\text{Ind}_F(\psi, \phi))^{\text{inv}}$;
- (e) $\text{Ind}_F(\psi', \phi') \circ (\psi, \phi) \simeq_{\text{alg}} \text{Ind}_F(\psi, \phi) \oplus \text{Ind}_F(\psi', \phi')$;
- (f) $\text{Ind}_F(\psi, \phi) \overline{\otimes} (\theta, \chi) \simeq_{\text{alg}} \text{Ind}_F(\psi, \phi) \overline{\otimes} \text{Ind}_F(\theta, \chi)$.

Proof, except of (f). Everything is straightforward, except (e). It is easy to check if $(\psi', \phi') = (u, u^{-1})$, where u is invertible. In this case

$$\text{Ind}_F(\psi, \phi) \xrightarrow{\langle 1_{\Omega_0} \oplus u, 1_{\Omega_0} \oplus u \rangle} \text{Ind}_F(\psi', \phi') \circ (\psi, \phi).$$

This special case implies the general case as follows. Let $\xi' = \begin{bmatrix} \xi'_{11} & \xi'_{12} \\ \xi'_{21} & \xi'_{22} \end{bmatrix} \approx \begin{bmatrix} & \phi' \\ \psi' & \end{bmatrix}$ be a connector element with inverse $\tilde{\xi}'$. Then

$$\left(\begin{bmatrix} \xi'_{21} & \xi'_{22} \\ \xi'_{11} & \xi'_{12} \end{bmatrix}, \begin{bmatrix} \tilde{\xi}'_{12} & \tilde{\xi}'_{11} \\ \tilde{\xi}'_{22} & \tilde{\xi}'_{21} \end{bmatrix} \right) \circ \left(\begin{bmatrix} \psi & \\ & \psi' \end{bmatrix}, \begin{bmatrix} \phi & \\ & \phi' \end{bmatrix} \right) \approx \left(\begin{bmatrix} \psi'\psi & \\ & 1_{\Omega_1} \end{bmatrix}, \begin{bmatrix} \phi'\phi & \\ & 1_{\Omega_1} \end{bmatrix} \right).$$

Consequently, $\text{Ind}_F(\psi, \phi) \oplus \text{Ind}_F(\psi', \phi') \simeq_{\text{alg}} \text{Ind}_F(\psi', \phi') \circ (\psi, \phi) \oplus \text{Ind}_F(1_{\Omega_1}, 1_{\Omega_1})$. \square

What prevents us from proving (f) is that we have not defined the product of Fredholm pairs yet. Thus, it remains to make a definition such that (f) would hold.

9.10. Evaluating it, we find that the connector term of $[F(\psi, \phi), 0_{\Omega_0} \oplus 1_{\Omega_1}] \overleftarrow{\otimes} [F(\theta, \chi), 0_{\Xi_0} \oplus 1_{\Xi_1}]$ is a connector term for the Fredholm pair

$$\left(\begin{bmatrix} (1_{\Omega_0} - \phi\psi) \otimes \theta & (2\phi - \phi\psi\phi) \otimes 1_{\Xi_1} \\ \psi \otimes 1_{\Xi_0} & (\psi\phi - 1_{\Omega_1}) \otimes \chi \end{bmatrix}, \begin{bmatrix} (1_{\Omega_0} - \phi\psi) \otimes \chi & (2\phi - \phi\psi\phi) \otimes 1_{\Xi_0} \\ \psi \otimes 1_{\Xi_1} & (\psi\phi - 1_{\Omega_1}) \otimes \theta \end{bmatrix} \right),$$

which, henceforth, can be chosen as the definition of $(\psi, \phi) \overleftarrow{\otimes} (\theta, \chi)$. Applying the composition

$$\left(\begin{bmatrix} 1_{\Omega_0 \times \Xi_1} & -\phi \otimes \theta \\ & -1_{\Omega_1 \times \Xi_0} \end{bmatrix}, \begin{bmatrix} 1_{\Omega_0 \times \Xi_1} & -\phi \otimes \theta \\ & -1_{\Omega_1 \times \Xi_0} \end{bmatrix} \right) \circ,$$

which, according to 9.9 (b) and (e), does not change the index up to algebraic homotopy, we obtain \approx

$$\left(\begin{bmatrix} 1_{\Omega_0} \otimes \theta & \phi \otimes 1_{\Xi_1} \\ -\psi \otimes 1_{\Xi_0} & (1_{\Omega_1} - \psi\phi) \otimes \chi \end{bmatrix}, \begin{bmatrix} (1_{\Omega_0} - \phi\psi) \otimes \chi & -\phi \otimes 1_{\Xi_0} \\ \psi \otimes 1_{\Xi_1} & 1_{\Omega_1} \otimes \theta \end{bmatrix} \right).$$

Hence, this is another sufficiently good definition for $(\psi, \phi) \overleftarrow{\otimes} (\theta, \chi)$. This latter one is essentially the same as [1] (2.6). Needless to say, many variants are possible.

Fredholm operators coming from geometry are more general. They do not act around $0_{\Omega_0} \oplus 1_{\Omega_1}$ but more general idempotents corresponding to vector bundles.

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