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UNITARIZATION OF SCHUR REPRESENTATIONS OF A POSET CORRESPONDING TO $\widetilde{E_8}$

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Abstract. We prove that every Schur representation of a poset corresponding to $\widetilde{E_8}$ can be unitarized with some character.

1. Introduction

A description of representations of quivers and posets in the category of linear spaces is related to numerous linear algebra problems, and works in this area have become classical, see references in [15] and elsewhere.

The problem of finding a description of representations of quivers and posets is also being studied for the category of Hilbert spaces by introducing additional conditions on the representations. In the category of Hilbert spaces, a description of representations of the corresponding algebras, calculation of the dimensions, and other results are given in [10, 14, 1] and elsewhere. In [9, 11], the authors studied locally scalar representations of quivers (the term was changed to orthoscalar in [9]). One can also consider orthoscalar representations of posets in the category of Hilbert spaces. In such a case, the study of orthoscalar representations of primitive posets can be reduced to a study of orthoscalar representations of the corresponding quivers and vice versa.

In this paper, we continue to study the relation between indecomposable linear representations of posets and their irreducible orthoscalar Hilbert representations, see [12, 15, 3, 17]. To an irreducible orthoscalar representation there corresponds a linear representation, but a linear representation corresponds to a set of orthoscalar representations with different characters; this set could also be empty. A linear representation π is unitarizable with a character χ if there exists an orthoscalar representation π' with χ, which is linearly isomorphic to π, see [15]. It is known [9] that an indecomposable representation of a poset is unitarizable with some character only if it is a Schur representation. If, for a fixed character, a unitary representation exists, then it is unique up to unitary equivalence. However, not every Schur representation can be unitarized with some character, see a counterexample in [18].

It was proved in [3] that all Schur representations of primitive finite type posets can be unitarized with some characters, and a description of such characters is given. It was proved that all Schur representations of $\widetilde{D_4}$ can be unitarized [12], and a description of the characters that admit a unitarization is obtained in [15]. This was carried out for the poset corresponding to $\widetilde{E_6}$ in [17], and for the poset corresponding to $\widetilde{E_7}$ in [18], except for giving a description of the characters.

In this paper, we show that all Schur representations of a poset corresponding to $\widetilde{E_8}$ can be unitarized with some characters.

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2. Auxiliary results

2.1. **Collections of linear subspaces.** We will be considering finite dimensional linear spaces over $\mathbb{C}$. Linear spaces, together with a system of subspaces, naturally make an additive category. To be more precise, objects of the category Sys_n are ordered collections $(V; V_1, \dots, V_n)$, where V_i are subspaces of V. Morphisms from $(V; V_1, \dots, V_n)$ to $(W; W_1, \dots, W_n)$ are linear mappings $\phi : V \to W$ such that $\phi(V_i) \subseteq W_i$. An isomorphism is a morphism that admits an inverse. An endomorphism is a morphism of an object into itself. Objects are called isomorphic or equivalent if there is an isomorphism between them. A direct sum of the objects $(V; V_1, \dots, V_n)$ and $(W; W_1, \dots, W_n)$ is $(V \oplus W; V_1 \oplus W_1, \dots, V_n \oplus W_n)$. An object is called indecomposable if it is not isomorphic to a sum of nonzero objects. An object is called Schur if it only has trivial endomorphisms. A Schur object is indecomposable but not every indecomposable object is Schur.

The problem of classification of indecomposable nonequivalent objects is important and has been extensively studied. For $n \leqslant 3$ there is a finite number of such objects, for $n = 4$ there are infinitely many of them but they admit a classification, see [2] and the references therein. If $n \geqslant 5$, the classification problem is wild.

2.2. **Collections of Hilbert spaces.** Let us now look at collections of Hilbert spaces. Consider the category $\mathrm{Sys}\mathcal{H}_n$ as a subcategory of Sys_n. The objects are collections $(V; V_1, \dots, V_n)$ such that V is endowed with a inner product. Morphisms from $(V; V_1, \dots, V_n)$ to $(W; W_1, \dots, W_n)$ are the ones for which $\phi(V_i) \subseteq W_i$ and also $\phi^*(W_i) \subseteq V_i$. One can show that two objects will be isomorphic if and only if there is a unitary operator ϕ such that $\phi(V_i) = W_i$. A direct product in the category is the orthogonal sum. It is clear that orthogonal decomposability implies linear decomposability but not vice versa. Note that, for this category, the Schur property is equivalent for a collection of orthogonal projections to be irreducible.

2.3. **Orthoscalarity.** The category $\mathrm{Sys}\mathcal{H}_n$ contains fewer morphisms so there are more equivalence classes and the description problem is more difficult. Indeed, if $n = 3$, the problem of describing indecomposable objects is already wild. However, the description problem becomes meaningful if one introduces additional conditions. A collection $(V; V_1, \dots, V_n)$ in $\mathrm{Sys}\mathcal{H}_n$ will be called orthoscalar with a character $\chi = (\alpha_0; \alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$, if

$$\alpha_1 P_1 + \cdots + \alpha_n P_n = \alpha_0 I,$$

where P_i are orthogonal projections onto the subspaces V_i.

All such collections make a category $\mathrm{Sys}\mathcal{H}_{\chi,n}$ that is a subcategory of $\mathrm{Sys}\mathcal{H}_n$. For this category, the problem of describing (unitarily) nonequivalent (orthogonally) indecomposable objects is like in the linear case. If $n \leqslant 3$, the number of such objects if finite for any fixed character χ. If $n = 4$, there exist characters for which the number of the objects is infinite but one can give a description of them for any such character. If $n = 5$ there are characters for which the description problem is wild.

It was shown in [12] that any Schur quadruple of linear spaces is linearly isomorphic to an orthoscalar quadruple with a character of the form $\chi = (\lambda; 1, 1, 1, 1)$.

2.4. **Unitarization of a collection of linear subspaces.** The category $\mathrm{Sys}\mathcal{H}_n$ is a subcategory of Sys_n, and to any orthoscalar collection there naturally corresponds a linear one. However, the converse correspondence is not single-valued. A object $\pi = (V; V_1, \dots, V_n)$ in Sys_n will be called unitarizable with a character $\chi = (\alpha_0; \alpha_1, \dots, \alpha_n)$ if there exists an object π' in $\mathrm{Sys}\mathcal{H}_{\chi,n}$, which is orthoscalar with a character χ, such that $\pi \simeq \pi'$ in Sys_n. In other words, V can be endowed with an inner product such that the collection π becomes orthoscalar with the character χ. There are two questions that arise.

(1) Which collections can be unitarized with some character ?
(2) What are the characters for which a given collection can be unitarized ?

It is known that an (orthogonally) indecomposable orthoscalar collection of subspaces must have the Schur property in the category of linear spaces [9]. Hence, an indecomposable collection can be unitarized with some character only if it has the Schur property.

For $n \leq 4$, answers to the above questions are obtained in [15].

2.5. **Representations of posets.** Let $\mathcal{N}$ be a finite poset consisting of $|\mathcal{N}| = n$ elements. One can consider the category $\mathrm{Sys}_{\mathcal{N}}$ of linear representations of this poset, namely, $\mathrm{Sys}_{\mathcal{N}}$ is a complete subcategory of Sys_n, the objects of which are collections of subspaces, $(V; V_1, \dots, V_n)$, such that $V_i \subseteq V_j$ if $i \prec j$.

The problem of classifying indecomposable nonequivalent objects in this category has been extensively studied, see [13, 5] and others. There is a list of posets which have only a finite number of indecomposable nonequivalent representations, and a list of posets with an infinite number of representations that admit a description.

Similarly to the linear case, one can consider Hilbert representations of posets, in particular, orthoscalar representations, i.e., the categories $\mathrm{Sys}\mathcal{H}_{\mathcal{N}}$ and $\mathrm{Sys}\mathcal{H}_{\chi,\mathcal{N}}$. In such a case, one obtains results for a classification of poset representations comparable to those obtained for the linear case. Some results for primitive finite type posets are obtained in [3].

2.6. **Stability of a collection of linear subspaces.** Let $\pi = (V; V_1, \dots, V_n)$ be a collection in Sys_n. The collection π will be called semistable with a character $\chi = (\alpha_0; \alpha_1, \dots, \alpha_n)$ if

$$\alpha_1 \dim V_i + \dots + \alpha_n \dim V_n = \alpha_0 \dim V,$$

$$\alpha_1 \dim(V_i \cap F) + \dots + \alpha_n \dim(V_n \cap F) \leqslant \alpha_0 \dim F,$$

for any subspace $F \subset V$, and we call it stable if all the inequalities are strict.

A collection $\pi \cap F = (F; V_1 \cap F, \dots, V_n \cap F)$ is called a subcollection of the collection π, $\dim(\pi \cap F)$ is called a subdimension of π.

There is a criterion for unitarization, see [6, 16] and a comment in [4].

Theorem. *A collection $\pi = (V; V_1, \dots, V_n)$ in Sys_n with Schur property is unitarizable with χ if and only if $\pi = (V; V_1, \dots, V_n)$ is stable with χ.*

2.7. **A description of representations of the poset that corresponds to $\widetilde{E}_8$.** The results given in this section can be found, e.g., in [2], see also the bibliography in [2].

Let $\pi = (P; X; Y_3, Y_4; Z_5, Z_6, Z_7, Z_8, Z_9)$ be a representation of the poset that corresponds to $\widetilde{E}_8$, that is, P is a finite dimensional linear space and $X \subset P$, $Y_4 \subset Y_3 \subset P$, $Z_9 \subset Z_8 \subset Z_7 \subset Z_6 \subset Z_5 \subset P$. If π is an indecomposable collection, then the value of the quadratic Tits form of the dimension of π equals either 0 or 1. If it equals 1, the dimension of π is called a real root, and such collections π are called a discrete series. There is only one indecomposable representation with such a dimension. Schur representations can be obtained in this case by using the simplest Coxeter functors. Existence of a unitarization with some character follows in this case directly from results in [8].

If the value of the Tits form equals 0, then the dimension of π is called an imaginary root, and such collections π are called a continuous series. In such a case, the dimension of π can only have the values $\sigma k = (6k; 3k; 4k, 2k; 5k, 4k, 3k, 2k, k)$, $k \in \mathbb{N}$. Here, an indecomposable collection of dimension σk will have the Schur property only if $k = 1$. A description of all Schur representations, in this case, can be obtained, see, e.g., [2], and is given by the following:

$\Gamma(1; -\lambda) = (< e_{123456} >; < e_1 + e_5, e_2 + e_6, e_4 >; < e_{1234} >, < e_2 + e_3, e_4 + e_2 >; < e_{12356} >, < e_{1356} >, < e_3 + \lambda e_1, e_{56} >, < e_{56} >, < e_5 + e_6 >)$, $\lambda \in \mathbb{C}, \lambda \neq 0, -1$;
$\Gamma_1(1; 0) = (< e_{123456} >; < e_1 + e_5, e_2 + e_6, e_4 >; < e_{1234} >, < e_2 + e_3, e_4 + e_2 >; < e_{12356} >, < e_{1356} >, < e_3 + e_1, e_{56} >, < e_{56} >, < e_5 >)$;

$\Gamma_2(1;0) = (< e_{123456} >; < e_1 + e_5, e_2 + e_6, e_4 >; < e_{1234} >, < e_2, e_4 + e_3 >; < e_{12356} >, < e_{1356} >, < e_3 + e_1, e_{56} >, < e_{56} >, < e_5 + e_6 >)$;
$\Gamma_3(1;0) = (< e_{123456} >; < e_1+e_5, e_2+e_6, e_4 >; < e_{1234} >, < e_2+e_3, e_4+e_2 >; < e_{12356} >, < e_{1356} >, < e_3, e_{56} >, < e_{56} >, < e_5 + e_6 >)$;
$\Gamma_1(1;1) = (< e_{123456} >; < e_1+e_5, e_2+e_6, e_4 >; < e_{1234} >, < e_2+e_3, e_4+e_2 >; < e_{12356} >, < e_{1356} >, < e_3 - e_1, e_{56} >, < e_{56} >, < e_5 + e_6 >)$;
$\Gamma_2(1;1) = (< e_{123456} >; < e_2 + e_5, 2e_3 + e_6, e_3 - e_1 >; < e_{1234} >, < e_2 + e_3, e_4 + e_2 >; < e_{12456} >, < e_{456}, e_1 + e_2 >, < e_{456} >, < e_{56} >, < e_6 >)$;
$\Gamma_1(1;\infty) = (< e_{123456} >; < e_1 + e_5, e_2 + e_6, e_4 >; < e_{1234} >, < e_2 + e_3, e_4 + e_2 >; < e_{12356} >, < e_{1356} >, < e_3 + e_1, e_{56} >, < e_{56} >, < e_6 >)$;
$\Gamma_2(1;\infty) = (< e_{123456} >; < e_1 + e_5, e_2 + e_6, e_4 >; < e_{1234} >, < e_3, e_4 + e_2 >; < e_{12356} >, < e_{1356} >, < e_3 + e_1, e_{56} >, < e_{56} >, < e_5 + e_6 >)$;
$\Gamma_3(1;\infty) = (< e_{123456} >; < e_1 + e_5, e_2 + e_6, e_4 >; < e_{1234} >, < e_2 + e_3, e_4 + e_2 >; < e_{12356} >, < e_{1356} >, < e_1, e_{56} >, < e_{56} >, < e_5 + e_6 >)$;
$\Gamma_4(1;\infty) = (< e_{123456} >; < e_2 + e_5, e_3 + e_6, e_1 >; < e_{1234} >, < e_2 + e_3, e_4 + e_2 >; < e_{12456} >, < e_{456}, e_1 + e_2 >, < e_{456} >, < e_{56} >, < e_6 >)$.

Here, for brevity, we denote the collection e_i, e_j, e_k by e_{ijk}.

3. Unitarization of the continuous series of Schur representations of $\widetilde{E_8}$.

The following theorem is the main result of this paper.

Theorem 3.1. *Any Schur representation of the continuous series of the poset that corresponds to $\widetilde{E_8}$ can be unitarized with some character.*

A proof of this theorem will be based on the following Propositions 3.2 and 3.3.

Proposition 3.2. *Let $\pi = (V; V_1, \dots, V_n)$ be a Schur collection of subspaces of a linear space, $V_i \neq 0$, and assume that it can be unitarized with some character. Then for all $V_{n+1} \subset V$, the collection of subspaces $\pi' = (V; V_1, \dots, V_n, V_{n+1})$ can also be unitarized with some character.*

Proof. Denote by $\sigma_i = \dim V_i$, $i = \overline{1, n+1}$, $\sigma_0 = \dim V$. Let $\pi = (V; V_1, \dots, V_n)$ be unitarizable with a character $\chi = (1; \alpha_1, \dots, \alpha_n)$. Then π is stable with χ and, hence, $\alpha_1\sigma_1 + \cdots + \alpha_n\sigma_n = \sigma_0$ and $\alpha_1 d_1 + \cdots + \alpha_n d_n < d_0$ for all subdimensions $(d_0; d_1, \dots d_n)$ of the collection π. Take $\beta_i = \alpha_i$, $i = \overline{2, n}$, $\beta_1 = \alpha_1 - \beta_{n+1}\frac{\sigma_{n+1}}{\sigma_1}$. Let us show that if β_{n+1} is taken sufficiently small so that $\beta_{n+1} < \alpha_1 \frac{\sigma_1}{\sigma_{n+1}}$ since $\beta_1 > 0$, then $\pi' = (V; V_1, \dots, V_n, V_{n+1})$ will be unitarizable with the character $(1; \beta_1, \dots, \beta_{n+1})$. For $(1; \beta_1, \dots, \beta_{n+1})$ to be unitarizable, it is necessary and sufficient that π' be stable with this character, that is, that $\beta_1\sigma_1 + \cdots + \beta_{n+1}\sigma_{n+1} = \sigma_0$ would hold for some choice of the numbers β_i, and that, for any subdimension $(d_0; d_1, \dots, d_{n+1})$ of the collection π', we would have $\beta_1 d_1 + \cdots + \beta_{n+1} d_{n+1} < d_0$.

If $D = \{(d_0^F; d_1^F, \dots, d_n^F) \mid F \subset V\}$ are all subdimensions of π, then possible subdimensions of π' will only be

$$\bigcup_{F \subset V} \{(d_0^F; d_1^F, \dots, d_n^F, 0), (d_0^F; d_1^F, \dots, d_n^F, 1), \dots, (d_0^F; d_1^F, \dots, d_n^F, \sigma_{n+1})\}.$$

In the worst case, these possible subdimensions will make all subdimensions of π', that is, β_i satisfy the inequalities $\beta_1 d_1 + \cdots + \beta_n d_n + \beta_{n+1}\sigma_{n+1} < d_0$ for all $d \in D$. This gives the conditions $(\alpha_1 - \beta_{n+1}\frac{\sigma_{n+1}}{\sigma_1})d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n + \beta_{n+1}\sigma_{n+1} < d_0$ that are verified for $d_1 = \sigma_1$ and are equivalent to $\beta_{n+1} < (d_0 - \sum_{i=1}^n \alpha_i d_i)/(\sigma_{n+1}(1 - \frac{d_1}{\sigma_1}))$ for $d_1 < \sigma_1$. Hence, if $\beta_{n+1} < \alpha_1 \frac{\sigma_1}{\sigma_{n+1}}$ and $\beta_{n+1} < (d_0 - \sum_{i=1}^n \alpha_i d_i)/(\sigma_{n+1}(1 - \frac{d_1}{\sigma_1}))$ for any $d \in D, d_1 < \sigma_1$, then for π' there must exist a unitarization with the character $(1; \beta_1, \dots, \beta_{n+1})$. $\square$

Remark. A result similar to the one stated in Proposition 3.2 was independently obtained in [4].

Proposition 3.3. *One can remove one subspace from any Schur representation of the continuous series of $\widetilde{E_8}$ in such a way that the obtained collection of subspaces will still have the Schur property.*

For a proof of this proposition, we will need the following lemma.

Lemma 3.4. *Let V be a finite dimensional vector space, $\dim V = n$, and A be an algebra of operators on this space. Let $\{< v_1 >, < v_2 >, \dots, < v_k >\}$ be a collection of one-dimensional subspaces invariant with respect to A such that the collection of vectors $\{v_1, v_2, \dots, v_{k-1}\}$ is linearly independent and $\{v_1, v_2, \dots, v_k\}$ is linearly dependent. Then any subspace W of the space $< v_1, v_2, \dots, v_k >$ is invariant with respect to A, and one-dimensional subspaces will have the same eigen value for any fixed $\phi \in A$.*

Proof. It follows from the conditions that V has a basis $\{e_1, \dots, e_n\}$ such that $\{< v_1 >, < v_2 >, \dots, < v_k >\} = \{< e_1 >, < e_2 >, \dots, < e_{k-1} >, < e_1 + e_2 + \dots + e_{k-1} >\}$. This easily implies that e_i must have the same eigen value for all $i = \overline{1, k-1}$ and any fixed $\phi \in A$. Hence, any subspace W of the space $< e_1, e_2, \dots, e_{k-1} >$ will be invariant, and one-dimensional subspaces will have the same eigen value as e_i. □

Remark. For the algebra A, the one-dimensional subspaces $< v_1 >, < v_2 >, \dots, < v_k >$ will be called connected if any subspace W of the space $< v_1, v_2, \dots, v_k >$ is invariant with respect to A (thus one-dimensional subspaces will have the same eigen value for any fixed $\phi \in A$). Note that the relationship of connectedness is "transitive" in some sense, that is, if $< v_1 >, < v_2 >, \dots, < v_k >$ are connected and $< w_1 >, < w_2 >, \dots, < w_l >$ are connected with $< v_1, v_2, \dots, v_k > \cap < w_1, w_2, \dots, w_l > \neq \emptyset$, then $< v_1 >, < v_2 >, \dots, < v_k >, < w_1 >, < w_2 >, \dots, < w_l >$ are connected.

Proof of Proposition 3.3. We use the description of representations given in Section 2.7.

1. For $\Gamma(1; -\lambda)$, $\lambda \in \mathbb{C}, \lambda \neq 0, -1$, $\Gamma_2(1; 0)$, $\Gamma_3(1; 0)$, $\Gamma_1(1; 1)$, $\Gamma_3(1; \infty)$, we can remove the subspace Z_7 (note that we get in this case the same system of subspaces when removing Z_7).

Indeed, let us show that the algebra of endomorphisms A of the obtained system of subspaces remains trivial. Recall that if the subspaces Z_1 and Z_2 are invariant with respect to an algebra of operators, then $Z_1 \cap Z_2$ and $Z_1 + Z_2$ will also be invariant.

In this case, the following subspaces will be invariant with respect to A:

$Z_5 \cap X = < e_1 + e_5, e_2 + e_6 >$,
$X \cap Y_3 = < e_4 >$,
$Y_3 \cap Z_5 = < e_{123} >$,
$< e_{123} > \cap Z_6 = < e_{13} >$,
$< e_{123} > \cap Y_4 = < e_2 + e_3 >$,
$(X + Z_8) \cap Y_4 = < e_4 + e_2 >$,
$< e_4 > + < e_4 + e_2 > = < e_{42} >$,
$< e_{42} > \cap < e_{123} > = < e_2 >$.

Since $< e_4 + e_2 >, < e_4 >, < e_2 >$ are invariant, by Lemma 3.4, $< e_4 >$ and $< e_2 >$ are connected.

$Y_4 \cap (< e_{13} > + < e_4 >) = < e_4 - e_3 >$,
$< e_4 > + < e_4 - e_3 > = < e_{43} >$,
$< e_{43} > \cap < e_{123} > = < e_3 >$.

Since $< e_4 - e_3 >, < e_4 >, < e_3 >$ are invariant, $< e_4 >$ and $< e_3 >$ are connected.

$(< e_2 > + Z_8) \cap X = < e_2 + e_6 >$,
$(< e_2 + e_6 > + < e_2 >) \cap Z_8 = < e_6 >$.

Since $< e_2 + e_6 >, < e_2 >, < e_6 >$ are invariant, $< e_2 >$ and $< e_6 >$ are connected.

$Z_6 \cap X = < e_1 + e_5 >$,

$< e_1 + e_5 > +Z_8 =< e_{156} >$,
$< e_{156} > \cap < e_{13} >=< e_1 >$,
$< e_1 > + < e_1 + e_5 >=< e_{15} >$,
$< e_{15} > \cap Z_8 =< e_5 >$.
Since $< e_1 + e_5 >, < e_1 >, < e_5 >$ are invariant, $< e_1 >$ and $< e_5 >$ are connected.
Since $< e_5 + e_6 >, < e_5 >, < e_6 >$ are invariant, $< e_5 >$ and $< e_6 >$ are connected.
Hence, $< e_1 >, < e_2 >, < e_3 >, < e_4 >, < e_5 >, < e_6 >$ are connected and thus A is trivial.

2. For $\Gamma_1(1; 0)$, $\Gamma_1(1; \infty)$, we can remove Z_9.

Similarly to the previous case, we see that $< e_4 >, < e_2 >, < e_3 >, < e_6 >$ are connected, and $< e_1 >, < e_5 >$ are also connected.

Since $Z_7 \cap (< e_3 > + < e_1 >) =< e_3 + e_1 >$ is invariant, we see that $< e_1 >$ and $< e_3 >$ are also connected. Hence, $< e_1 >, < e_2 >, < e_3 >, < e_4 >, < e_5 >, < e_6 >$ are connected, so the algebra of endomorphisms is trivial.

3. For $\Gamma_2(1; \infty)$, we can remove Z_6.

The algebra of endomorphisms, A, of the obtained system leaves the following subspaces invariant:

$Z_5 \cap X =< e_1 + e_5, e_2 + e_6 >$,
$X \cap Y_3 =< e_4 >$,
$Y_3 \cap Z_5 =< e_{123} >$,
$< e_{123} > \cap Y_4 =< e_3 >$,
$(X + Z_8) \cap Y_4 =< e_4 + e_2 >$,
$< e_4 > + < e_4 + e_2 >=< e_{42} >$,
$< e_{42} > \cap < e_{123} >=< e_2 >$.
Since $< e_4 + e_2 >, < e_4 >, < e_2 >$ are invariant, $< e_4 >$ and $< e_2 >$ are connected.
$(< e_2 > +Z_8) \cap X =< e_2 + e_6 >$,
$(< e_2 + e_6 > + < e_2 >) \cap Z_8 =< e_6 >$.
Since $< e_2 + e_6 >, < e_2 >, < e_6 >$ are invariant, $< e_2 >$ and $< e_6 >$ are connected.
$Z_7 \cap < e_{123} >=< e_1 + e_3 >$,
$< e_1 + e_3 > + < e_3 >=< e_{13} >$,
$(< e_{13} > +Z_8) \cap X =< e_1 + e_5 >$,
$< e_1 + e_5 > +Z_8 =< e_{156} >$,
$< e_{156} > \cap < e_{13} >=< e_1 >$,
$< e_1 > + < e_1 + e_5 >=< e_{15} >$,
$< e_{15} > \cap Z_8 =< e_5 >$.
Since $< e_1 + e_5 >, < e_1 >, < e_5 >$ are invariant, $< e_1 >$ and $< e_5 >$ are connected.
Since $< e_5 + e_6 >, < e_5 >, < e_6 >$ are invariant, $< e_5 >$ and $< e_6 >$ are connected.
Invariance of the subspaces $< e_1 + e_3 >, < e_1 >, < e_3 >$ shows that $< e_1 >$ and $< e_3 >$ are connected.

Hence, $< e_1 >, < e_2 >, < e_3 >, < e_4 >, < e_5 >, < e_6 >$ are connected, whence A is trivial.

4. For $\Gamma_4(1; \infty)$, we can remove Z_5.

The algebra A of endomorphisms of the obtained system has the following invariant subspaces:

$X \cap Y_4 =< e_1 >$, $Z_6+ < e_1 >=< e_{12456} >$,
that is, Z_5 is invariant. This means that the algebra of endomorphisms is trivial, since the system $\Gamma_4(1; \infty)$, together with Z_5, has trivial algebra of endomorphisms. □

Remark. Proposition 3.3 can be proved in another way. Since, after removing a subspace, we get a collection of subspaces that corresponds to a representation of the poset E_8, instead of proving that the obtained collection has the Schur property, we could prove that this collection is equivalent to one of Schur representations of E_8 using the description of representations of E_8, see, e.g., [5].

Proof. Proof of Theorem 3.1. Since any Schur representation of the poset E_8 can be unitarized with some character, see [3], this theorem is a direct corollary of Propositions 3.2 and 3.3. □

Remark. In the same way as $\widetilde{E_6}$ was considered in [17], one can try to describe the characters that allow a unitarization of representations for the posets $\widetilde{E_7}$ and $\widetilde{E_8}$. However, the complexity of the calculations of admissible characters increases significantly in these cases.

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References

1. S. Albeverio, V. Ostrovskyi, Yu. Samoilenko, *On functions on graphs and representations of a certain class of *-algebras*, J. Algebra **308** (2007), no. 2, 567–582.
2. P. Donovan and M. Freislich, *The Representation Theory of Finite Graphs and Associated Algebras*, Carleton Univ., Ottawa, 1974.
3. R. Grushevoy, K. Yusenko, *On the unitarization of linear representations of primitive partially ordered sets*, Oper. Theory Adv. Appl. **190** (2009), 279–294.
4. R. Grushevoy, K. Yusenko, *Unitarization of linear representations of non-primitive posets*, http://arxiv.org/pdf/0807.0155.
5. M. M. Kleiner, *Finite type partially ordered sets*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **28** (1972), 32–41.
6. A. A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), no. 3, 419–445.
7. S. Kruglyak, S. Popovich, Yu. Samoilenko, *The spectral problem and ∗-representations of algebras associated with Dynkin graphs*, J. Algebra Appl. **4** (2005), no. 6, 761–776.
8. S. A. Kruglyak, S. V. Popovich, and Yu. S. Samoilenko, *The spectral problem and algebras associated with extended Dynkin graphs.* I., Methods Funct. Anal. Topology **11** (2005), no. 4, 383–396.
9. S. A. Kruglyk, L. A. Nazarova, A. V. Roiter, *Orthoscalar representations of quivers in the category of Hilbert spaces*, Problems of the theory of representations of algebras and groups, 14, Zap. Nauchn. Sem. St.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) **338** (2006), 180–201.
10. S. A. Kruglyak, V. I. Rabanovich, Yu. S. Samoĭlenko, *On sums of orthogonal projections*, Funktsional. Anal. i Prilozhen. **36** (2002), no. 3, 20–35.
11. S. A. Kruglyak, A. V. Roiter, *Locally scalar representations of graphs in the category of Hilbert spaces*, Funktsional. Anal. i Prilozhen. **39** (2005), no. 2, 13–30.
12. Yu. P. Moskaleva and Yu. S. Samoilenko, *Systems of n subspaces and representations of ∗-algebras generated by projections*, Methods Funct. Anal. Topology **12** (2006), no. 1, 57–73.
13. L. A. Nazarova, A. V. Roiter, *Representations of partially ordered sets*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **28** (1972), 5–31.
14. V. L. Ostrovskyi, Yu. S. Samoilenko *On spectral theorems for families of linearly connected selfadjoint operators with prescribed spectra associated with extended Dynkin graphs*, Ukrain. Mat. Zh. **58** (2006), no. 11, 1556–1570. (Ukrainian)
15. Yu. S. Samoilenko, D. Yu. Yakymenko *On n-tuples of subspaces in linear and unitary spaces*, Methods Funct. Anal. Topology **15** (2009), no. 1, 383–396.
16. Yi Hu *Stable configurations of linear subspaces and quotient coherent sheaves*, Pure and Applied Mathematics Quartely **1** (2005), no. 1, 127–164.
17. D. Yu. Yakimenko *Unitarization of representations of a partially ordered set associated with a graph $\widetilde{E_6}$*, Ukrain. Mat. Zh. **61** (2009), no. 10, 1424–1433. (Russian)
18. D. Yu. Yakimenko *Unitarization of representations of the partially ordered set corresponding to the graph $\widetilde{E_7}$*, Ukrain. Mat. Zh. (to appear).

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