Methods of Functional Analysis and Topology
Vol. 16 (2010), no. 3, pp. 203–241

THE STRONG HAMBURGER MOMENT PROBLEM AND RELATED DIRECT AND INVERSE SPECTRAL PROBLEMS FOR BLOCK JACOBI-LAURENT MATRICES

YURIJ M. BEREZANSKY AND MYKOLA E. DUDKIN

Dedicated with great pleasure to F. H. Szafraniec on the occasion of his 70th birthday

Abstract. In this article we propose an approach to the strong Hamburger moment problem based on the theory of generalized eigenvectors expansion for a selfadjoint operator. Such an approach to another type of moment problems was given in our works earlier, but for strong Hamburger moment problem it is new. We get a sufficiently complete account of the theory of such a problem, including the spectral theory of block Jacobi-Laurent matrices.

1. Introduction

A theory of the moment problem is connected with generalized eigenvectors expansion approach, by means of which it is possible to investigate different situations. In this approach we firstly obtain a moment representation by applying the theory of eigenfunction expansion in generalized eigenvectors to the corresponding operators. For such vectors we get a simple equation depending on the moment problem under consideration,— a solution of this equation gives a form of the representation. The corresponding Parseval equality gives the moment representation itself.

After this we connect, with the considered moments, a Jacobi type three-diagonals block matrix the spectral measure of which is equal to the measure in the moment representation. The corresponding spectral theory for such a matrix gives further information about the considered moment problem.

Such an approach gives a possibility to investigate the following moment problems: classical, trigonometric, complex, matrix and different many-dimensional analogs of them, including infinite-dimensional cases (in many-dimensional situation it is necessary to investigate commuting families of Jacobi type operators), see [4, Ch. 7, 8], [5, Ch. 5, Section 2], [7, 8, 9, 11, 12, 13, 14].

This article is devoted to a demonstration of such an approach applied to an investigation of the strong Hamburger (strong) moment problem. We get a sufficiently complete account of the theory of the strong moment problem based on the spectral theory of selfadjoint operators and, in fact, independent of previous works in this direction.

It is necessary to note that an idea, similar to the above mentioned one, of investigating positive defined functions, moment problems etc. belongs to M. G. Krein (1946–1948, [22, 23]). He has constructed a Hilbert space by using the investigated positive definite kernel, and, to natural operators on this space and connected with investigated problem, he had applied the method of directed functionals he had created in those years. Yu. M. Berezansky in 1956 [2] had applied, to such operators, a general method of a

2000 *Mathematics Subject Classification.* Primary: 44A60, 47A57, 47A70.

Key words and phrases. Classical and strong moment problems, block three-diagonal matrix, eigenfunction expansion, generalized eigenvector.

generalized eigenfunctions expansion, which gave, in particular, the above mentioned results for the moment problems.

The strong Hamburger moment problem is formulated in the following way. We have a sequence $s = (s_n)_{n=-\infty}^{\infty}$ of real numbers s_n. What is the case where the numbers s_n are moments of some measure $d\rho(\lambda)$ on the Borel σ-algebra $\mathfrak{B}(\mathbb{R})$, i.e.,

$$s_n = \int_{\mathbb{R}} \lambda^n \, d\rho(\lambda), \quad n \in \mathbb{Z} := \{\dots, -1, 0, 1, \dots\}. \tag{1.1}$$

If the representation (1.1) holds true only for $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\} = \{0\} \cup \mathbb{N}$, we have the classical moment problem [1, 4, 31], an answer to it is well known,— a sequence $(s_n)_{n=0}^{\infty}$ is a moment sequence iff for an arbitrary finite sequence $(f_n)_{n=0}^{\infty}$ of complex numbers f_n the following inequality takes place:

$$\sum_{j,k=0}^{\infty} s_{j+k} f_j \bar{f}_k \geq 0. \tag{1.2}$$

In other words, the matrix $(s_{j+k})_{j,k=0}^{\infty}$ must be positive (i.e. nonnegative) definite.

From these definitions it follows that every strong moment sequence is a classical one, $\mathbb{Z} \supset \mathbb{N}_0$. It is possible to understand that, since we have representation (1.1) for $n \in \mathbb{N}_0$, we can extend it to $n \in \mathbb{Z}_- := \{\dots, -2, -1\}$ iff the measure $d\rho(\lambda)$ near the point 0 is "small", every integral for $n \in \mathbb{Z}$ must be exist.

Such a situation with a measure corresponds to reality but this problem is not simple and there are many papers have appeared starting in 1983–1984 that deal with a study of the strong Hamburger moment problem. We will not give here a corresponding list and only refer to the detailed survey [21] and to some articles closer connected with our work.

It is necessary to say that representation (1.1) takes place iff the condition (1.2) is fulfilled for arbitrary finite $(f_n)_{n=-\infty}^{\infty}$ with summation going from $-\infty$ to ∞. This result was published in 1984 in the work [20] but we would like to say that representation (1.1) and its equivalence to positivity of type (1.2) was obtained in 1965 by Yu. M. Berezansky in [4] (see also [3]) as a special case of a more general theorem (a more detailed account of this fact will be given in the Section 8 of this article). Note also, that the strong moment problem has appeared at first in the article of A. A. Nudelman [28].

Similarly to the classical moment problem, the same problems arise for its strong variant,— what are the cases where representation (1.1) is unique, if we have a nonuniqueness, — in what way is it possible to describe all measures $d\rho(\lambda)$ with a given moment sequence $s = (s_n)_{n=-\infty}^{\infty}$. Now, the so-called Laurent polynomials, i.e., finite linear combinations of λ^n, $n \in \mathbb{Z}$, become essential. Important questions now are: what is an analog of a Jacobi matrix connected with s and what spectral theory such matrices have, etc.?

Many of these questions are investigated in the articles cited in [21] and in [27]. But an approach to the corresponding problem was analytical, often without application of the corresponding natural tools of the spectral theory of operators. It is necessary to say that applying the theory of generalized eigenvectors and the corresponding results to Jacobi matrices and positive definite kernels ([4], Ch. 5, 7, 8) to such problem gives the very clear picture similar to the classical moment problem. Unfortunately, the corresponding authors used another ways.

For our investigations, important are the works [19, 33, 34] (note that these works do not also use the theory of generalized eigenvectors, etc.). For the Laurent polynomials, the Laurent-Jacobi matrices corresponding to strong moment problem are presented in [19]. Here, an operator generated by such a matrix was considered and some of its spectral properties were investigated. The articles [33, 34] generalized some results on strong moment problem from [21] on the matrix case. The author systematically used the operator theory and namely his works gave an insentive for writing this article. Also note some last works [16, 17] concerning the questions connected with strong moment problem which had some influence on our constructions.

The presentation of this work is as follows. In Section 2 we recall the main results about the generalized eigenvector expansion which are necessary for the following. Section 3 gives a proof of main theorem of representation (1.1) and some condition of uniqueness of the measure.

Section 4 is devoted to the construction and an investigation of a block Jacobi type matrix connected with (1.1), (1.2). It is necessary to note that our Jacobi-Laurent matrix is a block three-diagonal matrix, instead of five-diagonal numerical matrix in the previous investigations. The use of block matrices, often with blocks havin different dimension, instead of numerical ones is very convenient in the corresponding situations and were earlier proposed by the authors in [11, 12] for trigonometric and complex moment problems; see also [10, 26]. To devise such an approach it was very essential to consider results of the article [36] devoted to a complex moment problem. The convenience of block matrices consists in a more easy finding the relations between the objects of consideration, as it formally the case for the classical Jacobi matrix.

The main results of this Section are the Theorems 5, 6. Some parts of these theorems are published in articles [34, 35], but we stress that our proofs are practically a repeatetion of proofs from [11, 12] for similar problems.

Our Jacobi-Laurent matrix J is symmetric and has algebraic inverse J^{-1} which is also a block three-diagonal matrix with corresponding properties. There are still some problems which include the following: to describe such matrices J in an inner way similar to the case of five-diagonals unitary matrices (i.e. three-diagonal block). In the unitary case such a description was given by S. Verblunsky (see book [32]), we plan to give a corresponding description for J in separate article.

In the Section 5 we present a spectral theory for block Jacobi-Laurent matrices, including the direct and inverse spectral problems. The constructions are similar to classical Jacobi matrices [4], Ch. 7, and we use the generalized eigenfunction expansion. In Section 6 we consider the Jacobi-Laurent matrix J in a general case, when J generates only a Hermitian operator, not a selfadjoint one. We construct the corresponding theory similar to the case of classical Jacobi matrices. We outline the theory describing all selfadjoint extensions in the initial Hilbert space but do not give a complete account of the theory since, on the one hand, it would require too much space and, on the other hand, the constructions are very similar to the classical case of a Jacobi matrix given in the book [4], Ch. 7.

Let us also stress that in this article we do not consider selfadjoint extension of the operator generated by J that would act on a space larger that the initial space. So, *we consider only "orthogonal" spectral measures.*

Section 7 is devoted to a construction, from the initial moments s_n, $n \in \mathbb{Z}$, of a Jacobi-Laurent matrix and to a discussion of the connection between the initial measures $d\rho(\lambda)$ from (1.1) and the spectral measure of selfadjoint operators generated by J. This section clarifies how the theory of selfadjoint extensions of Jacobi-Laurent matrix gives a description of all solutions of problem (1.1).

In the Section 8 we do the following: 1) explain a connection with the former results of Yu. M. Berezansky [3, 4] and the strong moment problem; 2) outline a way of investigating the strong matrix moment problem using the approach of Sections 2–6 and the work [24] of M. G. Krein.

Note, that for the convenient of readers we often at first present a particular case of the corresponding theory. So, in the Section 4 we at first consider the most simple case of a bounded operator generated by a Jacobi-Laurent matrix, i.e., the case where the measure $d\rho(\lambda)$ has bounded support. Then we pass to the general case where the support is arbitrary, but the set of all functions $\mathbb{R} \ni \lambda \longmapsto \lambda^m$, $m \in \mathbb{Z}$, is total in the space $L^2(\mathbb{R}, d\rho(\lambda))$ (such measures can be exotic enough).

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and let A be a selfadjoint operator defined on $\mathrm{Dom}(A)$ in $\mathcal{H}$. Consider a rigging of $\mathcal{H}$

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset \mathcal{D}, \tag{2.1}$$

such that $\mathcal{H}_+$ is a Hilbert space topologically and quasinuclear embedded into $\mathcal{H}$ (topologically means densely and continuously; quasinuclear means that the inclusion operator is of Hilbert-Schmidt type); $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$ with respect to space $\mathcal{H}$; $\mathcal{D}$ is a linear, topological space, topologically embedded into $\mathcal{H}_+$.

The operator A is called standardly connected with the chain (2.1) if $\mathcal{D} \subset \mathrm{Dom}(A)$ and the restriction $A \upharpoonright \mathcal{D}$ acts from $\mathcal{D}$ into $\mathcal{H}_+$ continuously.

We formulate a short version of the projection spectral theorem (see [4], Ch. 5, [5], Ch. 3, [6], Ch. 15).

Theorem 1. *Let A be a selfadjoint operator defined on a separable Hilbert space $\mathcal{H}$ and standardly connected with the chain (2.1), where $\mathcal{D}$ is separable. Then there exist an operator-valued function $\Phi(\lambda)$ and a bounded Borel (general) spectral measure $d\sigma(\lambda)$ such that $\Phi(\lambda)$ is weakly measurable and is defined for almost all λ from the spectrum $s(A)$ of the operator A in the sense of the spectral measure $d\sigma(\lambda)$ and takes values in non-negative operators from $\mathcal{H}_+$ into $\mathcal{H}_-$, and for every λ its Hilbert-Schmidt norm satisfies the equality $|\Phi(\lambda)| \leq \mathrm{Tr}(\Phi(\lambda)) = 1$. Here Tr denotes the trace of corresponding operator. The function $\Phi(\lambda)$ and the measure $d\sigma(\lambda)$ give a representation of the expansion of the identity E of A,*

$$E(\Delta)f = \Big(\int_{\Delta} \Phi(\lambda)\, d\sigma(\lambda)\Big) f, \quad \Delta \in \mathfrak{B}(\mathbb{R}), \quad f \in \mathcal{H}_+, \tag{2.2}$$

and of the operator A,

$$Af = \Big(\int_{s(A)} \lambda\Phi(\lambda)\, d\sigma(\lambda)\Big) f, \quad f \in \mathrm{Dom}(A) \cap \mathcal{H}_+. \tag{2.3}$$

The set of values $\mathrm{Ran}(\Phi(\lambda)) \subset \mathcal{H}_-$ consists of a generalized eigenvector $\varphi(\lambda) \in \mathcal{H}_-$ of the operator A with the corresponding eigenvalue λ, i.e.,

$$(\varphi(\lambda), Af)_{\mathcal{H}} = \lambda(\varphi(\lambda), f)_{\mathcal{H}}, \quad \lambda \in \mathbb{R}, \quad f \in \mathcal{D}; \quad \varphi(\lambda) \neq 0. \tag{2.4}$$

In a general case, for the operator A appearing in the Theorem 1, it is possible to construct the expansion of almost arbitrary vector $f \in \mathcal{H}$ in the generalized eigenvectors of operator A in the form of a "Fourier transform" (see [4], Ch. 5, [5], Ch. 3, in particular, [6], Ch. 15, Section 3). But in a general situation the dimension of the vector, the Fourier transform $\hat{f}(\lambda)$, depends on the "multiplicity" of the eigenvalue λ and the corresponding formulas are not very effective. But in some special case of operators A, the language of the Fourier transform is very convenient and replaces formulas (2.2), (2.3).

Let A be some selfadjoint operator on $\mathcal{H}$, a vector $q \in \mathcal{H}$ is called cyclic if $q \in \mathrm{Dom}(A^n)$, $n \in \mathbb{N}$. Let A has an algebraically inverse operator A^{-1} (i.e., for $f \in \mathrm{Dom}(A)$), $A^{-1}Af = f$, $\mathrm{Dom}(A^{-1}) = \mathrm{Ran}(A)$). A cyclic vector q, is called double cyclic if $q \in \mathrm{Dom}(A^n)$, $n \in \mathbb{Z}$.

Theorem 2. *Let A be a selfadjoint operator such that all conditions of Theorem 1 are fulfilled. Assume that there exists a cyclic vector q for this operator (or double cyclic, in this case we assume that $A^{-1}f \in \mathcal{D}$ if $f \in \mathcal{D}$ and the algebraic inverse A^{-1} is defined on $\mathcal{D}$) which is generating in the following sense: $\forall n \in \mathbb{N}_0$ (or $\forall n \in \mathbb{Z}$) $A^n q \in \mathcal{D}$ and the set of such vectors is total in $\mathcal{D}$.*

Then the spectrum of A is simple and for every $\lambda \in s(A)$ the corresponding generalized eigenvector $\varphi(\lambda) \in \mathcal{H}_-$ exists and we can introduce the Fourier transform F,

$$\mathcal{D} \ni f \longmapsto (Ff)(\lambda) = \hat{f}(\lambda) := (f, \varphi(\lambda))_{\mathcal{H}} \in \mathbb{C}. \tag{2.5}$$

Instead of (2.2), (2.3) we have an equivalent representation: $\forall f, g \in \mathcal{D}$

$$(f, g)_{\mathcal{H}} = \int_{\mathbb{R}} f(\lambda)\overline{g(\lambda)}\, d\sigma(\lambda), \quad (\widehat{Af})(\lambda) = \lambda \hat{f}(\lambda), \quad \lambda \in s(A). \tag{2.6}$$

With the help of extension by continuity, definition (2.5) can be extended to all $f \in \mathcal{H}$, then $\hat{f}(\lambda) \in L^2(\mathbb{R}, d\sigma(\lambda))$. The first equality in (2.6) extends to $f, g \in \mathcal{H}$, becoming the Parseval equality. The second equality extends to $f \in \operatorname{Dom}(A)$ and shows that our operator is unitary equivalent to the operator of multiplication by λ on the space $L^2(\mathbb{R}, d\sigma(\lambda))$, acting on $\hat{f}(\lambda)$, $f \in \operatorname{Dom}(A)$.

Proof. This theorem, in the case of a cyclic vector, is proved in [6], Ch. 15, Theorem 3.2.

In the case of a double cyclic vector q, it is necessary to repeat the proof of this Theorem 3.2. Doing so there is only one place needed to be explained. Namely let $\varphi(\lambda) := P(\lambda)Jf_0 \in \mathcal{H}_-$ be the vector from the proof of this theorem. We have the following: $(\varphi(\lambda), q)_{\mathcal{H}} = 0$ (in [8] q was denoted by Ω).

It is necessary to show that $\varphi(\lambda) = 0$. This vector is a generalized eigenvector with the eigenvalue λ, i.e.,

$$(\varphi(\lambda), Af)_{\mathcal{H}} = \lambda(\varphi(\lambda), f)_{\mathcal{H}}, \quad f \in \mathcal{D}.$$

Therefore $\forall f \in \mathcal{D}$,

$$(\varphi(\lambda), f)_{\mathcal{H}} = (\varphi(\lambda), A(A^{-1}f))_{\mathcal{H}} = \lambda(\varphi(\lambda), A^{-1}f)_{\mathcal{H}}, \tag{2.7}$$

since, by conditions the theorem, $A^{-1}f \in \mathcal{D}$. From (2.7) we conclude that $\lambda \neq 0$: if $\lambda = 0$, then $(\varphi(\lambda), f)_{\mathcal{H}} = 0$, $f \in \mathcal{D}$, i.e., $\varphi(\lambda) = 0$ and the proof is finished.

So, let $\lambda \neq 0$. Using (2.7) we get

$$(\varphi(\lambda), A^{-1}f)_{\mathcal{H}} = \lambda^{-1}(\varphi(\lambda), f)_{\mathcal{H}}, \quad f \in \mathcal{D}. \tag{2.8}$$

By iterating (2.8) (note that $A^{-1}f \in \mathcal{D}$), we have

$$(\varphi(\lambda), A^{-n}f)_{\mathcal{H}} = \lambda^{-n}(\varphi(\lambda), f)_{\mathcal{H}}, \quad f \in \mathcal{D}, \quad n \in \mathbb{N}. \tag{2.9}$$

For nonnegative powers of A we evidently have that

$$(\varphi(\lambda), A^n q)_{\mathcal{H}} = \lambda^n(\varphi(\lambda), f)_{\mathcal{H}} = 0, \quad n \in \mathbb{N}_0. \tag{2.10}$$

Taking $f = q$ in (2.9) we conclude that (2.10) takes place for $n \in \mathbb{Z}$. But by conditions of the theorem, the set $\{A^n q, n \in \mathbb{Z}\}$ is total in $\mathcal{D}$. Therefore, this is also true in the double cyclic case $\varphi(\lambda) = 0$. □

We will use below in this article some conditions of selfadjointness, connected with the notion of a quasianalytic vector. We will recall these results. So for a Hermitian operator A defined on $\operatorname{Dom}(A)$ in $\mathcal{H}$, the vector $f \in \bigcap_{n=1}^{\infty} \operatorname{Dom}(A^n)$ is called quasianalytic [29, 30] if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\|A^n f\|_{\mathcal{H}}}} = \infty.$$

Theorem 3. *A closed Hermitian operator A is selfadjoint on the Hilbert space $\mathcal{H}$ iff the space $\mathcal{H}$ contains a total set of quasianalytic vectors.*

Versions of this theorem are published in [29, 30], see also [4], Ch. 8, Section 5. For the given form of it, see [6], Ch. 13, Section 9.

3. The strong Hamburger moment problem

A solution of the strong Hamburger moment problem is given in the next theorem.

Theorem 4. *A given sequence of real numbers $s = (s_n)_{n=-\infty}^{\infty} =: (s_n)$, $n \in \mathbb{Z}$, $s_n \in \mathbb{R}$ admits the representation*

$$s_n = \int_{\mathbb{R}} \lambda^n \, d\rho(\lambda), \quad n \in \mathbb{Z}, \tag{3.1}$$

with some Borel measure $d\rho(\lambda)$ iff it is positive definite, i.e.,

$$\sum_{j,k\in\mathbb{Z}} s_{j+k} f_j \bar{f}_k \geq 0 \tag{3.2}$$

for every finite sequences of complex numbers (f_j), $j \in \mathbb{Z}$, $f_j \in \mathbb{C}$.

The measure in representation (3.1) is unique if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty. \tag{3.3}$$

Proof. Necessity of the condition (3.2) is obvious. Indeed, if the sequence s has representation (3.1), then for an arbitrary finite sequence $f = (f_k)_{k\in\mathbb{Z}}$, $f_k \in \mathbb{C}$, we have

$$\sum_{j,k\in\mathbb{Z}} s_{j+k} f_j \bar{f}_k = \int_{\mathbb{R}} \Big| \sum_{n\in\mathbb{Z}} f_n \lambda^n \Big|^2 d\rho(\lambda) \geq 0. \tag{3.4}$$

Denote by l the linear space $\mathbb{C}^\infty$ of sequences $f = (f_j), j \in \mathbb{Z}$, $f_j \in \mathbb{C}$, and by l_{fin} its linear subspace consisting of finite sequences $f = (f_j), j \in \mathbb{Z}$, i.e., sequences such that $f_j \neq 0$ for only a finite number of j. Let δ_m, $m \in \mathbb{Z}$, be the δ-sequence, i.e., $\delta_m = (0, \ldots, \underset{m \ \text{place}}{0, 1}, 0, 0, \ldots)$. Then each $f \in l_{\text{fin}}$ has the representation $f = \sum_{m\in\mathbb{Z}} f_m \delta_m$.

Let us consider a linear operator J,

$$(Jf)_j = f_{j-1}, \quad j \in \mathbb{Z}; \quad \operatorname{Dom}(J) = l_{\text{fin}}. \tag{3.5}$$

The operator J is a "creation" type operator. For the δ-sequence we get

$$J\delta_m = \delta_{m+1}, \quad m \in \mathbb{Z}. \tag{3.6}$$

The operator J is Hermitian with respect to the (quasi)scalar product consistent with (3.4),

$$(f, g)_S = \sum_{j,k\in\mathbb{Z}} s_{j+k} f_j \bar{g}_k, \quad f, g \in l_{\text{fin}}. \tag{3.7}$$

Indeed

$$\begin{aligned}
(Jf, g)_S &= \sum_{j,k\in\mathbb{Z}} s_{j+k} (Jf)_j \bar{g}_k = \sum_{j,k\in\mathbb{Z}} s_{j+k} f_{j-1} \bar{g}_k \\
&= \sum_{j,k\in\mathbb{Z}} s_{j+k+1} f_j \bar{g}_k = \sum_{j,k\in\mathbb{Z}} s_{j+k} f_j \bar{g}_{k-1} \\
&= \sum_{j,k\in\mathbb{Z}} s_{j+k} f_j \overline{(Jg)}_k = (f, Jg)_S.
\end{aligned}$$

In the next step we use Theorem 2. For simplicity, we suppose that the given sequence $s = (s_n)$, $n \in \mathbb{Z}$, is nondegenerate, i.e., if $(f, f)_S = 0$ for $f \in l_{\text{fin}}$, then $f = 0$. The investigation in general case is more complicated, we will return to it at the end of this proof. So, (3.4) now defines some scalar product on l_{fin}. Let S be a Hilbert space constructed as the completion w.r.t. (3.4).

Consider the operator J (3.5). It is Hermitian and defined on the domain $\text{Dom}(J) = l_{\text{fin}}$ dense in S. Moreover it is real in S w.r.t. to usual passage from $f = (f_j)$, $j \in \mathbb{Z}$, to $\bar{f} = (\bar{f}_j)$, $j \in \mathbb{Z}$. Therefore it has equal deficiency numbers and can be extended to a selfadjoint operator in S.

We take and fix such an extension A. We will apply the general results of Section 2 to this operator A. But at first it is necessary to construct some rigging of the space S.

So, we will consider the following rigging:

$$(l_2(p))_{-,S} \supset S \supset l_2(p) \supset l_{\text{fin}}, \tag{3.8}$$

where $l_2(p)$ is a weighted l_2-type space (l_2 space on $\mathbb{Z}$) with a weight $p = (p_n), n \in \mathbb{Z}$, $p_n \geq 1$. The norm in $l_2(p)$ is given by $\|f\|^2_{l_2(p)} = \sum_{n\in\mathbb{Z}} |f_n|^2 p_n$; $(l_2(p))_{-,S} = \mathcal{H}_-$ is the negative space with respect to the positive space $l_2(p) = \mathcal{H}_+$ and the zero space $S = \mathcal{H}$. The space $l_{\text{fin}} = \mathcal{D}$ is provided with the coordinate-wise uniform finite convergence.

Lemma 1. *There exists a sufficiently fast increasing sequence p such that the embedding $l_2(p) \hookrightarrow S$ takes place and is quasinuclear.*

Proof. The inequality (3.2) means that the matrix $(K_{j,k}), j, k \in \mathbb{Z}$, where $K_{j,k} = s_{j+k}$, is nonnegative definite and, therefore,

$$|s_{j+k}|^2 = |K_{j,k}|^2 \leq K_{j,j} K_{k,k} = s_{2j} s_{2k}, \quad j, k \in \mathbb{Z}. \tag{3.9}$$

Let the weight $q = (q_j), j \in \mathbb{Z}$, $q_j \geq 1$, be such that $\sum_{j\in\mathbb{Z}} s_{2j} q_j^{-1} < \infty$. Then from (3.7) and (3.9) it follows that

$$\|f\|^2_S = \sum_{j,k\in\mathbb{Z}} s_{j+k} f_j \bar{f}_k \leq \Bigg(\sum_{j,k\in\mathbb{Z}} \frac{s_{2j}}{q_j} \Bigg)^2 \|f\|^2_{l_2(q)}, \quad f \in l_{\text{fin}}.$$

Therefore, $l_2(q) \hookrightarrow S$ topological. And if $\sum_{j\in\mathbb{Z}} q_j p_j^{-1} < \infty$, then $l_2(p) \hookrightarrow l_2(q)$ is quasinuclear. The composition $l_2(p) \hookrightarrow S$ of the quasinuclear and topological embedding is also a quasinuclear one. ☐

In the next step we use the rigging (3.8) to construct generalized eigenvectors. The inner structure of the space $(l_2(p))_{-,S}$ is complicated, because of the complicated structure of S. This is a reason to introduce a new auxiliary rigging.

$$l = (l_{\text{fin}})' \supset (l_2(p^{-1})) \supset l_2 \supset l_2(p) \supset l_{\text{fin}}, \tag{3.10}$$

where $l_2(p^{-1})$, $p^{-1} = (p_n^{-1})$, $n \in \mathbb{Z}$, is a negative space with respect to the positive space $l_2(p)$ and the zero space l_2. Chains (3.8) and (3.10) have the same positive space $l_2(p)$. The next general Lemma [8] establishes that the space $(l_2(p))_{-,S}$ is isometric to the space $l_2(p^{-1})$.

Lemma 2. *Suppose we have two riggings,*

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+, \quad \mathcal{F}_- \supset \mathcal{F} \supset \mathcal{F}_+ = \mathcal{H}_+, \tag{3.11}$$

with the equal positive spaces. Then there exists a unitary operator $U : \mathcal{H}_- \to \mathcal{F}_-$, $U\mathcal{H}_- = \mathcal{F}_-$, such that

$$(U\xi, f)_{\mathcal{F}} = (\xi, f)_{\mathcal{H}}, \quad \xi \in \mathcal{H}_-, \quad f \in \mathcal{H}_+ = \mathcal{F}_+. \tag{3.12}$$

This operator can be given as follows: $U = \mathbb{I}_{\mathcal{F}}^{-1} \mathbb{I}_{\mathcal{H}}$, where $\mathbb{I}_{\mathcal{F}}$ and $\mathbb{I}_{\mathcal{H}}$ are standard unitary maps in corresponding chains ($\mathbb{I}_{\mathcal{F}} \mathcal{F}_- = \mathcal{F}_+$, $\mathbb{I}_{\mathcal{H}} \mathcal{H}_- = \mathcal{H}_+$).

Proof. It is very simple. Namely, the standard operators $\mathbb{I}_{\mathcal{H}}\colon \mathcal{H}_- \longmapsto \mathcal{H}_+$, $\mathbb{I}_{\mathcal{F}}\colon \mathcal{F}_- \longmapsto \mathcal{F}_+$ are unitary operators between the indicated spaces. For these operators we have: $\forall \alpha \in \mathcal{H}_-$, $f \in \mathcal{H}_+$

$$(\alpha, f)_{\mathcal{H}} = (\mathbb{I}_{\mathcal{H}}\alpha, f)_{\mathcal{H}_+} = (\alpha, \mathbb{I}_{\mathcal{H}}^{-1} f)_{\mathcal{H}_-}, \quad (\mathbb{I}_{\mathcal{H}}\alpha, \beta)_{\mathcal{H}} = (\alpha, \mathbb{I}_{\mathcal{H}}\beta)_{\mathcal{H}}$$

and analogous equalities for the second rigging in (3.11). Using these equalities we get

$$\begin{aligned}(U\xi, f)_{\mathcal{F}} &= (\mathbb{I}_{\mathcal{F}}^{-1}\mathbb{I}_{\mathcal{H}}\xi, f)_{\mathcal{F}} = (\mathbb{I}_{\mathcal{H}}\xi, f)_{\mathcal{F}_+} \\ &= (\mathbb{I}_{\mathcal{H}}\xi, f)_{\mathcal{H}_+} = (\xi, f)_{\mathcal{H}}, \quad \xi \in \mathcal{H}_-, \quad f \in \mathcal{H}_+ = \mathcal{F}_+.\end{aligned}$$

□

Let us return to the operator A. It is some selfadjoint extension of J on the space S. It is easy to understand that the operator A is standardly connected with the rigging (3.8) but, instead of riggings (3.8), we use (3.10) and Lemma 2.

Let $\varphi(\lambda) \in (l_2(p))_{-,S}$ be a generalized eigenvector of the operator A in terms of the chain (3.8). So, in this case due to Theorem 2 and (2.4), we have

$$(\varphi(\lambda), Af)_S = \lambda(\varphi(\lambda), f)_S, \quad \lambda \in \mathbb{R}, \quad f \in l_{\text{fin}}. \tag{3.13}$$

Denote $P(\lambda) = U\varphi(\lambda) \in l_2(p^{-1})$, $P(\lambda) = (P_n(\lambda)), n \in \mathbb{Z}$; $\forall n \in \mathbb{Z}$ $P_n(\lambda) \in \mathbb{R}$ (here we apply Lemma 2 with $\mathcal{H}_- = (l_2(p))_{-,S}$ and $\mathcal{F}_- = l_2(p^{-1})$). Using (3.12) we can rewrite (3.13) in the form

$$(P(\lambda), Af)_{l_2} = \lambda(P(\lambda), f)_{l_2}, \quad \lambda \in \mathbb{R}, \quad f \in l_{\text{fin}}. \tag{3.14}$$

The corresponding Fourier transform (2.5) has the form

$$S \supset l_{\text{fin}} \ni f \to (Ff)(\lambda) = \hat{f}(\lambda) = (f, P(\lambda))_{l_2} \in L^2(\mathbb{R}, d\sigma(\lambda)). \tag{3.15}$$

Let us calculate $P(\lambda)$. The operator A is a selfadjoint extension of the operator J on S with $\operatorname{Dom}(J) = l_{\text{fin}}$ and acting on l_{fin} by the formula (3.5) and therefore (3.14) gives $\forall f \in l_{\text{fin}}$

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \lambda P_n(\lambda)\bar{f}_n &= \lambda(P(\lambda), f)_{l_2} = (P(\lambda), Af)_{l_2} \\ &= (P(\lambda), Jf)_{l_2} = (J^+P(\lambda), f)_{l_2} = \sum_{n\in\mathbb{Z}} P_{n+1}(\lambda)\bar{f}_n.\end{aligned} \tag{3.16}$$

Hence we have

$$\lambda P_n(\lambda) = P_{n+1}(\lambda), \quad n \in \mathbb{Z}. \tag{3.17}$$

Without loss of generality, we can take $P_0(\lambda) = 1,\ \lambda \in \mathbb{R}$. Then equalities (3.17) give

$$P_n(\lambda) = \lambda^n, \quad n \in \mathbb{Z}. \tag{3.18}$$

Thus the Fourier transform (3.15) finally has the form

$$S \supset l_{\text{fin}} \ni f \to (Ff)(\lambda) = \hat{f}(\lambda) = \sum_{n\in\mathbb{Z}} f_n\lambda^n \in L^2(\mathbb{R}, d\sigma(\lambda)), \tag{3.19}$$

and the Parseval equality (2.5) is as follows:

$$(f, g)_S = \int_{\mathbb{R}} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}\, d\sigma(\lambda), \quad f, g \in l_{\text{fin}}. \tag{3.20}$$

To construct the Fourier transform (3.15) and to verify formulas (3.16)–(3.20) it is necessary to note that for our operator A the algebraically inverse operator A^{-1} on l_{fin} exists and the vector $q = \delta_0 \in l_{\text{fin}}$ has the property $A^n q = J^n\delta_0 = \delta_n \in \mathcal{D}$, $n \in \mathbb{Z}$. This set is total in l_{fin} and Theorem 2 is applicable.

Parseval equality (3.20) immediately leads to representation (3.1). According to (3.18) and (3.19), $\hat{\delta}_n = \lambda^n$ and $\hat{\delta}_0 = 1$, and by (3.7), we get

$$s_n = (\delta_n, \delta_0)_S = (\hat{\delta}_n, \hat{\delta}_0)_{L^2(\mathbb{R}, d\sigma(\lambda))} = \int\limits_{\mathbb{R}} \lambda^n \, d\sigma(\lambda), \quad n \in \mathbb{Z},$$

i.e., (3.1) holds true with the measure $d\rho(\lambda) = d\sigma(\lambda)$.

If the operator J (3.5) is essentially selfadjoint on S we can take A to be the closure of J. In this case the measure $d\rho(\lambda)$ in representation (3.1) is unique.

So, to the finish the proof of our theorem it is only sufficient to prove that the condition (3.3) provides essential selfadjointness of J on the space S.

We will use the Theorem 3. Therefore it is necessary to prove that the operator J has a total set Q from the space S of quasianalytic vectors. (Note, that this fact is easily proved directly using the inequality (3.9)).

We put $Q = \{\delta_p, p \in \mathbb{Z}\}$. This set is total in S,— its linear envelope equals to l_{fin}. Let us prove that every such vector δ_p is quasianalytic. According to (3.5)–(3.7) we can write $\|J^n \delta_p\|_S^2 = \|\delta_{p+n}\|_S^2 = s_{2p+2n}$, $n \in \mathbb{N}$, $p \in \mathbb{Z}$. We have

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\|J^n \delta_p\|}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2p+2n}}}, \tag{3.21}$$

But since ([15], page 106 and also [25]), the series $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2p+2n}}}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}}$ are either convergent or divergent simultaneously, so equality (3.21) and condition (3.3) give that the vector δ_p is quasianalytic.

We have proved our theorem in the main nondegenerate case.

Consider the situation when the quadratic form is degenerate, i.e., there exists a finite nonzero $f = (f_j)$, $j \in \mathbb{Z}$, such that

$$\sum_{j,k \in \mathbb{Z}} s_{j+k} f_j \bar{f}_k = 0. \tag{3.22}$$

In this case expression (3.7) gives a quasiscalar product and for construction of the space S it is necessary to take the factor space of l_{fin} by all such f and after this to construct the completion. The operator J is Hermitian w.r.t. the quasiscalar product, therefore it is correctly defined on our S and is Hermitian w.r.t. the introduced scalar product. After this it is necessary to repeat the given above scheme of the proof. A detailed account of the corresponding constructions connected with rigging (2.1) (standard connection etc.) is given in [5], Ch. 7, Section 5. □

Remark 1. If the strong moment problem is degenerate, then the measure $d\rho(\lambda)$ from (3.1) is defined uniquely and concentrated on a finite number of points on $\mathbb{R}$. In fact, let condition (3.22) be fulfilled and for s_n representation (3.1) holds true. Substituting (3.1) into (3.22) we see that for the nonnegative Laurent polynomial $F(\lambda) = \big|\sum\limits_{j \in \mathbb{Z}} \lambda^j f_j\big|^2$, where the sequence $f = (f_j)$, $j \in \mathbb{Z}$, is finite, we have $\int\limits_{\mathbb{R}} F(\lambda)\, d\rho(\lambda) = 0$. But such a situation is possible only if the measure $d\rho(\lambda)$ is supported by a finite set of zeros of $F(\lambda)$. The operator J now is, of course, essentially selfadjoint.

Remark 2. It is necessary to explain why the measure $d\rho(\lambda)$ in (3.1), for given s_n, $n \in \mathbb{Z}$, is defined uniquely only in case of essential selfadjointness of the operator J. For this it is necessary to note that this operator can be represented as a block Jacobi-Laurent matrix on the space $\mathbf{l}_2$ of type l_2 (see Section 7). Elements of this matrix are calculated by only using s_n, $n \in \mathbb{Z}$. The measure $d\rho(\lambda)$ is always a spectral measure of the corresponding operator on $\mathbf{l}_2$, therefore this measure is unique iff this operator (i.e. J) is essential seladjoint on $\mathbf{l}_2$.

Note, that in this article we consider only "orthogonal" measure $d\rho(\lambda)$ in (3.1), i.e., a measure constructed by means by a selfadjoint extension of the operator J on the space S.

The following 3 sections will be devoted to an exposition of a spectral theory for block Jacobi-Laurent matrices connected with the strong moment problem.

4. The orthogonalization procedure and the construction of a three-diagonal block matrix connected with the strong moment problem

We at first propose some orthogonalization procedure and construction of a three-diagonal block matrix of the selfadjoint operator related to the corresponding strong Hamburger moment problem.

Instead of the usual space l_2 of sequences $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathbb{C}$, on which the ordinary Jacobi matrix acts, we will use the "double" space l_2 which, by definition, is

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus, \ \cdots, \quad \mathcal{H}_0 = \mathbb{C}^1, \quad \mathcal{H}_1 = \mathcal{H}_2 = \cdots \mathbb{C}^2. \tag{4.1}$$

Our three-diagonal matrices act in the space in (4.1). Of course, this space is equal to the space l_2 but on the $\mathbb{Z}$, i.e., the space of sequences $f = (f_n)_{n=-\infty}^{\infty}$, $f_n \in \mathbb{C}^1$. But its representation of the form (4.1) is more convenient for us.

Let $d\rho(\lambda)$ be a Borel measure on $\mathbb{R}$ with *bounded support* and $L^2 = L^2(\mathbb{R}, d\rho(\lambda))$ the space of complex square integrable functions defined on $\mathbb{R}$. We suppose that the Borel measure $d\rho(\lambda)$ is such that all the functions $\mathbb{R} \ni \lambda \longmapsto \lambda^m$, $\quad m \in \mathbb{Z}_- := \{\cdots, -2, -1\}$, belong to L^2, and all the functions λ^m, $m \in \mathbb{Z}$, are linearly independent.

In order to find an analog of the usual Jacobi matrix J we need to choose an order for the orthogonalization in L^2 applied to the family of the linear independent functions

$$\mathbb{R} \ni \lambda \longrightarrow \lambda^m, \quad m \in \mathbb{Z}. \tag{4.2}$$

We use the following order for the orthogonalization via the Gram-Schmidt procedure (such an order is same as in [19], compare also with corresponding pictures from [11, 12]):

$$\lambda^0; \quad \lambda^{-1}, \lambda^1; \quad \lambda^{-2}, \lambda^2; \quad \dots \quad ; \quad \lambda^{-n}, \lambda^n; \quad \dots \tag{4.3}$$

Applying the Gram-Schmidt orthogonalization procedure to (4.3) with real coefficients (see, for example, [6], Ch. 7) we obtain an orthonormal polynomial system in the space L^2 (w.r.t. λ and λ^{-1}, the so-called Laurent polynomials) indexed in the following way:

$$P_{0;0}(\lambda); \quad P_{1;0}(\lambda), P_{1;1}(\lambda); \quad P_{2;0}(\lambda), P_{2;1}(\lambda); \quad \dots; \quad P_{n;0}(\lambda), P_{n;1}(\lambda); \ \dots \tag{4.4}$$

where each polynomial has the form $P_{n;\alpha}(\lambda) = k_{n;\alpha}\lambda^{(-1)^{\alpha+1}n} + \cdots$, $n \in \mathbb{N}$, $\alpha = 0, 1$, $k_{n;\alpha} > 0$; here $+\cdots$ denotes the previous part of the corresponding polynomial; $P_0(\lambda) = P_{0;0}(\lambda) = 1$. In such a way, $P_{n;\alpha}$ is some linear combination of

$$\begin{gathered} \{1;\ \lambda^{-1}, \lambda^1;\ \lambda^{-2}, \lambda^2;\ \dots;\ \lambda^{-(n-1)}, \lambda^{(n-1)};\ \dots; \lambda^{-n}\} \ \text{ for } \ \alpha = 0, \\ \{1;\ \lambda^{-1}, \lambda^1;\ \lambda^{-2}, \lambda^2;\ \dots;\ \lambda^{-(n-1)}, \lambda^{(n-1)}; \dots; \lambda^{-n}, \lambda^n\} \ \text{ for } \ \alpha = 1. \end{gathered} \tag{4.5}$$

Since *the family (4.2) is total in the space L^2 even for $m \in \mathbb{N}_0$*, the sequence (4.4) is an orthonormal basis in this space.

Denote by $\mathcal{P}_{n;\alpha}$ the real subspace spanned by the elements $P_{n;\alpha}$, $\forall n \in \mathbb{N}$, $\alpha = 0, 1$, from (4.5). It is clear that $\forall n \in \mathbb{N}$ we have

$$\begin{gathered} \mathcal{P}_{0;0} \subset \mathcal{P}_{1;0} \subset \mathcal{P}_{1;1} \subset \mathcal{P}_{2;0} \subset \mathcal{P}_{2;1} \subset \cdots \subset \mathcal{P}_{n;0} \subset \mathcal{P}_{n;1} \subset \cdots, \\ \mathcal{P}_{n;\alpha} = \{P_{0;0}(\lambda)\} \oplus \{P_{1;0}(\lambda)\} \oplus \{P_{1;1}(\lambda)\} \oplus \{P_{2;0}(\lambda)\} \oplus \{P_{2;1}(\lambda)\} \oplus \cdots \oplus \{P_{n;\alpha}(\lambda)\}, \end{gathered} \tag{4.6}$$

where $\{P_{m;\alpha}(\lambda)\}$, $m \in \mathbb{N}$, $\alpha = 0, 1$, denotes one dimensional real space spanned by $P_{m;\alpha}(\lambda)$; $\mathcal{P}_{0;0} = \mathbb{R}$.

As was mentioned above, for the next investigation we need, instead of the space l_2, the complex Hilbert space (4.1). Each vector $f \in \mathbf{l}_2$ has the form $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$, and consequently $\forall f, g \in \mathbf{l}_2$

$$\|f\|^2_{\mathbf{l}_2} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} < \infty, \quad (f, g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n}.$$

For $n = 0$, the vector $f_0 \in \mathcal{H}_0$ has, in the standard orthonormal basis $\{e_{0;0}\}$ of the space $\mathbb{C}^1$, a representation $f_{0;0}$, hence $f_0 = (f_{0;0})$. For $n \in \mathbb{N}$ coordinates of the vector $f_n \in \mathcal{H}_n$, in corresponding orthonormal basis $\{e_{n;0},\ e_{n;1}\}$ in the space $\mathbb{C}^2$, are denoted by $(f_{n;0}, f_{n;1})$ and, hence, we have $f_n = (f_{n;0}, f_{n;1})$. By the way, it is clear that the space $\mathbf{l}_2$ is isometric to some subspace in $l_2 \oplus l_2$.

Using the orthonormal system (4.4) one can define a mapping of $\mathbf{l}_2$ into L^2. We put $\forall n \in \mathbb{N}_0$ and $\forall \lambda \in \mathbb{R}$, $P_n(\lambda) = (P_{n;0}, P_{n;1}(\lambda)) \in \mathcal{H}_n$. Then

$$\mathbf{l_2} \ni f = (f_n)_{n=0}^{\infty} \longmapsto (If)(\lambda) := \hat{f}(\lambda) = \sum_{n=0}^{\infty} (f_n, P_n(\lambda))_{\mathcal{H}_n} \in L^2. \tag{4.7}$$

Since for $n \in \mathbb{N}_0$ we get

$$(f_n, P_n(\lambda))_{\mathcal{H}_n} = f_{n;0} P_{n;0}(\lambda) + f_{n;1} P_{n;1}(\lambda)$$

and

$$\|f\|^2_{\mathbf{l}_2} = \|(f_{0;0}, f_{1;0}, f_{1;1}, f_{2;0}, f_{2;1}, \ldots, f_{n;0}, f_{n;1}, \ldots)\|^2_{\mathbf{l}_2},$$

we see that (4.7) is a mapping of the space $\mathbf{l}_2$ into L^2, and the use of the orthonormal system (4.4) shows that this mapping is isometric. The image of $\mathbf{l}_2$ under the mapping (4.7) coincides with the space L^2 because, due to our assumption, system (4.4) is an orthonormal basis in L^2 (Laurent polynomial basis). Therefore the mapping (4.7) is a unitary transformation I that acts from $\mathbf{l}_2$ onto L^2.

Let A be an arbitrary linear operator defined on $\mathrm{Dom}(A) = \mathbf{l}_{\mathrm{fin}} \subset \mathbf{l}_2$, where $\mathbf{l}_{\mathrm{fin}}$ denotes the set of finite vectors from $\mathbf{l}_2$. It is possible to construct a corresponding operator matrix $(a_{j,k})_{j,k=0}^{\infty}$, where for each $j, k \in \mathbb{N}_0$ the element $a_{j,k}$ is an operator from $\mathcal{H}_k$ into $\mathcal{H}_j$, so that $\forall f, g \in \mathrm{Dom}(A) = \mathbf{l}_{\mathrm{fin}} \subset \mathbf{l}_2$ we have

$$(Af)_j = \sum_{k=0}^{\infty} a_{j,k} f_k, \quad j \in \mathbb{N}_0, \quad (Af, g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} (a_{j,k} f_k, g_j)_{\mathcal{H}_j}. \tag{4.8}$$

To prove of (4.8) we only need to write the usual matrix of the operator A in the space $\mathbf{l}_2$ using the basis

$$(e_{0;0};\ e_{1;0}, e_{1;1};\ e_{2;0}, e_{2;1};\ \ldots;\ e_{n;0}, e_{n;1},\ \ldots), \quad e_{0;0} = 1. \tag{4.9}$$

Then $a_{j,k}$, for each $j, k \in \mathbb{N}_0$, is an operator $\mathcal{H}_k \longrightarrow \mathcal{H}_j$ that has the matrix representation

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{\mathbf{l}_2}, \tag{4.10}$$

where $\alpha = 0, 1$ and $\beta = 0, 1$. We will write: $a_{j,k} = (a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$, $j, k \in \mathbb{N}$ (including cases: $a_{0,1} = (a_{0,1;\alpha,\beta})_{\alpha,\beta=0}^{0,1}$, $a_{1,0} = (a_{1,0;\alpha,\beta})_{\alpha,\beta=0}^{1,0}$ and $a_{0,0} = (a_{0,0;\alpha,\beta})_{\alpha,\beta=0}^{0,0} = a_{0,0;0,0}$).

Note that the first formula from (4.8) takes place for $f \in \mathbf{l}_{\mathrm{fin}}$; in the second formula $f \in \mathbf{l}_{\mathrm{fin}}$, $g \in \mathbf{l}_2$.

Let us consider the image $\hat{A} = IAI^{-1} : L^2 \longrightarrow L^2$ of the above operator $A : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$ under the mapping I (4.7). Its matrix in the basis (4.4),

$$(P_{0;0}(\lambda);\ P_{1;0}(\lambda), P_{1;1}(\lambda);\ P_{2;0}(\lambda), P_{2,1}(\lambda);\ \ldots;\ P_{n;0}(\lambda), P_{n;1}(\lambda);\ \ldots),$$

is equal to the usual matrix of the operator A understanding as an operator: $\mathbf{l}_2 \longrightarrow \mathbf{l}_2$ in the corresponding basis (4.9). Using (4.10) and the above mentioned procedure, we get the operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $A : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$. *By definition, this matrix is also the operator matrix of* $\hat{A} : L^2 \longrightarrow L^2$. It is clear that we can take an arbitrary essentially selfadjoint operator on L^2 to be the operator $\hat{A}$.

Return now to the objects connected with our measure $d\rho(\lambda)$ and sequences (4.3), (4.4).

Lemma 3. *For the polynomials $P_{n;\alpha}(\lambda)$ (4.4) and the subspaces $\mathcal{P}_{m,\beta}$ (4.6), the following relations hold:*

$$\begin{aligned} \lambda P_{0;0}(\lambda) &= \lambda \in \mathcal{P}_{1;1}, \\ \lambda P_{n;0}(\lambda) &\in \mathcal{P}_{n;1}, \\ \lambda P_{n;1}(\lambda) &\in \mathcal{P}_{n+1;1}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.11}$$

Proof. According to (4.4), the polynomial $P_{n;\alpha}(\lambda)$, $n \in \mathbb{N}$, is equal to some linear combination of $\{1; \lambda^{-1}, \lambda^1; \ldots; \lambda^{-(n-1)}, \lambda^{n-1}, \lambda^{(-1)^{\alpha+1}n}\}$. Hence, multiplying by λ we obtain a linear combination of $\{\lambda;\, 1,\, \lambda^2; \lambda^{-1}, \lambda^3; \ldots; \lambda^{-(n-2)}, \lambda^n, \lambda^{(-1)^{\alpha+1}n+1}\}$ } and such a linear combination belongs to $\mathcal{P}_{n;1}$ for $\alpha = 0$ and to $\mathcal{P}_{n+1;1}$ for $\alpha = 1$. The first inclusion in (4.11) is trivial. □

Lemma 4. *Let $\hat{A}$ be the operator (bounded and selfadjoint) of multiplication by λ in the space L^2,*

$$L^2 \ni \varphi(\lambda) \longmapsto (\hat{A}\varphi)(\lambda) = \lambda\varphi(\lambda) \in L^2.$$

The operator real matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $\hat{A}$ (i.e. of $A = I^{-1}\hat{A}I$) has a three-diagonal structure: $a_{j,k} = 0$ for $|j-k| > 1$.

Proof. Using (4.10) for $e_{n;\gamma} = I^{-1}P_{n;\gamma}(\lambda)$, $n \in \mathbb{N}_0$; $\gamma = 0, 1$, we have $\forall j, k \in \mathbb{N}_0$

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{\mathbf{l}_2} = \int_{\mathbb{R}} \lambda P_{k;\beta}(\lambda) P_{j;\alpha}(\lambda)\, d\rho(\lambda), \tag{4.12}$$

where $\alpha, \beta = 0, 1$. From (4.11), $\lambda P_{k;\alpha} \in \mathcal{P}_{k+1;\alpha}$. According to (4.6), the integral in (4.12) is equal to zero for $j > k+1$ and for each $\alpha = 0, 1$.

On the other hand, in integral (4.12) we can multiply by λ the polynomial $P_{j;\alpha}(\lambda)$. Therefore as earlier we conclude that this integral is equal to zero for $k > j+1$ and for each $\beta = 0, 1$.

As a result the integral in (4.12), i.e., the coefficients $a_{j,k;\alpha,\beta}$, $j, k \in \mathbb{N}_0$, are equal to zero for $|j-k| > 1$; $\alpha, \beta = 0, 1$. (In the previous considerations it was necessary to take into account that $e_{0;0} = I^{-1}P_{0;0}(\lambda) = 1$). □

In such a way the matrix $(a_{j,k})_{j,k=0}^{\infty}$ of our operator $\hat{A}$ of multiplication has a three-diagonal block structure

$$\begin{bmatrix} a_{0,0} & a_{0,1} & 0 & 0 & 0 & \ldots \\ a_{1,0} & a_{1,1} & a_{1,2} & 0 & 0 & \ldots \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & \ldots \\ 0 & 0 & a_{3,2} & a_{3,3} & a_{3,4} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{4.13}$$

From (4.12) we conclude that the following symmetry takes place:

$$a_{j,k;\alpha,\beta} = a_{k,j;\beta,\alpha}, \quad j, k \in \mathbb{N}_0, \quad \alpha, \beta = 0, 1. \tag{4.14}$$

A more careful analysis of expressions (4.12) allows to find which of the elements of the matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$ are zero and which are not in the general case for $|j-k| \le 1$.

We can also describe properties of the matrix with respect to permutation of the indexes j, k, and α, β.

Lemma 5. *Let $(a_{j,k})_{j,k=0}^{\infty}$ be the operator matrix (4.13) for our operator of multiplication by λ in L^2. Now $a_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$; $a_{j,k} = (a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$ are matrices of the operators $a_{j,k}$ in the corresponding standard orthonormal basis. Then $\forall j \in \mathbb{N}$*

$$a_{j,j+1;0,0} = a_{j,j+1;0,1} = 0,$$
$$a_{j+1,j;0,0} = a_{j+1,j;1,0} = 0. \tag{4.15}$$

If we choose another order inside each pair $\{\lambda^{-m}, \lambda^m\}$, from (4.5) then Lemma 5 is not true but it will also be possible to describe zeros of the matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$. Such matrices $(a_{j,k})_{j,k=0}^{\infty}$ have also a three-diagonal block structure and have zeros but in other places.

Proof. According to (4.12) we have, for $j \in \mathbb{N}$,

$$a_{j,j+1;0,0} = \int_{\mathbb{R}} \lambda P_{j+1;0}(\lambda) P_{j;0}(\lambda)\, d\rho(\lambda),$$
$$a_{j,j+1;0,1} = \int_{\mathbb{R}} \lambda P_{j+1;1}(\lambda) P_{j;0}(\lambda)\, d\rho(\lambda). \tag{4.16}$$

In the first integral in (4.16), according to (4.11), $\lambda P_{j;0}(\lambda) \in \mathcal{P}_{j;1}$ but $P_{j+1;0}(\lambda)$ is orthogonal to the last set (see (4.6)). Therefore this integral is equal to zero. Analogously, the second integral in (4.16) also equals zero: it is necessary to use orthogonality of $P_{j+1;1}(\lambda)$ to $\mathcal{P}_{j;1}$.

So, the first two equalities in (4.15) are fulfilled. The second equalities are fulfilled according to (4.14). □

The above shows that the $(2) \times (2)$-matrices in (4.13), $a_{j,j+1}$ and $a_{j+1,j}$, $j \in \mathbb{N}$, have the first rows and columns, respectively, equal to zero. Taking into account (4.13) we can conclude that the selfadjoint matrix of the multiplication operator by λ is a five-diagonal usual scalar matrix, i.e., in the usual basis of some subspace of $l_2 \oplus l_2$.

Lemma 6. *The following elements of the matrix $(a_{j,k})_{j,k=0}^{\infty}$ (4.13) are positive:*

$$a_{0,1;0,1}, \quad a_{1,0;1,0},$$
$$a_{j,j+1;1,1}, \quad a_{j+1,j;1,1}, \quad j \in \mathbb{N}. \tag{4.17}$$

Proof. The symmetry (4.14) shows that it is sufficient to show positivity of the second and the forth elements in (4.17). We start with $a_{1,0;1,0}$. Denote by $P'_{1;1}(\lambda)$ the non normalized vector $P_{1;1}(\lambda)$, (obtained from the Gram-Schmidt orthogonalization procedure but not normalized). According to (4.3) and (4.4) we have

$$P'_{1;1}(\lambda) = \lambda - (\lambda, P_{1;0}(\lambda))_{L^2} P_{1;0}(\lambda) - (\lambda, 1)_{L^2}.$$

Therefore using (4.12) we get

$$\begin{aligned} a_{1,0;1,0} &= \int_{\mathbb{R}} \lambda P_{1;1}(\lambda)\, d\rho(\lambda) = \|P'_{1;1}(\lambda)\|_{L^2}^{-1} \int_{\mathbb{R}} \lambda P'_{1;1}(\lambda)\, d\rho(\lambda) \\ &= \|P'_{1;1}(\lambda)\|_{L^2}^{-1} \int_{\mathbb{R}} \lambda(\lambda - (\lambda, P_{1;0}(\lambda))_{L^2} P_{1;0}(\lambda) - (\lambda, 1)_{L^2})\, d\rho(\lambda) \\ &= \|P'_{1;1}(\lambda)\|_{L^2}^{-1} (\|\lambda\|_{L^2}^2 - |(\lambda, P_{1;0}(\lambda))_{L^2}|^2 - |(\lambda, 1)_{L^2}|^2). \end{aligned} \tag{4.18}$$

The positiveness of the expression (4.18) follows from the Parseval equality for the decomposition of the function $\lambda \in L^2$ with respect to the orthonormal basis (4.4) in the space L^2. Namely,

$$|(\lambda, 1)_{L^2}|^2 + |(\lambda, P_{1;0}(\lambda))_{L^2}|^2 + |(\lambda, P_{1;1}(\lambda))_{L^2}|^2 + \cdots = \|\lambda\|^2_{L^2} \quad (1 = P_{0;0}(\lambda)).$$

Let us now pass to the proof of positivity of $a_{j+1,j;1,1}$, where $j \in \mathbb{N}$.

From (4.12) we have

$$a_{j+1,j;1,1} = \int\limits_{\mathbb{R}} \lambda P_{j;1}(\lambda) P_{j+1;1}(\lambda)\, d\rho(\lambda). \tag{4.19}$$

According to (4.4) and (4.6),

$$P_{j;1}(\lambda) = k_{j;1}\lambda^j + R_{j;0}(\lambda), \tag{4.20}$$

where $R_{j;0}(\lambda)$ is some polynomial from $\mathcal{P}_{j;0}$ and $k_{j;1} > 0$. Multiply expression (4.20) by λ we get

$$\lambda P_{j;1}(\lambda) = k_{j;1}\lambda^{j+1} + \lambda R_{j;0}(\lambda), \quad \lambda R_{j;0}(\lambda) \in \mathcal{P}_{j;1} \tag{4.21}$$

(now it is necessary to use the second inclusion from (4.11) and (4.6)).

Analogously to (4.20) we have

$$P_{j+1;1}(\lambda) = k_{j+1;1}\lambda^{j+1} + R_{j+1;0}(\lambda), \quad R_{j+1;0}(\lambda) \in \mathcal{P}_{j+1;0}, \quad k_{j+1;1} > 0. \tag{4.22}$$

Find λ^{j+1} from (4.22) and substitute it into (4.21). We get

$$\begin{aligned} \lambda P_{j;1}(\lambda) &= \frac{k_{j;1}}{k_{j+1;1}}(P_{j+1;1}(\lambda) - R_{j+1;0}(\lambda)) + \lambda R_{j;0}(\lambda) \\ &= \frac{k_{j;1}}{k_{j+1;1}} P_{j+1;1}(\lambda) - \frac{k_{j;1}}{k_{j+1;1}} R_{j+1;0}(\lambda) + \lambda R_{j;0}(\lambda). \end{aligned} \tag{4.23}$$

The second two terms in (4.23) belong to $\mathcal{P}_{j+1;0}$ and $\mathcal{P}_{j;1}$ respectively and are in any cases orthogonal to $P_{j+1;1}(\lambda)$. Therefore the substitution of the expression (4.23) into (4.19) gives $a_{j,j+1;1,1} = \frac{k_{j;1}}{k_{j+1;1}} > 0$. $\square$

In what follows we will use the usual, well known notations for elements $a_{j,k}$ of the Jacobi matrix (4.13):

$$\begin{aligned} a_n &= a_{n+1,n} &:& \quad \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ b_n &= a_{n,n} &:& \quad \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ c_n &= a_{n,n+1} &:& \quad \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{4.24}$$

All previous investigation are summarized in the following theorem.

Theorem 5. *The bounded selfadjoint operator $\hat{A}$ of multiplication by λ in the space L^2 in the orthonormal basis (4.4) of polynomials has the form of a three-diagonal block Jacobi type symmetric matrix $J = (a_{j,k})_{j,k=0}^{\infty}$ that acts on the space (4.1),*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus, \quad \cdots, \quad \mathcal{H}_0 = \mathbb{C}^1, \quad \mathcal{H}_n = \mathbb{C}^2, \quad n \in \mathbb{N}. \tag{4.25}$$

In notation (4.24), this matrix has the form

$$
J = \left[\begin{array}{c|cc|cc|cc|cc|c}
*b_0 & * & c_0 \; + & & & & & & & \cdots \\
\hline
* & * & * & 0 & 0 & & & & & \\
a_0 & \multicolumn{2}{c|}{b_1} & \multicolumn{2}{c|}{c_1} & & & 0 & & \cdots \\
+ & * & * & * & + & & & & & \\
\hline
 & 0 & * & * & * & 0 & 0 & & & \\
 & \multicolumn{2}{c|}{a_1} & \multicolumn{2}{c|}{b_2} & \multicolumn{2}{c|}{c_2} & & & \cdots \\
 & 0 & + & * & * & * & + & & & \\
\hline
 & & & 0 & * & * & * & 0 & 0 & \\
 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{a_2} & \multicolumn{2}{c|}{b_3} & \multicolumn{2}{c|}{c_3} & \cdots \\
 & & & 0 & + & * & * & * & + & \\
\hline
\vdots & \multicolumn{2}{c}{\vdots} & \multicolumn{2}{c}{\vdots} & \multicolumn{2}{c}{\vdots} & \multicolumn{2}{c}{\vdots} & \ddots
\end{array}\right]. \tag{4.26}
$$

In (4.26) $b_0 = b_{0;0,0}$ *is a* 1×1*-matrix, i.e., a scalar;* b_n *is a* 2×2*-matrix,* $b_n = (b_{n;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$, $\forall n \in \mathbb{N}$*;* a_0 *is a* 1×2*-matrix,* $a_0 = (a_{0;\alpha,\beta})_{\alpha,\beta=0}^{1,0}$*;* a_n *is a* 2×2*-matrix,* $a_n = (a_{n;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$ $\forall n \in \mathbb{N}$*;* c_0 *is a* 2×1*-matrix,* $c_0 = (c_{0;\alpha,\beta})_{\alpha,\beta=0}^{0,1}$*;* c_n *is a* 2×2*-matrix,* $c_n = (c_{n;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$ $\forall n \in \mathbb{N}$*. In these matrices* a_n *and* c_n *some elements are always equal to zero,*

$$a_{n;0,0} = a_{n;1,0} = 0, \quad c_{n;0,0} = c_{n;0,1} = 0, \quad \forall n \in \mathbb{N}. \tag{4.27}$$

Some other their elements are positive, namely,

$$\begin{aligned} &a_{0;1,0}, c_{0;0,1} > 0, \\ &a_{n;1,1}, c_{n;1,1} > 0, \quad n \in \mathbb{N}. \end{aligned} \tag{4.28}$$

Thus, it is possible to say, that $\forall n \in \mathbb{N}$ *every left column the matrices* a_n *(starting with* $n = 1$*) and every top row of the matrices* c_n *(starting from the* $n = 1$*) consist of zero elements. All positive elements in (4.26) are denoted by* $+$*.*

So, the matrix (4.26), in the scalar form, is five-diagonal of the indicated structure. It is symmetric in basis (4.4), $b_{n;\alpha,\beta} = b_{n;\beta,\alpha}$, $c_{n;\alpha,\beta} = a_{n;\beta,\alpha}$, $n \in \mathbb{N}_0$, $\alpha, \beta = 0, 1$*.*

For the considered operator $A = I^{-1}\hat{A}I$*, we have* $\forall f, g \in \operatorname{Dom}(A) = \mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$

$$(Af)_n = (Jf)_n = a_{n-1}f_{n-1} + b_n f_n + c_n f_{n+1}, \quad n \in \mathbb{N}_0, \quad f_{-1} := 0. \tag{4.29}$$

We want to make simple enough but essential remarks concerning the operator A generated in (4.25) by the matrix J (4.26) in the case where such an operator is unbounded and corresponding inverse operator A^{-1} exists.

Remark 3. Assume that the measure $d\rho(\lambda)$ discussed in the beginning of Section 4 has an arbitrary support in $\mathbb{R}$ and all the functions

$$\mathbb{R} \ni \lambda \longmapsto \lambda^m, \quad m \in \mathbb{Z} \tag{4.30}$$

belong to $L^2(\mathbb{R}, d\rho(\lambda)) = L^2$ and are linearly independent. In this case we can repeat all constructions (4.3)–(4.29), but now the operator A defined on the set $\mathbf{l}_{\text{fin}}$ in the space $\mathbf{l}_2$ is only Hermitian and symmetric (see (4.14)) and therefore has equal deficiency numbers. Moreover *in what follows we will assume that the set of functions (4.30) is total in* L^2.

Remark 4. Consider the one-to-one mapping between $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}$

$$\mathbb{R}_0 := \mathbb{R} \setminus \{0\} \ni \lambda \longmapsto \mu = \lambda^{-1} =: \varphi(\lambda) \in \mathbb{R}. \tag{4.31}$$

We will assume that our given measure $d\rho(\lambda)$ on $\mathbb{R}$ is such that the point 0 does not belong to its support, $0 \notin \operatorname{supp}(d\rho(\lambda))$. In such a case, the mapping (4.31) takes this measure into the Borel measure $d\sigma(\mu)$ on $\mathbb{R}$ (i.e. $\forall\alpha \in \mathfrak{B}(\mathbb{R})$, $\sigma(\alpha) = \rho(\varphi^{-1}(\alpha))$). For an arbitrary function $\mathbb{R}_0 \ni \lambda \longmapsto F(\lambda) \in \mathbb{C}$, we have

$$\int_{\mathbb{R}} F(\mu^{-1})\, d\sigma(\mu) = \int_{\mathbb{R}} F(\lambda)\, d\rho(\lambda); \quad F(\mu^{-1}) =: (I_\varphi F)(\mu). \tag{4.32}$$

Introduce the Hilbert space of complex-valued functions of μ on $\mathbb{R}$, $L^2(\mathbb{R}, d\sigma(\mu)) = L^2_\varphi$. Then from (4.32) we conclude that this space is unitary equivalent to the space $L^2(\mathbb{R}, d\rho(\lambda)) = L^2$ and the corresponding unitary operator $I_\varphi : L^2 \longmapsto L^2_\varphi$ is given by the last expression in (4.32).

Let the operator A constructed on $\mathbf{l}_2$ according to (4.29) vie the matrix J be essentially selfadjoint and invertible. Then its L^2-image, i.e., the operator $\hat{A}$ of multiplication by λ defined at first on linear combinations of functions (4.30) is also invertible and $0 \notin \operatorname{supp}(d\rho(\lambda))$. This inverse operator $\hat{A}^{-1}$, as an operator on $\mathbf{l}_2$, generates by the algebraic inverse matrix J^{-1}.

The mapping (4.31) shows that this matrix in the space $\mathbf{l}_2$, constructed as above but using the measure $d\sigma(\mu)$, has also the form (4.26), but with another polynomials (4.4). Of course, it is easy to calculate this matrix in the previous basis connected with $d\rho(\lambda)$, i.e., the matrix J^{-1} which we denote now by K.

Let us stress that the construction (4.31), (4.32) gives a possibility to find interesting examples of measures $d\rho(\lambda)$ for which the set (4.30) is total in $L^2(\mathbb{R}, d\rho(\lambda))$.

It is very interesting and unexpected that this matrix $J^{-1} = K$ inverse to the three-diagonal matrix J is also three-diagonal. This result is a consequence of the method the basis (4.4) constructed, these polynomial are linear combinations of λ^m and λ^{-n}, $m, n \in \mathbb{N}_0$.

We will accurately prove corresponding results since the form of the matrix K is slightly different from J. At first, instead of Lemma 3, we have the following.

Lemma 7. *For the polynomials $P_{n;\alpha}(\lambda)$, (4.4) and the subspaces $\mathcal{P}_{m,\beta}$ the following relations hold:*

$$\begin{aligned} \lambda^{-1}P_{0;0}(\lambda) = \lambda^{-1} &\in \mathcal{P}_{1;0}, \\ \lambda^{-1}P_{n;0}(\lambda) &\in \mathcal{P}_{n+1;0}, \\ \lambda^{-1}P_{n;1}(\lambda) &\in \mathcal{P}_{n+1;0}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.33}$$

Proof. It is similar to the proof of Lemma 3 and follows from (4.5), (4.6) and (4.7). □

Denote by $(p_{j,k})_{j,k=0}^{\infty}$, $(p_{j,k;\alpha,\beta})_{\alpha,\beta=0,0}^{1,1}$ the operator matrix K of the operator of multiplication by λ^{-1} in the previous space $L^2(\mathbb{R}, d\rho(\lambda)) = L^2$; this matrix is constructed as earlier (see (4.8), (4.9), (4.10)) using the basis (4.4).

We can restate now Lemma 4 for K. Using (4.33) we assert that the integrals

$$p_{j,k;\alpha,\beta} = \int_{\mathbb{R}} \lambda^{-1}P_{k;\beta}(\lambda)P_{j;\alpha}(\lambda)\, d\rho(\lambda), \quad j,k \in \mathbb{N}_0, \quad \alpha,\beta = 0,1, \tag{4.34}$$

are equal to zero if $|j-k| > 1$. So, our matrix K is of type (4.13). Of course, $K = J^{-1}$ is selfadjoint and symmetric: the equality (4.14) is fulfilled for $p_{j,k}$.

Instead of Lemma 5 we have some other equalities.

Lemma 8. *For the elements $p_{j,k} = (p_{j,k;\alpha,\beta})_{\alpha,\beta=0,0}^{1,1}$, of the matrix $(p_{j,k})_{j,k=0}^{\infty} = K$, we have equalities*

$$\begin{aligned} p_{0,1;0,1} &= p_{1,0;1,0} = 0, \\ p_{j,j+1;0,1} &= p_{j,j+1;1,1} = 0, \\ p_{j+1,j;1,0} &= p_{j+1,j;1,1} = 0, \quad j \in \mathbb{N}. \end{aligned} \tag{4.35}$$

Proof. It is also similar to the proof of Lemma 5 but with a use of (4.33) instead of (4.11). For example, we have, according to (4.34), that

$$p_{j,j+1;0,1} = \int_{\mathbb{R}} \lambda^{-1} P_{j+1,1}(\lambda) P_{j;0}(\lambda)\, d\rho(\lambda), \quad j \in \mathbb{N}_0. \tag{4.36}$$

Using the second inclusion from (4.33) we get that $\lambda^{-1}P_{j;0}(\lambda) \in \mathcal{P}_{j+1;0}$, therefore this function is orthogonal to $P_{j+1;1}(\lambda)$ and integral (4.36) is equal to zero. Analogously, using the third inclusion from (4.33) we get that $p_{j,j+1;1,1} = 0$. The rest of equalities in (4.35) are valid thanks to symmetry of K. □

An analog of Lemma 6 is the following assertion.

Lemma 9. *The following elements of the matrix $(p_{j,k})_{j,k=0}^{\infty}$ are positive:*

$$\begin{gathered} p_{0,1;0,0}, \quad p_{1,0;0,0}, \\ p_{j,j+1;0,0}, \quad p_{j+1,j;0,0}, \quad j \in \mathbb{N}. \end{gathered} \tag{4.37}$$

Proof. As earlier, we start from $p_{1,0;0,0}$. Denote by $P'_{1;0}(\lambda)$ the non normalized vector $P_{1;0}(\lambda)$, and we get $P'_{1;0}(\lambda) = \lambda^{-1} - (\lambda^{-1}, 1)_{L^2}$. Therefore, as in (4.18) with the help of Parseval equality for λ^{-1},

$$\begin{aligned} p_{1,0;0,0} &= \int_{\mathbb{R}} \lambda^{-1} P_{1;0}(\lambda)\, d\rho(\lambda) = \|P'_{1;0}(\lambda)\|_{L^2}^{-1} \int_{\mathbb{R}} \lambda^{-1}(\lambda^{-1} - (\lambda^{-1}, 1)_{L^2})\, d\rho(\lambda) \\ &= \|P'_{1;0}(\lambda)\|_{L^2}^{-1} (\|\lambda^{-1}\|_{L^2}^2 - |(\lambda^{-1}, 1)_{L^2}|^2) > 0. \end{aligned}$$

As in (4.19), consider the forth element from (4.37). We have

$$p_{j+1,j;0,0} = \int_{\mathbb{R}} \lambda^{-1} P_{j;0}(\lambda) P_{j+1;0}(\lambda)\, d\rho(\lambda), \quad j \in \mathbb{N}. \tag{4.38}$$

According to (4.4) and (4.6) we have

$$P_{j;0}(\lambda) = k_{j;0}\lambda^{-j} + R_{j-1;1}(\lambda), \tag{4.39}$$

where $R_{j-1;1}(\lambda)$ is some polynomial from $\mathcal{P}_{j-1;1}$, $k_{j;0} > 0$. Multiply (4.39) by λ^{-1} and get

$$\lambda^{-1}P_{j;0}(\lambda) = k_{j;0}\lambda^{-(j+1)} + \lambda^{-1}R_{j-1;1}(\lambda), \quad \lambda^{-1}R_{j-1;1}(\lambda) \in \mathcal{P}_{j-1;0}. \tag{4.40}$$

Analogously to (4.39) we have

$$P_{j+1;0}(\lambda) = k_{j+1;0}\lambda^{-(j+1)} + R_{j;1}(\lambda), \quad R_{j;1}(\lambda) \in \mathcal{P}_{j;1}; \quad k_{j+1;0} > 0. \tag{4.41}$$

Find $\lambda^{-(j+1)}$ from (4.41) and substitute it into (4.40),

$$\begin{aligned} \lambda^{-1}P_{j;0}(\lambda) &= \frac{k_{j;0}}{k_{j+1;0}}(P_{j+1;0}(\lambda) - R_{j;1}(\lambda)) + \lambda^{-1}R_{j-1;1}(\lambda) \\ &= \frac{k_{j;0}}{k_{j+1;0}}P_{j+1;0}(\lambda) - \frac{k_{j;0}}{k_{j+1;0}}R_{j;1}(\lambda) + \lambda^{-1}R_{j-1;1}(\lambda). \end{aligned} \tag{4.42}$$

In this expression, the second two terms belong to $\mathcal{P}_{j;1}$ and $\mathcal{P}_{j-1;0}$, respectively, and are orthogonal to $P_{j+1;0}(\lambda)$. After substitution of (4.42) into (4.38) we get $p_{j+1,j;0,0} > 0$.

Positivity of the rest of elements from (4.37) follows from symmetry of K. □

The results of the last considerations give the following theorem.

Theorem 6. *Let the measure $d\rho(\lambda)$ be such that the operator of multiplication $\hat{A}$ by λ^{-1} is selfadjoint and invertible in L^2. Then the bounded operator inverse to $\hat{A}$ is generated in the space $\mathbf{l}_2$ (4.25) by the three-diagonal block Jacobi type symmetric matrix $J^{-1} = K$ of the form analogous to (4.26),*

$$K = (p_{j,k})_{j,k=0}^{\infty}, \quad p_n := p_{n+1,n}, \quad q_n := p_{n,n}, \quad r_n := p_{n,n+1}, \quad n \in \mathbb{N}_0;$$

$$(4.43)\quad J^{-1} = K = \left[\begin{array}{c|cc|cc|cc|cc|c}
*q_0 & + \;\; r_0 & 0 & & & & & & & \cdots \\
+ & * & * & + & 0 & & & & & \\
p_0 & q_1 & & r_1 & & & & 0 & & \cdots \\
0 & * & * & * & 0 & & & & & \\
 & + & * & * & * & + & 0 & & & \\
 & p_1 & & q_2 & & r_2 & & & & \cdots \\
 & 0 & 0 & * & * & * & 0 & & & \\
 & & & + & * & * & * & + & 0 & \\
 & 0 & & p_2 & & q_3 & & r_3 & & \cdots \\
 & & & 0 & 0 & * & * & * & 0 & \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}\right].$$

So, we have

$$(4.44)\qquad \begin{gathered} p_{0;1,0} = r_{0;0,1} = 0, \quad p_{n;1,0} = p_{n;1,1} = r_{n;0,1} = r_{n;1,1} = 0, \quad n \in \mathbb{N}, \\ p_{n;0,0}, r_{n;0,0} > 0, \quad n \in \mathbb{N}_0; \end{gathered}$$

$$(4.45)\qquad \begin{gathered} (J^{-1}f)_n = (Kf)_n = p_{n-1}f_{n-1} + q_n f_n + r_n f_{n+1}, \\ n \in \mathbb{N}_0, \quad f_{-1} := 0, \quad f \in \mathbf{l}_2. \end{gathered}$$

We will consider now some simple but essential for us generalization of last results.

Namely, we will assume that the operator $\hat{A}$ of multiplication by λ is selfadjoint in L^2, but bounded $\hat{A}^{-1}$, possibly, does not exist. Recall that all functions $\mathbb{R} \ni \lambda \longmapsto \lambda^m$, $m \in \mathbb{Z}_-$, belong to L^2 (according to Remark 3).

Therefore the operator in L^2 of multiplication by λ^{-1} exists and is defined on those functions F from L^2 for which $\lambda^{-1}F(\lambda) \in L^2$. The set of such functions is linear and, of course, dense in L^2. This operator is (algebraically) inverse to $\hat{A}$, we denote it also by $\hat{A}^{-1}$. So, we can formulate the following assertion. *Consider the general situation when all the functions (4.30) belong to L^2 and the corresponding operator $\hat{A}$ of multiplication by λ is selfadjoint. Then $\hat{A}$ has an algebraically inverse operator $\hat{A}^{-1}$ with domain* $\mathrm{Dom}(\hat{A}^{-1}) \supset \mathrm{Ran}(\hat{A})$ dense in L^2,

$$(4.46)\qquad \begin{gathered} (\hat{A}^{-1}\hat{A})F = \lambda^{-1}(\lambda F(\lambda)) = F(\lambda), \quad F \in \mathrm{Dom}(\hat{A}), \\ \mathrm{Dom}(\hat{A}^{-1}) = \{G \in L^2 \mid \lambda^{-1}G(\lambda) \in L^2\}. \end{gathered}$$

Theorem 7. *Introduce, into L^2, the Laurent polynomials basis (4.4) and transfer this space into $\mathbf{l}_2$. The operator above $\hat{A}^{-1}$ can be rewritten as an operator on $\mathbf{l}_2$ generated by the matrix K (4.43) on the set $\mathbf{l}_{\mathrm{fin}}$. This matrix has properties (4.44) and acts, of course, by rule (4.45).*

We will call it the algebraically inverse matrix $J^{-1} = K$ to J, $J^{-1}Jf = f = JJ^{-1}f$, $f \in \mathbf{l}_{\mathrm{fin}}$.

Proof. At first we note that every Laurent polynomial belongs to $\mathrm{Dom}(\hat{A}^{-1})$. Indeed such a polynomial is a linear combination of a finite number of functions λ^m, $m \in \mathbb{Z}$. But according to the definition of $\mathrm{Dom}(\hat{A}^{-1})$ (4.46) $\forall m \in \mathbb{Z}$, $\lambda^m \in \mathrm{Dom}(\hat{A}^{-1})$ since $\lambda^{-1}\lambda^m \in L^2$.

Therefore we can construct the matrix K of type (4.10) for the operator $\hat{A}^{-1}$ using polynomials (4.4). This matrix has the structure (4.43) with properties (4.44) since we can repeat for this matrix Lemmas 4, 8, 9,— all integrals of type (4.34) now exist. The last equality in the formulation of the theorem follows from (4.46) and, for example, from the remarks made about (4.31). □

5. The direct and inverse spectral problems related to a three-diagonal block Jacobi-Laurent matrix

In this Section we will consider, on the space $\mathbf{l}_2$ (4.1), (4.25), the operator $\mathbf{J}$ generated by the matrix J (4.26) with the conditions (4.27), (4.28) on its elements, noted in the Theorem 5. Moreover, we will demand that algebraically inverse matrix J^{-1} exist and satisfy the conditions of Theorem 6.

At first we recall some general facts concerning a rigging for the case of space $\mathbf{l}_2$ and eigenfunction expansion for selfadjoint operators acting on this space. In addition to the space $\mathbf{l}_2$ we consider its rigging

$$(\mathbf{l}_{\mathrm{fin}})' \supset \mathbf{l}_2(p^{-1}) \supset \mathbf{l}_2 \supset \mathbf{l}_2(p) \supset \mathbf{l}_{\mathrm{fin}}, \tag{5.1}$$

where $\mathbf{l}_2(p)$ is a weighted $\mathbf{l}_2$-space with a weight $p = (p_n)_{n=0}^{\infty}$, $p_n \geq 1$, $(p^{-1} = (p_n^{-1})_{n=0}^{\infty})$. In our case, $\mathbf{l}_2(p)$ is the Hilbert space of sequences $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$ for which we have

$$\|f\|^2_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} p_n, \quad (f, g)_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n} p_n.$$

The space $\mathbf{l}_2(p^{-1})$ is defined analogously; recall that $\mathbf{l}_{\mathrm{fin}}$ is the space of finite sequences and $(\mathbf{l}_{\mathrm{fin}})'$ is the space conjugate to $\mathbf{l}_{\mathrm{fin}}$ and equal to the space $\mathbf{l}$ of all sequences $f = (f_n)_{n=1}^{\infty}$, $f_n \in \mathcal{H}_n$. It is easy to show that the embedding $\mathbf{l}_2(p) \hookrightarrow \mathbf{l}_2$ is quasinuclear if $\sum_{n=0}^{\infty} p_n^{-1} < \infty$ (see, for example, [4], Ch. 5; [6], Ch. 14).

Let A be an arbitrary selfadjoint operator standardly connected with the chain (5.1). According to the projection spectral theorem (see Section 2) such an operator has a representation

$$Af = \int_{\mathbb{R}} \lambda \Phi(\lambda)\, d\sigma(\lambda) f, \quad f \in \mathbf{l}_2, \tag{5.2}$$

where $\Phi(\lambda) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is the operator of generalized projection and $d\sigma(\lambda)$ is the spectral measure. For every $f \in \mathbf{l}_{\mathrm{fin}}$, the projection $\Phi(\lambda) f \in \mathbf{l}_2(p^{-1})$ is a generalized eigenvector of the operator A with corresponding eigenvalues λ. For all $f, g \in \mathbf{l}_{\mathrm{fin}}$ we have the Parseval equality

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} (\Phi(\lambda) f, g)_{\mathbf{l}_2}\, d\sigma(\lambda); \tag{5.3}$$

after extending it by continuity, the equality (5.3) takes place for $\forall f, g \in \mathbf{l}_2$.

Let us denote by π_n the operator of orthogonal projection in $\mathbf{l}_2$ on $\mathcal{H}_n$, $n \in \mathbb{N}_0$. Hence $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously on the spaces $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$ (but possibly has the norm which is not equal to one).

Let us consider the operator matrix $(\Phi_{j,k}(\lambda))_{j,k=0}^{\infty}$, where

$$\Phi_{j,k}(\lambda) = \pi_j \Phi(\lambda) \pi_k \ : \ \mathbf{l}_2 \longrightarrow \mathcal{H}_j \quad (\text{or} \quad \mathcal{H}_k \longrightarrow \mathcal{H}_j). \tag{5.4}$$

The Parseval equality (5.3) can be rewritten as follows: $\forall f, g \in \mathbf{l}_2$

$$(f,g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (\Phi(\lambda)\pi_k f, \pi_j g)_{\mathbf{l}_2}\, d\sigma(\lambda)$$
$$= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (\pi_j\Phi(\lambda)\pi_k f, g)_{\mathbf{l}_2}\, d\sigma(\lambda) \tag{5.5}$$
$$= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (\Phi_{j,k}(\lambda) f_k, g_j)_{\mathbf{l}_2}\, d\sigma(\lambda).$$

Let us now pass to a study of a more special selfadjoint operator A that acts on the space $\mathbf{l}_2$. Namely, let $A = \mathbf{J}$ where $\mathbf{J}$ is the closed operator generated on the space $\mathbf{l}_2$ by matrix (4.26) with conditions (4.27), (4.28) by the rule

$$\mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f \longmapsto \mathbf{J}f := Jf \in \mathbf{l}_2. \tag{5.6}$$

We will also assume that $\mathbf{J}$ is *selfadjoint*. From (5.6), (4.29) it is easy to conclude that our operator $\mathbf{J}$ is standardly connected with chain (5.1). So, the above stated results of type (5.2) – (5.5) can be applied to the operator $\mathbf{J}$.

Additionally we will demand *that the matrix J has an algebraically inverse matrix J^{-1} which satisfies the conditions of Theorem 6: (4.44), (4.45).*

Existence of such J^{-1} is a very essential condition. Such a matrix J we will call (selfadjoint) Jacobi-Laurent matrix.

Our first aim is to rewrite the Parseval equality (5.5) for our $A = \mathbf{J}$ in terms of generalized eigenvectors of $\mathbf{J}$. We prove the following essential lemma.

Lemma 10. *Let $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}$, $\varphi_n(\lambda) \in \mathcal{H}_n$, $\lambda \in \mathbb{R}$, be a generalized eigenvector from $(\mathbf{l}_{\mathrm{fin}})'$ of the operator $\mathbf{J}$ constructed from the selfadjoint Jacobi-Laurent matrix J. Multiplying of $\varphi(\lambda)$ by a scalar constant (depending on λ) we can obtain that $\varphi_0(\lambda) = \varphi_0$ is independent of λ.*

We assert that $\varphi(\lambda)$, $\forall \lambda \in \mathbb{R}$, is a solution from $(\mathbf{l}_{\mathrm{fin}})'$ of the difference equation

$$\begin{gathered}(J\varphi(\lambda))_n = a_{n-1}\varphi_{n-1}(\lambda) + b_n\varphi_n(\lambda) + c_n\varphi_{n+1}(\lambda) = \lambda\varphi_n(\lambda),\\ n \in \mathbb{N}_0, \quad \varphi_{-1}(\lambda) = 0,\end{gathered} \tag{5.7}$$

and has the following representation:

$$\varphi_n(\lambda) = Q_n(\lambda)\varphi_0; \quad Q_0(\lambda) = 1, \quad Q_n(\lambda) = (Q_{n;0}, Q_{n;1}), \quad n \in \mathbb{N}. \tag{5.8}$$

Here $Q_{n;\alpha}$, $\alpha = 0,1$, are Laurent polynomials of λ, λ^{-1} and these polynomials have the form

$$Q_{n;\alpha}(\lambda) = l_{n;\alpha}\lambda^{(-1)^{\alpha+1}n} + w_{n;\alpha}(\lambda), \quad n \in \mathbb{N}, \quad \alpha = 0,1. \tag{5.9}$$

In (5.9), $l_{n;\alpha} > 0$ and $w_{n;\alpha}(\lambda)$ is some linear combination of λ^j with real coefficients, $j \in \{0, -1, 1, -2, 2, \cdots, -(n-1), -\alpha n\}$, i.e., it belongs to $\mathcal{P}_{n-1;1}$ if $\alpha = 0$ and to $\mathcal{P}_{n;0}$ if $\alpha = 1$.

Proof. At first recall that, by definition, $\varphi(\lambda) \in (\mathbf{l}_{\mathrm{fin}})' = \mathbf{l}$ is a generalized eigenvector with eigenvalue λ for the operator $\mathbf{J}$ standardly connected with rigging (5.1) if the following equality is true:

$$(\varphi(\lambda), Jf)_{\mathbf{l}_2} = (\varphi(\lambda), \mathbf{J}f)_{\mathbf{l}_2} = \lambda(\varphi(\lambda), f)_{\mathbf{l}_2}, \quad f \in \mathbf{l}_{\mathrm{fin}}. \tag{5.10}$$

Using (4.29) and arbitrariness of f we conclude from (5.10) that

$$\begin{gathered}(J\varphi(\lambda))_n = a_{n-1}\varphi_{n-1}(\lambda) + b_n\varphi_n(\lambda) + c_n\varphi_{n+1}(\lambda) = \lambda\varphi_n(\lambda),\\ n \in \mathbb{N}_0, \quad \varphi_{-1}(\lambda) = 0.\end{gathered} \tag{5.11}$$

For the matrix J^{-1} (4.43) we also have an analogous equality. Namely, using Theorem 7 (equality $JJ^{-1}f = f$, $f \in \mathbf{l}_{\text{fin}}$) we get $\forall f \in \mathbf{l}_{\text{fin}}$

$$(5.12)\qquad \begin{gathered}(\varphi(\lambda), f)_{\mathbf{l}_2} = (\varphi(\lambda), JJ^{-1}f)_{\mathbf{l}_2} = \lambda(\varphi(\lambda), J^{-1}f)_{\mathbf{l}_2}, \quad \text{i.e.}\\ (\varphi(\lambda), J^{-1}f)_{\mathbf{l}_2} = \lambda^{-1}(\varphi(\lambda), f)_{\mathbf{l}_2}.\end{gathered}$$

Note that the matrix J^{-1} is three-diagonal and, therefore, $J^{-1}f \in \mathbf{l}_{\text{fin}}$ and the second equality in (5.12) follows from (5.10).

Analogously to (5.11) the last equality in (5.12) and (4.43) give $\forall \lambda \in \mathbb{R} \setminus \{0\}$

$$(5.13)\qquad \begin{gathered}(J^{-1}\varphi(\lambda))_n = p_{n-1}\varphi_{n-1}(\lambda) + q_n\varphi_n(\lambda) + r_n\varphi_{n+1}(\lambda) = \lambda^{-1}\varphi_n(\lambda),\\ n \in \mathbb{N}_0, \quad \varphi_{-1}(\lambda) = 0.\end{gathered}$$

We give at first some explanation for the subsequent calculations. By adding two equalities (5.11) and (5.13) we get

$$(5.14)\qquad ((J + J^{-1})\varphi(\lambda))_n = (\lambda + \lambda^{-1})\varphi_n(\lambda), \quad n \in \mathbb{N}_0, \quad \varphi_{-1}(\lambda) = 0.$$

The matrix $J + J^{-1}$ is also a block three-diagonal acting on the space $\mathbf{l}_2$. But from (4.26)–(4.28) and (4.43), (4.44) we see that its blocks on two off-diagonals are 2×2 invertible matrices for $n \in \mathbb{N}$. Such a form of the matrix $J + J^{-1}$ actually shows that similarly to the classical Jacobi matrices case we can using (5.14) step by step find a generalized eigenvector $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}$ and $\forall n \in \mathbb{N}$ $\varphi_n(\lambda)$ is a polynomial w.r.t $\lambda + \frac{1}{\lambda}$ i.e. is a Laurent polynomial. But for us it is essential to get for every $\varphi_n(\lambda)$ a more exact representation (5.8), (5.9). Therefore we give below more precise calculation.

Consider equalities (5.11), (5.13) for $n = 1$. We have

$$\begin{aligned}b_0\varphi_0 + c_0\varphi_1(\lambda) &= \lambda\varphi_0,\\ q_0\varphi_0 + r_0\varphi_1(\lambda) &= \lambda^{-1}\varphi_0,\end{aligned}$$

i.e.,

$$\begin{aligned}c_{0;0,0}\varphi_{1;0}(\lambda) + c_{0;0,1}\varphi_{1;1}(\lambda) =&(\lambda - b_0)\varphi_0,\\ r_{0;0,0}\varphi_{1;0}(\lambda) + r_{0;0,1}\varphi_{1;1}(\lambda) =&(\lambda^{-1} - q_0)\varphi_0.\end{aligned}$$

The last two equalities can be regarded as a linear system of equations with respect to the unknowns $\varphi_{1;0}(\lambda)$, $\varphi_{1;1}(\lambda)$; $\varphi_0 \in \mathbb{C}$ is given. According to (4.26), (4.43) we have, for matrix of this system and its solutions, that

$$(5.15)\qquad \begin{gathered}D_1 = \begin{bmatrix} c_{0;0,0} & c_{0;0,1} \\ r_{0;0,0} & r_{0;0,1} \end{bmatrix} = \begin{bmatrix} * & + \\ + & 0 \end{bmatrix},\\ \Delta_1 = \text{Det}D_1 = |D_1| = -r_{0;0,0}c_{0;0,1} < 0,\\ \varphi_{1;0}(\lambda) = \Delta_1^{-1}\begin{vmatrix} (\lambda - b_0)\varphi_0 & + \\ (\lambda^{-1} - q_0)\varphi_0 & 0 \end{vmatrix}, \quad \varphi_{1;1}(\lambda) = \Delta_1^{-1}\begin{vmatrix} * & (\lambda - b_0)\varphi_0 \\ + & (\lambda^{-1} - a_0)\varphi_0 \end{vmatrix}.\end{gathered}$$

It is clear that these two functions have the required form (5.7).

Let $n \in \mathbb{N}$. Taking the equality (5.13) for coordinate 0 and equality (5.11) for coordinate 1 we get

$$(5.16)\qquad \begin{aligned}(p_{n-1}\varphi_{n-1}(\lambda))_0 + (q_n\varphi_n(\lambda))_0 + (r_n\varphi_{n+1}(\lambda))_0 =&\lambda^{-1}\varphi_{n;0}(\lambda),\\ (a_{n-1}\varphi_{n-1}(\lambda))_1 + (b_n\varphi_n(\lambda))_1 + (c_n\varphi_{n+1}(\lambda))_1 =&\lambda\varphi_{n;1}(\lambda).\end{aligned}$$

We can rewrite the equalities (5.16) in the following way:

$$(5.17)\qquad \begin{aligned}\begin{bmatrix} r_{n;0,0} & r_{n;0,1} \\ c_{n;1,0} & c_{n;1,1} \end{bmatrix}\varphi_{n+1}(\lambda) = \begin{bmatrix} + & 0 \\ * & + \end{bmatrix}\varphi_{n+1}(\lambda)&\\ = (\lambda^{-1}\varphi_{n;0}(\lambda) - (p_{n-1}\varphi_{n-1}(\lambda))_0 - (q_n\varphi_n(\lambda))_0,&\\ \lambda\varphi_{n;1}(\lambda) - (a_{n-1}\varphi_{n-1}(\lambda))_1 - (b_{n-1}\varphi_{n-1}(\lambda))_1),&\end{aligned}$$

i.e.

$$\varphi_{n+1;0}(\lambda) = \frac{1}{r_{n;0,0}}\left(\lambda^{-1}\varphi_{n;0}(\lambda) - (p_{n-1}\varphi_{n-1}(\lambda))_0 - (q_n\varphi_n(\lambda))_0\right),$$
$$\varphi_{n+1;1}(\lambda) = \frac{1}{c_{n;1,1}}\left(\lambda\varphi_{n;1}(\lambda) - (a_{n-1}\varphi_{n-1}(\lambda))_1 - (b_{n-1}\varphi_{n-1}(\lambda))_1 - c_{n;1,0}\varphi_{n+1;0}(\lambda)\right), \quad n \in \mathbb{N}.$$

We can now use the induction: $\varphi_1(\lambda)$ according to (5.15) has the form (5.8), (5.9); φ_0 is also such. Let for $n \in \mathbb{N}$ $\varphi_{n-1}(\lambda)$, $\varphi_n(\lambda)$ have the required form (5.8), (5.9). Then from second equality in (5.17) it is easy to see that $\varphi_{n+1;0}(\lambda)$ has the form (5.8), (5.9). The last equality in (5.17) shows that the same situation is also for $\varphi_{n+1;1}(\lambda)$. □

In what follows, it will be convenient to look at $Q_n(\lambda)$, $\forall n \in \mathbb{N}_0$, with fixed λ as a linear operator that acts $\forall n \in \mathbb{N}$ from $\mathbb{R}^1$ into $\mathbb{R}^2$, i.e., $\mathbb{R}^1 \ni \varphi_0 \longmapsto Q_n(\lambda)\varphi_0 \in \mathbb{R}^2$, and into $\mathbb{R}^1$ if $n = 0$. This operator is standardly extended to the corresponding complex space. As a result we can write

$$(5.18) \qquad \begin{gathered} \mathcal{H}_0 \ni \varphi_0 \longmapsto Q_n(\lambda)\varphi_0 \in \mathcal{H}_n, \\ Q_n^*(\lambda) = (Q_n(\lambda))^* \ : \ \mathcal{H}_n \longmapsto \mathcal{H}_0, \quad n \in \mathbb{N}_0, \quad \lambda \in \mathbb{R}. \end{gathered}$$

We also regard $Q_n(\lambda)$ as an vector-valued Laurent polynomial of $\lambda \in \mathbb{R}$ with real coefficients.

Using these polynomials $Q_n(\lambda)$ we construct the following representation for $\Phi_{j,k}(\lambda)$, introduced by (5.4).

Lemma 11. *The operator $\Phi_{j,k}(\lambda)$, $\forall \lambda \in \mathbb{R}$, has the following representation:*

$$(5.19) \qquad \Phi_{j,k}(\lambda) = Q_j(\lambda)\Phi_{0,0}(\lambda)Q_k^*(\lambda) : \mathcal{H}_k \longrightarrow \mathcal{H}_j, \quad j, k \in \mathbb{N}_0,$$

where $\Phi_{0,0}(\lambda) \geq 0$ is a scalar.

Proof. For a fixed $k \in \mathbb{N}_0$ and arbitrary fixed $x \in \mathcal{H}_k \subset \mathbf{l}_2$, the vector $\varphi(\lambda) = (\varphi_j(\lambda))_{j=0}^{\infty}$, where

$$\varphi_j(\lambda) = \Phi_{j,k}(\lambda)x = \pi_j\Phi(\lambda)\pi_k x \in \mathcal{H}_j, \quad \lambda \in \mathbb{R},$$

is a generalized solution in $(\mathbf{l}_{\text{fin}})'$ of the equation $J\varphi(\lambda) = \lambda\varphi(\lambda)$, since $\Phi(\lambda)$ is a projection onto generalized eigenvectors of the operator A with corresponding eigenvalues λ. Therefore, $\forall g \in \mathbf{l}_{\text{fin}}$ we have $(\varphi, Jg)_{\mathbf{l}_2} = \lambda(\varphi, g)_{\mathbf{l}_2}$. Hence, it follows that $\varphi = \varphi(\lambda) \in \mathbf{l}_2(p^{-1})$ exists as a usual solution of the equation $J\varphi(\lambda) = \lambda\varphi(\lambda)$ with the initial condition $\varphi_0(\lambda) = \pi_0\Phi(\lambda)\pi_k x \in \mathcal{H}_0$.

Using Lemma 10 and due to (5.8) we obtain that

$$(5.20) \qquad \Phi_{j,k}(\lambda)x = Q_j(\lambda)(\Phi_{0,k}(\lambda)x), \quad \text{i.e.} \quad \Phi_{j,k}(\lambda) = Q_j(\lambda)\Phi_{0,k}(\lambda), \quad j \in \mathbb{N}_0.$$

The operator $\Phi(\lambda) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is formally selfadjoint on $\mathbf{l}_2$ (see Section 2). Hence, according to (5.4) we get

$$(5.21) \qquad (\Phi_{j,k}(\lambda))^* = (\pi_j\Phi(\lambda)\pi_k)^* = \pi_k\Phi(\lambda)\pi_j = \Phi_{k,j}(\lambda), \quad j, k \in \mathbb{N}_0.$$

For a fixed $j \in \mathbb{N}_0$ from (5.21) and the previous discussion, it follows that the vector

$$\psi(\lambda) = (\psi_k(\lambda))_{k=0}^{\infty}, \quad \psi_k(\lambda) = \Phi_{k,j}(\lambda)y = (\Phi_{j,k}(\lambda))^*y, \quad y \in \mathcal{H}_j,$$

is a usual solution of the equations $J\psi(\lambda) = \lambda\psi(\lambda)$ with the initial condition $\psi_0(\lambda) = \Phi_{0,j}(\lambda)y = (\Phi_{j,0}(\lambda))^*y$.

Again using Lemma 10 and arbitrariness of y we obtain a representation of the type (5.20),

$$(5.22) \qquad \Phi_{0,k}(\lambda) = Q_k(\lambda)\Phi_{0,j}(\lambda), \quad k \in \mathbb{N}_0.$$

Taking into account (5.3) and (5.22) we get

$$\Phi_{0,k}(\lambda) = (\Phi_{k,0}(\lambda))^* = (Q_k(\lambda)\Phi_{0,0}(\lambda))^* = \Phi_{0,0}(\lambda)(Q_k(\lambda))^*, \quad k \in \mathbb{N}_0. \tag{5.23}$$

Here we used that $\Phi_{0,0}(\lambda) \geq 0$, which follows from (5.3) and (5.4)). Substituting (5.23) into (5.20) we obtain (5.19). □

Now it is possible to rewrite the Parseval equality (5.5) in a more concrete form. To this end, we substitute the expression (5.19) for $\Phi_{j,k}(\lambda)$ into (5.5) and get that $\forall f, g \in \mathbf{l}_{\text{fin}}$

$$\begin{aligned} (f, g)_{\mathbf{l}_2} &= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (\Phi_{j,k}(\lambda) f_k, g_j)_{\mathbf{l}_2}\, d\sigma(\lambda) \\ &= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (Q_j(\lambda)\Phi_{0,0}(\lambda) Q_k^*(\lambda) f_k, g_j)_{\mathbf{l}_2}\, d\sigma(\lambda) \\ &= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}} (Q_k^*(\lambda) f_k, Q_j^*(\lambda) g_j)_{\mathbf{l}_2}\, d\rho(\lambda) \\ &= \int_{\mathbb{R}} \Big(\sum_{k=0}^{\infty} Q_k^*(\lambda) f_k\Big) \overline{\Big(\sum_{j=0}^{\infty} Q_j^*(\lambda) g_j\Big)}\, d\rho(\lambda), \\ d\rho(\lambda) &= \Phi_{0,0}(\lambda)\, d\sigma(\lambda). \end{aligned} \tag{5.24}$$

Introduce the Fourier transform $\widehat{\ }$ induced by the selfadjoint operator $A = \mathbf{J}$ on the space $\mathbf{l}_2$ as a unitary map from $\mathbf{l}_2$ into $L^2(\mathbb{R}, d\rho(\lambda)) = L^2$. At first it is defined on $\mathbf{l}_{\text{fin}}$ by (5.18)

$$\mathbf{l}_2 \supset \mathbf{l}_{\text{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(\lambda) = \sum_{n=0}^{\infty} Q_n^*(\lambda) f_n \in L^2(\mathbb{R}, d\rho(\lambda)). \tag{5.25}$$

Hence, (5.24) gives the Parseval equality in a final form

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}\, d\rho(\lambda), \quad f, g \in \mathbf{l}_{\text{fin}}. \tag{5.26}$$

Extending (5.26) by continuity, it becomes valid $\forall f, g \in \mathbf{l}_2$.

The polynomials $Q_n^*(\lambda) = (Q_{n;0}(\lambda), Q_{n;1}(\lambda)) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$ are in some sense orthonormal. This orthogonality follows from (5.25) and (5.26). Namely, we take $f = (0, \dots, 0, f_j, 0, \dots)$, $f_j \in \mathcal{H}_j$, $g = (0, \dots, 0, g_k, 0, \dots)$, $g_k \in \mathcal{H}_k$ in (5.25) and (5.26). Then

$$\begin{gathered} \int_{\mathbb{R}} (Q_j^*(\lambda) f_j)\overline{(Q_k^*(\lambda) g_k)}\, d\rho(\lambda) = \delta_{j,k}(f_j, g_j)_{\mathcal{H}_j}, \\ f_j \in \mathcal{H}_j, \quad g_k \in \mathcal{H}_k, \quad j, k \in \mathbb{N}_0. \end{gathered} \tag{5.27}$$

Rewrite the equality (5.27) in a more simple form. To do this, we first note that according to (5.18) and (5.8) we have for $n \in \mathbb{N}$ and $f_n = (f_{n,0}, f_{n,1}) \in \mathcal{H}_n$, $\lambda \in \mathbb{R}$, that

$$Q_n^*(\lambda) f_n = Q_{n;0}(\lambda) f_{n;0} + Q_{n;1}(\lambda) f_{n;1}, \quad Q_0^*(\lambda) = 1. \tag{5.28}$$

Taking, in (5.27), $f_j = e_\alpha$ and $g_k = e_\beta$, $\alpha, \beta = 0, 1$, we get from (5.27), (5.28) the final relation of orthogonality,

$$\begin{gathered} \int_{\mathbb{R}} Q_{j;\alpha}(\lambda) Q_{k;\beta}(\lambda)\, d\rho(\lambda) = \delta_{j,k}\delta_{\alpha,\beta}, \\ j, k \in \mathbb{N}_0, \quad \alpha, \beta = 0, 1; \quad Q_{0,0}(\lambda) = Q_0(\lambda) = 1, \quad \lambda \in \mathbb{R}. \end{gathered} \tag{5.29}$$

Let us remark that due to (5.28) the Fourier transform (5.25) can be rewritten as $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$

$$\hat{f}(\lambda) = f_{0;0} + \sum_{n=1}^{\infty}\sum_{\alpha=0}^{1} Q_{n;\alpha}(\lambda) f_{n;\alpha}, \quad \lambda \in \mathbb{R}.$$

The stated above results of this Section can be formulated as the following spectral theorem for our operator $A = \mathbf{J}$.

Theorem 8. *Consider the space (4.25)*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_0 = \mathbb{C}^1, \quad \mathcal{H}_n = \mathbb{C}^2, \quad n \in \mathbb{N}, \tag{5.30}$$

and the linear selfadjoint operator $A = \mathbf{J}$ which is defined on finite vectors $\mathbf{l}_{\mathrm{fin}}$ by a block three-diagonal Jacobi-Laurent matrix J. So, we suppose that J, being of the form (4.26), (4.27), (4.28), generates a selfadjoint operator $\mathbf{J}$ and for J the algebraically inverse matrix J^{-1} exists and has the form (4.43), (4.44).

The eigenfunction expansion of the operator $\mathbf{J}$ has the following form. According to Lemma 10 we represent, starting with $\varphi_0 \in \mathbb{R}$, the solution $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}$, $\varphi_n(\lambda) \in \mathcal{H}_n$, of equations (5.7) for $\lambda \in \mathbb{R}$

$$\varphi_n(\lambda) = Q_n(\lambda)\varphi_0 = (Q_{n;0}(\lambda), Q_{n;1}(\lambda))\varphi_0.$$

Here $Q_0(\lambda) = 1$ and $Q_{n;\alpha}(\lambda)$, $\alpha = 0, 1$, $n \in \mathbb{N}$, are real polynomials of λ and λ^{-1} of the form (5.8), (5.9). The corresponding Fourier transform has the form

$$\begin{aligned} &\mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(\lambda) \\ &\quad = f_{0;0} + \sum_{n=1}^{\infty} Q_n^*(\lambda) f_n = Q_{0;0}(\lambda) f_{0;0} + \sum_{n=1}^{\infty}\sum_{\alpha=0}^{1} Q_{n;\alpha}(\lambda) f_{n;\alpha} \in L^2(\mathbb{R}, d\rho(\lambda)). \end{aligned} \tag{5.31}$$

Here $Q_n^*(\lambda) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ is an adjoint to the operator $Q_n(\lambda) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$, $d\rho(\lambda)$ is the Borel probability spectral measure of $\mathbf{J}$.

The Parseval equality has the following form: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}\, d\rho(\lambda), \quad (Jf, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} \lambda \hat{f}(\lambda)\overline{\hat{g}(\lambda)}\, d\rho(\lambda). \tag{5.32}$$

Formulas (5.31) and (5.32) are extended by continuity to $\forall f, g \in \mathbf{l}_2$, the operator (5.31) now is unitary, which maps $\mathbf{l}_2$ onto the whole $L^2(\mathbb{R}, d\rho(\lambda))$.

The polynomials $Q_{n;\alpha}(\lambda)$, $n \in \mathbb{N}$, $\alpha = 0, 1$, and $Q_{0;0}(\lambda) = 1$, form an orthonormal system in $L^2(\mathbb{R}, d\rho(\lambda))$ in the sense of (5.29), total in this space.

The last theorem solves the direct spectral problem for the selfadjoint operator $A = \mathbf{J}$ which is generated on the space $\mathbf{l}_2$ by the selfadjoint Jacobi-Laurent matrix J of the form (4.26) with an algebraically inverse matrix J^{-1} (4.43). [1]

Let us pass to the corresponding inverse spectral problem. Roughly speaking, this problem is the following. We know the spectral measure $d\rho(\lambda)$ of the operator $\mathbf{J}$ on the space $\mathbf{l}_2$ generated by a selfadjoint Jacobi-Laurent matrix J. In what way we can find the matrix J?

The following theorem is a solution of this problem.

[1] *A proofreading remark.* Let us stress that for a selfadjoint operator on the space (5.30), the spectral measure generated by the matrix J of the structure (4.26) must be a 2×2-matrix measure [38]. It is very essential that in our case this measure is a scalar measure. This fact is a consequence of situation: for our operator $\mathbf{J}$ the algebraically inverse operator exists and therefore we can prove Lemma 10.

Theorem 9. *The spectral measure $d\rho(\lambda)$ of the selfadjoint operator $\mathbf{J}$ generated on the space $\mathbf{l}_2$ by a selfadjoint Jacobi-Laurent matrix J has the following properties:*

1. The measure $d\rho(\lambda)$ is a Borel probability measure on $\mathbb{R}$;

2. All the functions

$$\mathbb{R} \ni \lambda \longmapsto \lambda^m, \quad m \in \mathbb{Z}, \tag{5.33}$$

belong to the space $L^2(\mathbb{R}, d\rho(\lambda))$ and are linearly independent in this space;

3. The set of functions (5.33) is total in the space $L^2(\mathbb{R}, d\rho(\lambda))$.

Conversely, for every given measure $d\rho(\lambda)$ with properties 1—3 it is possible to construct a selfadjoint Jacobi-Laurent matrix J for which this measure is spectral.

For the construction of the matrix J it is necessary to repeat the constructions of Section 4: by orthogonalization of (4.3) we find $P_{n;\alpha}(\lambda)$, (4.4) and then apply the formulas (4.12) and (4.34). If we start from the spectral measure $d\rho(\lambda)$ of the operator $\mathbf{J}$, the last procedure gives the initial selfadjoint Jacobi-Laurent matrix J.

Proof. Let $\mathbf{J}$ be a given selfadjoint operator constructed from J and $d\rho(\lambda)$ be its spectral measure. By definition property 1 is true. From Theorem 8 and formula (5.9) it follows that every function (5.33) is some linear combination with real coefficients of the functions (5.9) and leading coefficient $l_{n;\alpha}$ at the function $\lambda^{(-1)^{\alpha+1}n}$ in $Q_{n;\alpha}(\lambda)$, is positive. The functions $Q_{n;\alpha}(\lambda)$, $n \in \mathbb{N}_0$, $\alpha = 0, 1$, belong to $L^2(\mathbb{R}, d\rho(\lambda)) = L^2$ and are orthogonal (see (5.29)), therefore all functions (5.33) are from L^2 and linearly independent, i.e., the property 2 is true.

Consider property 3. From the second equality in (5.32) and unitarity of the Fourier transform (5.31) between $\mathbf{l}_2$ and L^2, we conclude that our selfadjoint operator $\mathbf{J}$ is unitary equivalent to the closed operator $\hat{A}$ of multiplication by λ in the space L^2, at first defined on linear combinations of the functions $Q_{n;\alpha}$, $n \in \mathbb{N}_0$, $\alpha = 0, 1$. These functions form a total set in L^2. Using the representation (5.9) we conclude that set (5.33) is also total in L^2.

The other statement follows directly from results of Section 4,5. It is necessary only to note that the Laurent polynomials $P_{n;\alpha}(\lambda)$ and the polynomials $Q_{n;\alpha}(\lambda)$ from Theorem 8 are the same,— both classes of these polynomials are constructed from polynomials (4.3) or (5.9) and are orthonormal. ☐

In what follows the Laurent polynomials $Q_{n;\alpha}(\lambda)$ for J will be denoted in a more standard way, $P_{n;\alpha}(\lambda)$, $n \in \mathbb{N}_0$, $\alpha, \beta = 0, 1$.

6. Consideration of Hermitian block Jacobi-Laurent type matrices

This section is a development of the last part of Section 4. Here we consider the matrices J and J^{-1} of the form (4.26) and (4.43) for which the conditions (4.27), (4.28) and (4.44) are fulfilled but the corresponding to J operator $\mathbf{J}$ defined on the space $\mathbf{l}_2$ (5.30) by relation (5.27) on $\mathbf{l}_{\text{fin}}$ is only Hermitian. In this case the set of functions (5.33) (i.e. (4.30)) is, as earlier, total in $L^2(\mathbb{R}, d\rho(\lambda))$. Previous constructions of course, can be carried out (including the construction of algebraically inverse matrix J^{-1}) but other results of Section 5 connected with selfadjointness of $\mathbf{J}$ are not fulfilled.

So, we consider the matrices J, J^{-1} of the type (4.26), (4.43) with conditions (4.27), (4.28), (4.44). It is assumed that the matrix J^{-1} is algebraically inverse to J: $J^{-1}Jf = JJ^{-1}f = f$, $f \in \mathbf{l}_{\text{fin}}$. Then the operator $\mathbf{J}$, constructed by (5.6) on $\mathbf{l}_2$ is only Hermitian. Such a matrix J will be called Hermitian block Jacobi-Laurent type matrix.

At first we will formulate and prove some simple but essential theorem.

Theorem 10. *The deficiency indexes of* $\mathbf{J}$ *may be* $(0,0)$ *or* $(1,1)$. *Let* $z \in \mathbb{C} \setminus \mathbb{R}$ *and* $P(z) = (P_n(z))_{n=0}^{\infty}$, $P_0(z) = 1 \in \mathbb{C}^1$, $P_n(z) \in \mathbb{C}^2$, $n \in \mathbb{N}$, *be a solution of the difference* *equation*

$$JP(z) = zP(z), \quad \textit{i.e.} \tag{6.1}$$
$$a_{n-1}P_{n-1}(z) + b_nP_n(z) + c_nP_{n+1}(z) = zP_n(z), \quad n \in \mathbb{N}_0, \quad P_{-1}(z) = 0.$$

Consider the series

$$\sum_{n=0}^{\infty} \|P_n(z)\|^2_{\mathcal{H}_n}. \tag{6.2}$$

If this series is divergent for some $z \in \mathbb{C} \setminus \mathbb{R}$, then it is divergent for every $z \in \mathbb{C} \setminus \mathbb{R}$ and the deficiency indices of $\mathbf{J}$ are $(0,0)$. If it is convergent for some such z (and therefore for every z) then the deficiency indices of $\mathbf{J}$ are $(1,1)$.

Proof. At first we note that the proof of existence of a solution $P(z)$ of equation (6.1) is the same as in Lemma 10 with a replacement of λ with z. The proof of this lemma does not depend on selfadjointness of $\mathbf{J}$, it is necessary to use that J^{-1} is algebraically inverse to J.

As earlier, $P_n(z)$ must be of the form (5.8), (5.9) with $Q_{n;\alpha}(\lambda)$ being replaced with $P_{n;\alpha}(\lambda)$. So, we have

$$P_0(z) = 1,$$
$$P_n(z) = (P_{n;0}(z), P_{n;1}(z)),$$
$$P_{n;\alpha}(z) = l_{n;\alpha} z^{(-1)^{\alpha+1}n} + w_{n;\alpha}(z), \quad n \in \mathbb{N}, \quad \alpha = 0, 1;$$

and the structure of the polynomial $P_{n;\alpha}(z)$ is the same as $Q_{n;\alpha}(z)$ in (5.9).

The operator $\mathbf{J}$ generated by J on $\mathbf{l}_{\text{fin}}$ is Hermitian, let $\mathbf{J}^*$ be its adjoint in $\mathbf{l}_2$. Let $z \in \mathbb{C} \setminus \mathbb{R}$, consider the corresponding to $\bar{z}$ deficiency space $N_{\bar{z}}$ of operator $\mathbf{J}$, i.e., the subspace of $\mathbf{l}_2$ consisting of vectors $g \in \mathbf{l}_2$ for which

$$((\mathbf{J} - \bar{z}1)f, g)_{\mathbf{l}_2} = ((J - \bar{z}1)f, g)_{\mathbf{l}_2} = 0, \quad f \in \mathbf{l}_{\text{fin}}. \tag{6.3}$$

Since f in (6.3) is finite, we can move the matrix $J - \bar{z}1$ to g in (6.3). As a result we get $\forall f \in \mathbf{l}_{\text{fin}}$,

$$(f, (J - z1)g) = 0, \quad \text{i.e.} \quad Jg = \bar{z}g \tag{6.4}$$
$$\text{or} \quad a_{n-1}g_{n-1} + b_ng_n + c_ng_{n+1} = zg_n, \quad n \in \mathbb{N}_0, \quad g_{-1} = 0$$

(note that the elements of J are real and $\mathbf{l}_{\text{fin}}$ is dense in $\mathbf{l}_2$). So, $g = (g_n)_{n=0}^{\infty}$ is a solution of the difference equation in (6.4).

From the above considerations, it follows that this solution has the form

$$(g_n)_{n=0}^{\infty} = (g_0 P_n(z))_{n=0}^{\infty}. \tag{6.5}$$

In the case where the series (6.2) is divergent, the sequence (6.5) belongs to $\mathbf{l}_2$ iff $g_0 = 0$, i.e., if $g = 0$. Then the operator $\mathbf{J}$ is selfadjoint and its deficiency numbers are $(0,0)$.

Let series (6.2) be convergent. Then sequence (6.5) belongs to $\mathbf{l}_2$ and the set of such sequences is one-dimensional, i.e., $\dim(N_{\bar{z}}) = 1$. Since the elements of J are real, $N_{\bar{z}} = N_z$ and the deficiency numbers are $(1,1)$. □

Let $\mathbf{J}$ be the above introduced operator in the case of convergent series (6.2) $\forall z \in \mathbb{C} \setminus \mathbb{R}$, i.e., this operator is only Hermitian. We can take its some selfadjoint extension $\tilde{\mathbf{J}}$ in the space $\mathbf{l}_2$. It is easy, for this extension, to repeat all constructions of Section 5 and to prove Theorems 8, 9. But in this "indeterminate" case of J, the spectral measure $d\rho(\lambda)$ depends on the extension $\tilde{\mathbf{J}}$. One of our nearest aims is to give a description of all such measures corresponding to the strong moment problem, i.e., to operators that have (possibly densely defined) inverses.

Introduce by (6.1) Laurent polynomials "of the first kind" as an analog of classical polynomials of the first kind in the usual theory of Jacobi matrices. In our Laurent case it is possible and useful to introduce an analog of polynomials of the second kind.

Such Laurent polynomials $Q_n(z)$ of the second kind are introduced analogously to the classical case by the formula

$$Q_n(z) = \int_{\mathbb{R}} \frac{P_n(\lambda) - P_n(z)}{\lambda - z}\, d\rho(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad n \in \mathbb{N}_0. \tag{6.6}$$

Here $d\rho(\lambda)$ is the spectral measure of some fixed selfadjoint extension of the operator $\mathbf{J}$ in the space $\mathbf{l}_2$ or of the operator $\mathbf{J}$ if it is selfadjoint; we integrate in (6.6) the vector-valued function.

Lemma 12. *The sequence $Q(z) = (Q_n(z))_{n=0}^{\infty}$, $z \in \mathbb{C} \setminus \mathbb{R}$, where $Q_n(z)$ is given by formula (6.6), is a solution of the following difference equations with the indicated below initial data,*

$$\begin{gathered} a_{n-1}Q_{n-1}(z) + b_nQ_n(z) + c_nQ_{n+1}(z) = zQ_n(z), \\ p_{n-1}Q_{n-1}(z) + q_nQ_n(z) + r_nQ_{n+1}(z) = z^{-1}Q_n(z), \quad n \in \mathbb{N}, \\ Q_0(z) = 0, \quad Q_1(z) = (c_{0;0,1}^{-1}s_{-1}z^{-1}, c_{0;0,1}^{-1}(1 - c_{0;0,0}r_{0;0,0}^{-1}s_{-1}z^{-1})), \\ s_{-1} = \int_{\mathbb{R}} \lambda^{-1}\, d\rho(\lambda); \quad z \in \mathbb{C}. \end{gathered} \tag{6.7}$$

At first we note a simple general fact which we have actually exploited earlier.

Lemma 13. *Let $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n = \mathbb{C}^2$, $n \in \mathbb{N}$, $z \in \mathbb{C} \setminus \{0\}$ is some solution of the first equation in (6.7). Then it is a solution of the second equation in (6.7) (an analogous fact is also true for equations (5.11), (5.13)).*

Proof. Indeed, extend the sequence $\varphi(z) = (\varphi_n(z))_{n=1}^{\infty}$ to $\mathbb{N}_0$ by setting $\varphi_0(z) = 0$. Then such an extended sequence $\varphi'(z)$ is, according to (6.7), a solution of the equation $J\varphi'(z) = z\varphi'(z)$. The matrix J^{-1} is an algebraic inverse to J and is also block three-diagonal. Therefore, we can write $J^{-1}J = JJ^{-1} = 1$, i.e. $J^{-1}Jf = f$, $f \in \mathbf{l}$. Taking $f = \varphi'(z)$ we get $\varphi'(z) = J^{-1}J\varphi'(z) = zJ^{-1}\varphi'(z)$, that is the second equality from (6.7). □

Proof of Lemma 12. Since $P_0(z) = 1$, we get from (6.6) that

$$Q_0(z) = 0, \quad z \in \mathbb{C}. \tag{6.8}$$

Calculate $Q_{1;0}(z)$. From (5.15) and (6.6) we conclude that $\forall z \in \mathbb{C}$

$$\begin{aligned} P_{1;0}(z) &= \frac{1}{r_{0;0,0}}(z^{-1} - q_0), \\ Q_{1;0}(z) &= \frac{1}{r_{0;0,0}} \int_{\mathbb{R}} (\lambda^{-1} - z^{-1})(\lambda - z)^{-1}\, d\rho(\lambda) = \frac{1}{r_{0;0,0}} \int_{\mathbb{R}} \lambda^{-1}\, d\rho(\lambda) z^{-1}. \end{aligned} \tag{6.9}$$

Calculate $Q_{1;1}(z)$. From (5.15) we get

$$P_{1;1}(z) = -\frac{c_{0;0,0}}{r_{0;0,0}c_{0;0,1}}(z^{-1} - q_0) + \frac{1}{c_{0;0,1}}(z - b_0), \quad z \in \mathbb{C}. \tag{6.10}$$

From (6.6) and (6.10) we conclude after a simple calculation that

$$Q_{1;1}(z) = \frac{1}{c_{0;0,1}} - \frac{c_{0;0,0}}{r_{0;0,0}c_{0;0,1}} \int_{\mathbb{R}} \lambda^{-1}\, d\rho(\lambda) z^{-1}, \quad z \in \mathbb{C}.$$

Formulas (6.8)–(6.10) give the initial data in (6.7).

To prove the lemma, it is necessary to check that $Q_n(z)$, $n \in \mathbb{N}$, $Q_0(z) = 0$, satisfy the two equations from (6.7). At first we consider the first equation. Since $Q_0(z) = 0$ the left-hand side of this equation is equal to $(JQ(z))_n$, $n \in \mathbb{N}$. But according to (6.6) and (6.1) we have $\forall z \in \mathbb{C} \setminus \mathbb{R}$, $n \in \mathbb{N}$,

$$
\begin{aligned}
(JQ(z))_n &= \int\limits_{\mathbb{R}} \frac{(JP(\lambda))_n - (JP(z))_n}{\lambda - z}\, d\rho(\lambda) \\
&= \int\limits_{\mathbb{R}} \frac{\lambda P_n(\lambda) - z P_n(z)}{\lambda - z}\, d\rho(\lambda) \\
&= z \int\limits_{\mathbb{R}} \frac{P_n(\lambda) - P_n(z)}{\lambda - z}\, d\rho(\lambda) + \int\limits_{\mathbb{R}} P_n(\lambda)\, d\rho(\lambda) \\
&= z Q_n(z) + \int\limits_{\mathbb{R}} P_n(\lambda)\, d\rho(\lambda).
\end{aligned} \tag{6.11}
$$

According to (5.29) the last integral in (6.11) is equal to zero, therefore (6.11) gives $(JQ(z))_n = zQ_n(z)$, $n \in \mathbb{N}$. This means that the first equality in (6.7) is fulfilled.

The second equality in (6.7) is fulfilled on the basis of Lemma 13. □

As a result, for the Laurent polynomials $Q_n(z)$ of the second kind we have the situation similar to the case of the first kind,— their sequence is a solution for $n = 1, 2, \ldots$ of system (6.7) with the given in (6.7) initial data $Q_0(z) = 0$, $Q_1(z)$. These polynomial (in the general case) are not orthonormal in the space $L^2(\mathbb{R}, d\rho(\lambda))$ w.r.t. the spectral measure $d\rho(\lambda)$ generated by some selfadjoint extension $\tilde{\mathbf{J}}$ of the operator $\mathbf{J}$ in the space $\mathbf{l}_2$ (this situation is similar to the classical Jacobi matrices).

Polynomials of the second kind are necessary for a description of the set of all spectral measures generated by all selfadjoint extensions in $\mathbf{l}_2$ of operator $\mathbf{J}$. They can be found step-by-step as solution of system (6.7) from $Q_0(z)$, $Q_1(z)$. For the polynomials $Q_n(z)$, we can prove an analog of Lemma 10 about their precise structure, but such results are not necessary for us and we are restricted to the following rough result.

Lemma 14. *Every Laurent polynomial of the second kind has the form*

$$
\begin{aligned}
&Q_0(z) = (0, 0), \\
&Q_n(z) = (Q_{n;0}(z), Q_{n;1}(z)) \in \mathbb{C}^2, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad n \in \mathbb{N},
\end{aligned} \tag{6.12}
$$

where $\forall \alpha = 0, 1$, $Q_{n;\alpha}(z)$, *is a linear combination with real coefficients of* 1, z^{-1}, z, $\ldots$, z^{-n}, z^n.

Proof. The vectors $Q_n(z) \in \mathbb{C}^2$, $n \in \mathbb{N}$, are solutions of the system of difference equations (6.7). Here we can assume that $a_0 = p_0 = 0$, since $Q_0(z) = 0$. By adding the equality (6.7) we get

$$
\begin{aligned}
&(a_{n-1} + p_{n-1}) Q_{n-1}(z) + (b_n + q_n) Q_n(z) + (c_n + r_n) Q_{n+1}(z) \\
&\qquad = (z + z^{-1}) Q_n(z), \quad n \in \mathbb{N}, \quad a_0 + p_0 = 0.
\end{aligned} \tag{6.13}
$$

The last equation can be rewritten as some equation with three-diagonal block (2×2) matrices acting on the space $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots$. The off-diagonal matrices $a_n + p_n$ and $c_n + r_n$ are invertible according to (4.26), (4.27), (4.28) and (4.43) (4.44), have positive numbers on the main diagonals and zero on places 1,0 and 0,1 respectively.

Therefore we can find a solution of (6.13) step-by-step starting with $Q_1(z)$, which has the form $(C_1 z^{-1}, C_2 + C_3 z^{-1})$, $C_1, C_2, C_3 \in \mathbb{R}$. Every $n + 1$ step gives a multiplication of $Q_n(z)$ by $z + \frac{1}{z}$ plus some linear combinations of $1, z^{-1}, z, \ldots, z^{-n}, z^n$. From this, our Lemma follows. □

Let us now pass to a description of the spectral measures $d\rho(\lambda)$ corresponding to the operator $\tilde{\mathbf{J}}$ acting in the space $\mathbf{l}_2$. Let $R_z : \mathbf{l}_2 \to \mathbf{l}_2$, $z \in \mathbb{C}\setminus\mathbb{R}$, be the resolvent of operator $\tilde{\mathbf{J}}$. The representation (4.8), (4.10) for the resolvent using the mapping (5.2), (5.19) and (5.31), (5.32) can be written in the form

$$
\begin{aligned}
(R_z f)_j &= \sum_{k=0}^{\infty} R_{z;j,k} f_k,\\
R_{z;j,k;\alpha,\beta} &= (R_z e_{k,\beta}, e_{j,\alpha})_{\mathbf{l}_2} = \Big(\Big(\int_{\mathbb{R}} \frac{\Phi(\lambda)}{\lambda - z}\, d\sigma(\lambda)\Big) e_{k;\beta}, e_{j,\alpha}\Big)_{\mathbf{l}_2}\\
&= \int_{\mathbb{R}} \frac{P_{k;\beta}(\lambda) - P_{j;\alpha}(\lambda)}{\lambda - z}\, d\rho(\lambda),\\
& z \in \mathbb{C}\setminus\mathbb{R}, \quad j,k \in \mathbb{N}_0, \quad \alpha,\beta = 0,1; \quad f \in \mathbf{l}_2.
\end{aligned} \tag{6.14}
$$

Introduce the *Weyl function*

$$
m(z) = R_{z;0,0;0,0} = (R_z e_{0;0}, e_{0;0})_{\mathbf{l}_2} = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\rho(\lambda), \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{6.15}
$$

The classical fact asserts that the function $m(z)$ defines the measure $d\rho(\lambda)$ uniquely, therefore, instead of the measure $d\rho(\lambda)$ it is sufficient to get a description of all functions $m(z)$. For this aim we at first give some simple but essential formulas.

So, using (6.14), (6.15) and the definition (6.6) we can write: $\forall z \in \mathbb{C}\setminus\mathbb{R}$, $j \in \mathbb{N}_0$, $\alpha = 0,1$

$$
\begin{aligned}
(R_z e_{0;0})_{j;\alpha} = R_{z;j,0;\alpha,0} &= \int_{\mathbb{R}} \frac{P_{j;\alpha}(\lambda)}{\lambda - z}\, d\rho(\lambda)\\
&= \int_{\mathbb{R}} \frac{P_{j;\alpha}(\lambda) - P_{j;\alpha}(z)}{\lambda - z}\, d\rho(\lambda) + P_{j;\alpha}(z) \int_{\mathbb{R}} \frac{d\rho(\lambda)}{\lambda - z} = Q_{j;\alpha} z + m(z) P_{j;\alpha}(z).
\end{aligned}
$$

Another form of this equality is

$$
(R_z e_{0;0})_j = Q_j(z) + m(z) P_j(z), \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad j \in \mathbb{N}_0. \tag{6.16}
$$

Using the Hilbert identity and (6.16) we get for arbitrary $z, \zeta \in \mathbb{C}\setminus\mathbb{R}$, $z \neq \bar{\zeta}$ that

$$
\begin{aligned}
\frac{\overline{m(\zeta)} - m(z)}{\bar{\zeta} - z} &= \Big(\frac{R_{\bar{\zeta}} - R_z}{\bar{\zeta} - z} e_{0;0}, e_{0;0}\Big)_{\mathbf{l}_2} = \big(R_{\bar{\zeta}} R_z e_{0;0}, e_{0;0}\big)_{\mathbf{l}_2}\\
&= \big(R_{\zeta}^* R_z e_{0;0}, e_{0;0}\big)_{\mathbf{l}_2} = (R_z e_{0;0}, R_{\zeta} e_{0;0})_{\mathbf{l}_2}\\
&= \sum_{j=0}^{\infty} \big(Q_j(z) + m(z) P_j(z), Q_j(\zeta) z + m(\zeta) P_j(\zeta)\big)_{\mathcal{H}_j}.
\end{aligned} \tag{6.17}
$$

Let $\zeta = z$, then (6.17) gives

$$
\frac{\overline{m(z)} - m(z)}{\bar{z} - z} = \sum_{j=0}^{\infty} \| Q_j(z) + m(z) P_j(z) \|^2_{\mathcal{H}_j}, \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{6.18}
$$

From this equality we can conclude some simple but essential fact which we can formulate, at first, as the following theorem.

Theorem 11. *Consider two series* $\forall z \in \mathbb{C}\setminus\mathbb{R}$

$$\sum_{j=0}^{\infty} \|P_j(z)\|^2_{\mathcal{H}_j}, \quad \sum_{j=0}^{\infty} \|Q_j(z)\|^2_{\mathcal{H}_j}. \tag{6.19}$$

These series are simultaneously divergent or convergent. In the first case, the operator $\mathbf{J}$ *is selfadjoint in* $\mathbf{l}_2$*, in the second case it is not selfadjoint and the measure* $d\rho(\lambda)$*, appeared above, is the spectral measure of some its selfadjoint extension in the space* $\mathbf{l}_2$*.*

In the second case both series (6.19) are convergent uniformly on every bounded domain in $\mathbb{C}$ *which is locates at a positive distance from the real axis.*

Proof. For every $z \in \mathbb{C} \setminus \mathbb{R}$, we have $m(z) \neq 0$ and the series (6.18) is convergent. Therefore, if one of the series (6.19) is divergent then the other must also be divergent.

Let both series in (6.19) be convergent. It is necessary to prove that this convergence is uniformly in the domain G pointed out in the theorem.

Since the left-hand side of equality (6.18) is a continuous function of $z \in G$ and every summand in (6.18) is a continuous nonnegative function, by Dini's theorem, this series convergent uniformly on G.

In our case, the operator $\mathbf{J}$ is not selfadjoint in $\mathbf{l}_2$. The above measure $d\rho(\lambda)$ and the function $m(z)$ were connected with some fixed extension of $\mathbf{J}$ to a selfadjoint operator in $\mathbf{l}_2$. Take another its extension (such an extension exists since the deficiency indices of $\mathbf{J}$ are $(1,1)$). Let $d\rho_1(\lambda)$ and the function $m_1(z)$ the spectral measure and the Weyl function corresponding to this extension; $m_1(z) \neq m(z)$, $z \in \mathbb{C}\setminus\mathbb{R}$. Here we need to make the following general remark: *The Laurent polynomials of the first and second kind do not depend on the corresponding spectral measure* $d\rho(\lambda)$*,— they are solutions of equations (5.11), (5.13) with* λ *replaced with* z *and the corresponding initial data which does not depend on* $d\rho(\lambda)$. Let us explain the last assertion. For $P_n(z)$ we have the initial data $P_{-1}(z) = 0$, $P_0(z) = 1$, $n \in \mathbb{N}_0$. For $Q_n(z)$ the initial data are given in (6.7), $n \in \mathbb{N}_0$. These expressions contain s_{-1}, but using (4.34) we conclude that $s_{-1} = q_0$. So, the initial data for $Q_n(z)$ is defined only by elements of the matrices J, J^{-1}.

Consider the equality (6.18) for $m_1(z)$. As above the corresponding to $m_1(z)$ series in (6.18) converges uniformly in G. We have $\forall n \in \mathbb{N}_0$

$$\begin{aligned}&\sum_{j=0}^{n} \|(Q_j(z) + m(z)P_j(z)) - (Q_j(z) + m_1(z)P_j(z))\|^2_{\mathcal{H}_j} \\ &\qquad\qquad = |m(z) - m_1(z)| \sum_{j=0}^{n} \|P_j(z)\|^2_{\mathcal{H}_j}, \quad z \in \mathbb{C} \setminus \mathbb{R}.\end{aligned} \tag{6.20}$$

The left part in (6.20) converges for $n \longrightarrow \infty$ uniformly on G, since such convergence takes place for the series in (6.18) for $m(z)$ and $m_1(z)$. Therefore such convergence must also be for the right-hand side of (6.20), i.e., for the first series in (6.19). The uniform convergence for the second series in (6.19) follows from such convergence of (6.18) and the first series. $\square$

Note, that it is possible to prove [35] that in the second case the series (6.19) are uniformly convergent also on every bounded closed domain in $\mathbb{C} \setminus \{0\}$.

By means of an easy calculation we can rewrite the equality (6.18) in the form

$$\begin{aligned}&\Big(\sum_{j=0}^{\infty} \|P_j(z)\|^2_{\mathcal{H}_j}\Big)|m(z)|^2 + \Big(\frac{1}{z-\bar z} + \sum_{j=0}^{\infty}(Q_j(z), P_j(z))_{\mathcal{H}_j}\Big)\overline{m(z)} \\ &\quad + \overline{\Big(\frac{1}{z-\bar z} + \sum_{j=0}^{\infty}(Q_j(z), P_j(z))_{\mathcal{H}_j}\Big)} m(z) + \sum_{j=0}^{\infty} \|Q_j(z)\|^2_{\mathcal{H}_j} = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.\end{aligned} \tag{6.21}$$

The last equality (6.21) means that the point $m(z)$ lies on some circle in the complex plane. We can now repeat all conclusions from ([4], Ch. 7, § 1, pp. 524–531) practically without essential changes. These facts in [4] give a description of all spectral measures $d\rho(\lambda)$ for the classical Jacobi matrix in the indeterminate case, when it is only Hermitian, not selfadjoint.

Such reasoning gives, in our case, the following result.

Theorem 12. *Let for a Jacobi-Laurent matrix J, the indeterminate case takes place. Fix some point $z \in \mathbb{C}\setminus\mathbb{R}$ and consider the point $m(z) \in \mathbb{C}$ which is defined by the integral*

$$m(z) = \int\limits_{\mathbb{R}} \frac{d\rho(\lambda)}{\lambda - z}, \tag{6.22}$$

where $d\rho(\lambda)$ is the spectral measure of some selfadjoint extension in the space $\mathbf{l}_2$ of the operator $\mathbf{J}$. Taking different such spectral measures $d\rho(\lambda)$ we assert that the point (6.22) $m(z)$ passes completely some circle K_z (an analog of the Weyl-Hamburger circle) with following center $O(z)$ and the radius $R(z)$:

$$\begin{aligned} O(z) &= -\frac{1}{\sum\limits_{j=0}^{\infty} \|P_j(z)\|^2_{\mathcal{H}_j}} \left(\frac{1}{z-\bar{z}} + \sum_{j=0}^{\infty} (Q_j(z), P_j(z))_{\mathcal{H}_j} \right), \\ R(z) &= \left(|z-\bar{z}| \sum_{j=0}^{\infty} \|P_j(z)\|^2_{\mathcal{H}_j} \right)^{-1}. \end{aligned} \tag{6.23}$$

It is possible, as for classical Jacobi matrices, to give some procedure of finding the measure $d\rho(\lambda)$ from a given point on the circle K_z (6.23).

But we will not present in this article corresponding considerations.

It is clear that the above developed approach to the spectral theory of the Jacobi-Laurent matrices is similar to the classical Jacobi matrices (see, e.g. [4], Ch. 7, § 1,2) and, therefore, many results of the latter theory can be transferred to our case. Let us stress once more that in developed above approach we have used, instead of the ordinary space l_2 on $\mathbb{N}_0$, the space $\mathbf{l}_2$ of $\mathbb{C}^2$ vector-valued sequences $(f_n)_{n=0}^{\infty}$ but its the first coordinate is one-dimensional: $f_0 \in \mathbb{C}^1$. This particularity of the space $\mathbf{l}_2$ gives the possibility to use a *scalar spectral measure* $d\rho(\lambda)$: roughly speaking $\forall \alpha \in \mathfrak{B}(\mathbb{R})$, $P(\alpha) = (E(\alpha)e_{0,0}, e_{0,0})_{\mathbf{l}_2}$, where $e_{0,0} = (1, 0, 0, \ldots) \in \mathbf{l}_2$ and E is an expansion of identity.

We will finish this Section by discussing an analog of the classical theorem of Carleman for Jacobi-Laurent matrices (see [4], Ch. 7, Theorems 1.3, 2.9 for the classical and matrix cases).

Theorem 13. *The operator $\mathbf{J}$ is selfadjoint in $\mathbf{l}_2$ if for the matrix J (4.26) the following condition is fulfilled:*

$$\sum_{n=0}^{\infty} \frac{1}{\|a_n\|} = \infty. \tag{6.24}$$

Proof. It is easy to calculate that the following Green's formula takes place (compare, e.g. with [4], Ch. 7, formulas (1.4), (2.24)). For arbitrary sequences $f = (f_n)_{n=0}^{\infty}$, $g = (g_n)_{n=0}^{\infty}$, $f_n, g_n \in \mathcal{H}_n$, we have (assuming that $f_{-1} = g_{-1} = 0$)

$$\begin{aligned} \sum_{j=m}^{n} \big(((Jf)_j, g_j)_{\mathcal{H}_j} - (f_j, (Jg)_j)_{\mathcal{H}_j} \big) &= [(f_n, c_n g_{n+1})_{\mathcal{H}_n} - (c_n f_{n+1}, g_n)_{\mathcal{H}_n}] \\ &\quad - [(f_m, a_{m-1} g_{m-1})_{\mathcal{H}_m} - (a_{m-1} f_{m-1}, g_m)_{\mathcal{H}_m}], \quad m, n \in \mathbb{N}_0, \quad n > m. \end{aligned} \tag{6.25}$$

Let us explain that if $m > 0$, then all $\mathcal{H}_j$ in (6.25) are equal to $\mathbb{C}^2$ and the calculation in (6.25) is simple. For the case $m = 0$, we write at first the equality (6.25) for $m = 1$ and then it is sufficient to add to every side corresponding sides of the trivial equality

$$\begin{aligned}((Jf)_0, g_0)_{\mathbb{C}^1} - (f_0, (Jg)_0)_{\mathbb{C}^1} &= (b_0 f_0 + c_0 f_1, g_0)_{\mathbb{C}^1} - (f_0, b_0 g_0 + c_0 g_1)_{\mathbb{C}^1} \\ &= (c_0 f_1, g_0)_{\mathbb{C}^1} - (f_0, c_0 g_1)_{\mathbb{C}^1} = (f_1, a_0 g_0)_{\mathbb{C}^2} - (a_0 f_0, g_1)_{\mathbb{C}^2}.\end{aligned}$$

As a result, the last square brackets becomes zero and we get (6.25) for the indicated m, n.

Let $z \in \mathbb{C} \setminus \mathbb{R}$ be fixed. Consider the Laurent polynomials of the first kind $P_n(z)$, $n \in \mathbb{N}_0$. For the sequences $(f_n)_{n=0}^\infty = (g_n)_{n=0}^\infty = (P_n(z))_{n=0}^\infty = P(z)$, the equality (6.25) for $m = 0$ gives (we use (6.1)): $\forall n \in \mathbb{N}$

$$\begin{aligned}(z - \bar{z}) \sum_{j=0}^{n} \|P_j(z)\|_{\mathcal{H}_j}^2 &= \sum_{j=0}^{n} \big(((JP(z))_j, P_j(z))_{\mathcal{H}_j} - (P_j(z), (JP(z))_j)_{\mathcal{H}_j}\big) \\ &= (P_n(z), c_n P_{n+1}(z))_{\mathcal{H}_n} - (c_n P_{n+1}(z), P_n(z))_{\mathcal{H}_n}\,.\end{aligned} \tag{6.26}$$

Since $P_0(z) = 1$ we get from (6.26) that

$$|z - \bar{z}| \le |z - \bar{z}| \sum_{j=0}^{n} \|P_j(z)\|_{\mathcal{H}_j}^2 \le 2\|c_n\| \|P_n(z)\|_{\mathbb{C}^2} \|P_{n+1}(z)\|_{\mathbb{C}^2},$$

therefore,

$$\|c_n\|^{-1} \le 2\|P_n(z)\|_{\mathbb{C}^2} \|P_{n+1}(z)\|_{\mathbb{C}^2} |z - \bar{z}|^{-1}.$$

Using this inequality, (6.24), and the identity $\|c_n\| = \|a_n^*\| = \|a_n\|$ we get

$$\begin{aligned}\infty = \sum_{n=1}^{\infty} \frac{1}{\|a_n\|} &\le 2|z - \bar{z}|^{-1} \sum_{n=1}^{\infty} \|P_n(z)\|_{\mathbb{C}^2} \|P_{n+1}(z)\|_{\mathbb{C}^2} \\ &\le 2|z - \bar{z}|^{-1} \Big(\sum_{n=1}^{\infty} \|P_n(z)\|_{\mathbb{C}^2}^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} \|P_{n+1}(z)\|_{\mathbb{C}^2}^2\Big)^{1/2} \\ &\le 2|z - \bar{z}|^{-1} \sum_{n=0}^{\infty} \|P_n(z)\|_{\mathcal{H}_n}^2.\end{aligned}$$

Therefore the series (6.2) is divergent and our operator **J** is selfadjoint. ☐

7. A connection between the strong moment problem and spectral theory of Jacobi-Laurent matrices

For the classical moment problem, a connection between such a problem and spectral theory of operators and Jacobi matrices is well known (see, e.g. [1], Ch. 4). A more precise and a solid description of such a connection with Jacobi matrices is given in [4], Ch. 8, Section 5, Subsections 4,5. We repeat now it in some another terms.

Let $s = (s_n)_{n=0}^\infty$, $s_n \in \mathbb{R}$, be some sequence of real numbers. As it was said in the Introduction, this sequence is called a moment sequence if the Borel measure $d\rho(\lambda)$ on $\mathbb{R}$ exists such that

$$s_n = \int_{\mathbb{R}} \lambda^n \, d\rho(\lambda), \quad n \in \mathbb{N}_0. \tag{7.1}$$

The classical theorem about moments s_n asserts that representation (7.1) exists iff the sequence s is positive definite, i.e., for an arbitrary finite sequence $f = (f_n)_{n=0}^\infty$, $f_n \in \mathbb{C}$, we have

$$\sum_{j,k=0}^{\infty} s_{j+k} f_j \bar{f}_k \ge 0.$$

So, let us have a moment sequence s with representation (7.1). Consider the space $L^2(\mathbb{R}, d\rho(\lambda)) = L^2$ and the operator $\hat{A}$ of multiplication on λ in this space, $(\hat{A}F)(\lambda) = \lambda F(\lambda)$, where $F \in L^2$ belongs to $\mathrm{Dom}(\hat{A})$ that consists of all ordinary polynomials of λ with complex coefficients. Every such polynomial $F(\lambda)$ belongs to L^2 and $\hat{A}F$ is well defined, since all the functions

$$\lambda^m, \quad m \in \mathbb{N}_0, \tag{7.2}$$

are summable according to (7.1). We assume that they are linearly independent in L^2 (i.e. we consider the nondegenerate moment problem (7.1)) and the set (7.2) is total in L^2. The defined above operator $\hat{A}$ always is Hermitian.

Apply the usual Gram-Schmidt procedure of orthogonalization with real coefficients to the functions $1, \lambda, \lambda^2, \ldots$. We get, as a result, the sequence of polynomials

$$P_0(\lambda) = 1, P_1(\lambda), P_2(\lambda), \ldots$$

(polynomials of the first kind) which make an orthonormal basis in L^2. Then we can go, from L^2, to the ordinary space l_2 of sequences $(f_n)_{n=0}^{\infty}$, $f_n \in \mathbb{C}^1$. Our operator $\hat{A}$ becomes, as it is easy to understand, a usual symmetric Jacobi matrix

$$\begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & \ldots \\ a_0 & b_1 & a_1 & 0 & 0 & \ldots \\ 0 & a_1 & b_2 & a_2 & 0 & \ldots \\ 0 & 0 & a_2 & b_3 & a_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{7.3}$$

acting in the space l_2 at first on vectors from $l_{\mathrm{fin}} \subset l_2$. Its real elements a_n, b_n are calculated by classical formulas of type (4.2). It is essential, as it easy to understand, that *the formulas for a_n, b_n can be rewritten in terms of the given moments s_n, $n \in \mathbb{N}_0$ as follows: we orthogonalize the sequence λ^n, $n \in \mathbb{N}_0$.*

So, we can construct the Jacobi matrix J immediately by giving moments s_n, $n \in \mathbb{N}_0$.

In the case of essential selfadjointness of $\hat{A}$, the operator $\mathbf{J}$ generated in l_2 by J (7.3) on l_{fin} is selfadjoint (after being closed) and its spectral measure is equal to a given earlier $d\rho(\lambda)$. Conversely, let us first have some matrix J and construct the corresponding operator $\mathbf{J}$ in l_2 and let it be selfadjoint with some spectral measure $d\rho(\lambda)$. Then for the numbers s_n, constructed from $(a_n, b_n)_{n=0}^{\infty}$ by a procedure inverse to the one mentioned above, we get a representation (7.1) (it is necessary to take $\mathbf{J}^n$; $\forall n \in \mathbb{N}_0$, $s_n = (\mathbf{J}^n e_0, e_0)_{l_2}$. $e_0 = (1, 0, 0, \ldots)$).

In the case of nonselfadjointness of $\hat{A}$ in L^2, it is necessary to take selfadjoint extensions of operator $\mathbf{J}$ in l_2 (or extend it to a larger space). For every such an extension we have the measure $d\rho(\lambda)$ in (7.1), but this measure is not unique for a given s. Every such so-called "orthogonal" measure is defined by some selfadjoint extension of $\mathbf{J}$ in l_2. So, if we apply the description of all extensions in l_2 (it exists in the classical theory of Jacobi matrices), as a result we get a description of all "orthogonal" measures $d\rho(\lambda)$, which give a representation for given moment sequence $s = (s_n)_{n=0}^{\infty}$.

This classical scheme can be easily repeated for the considered strong moment sequence $s = (s_n)_{n=-\infty}^{\infty}$ (3.1) and the Jacobi-Laurent matrices (4.26), (4.43). Assume for simplicity that the sequence s is nondegenerate. We prove the following result.

Theorem 14. *Let $s = (s_n)$, $n \in \mathbb{Z}$, be a strong moment sequence and (3.1) its representation by an integral with some Borel measure $d\rho(\lambda)$. We assume that the measure in (3.1) is such that the set $\{\lambda^m,\ m \in \mathbb{Z}\}$ is total in $L^2(\mathbb{R}, d\rho(\lambda))$.*

Connect with s a Jacobi-Laurent matrix J by the rule which will be described below.

Consider now in the space $\mathbf{l}_2$ (5.30) the operator $\mathbf{J}$ constructed by (5.6). Assume that this operator is essentially selfadjoint. Then its spectral measure equals the measure $d\rho(\lambda)$ in the representation (3.1).

If the operator $\mathbf{J}$ is only Hermitian, then for every its selfadjoint extension in $\mathbf{l}_2$, the corresponding spectral measure is one of the measures $d\rho(\lambda)$ in representation (3.1). Taking all such extensions (see Theorem 12) we get all measures in (3.1) for the fixed s. The correspondence between the extensions and the measures is one-to-one.

The rule of constructing the matrix J from s is the following. Introduce the finite-dimensional matrices and their determinants for $k \in \mathbb{N}_0$,

$$H_k^{(l)} = \begin{bmatrix} s_l & s_{l+1} & \dots & s_{l+k} \\ s_{l+1} & s_{l+2} & \dots & s_{l+k+1} \\ \dots & \dots & \dots & \dots \\ s_{l+k} & s_{l+k+1} & \dots & s_{l+2k} \end{bmatrix}, \qquad \begin{aligned} g_k &:= \mathrm{Det} H_{2k}^{(-2k)} \geq 0, \\ h_k &:= \mathrm{Det} H_{2k-1}^{(-2k)} \geq 0, \end{aligned}$$

and $\forall n \in \mathbb{N}$ write the Laurent polynomials $R_n(\lambda) = (R_{n;0}(\lambda), R_{n;0}(\lambda)) \in \mathbb{C}^2$, $\lambda \in \mathbb{R}$, where

$$R_{n;0}(\lambda) = -(g_n h_{n-1})^{1/2} \mathrm{Det} \begin{bmatrix} s_{-2n-1} & s_{-2n} & \dots & s_0 \\ s_{-2n} & s_{-2n+1} & \dots & s_1 \\ \dots & \dots & \dots & \dots \\ s_{-1} & s_0 & \dots & s_{2n} \\ \lambda^{-n-1} & \lambda^{-n} & \dots & \lambda^n \end{bmatrix},$$

$$(7.4) \qquad R_{n;1}(\lambda) = (g_n h_n)^{1/2} \mathrm{Det} \begin{bmatrix} s_{-2n} & s_{-2n+1} & \dots & s_0 \\ s_{-2n+1} & s_{-2n+2} & \dots & s_1 \\ \dots & \dots & \dots & \dots \\ s_{-1} & s_0 & \dots & s_{2n-1} \\ \lambda^{-n} & \lambda^{-n+1} & \dots & \lambda^n \end{bmatrix},$$

$$R_{0;0}(\lambda) = 1.$$

On the other hand, on the linear set L of Laurent polynomials introduce a scalar product $(\cdot,\cdot)_S$ by putting $\forall R, T \in L$,

$$(7.5) \qquad (R,T)_S = \sum_{j,k \in \mathbb{Z}} s_{j+k} r_j \bar{t}_k, \quad R(\lambda) = \sum_{j \in \mathbb{Z}} r_j \lambda^j, \quad T(\lambda) = \sum_{k \in \mathbb{Z}} t_k \lambda^k.$$

The elements $a_{j,k;\alpha,\beta}$ of the matrix J are introduced by defining

$$(7.6) \qquad a_{j,k;\alpha,\beta} = (\lambda R_{k;\beta}(\lambda), R_{j;\alpha}(\lambda))_S, \quad j,k \in \mathbb{N}_0, \quad \alpha,\beta = 0,1.$$

Proof. At first we note representation (3.1) shows that the scalar product (7.5) is equal to the scalar product in $L^2(\mathbb{R}, d\rho(\lambda))$ (the nondegeneracy of s gives that $(\cdot,\cdot)_S$ is a scalar product, not quasiscalar). Therefore formulas (7.6) are the same as (4.12).

The condition of totality in $L^2(\mathbb{R}, d\rho(\lambda))$ of the set (4.30) for $m \in \mathbb{N}_0$ has been assumed. Therefore, we can apply to our case the constructions of Section 4,5. It is clear that our Laurent polynomials (7.4) are equal to polynomials of the first kind $P_{n;\alpha}(\lambda)$,— formulas (7.4) can be easily obtained by calculating orthonormal polynomials via the Gram-Schmidt procedure from sequence (4.3) and using the notations (3.1) for the integrals (such formulas can be found, for example, in [21]). As it follows from Theorem 8, every function $\mathbb{R} \ni \lambda \longmapsto \lambda^m$, $m \in \mathbb{Z}$, belongs to $L^2(\mathbb{R}, d\rho(\lambda))$ w.r.t. any spectral measure $d\rho(\lambda)$. Therefore, every selfadjoint extension in $\mathbf{l}_2$ of the operator $\mathbf{J}$ constructed from J of type (7.6) gives some spectral measure $d\rho(\lambda)$ for which the representation (3.1) holds. Since we can describe such extensions (see Theorem 12 and the corresponding discussion in Section 6), we can get a description of all measures from (3.1) with a given s and such that the set (4.30) for $m \in \mathbb{N}_0$ is total in $L^2(\mathbb{R}, d\rho(\lambda))$. ∎

8. Two addition facts

In this short Section we will touch upon two types of results: 1) In what manner the representation (1.1), i.e., the main result of Theorem 4 and the corresponding result of works [20, 21, 27] is a particular case of one old general theorem, published in [4]; 2) We explain the way of getting results of type Sections 2–7 for matrix strong moment problem.

1) We will use in 1) notations similar to the book [4]. Consider, for a sequences of complex numbers $f = (f_n)$, $n \in \mathbb{Z}$, the difference expression of order $r \in \mathbb{N}$,

$$(Lf)_j = \sum_{\alpha=r_-}^{r_+} a_{j,\alpha} f_{j+\alpha}, \quad j \in \mathbb{Z}. \tag{8.1}$$

Here $a_{j,\alpha} \in \mathbb{C}$, the expansion $r = r_- + r_+$, $r_-, r_+ \in \mathbb{N}_0$, is fixed. Let $K = (K_{j,k})$, $j,k \in \mathbb{Z}$, $K_{j,k} \in \mathbb{C}$, be a some positive definite kernel (a matrix), i.e., the following inequality for an arbitrary finite sequence $f = (f_j)$, $j \in \mathbb{Z}$, takes place:

$$\sum_{j,k\in\mathbb{Z}} K_{j,k} f_j \bar{f}_k \geq 0. \tag{8.2}$$

Such a kernel generates, on the set l_{fin} of finite sequences, a (quasi) scalar product $(\cdot,\cdot)_K$ and a norm $\|\cdot\|_K$ for which $\|f\|_K^2$ is given by (8.2). The corresponding Hilbert space will be denoted by H_K.

Let the kernel K and the expression L be $*$-commuting, i.e., the following equality takes place on $\mathbb{Z} \times \mathbb{Z}$:

$$L_{(j)} K = \bar{L}_{(k)} K. \tag{8.3}$$

Here $(j)((k))$ means that L of the form (8.1) acts on $K = (K_{j,k})$, $j,k \in \mathbb{Z}$, at the index $j(k)$.

The main result of the theory of positive definite kernels K which are $*$-commuting with the expression L is a representation of K by an integral on fundamental solution of the difference equation

$$(Lf)_j = \lambda f_j, \quad j \in \mathbb{N}; \quad \lambda \in \mathbb{R}. \tag{8.4}$$

Recall that a system of r solutions on $j \in \mathbb{N}$ of (8.4), $\chi_{j;\alpha}(\lambda)$, $\alpha = 0,1,\ldots,r-1$, is called fundamental if these solutions satisfy the following initial data with some fixed $p \in \mathbb{Z}$:

$$\chi_{j;\alpha}(\lambda) = \delta_{j,p-r_-+\alpha}, \quad j = p - r_-, \ldots, p + r_+ - 1, \quad \alpha = 0,1,\ldots,r-1$$

with standard notation $\delta_{m,n}$ (the fundamental system of solution "with initial data near the point p").

It is easy to give conditions on the coefficients $a_{j,\alpha}$ of expression L (8.1) which guarantee existence of such fundamental solutions. We assume that such solutions exist for $p = 0$.

The following theorem is true (see [4], Ch. 8, Theorem 5.1).

Theorem 15. *A positive definite kernel $K = (K_{j,k})$, $j,k \in \mathbb{Z}$, has the following representation by the fundamental system $\chi_{j;\alpha}(\lambda)$, $j \in \mathbb{Z}$, $\alpha = 0,1,\ldots,r-1$, $\lambda \in \mathbb{R}$,*

$$K_{j,k} = \int_{\mathbb{R}} \sum_{\alpha,\beta=0}^{r-1} \chi_{j;\alpha}(\lambda) \overline{\chi_{k;\beta}(\lambda)}\, d\rho_{\alpha,\beta}(\lambda), \quad j,k \in \mathbb{Z}, \tag{8.5}$$

with a nonnegative $r \times r$-matrix measure $d\rho(\lambda) = (d\rho_{\alpha,\beta}(\lambda))_{\alpha,\beta=0}^{r-1}$ on $\mathfrak{B}(\mathbb{R})$ iff K and L are connected by equality (8.3). The measure in (8.5) is defined by K uniquely iff the closure of the operator in the space H_K, $l_{\text{fin}} \ni f \longmapsto Lf \in l_{\text{fin}}$ is maximal.

The proof of this theorem is similar to the proof of Theorem 4 and is based on Theorem 1; it is given in [4].

The representation (3.1) is a particular case of Theorem 15,— it is necessary to take L (8.1) of the form

$$(Lf)_j = f_{j-1}, \quad j \in \mathbb{Z}, \qquad \text{(i.e.} \quad p = 0, \quad r = 1, \quad r_- = 1, \quad r_+ = 0, \quad a_{j,\alpha} = 1, \quad j \in \mathbb{Z}, \quad \alpha = 0). \tag{8.6}$$

Now $\forall j, k \in \mathbb{Z}$, $K_{j,k} = s_{j+k}$, condition (8.3) for L (8.6) and for this K is fulfilled. Now $r = 1$, $\chi_{j;0}(\lambda) = \lambda^j$, $j \in \mathbb{Z}$, and the matrix measure $d\rho(\lambda) = d\rho_{0,0}(\lambda)$ is an ordinary probability measure. The representation (8.5) becomes (3.1).

Note that this partial case was not considered in [4], but a similar situation for the ordinary classical moment problem was considered,— in [4], Ch. 8, Section 5, Subsection 4, it was explained that this moment problem is a particular case of an analog of Theorem 15, Theorem 5.2 from [4], Ch. 8.

2) Let us now consider a strong matrix moment problem. We fix some separable Hilbert space H and consider an operator-valued measure $d\rho(\lambda)$ on $\mathbb{R}$. More exactly we have the mapping $\mathfrak{B}(\mathbb{R}) \ni \Delta \longmapsto \rho(\Delta)$, where $\rho(\Delta)$ is a nonnegative operator on H, $\rho(\mathbb{R}) = 1$ and $\forall u \in H$ $(\rho(\Delta)u, u)_H$ is an ordinary measure on $\mathfrak{B}(\mathbb{R})$.

It is easy to introduce the integral $\int_{\mathbb{R}} F(\lambda)\, d\rho(\lambda)$ for the scalar-valued function $\mathbb{R} \ni \lambda \longmapsto F(\lambda) \in \mathbb{C}$, the value of this integral is an operator in H. This operator can be bounded or not. For a more detailed account of these questions see, e.g., [4], Ch. 7. In what follows the set of all bounded operators in H we will denoted by $L(H)$.

The strong operator moment problem is formulated as follows: let $s = (s_n)$, $n \in \mathbb{Z}$, be some sequence of bounded operators in the space H, i.e., $s_n \in L(H)$, $n \in \mathbb{Z}$. These operators are called moments if we have a representation of type (3.1),

$$s_n = \int_{\mathbb{R}} \lambda^n \, d\rho(\lambda) \in L(H), \quad n \in \mathbb{Z}, \tag{8.7}$$

but with measure $d\rho(\lambda)$, which is an introduced above, is an operator-valued measure. If in (8.7) only $n \in \mathbb{N}_0 \subset \mathbb{Z}$, then we have the classical operator moment problem. If $\dim(H) < \infty$, then the corresponding problem are called matrix.

There are many article were devoted to an investigation of the operator moment problem, but in a majority of them, the authors investigate the matrix and the classical cases, not the strong problem. We mention here only some of these works, [24, 4, 18, 33, 34, 35, 37].

In this article we only demonstrate the above approach, discussed in Sections 3–7. Analogously to the condition (3.2) of positivity, we will say that a sequence $s = (s_n)$, $n \in \mathbb{Z}$, $s_n \in L(H)$, is positive definite, if for an arbitrary finite sequence $f = (f_n)$, $n \in \mathbb{Z}$, $f_n \in H$, we have

$$\sum_{j,k \in \mathbb{Z}} (s_{j+k} f_k, f_j)_H \geq 0 \tag{8.8}$$

(for $H = \mathbb{C}^1$ this condition is the same as (3.2)).

We formulate now a generalization of Theorem 4 in the simplest case where H is finite dimensional, $H = \mathbb{C}^d$, $d \in \mathbb{N}$, and in addition, every matrix s_n, $n \in \mathbb{Z}$, acting in the space $\mathbb{C}^d$, has real elements (in every case it is Hermitian, this follows from (8.8), but we demand that it be real). The following result takes place.

Theorem 16. *A matrix sequence $s = (s_n)$, $n \in \mathbb{Z}$, $s_n \in L(\mathbb{C}^d)$, with real matrix elements is a strong moment sequence, i.e., representation (8.7) with some operator-valued measure*

$d\rho(\lambda)$ with real matrix valued $\rho(\Delta)$, $\Delta \in \mathfrak{B}(\mathbb{R})$, holds iff it is a positive definite sequence, i.e., the condition (8.8) is fulfilled.

Proof. It is similar to the proof of Theorem 4 and we only indicate necessary changes.

Introduce the linear set $l_{\text{fin}}(H) = l_{\text{fin}}(\mathbb{C}^d)$ of finite sequences $f = (f_n)$, $n \in \mathbb{Z}$, of vectors $f_n \in H = \mathbb{C}^d$. According to (8.8) we introduce into $l_{\text{fin}}(H)$ a (quasi)scalar product,

$$(f, g)_{S(H)} := \sum_{j,k \in \mathbb{Z}} (s_{j+k} f_k, g_j)_H, \quad f, g \in l_{\text{fin}}(H).$$

Let $S(H)$ be the corresponding Hilbert space (with factorization of the set $l_{\text{fin}}(H)$ if needed). In our case this space is "real": $\forall f \in S(H)$ the mapping $f \longmapsto \bar{f} \in S(H)$ exists and $\forall f, g \in S(H)$, $(\bar{f}, \bar{g})_{S(H)} = \overline{(f, g)_{S(H)}}$, $\bar{\bar{f}} = f$. This fact follows since s_n, $n \in \mathbb{Z}$, is real.

As in Section 3 we define on $f \in l_{\text{fin}}(H)$ the operator of type (3.5):

$$(Jf)_j = f_{j-1}, \quad j \in \mathbb{Z}; \quad \text{Dom}(J) = l_{\text{fin}}(H).$$

As in (3.7) we check that this operator J is Hermitian in the space $S(H)$.

Riggings of type (3.8) and (3.10) are constructed similarly to the proof of Theorem 4, since $H = \mathbb{C}^d$ is finite-dimensional, the imbedding of type $l_2(p) \hookrightarrow l_2$ can be made quasinuclear by taking the sequence (p_n), $n \in \mathbb{Z}$, to be increasing to ∞ sufficiently fast.

The other parts of the proof of our theorem is similar to the proof of Theorem 4. At first we take some selfadjoint extension A in the space $S(H)$ of the operator J. This is possible, since the deficiency numbers of J are equal. Then it is necessary to use some generalization of Theorem 2,— instead of one vector q it is necessary to take d vectors of type $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1) \in \mathbb{C}^d$. The spectral measure $d\sigma(\lambda)$ must now be matrix-valued. It is possible also to apply, instead of such a variant of Theorem 2, directly Theorem 1. □

If the matrices s_n, $n \in \mathbb{Z}$, are only Hermitian, it is necessary to add to the conditions of Theorem 16 the equality of the deficiency numbers of the operator J.

Remark 5. The situation if $\dim(H) = \infty$ is more complicated. For a construction of quasinuclear rigging of type $l_2(p) \hookrightarrow l_2$ (the spaces of sequences (f_n), $n \in \mathbb{Z}$) now an increase of the weight $p = (p_n)$, $n \in \mathbb{Z}$, to ∞ is not sufficient. It is necessary to take, for the space of type $l_2(p)$, (l_2) that are sequences of vectors from H_1 (H_2) and the following embedding $H_1 \hookrightarrow H_2$ must be quasinuclear. Such a situation requires a change in the formulation of a theorem like Theorem 16.

An analog of Theorem 16, of course, is true for the classical matrix moment problem. In this case, the corresponding spectral theory for block Jacobi matrix is partially constructed (see [4], Ch. 7). Now the situation is more complicated,— the deficiency numbers can be equal to $p \in \mathbb{N}_0$ (see [18]).

For the strong matrix moment problem, some results similar to the ones obtained in Sections 3–7 are contained in the works [33, 34, 35, 37].

Acknowledgments. The authors are very grateful to V. A. Derkach for essential remarks.

References

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner, New York, 1965. (Russian edition: Fizmatgiz, Moscow, 1961).
2. Yu. M. Berezanskiĭ, *Generalization of Bochner's theorem to expansions according to eigenfunctions of partial differential equations*, Dokl. Akad. Nauk SSSR **110** (1956), 893–896. (Russian)
3. Yu. M. Berezanskiĭ, *Representation of positive definite kernels by eigenfunctions of differential equations*, Mat. Sb. **47(89)** (1959), 145–176. (Russian)

4. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965).
5. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Vols. 1, 2, Kluwer Acad. Publ., Dordrecht—Boston—London, 1995. (Russian edition: Naukova Dumka, Kiev, 1988).
6. Yu. M. Berezansky, Z. G. Sheftel and G. F. Us, *Functional Analysis*, Vols. 1, 2, Birkhäuser Verlag, Basel—Boston—Berlin, 1996. (Russian edition: Vyshcha shkola, Kiev, 1990).
7. Yu. M. Berezansky, Yu. G. Kondratiev, T. Kuna, and E. Lytvynov, *On a spectral representation for correlation measures in configuration space analysis*, Methods Funct. Anal. Topology **5** (1999), no. 4, 87–100.
8. Yu. M. Berezansky, *Some generalizations of the classical moment problem*, Integr. Equ. Oper. Theory **44** (2002), no. 3, 255–289.
9. Yu. M. Berezansky, *The generalized moment problem associated with correlation measures*, Funktsional. Anal. i Prilozhen. **37** (2003), no. 4, 86–91. (Russian); English transl. Funct. Anal. Appl. **37** (2003), no. 4, 311–315.
10. Yu. M. Berezansky and M. E. Dudkin, *The complex moment problem in the exponential form*, Methods Funct. Anal. Topology **10** (2004), no. 4, 1–10.
11. Yu. M. Berezansky and M. E. Dudkin, *The direct and inverce spectral problems for the block Jacobi type unitary matrices*, Methods Funct. Anal. Topology **11** (2005), no. 4, 327–345.
12. Yu. M. Berezansky and M. E. Dudkin, *The complex moment problem and direct and inverse spectral problems for block Jacobi type bounded normal matrices*, Methods Funct. Anal. Topology **12** (2006), no. 1, 1–31.
13. Yu. M. Berezansky and M. E. Dudkin, *On the complex moment problem*, Math. Nachr. **280** (2007), no. 1-2, 60–73.
14. Yu. M. Berezansky and D. A. Mierzejewski, *The investigation of generalized moment problem associated with correlation measures*, Methods Funct. Anal. Topology **13** (2007), no. 2, 124–151.
15. T. Carleman, *Les fonctions quasi analytiques*, Gauthier-Villars, Paris, 1926.
16. M. S. Derevyagin and V. A. Derkach, *Spectral problems for generalized Jacobi matrices*, Linear Algebra Appl. **382** (2004), 1–24.
17. M. S. Derevyagin and V. A Derkach, *On the convergence of Padé approximants of generalized Nevanlinna functions*, Trudy Moskov. Mat. Obshch. **68** (2007), 133–182 (Russian); English transl. Moscow Math. Soc. **68** (2007), 119–162.
18. Yu. M. Dyukarev, *Deficiency numbers of symmetric operators generated by block Jacobi matrices*, Mat. Sb. **197** (2006), no. 8, 73–100 (Russian); English transl. Sb. Math. **197** (2006), no. 8, 1177–1203.
19. E. Hendriksen, C. Nijhuis, *Laurent-Jacobi matrices and the strong Hamburger moment problem*, Proceedings of the International Conference on Rational Approximation, ICRA99 (Antwerp). Acta Appl. Math. **61** (2000), no. 1–3, 119–132.
20. W. B. Jones, W. J. Thron and O. Njåstad, *Orthogonal Laurent polynomials and strong Hamburger moment problem*, J. Math. Anal. Appl. **98** (1984), no. 2, 528–554.
21. W. B. Jones and O. Njåstad, *Orthogonal Laurent polynomials and strong moment theorey: a survey.* Continued fractions and geometric function theory (CONFUN) (Trondheim, 1997), J. Comput. Appl. Math. **105** (1999), no. 1-2, 51–91.
22. M. G. Krein, *On a general method of decomposing Hermite-positive nuclei into elementary products*, Doklady Acad. Sci. URSR **53** (1946), 3–6. (Russian)
23. M. G. Kreĭn, *On Hermitian operators with directed functionals*, Akad. Nauk Ukrain. RSR. Zbirnyk Prac' Inst. Mat. (1948), no. 10, 83–106. (Russian)
24. M. G. Kreĭn, *Infinite J-matrices and matrix moment problem*, Doklady Acad. Sci. SSSR **69** (1949), no. 2, 125–128. (Russian)
25. S. Mandelbrojt, *Séries Adhérentes. Régularisation des Suites. Applications*, Gauthier-Villars, Paris, 1952.
26. O. Mokhonko and S. Dyachenko, *Dimension stabilization effect for the block Jacobi-type matrix of a bounded normal operator with the spectrum on an algebraic curve*, Methods Funct. Anal. Topology **16** (2010), no.1, 28–41.
27. O. Njåstad, *Solutions of the strong Hamburger moment problem*, J. Math. Anal. Appl. **197** (1996), no.1, 227–248.
28. A. A. Nudelman, *On the application of completely and absolutely monotone sequences to the problem of moments*, Uspekhi Mat. Nauk **8** (1953), no. 6(58), 119–124. (Russian)
29. A. E. Nussbaum, *Quasi-analytic vectors*, Ark. Mat. **6** (1965), no. 10, 179–191.
30. A. E. Nussbaum, *A note on quasianalytic vectors*, Studia Math. **33** (1969), 305–309.

31. B. Simon, *The classical moment problem as a self-adjoint finite differential operator*, Advances Math. **137** (1998), 82–203.
32. B. Simon, *Orthogonal Polynomials on the Unite Circle*, Part 1: *Classical Theory*; Part 2: *Spectral Theory*, AMS Colloquium Series, Amer. Math. Soc., Providence, RI, 2005.
33. K. K. Simonov, *Strong matrix moment problem of Hamburger*, Methods Funct. Anal. Topology **12** (2006), no. 2, 183–196.
34. K. K. Simonov, *Orthogonal matrix Laurent polynomials*, Mat. Zametki **79** (2006), no. 2, 316–320 (Russian); English transl. Math. Notes **79** (2006), no. 1-2, 291–295.
35. K. Simonov, *Orthogonal matrix polynomials of Laurent on the real line*, Ukr. Mat. Visn. **3** (2006), no. 2, 275–299 (Russian); English transl. Ukr. Math. Bull. **3** (2006), no. 2, 267–290.
36. J. Stochel and F. H. Szafraniec, *The complex moment problem and subnormality: a polar decomposition approach*, J. Funct. Anal. **159** (1998), 432–491.
37. S. M. Zagorodnyuk, *On the strong matrix Hamburger moment problem*, Ukrainian Math. J. **62** (2010), no. 4, 471–487.
38. Yu. M. Berezansky, *The integration of double-infinite Toda lattice by means of inverse spectral problem and related questions*, Methods Funct. Anal. Topology **15** (2009), no. 2, 101–136.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: berezan@mathber.carrier.kiev.ua

National Technical University of Ukraine (KPI), 37 Peremogy Av., Kyiv, 03056, Ukraine
E-mail address: dudkin@imath.kiev.ua

Received 12/03/2009; Revised 25/05/2010