

## A DESCRIPTION OF ALL SOLUTIONS OF THE MATRIX HAMBURGER MOMENT PROBLEM IN A GENERAL CASE

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**ABSTRACT.** We describe all solutions of the matrix Hamburger moment problem in a general case (no conditions besides solvability are assumed). We use the fundamental results of A. V. Shtraus on the generalized resolvents of symmetric operators. All solutions of the truncated matrix Hamburger moment problem with an odd number of given moments are described in an "almost nondegenerate" case. Some conditions of solvability for the scalar truncated Hamburger moment problem with an even number of given moments are given.

### 1. INTRODUCTION

The main aim of this investigation is to obtain a description of all solutions of the matrix Hamburger moment problem. Recall that the matrix Hamburger moment problem consists of finding a left-continuous non-decreasing matrix function  $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$  on  $\mathbb{R}$ ,  $M(-\infty) = 0$ , such that

$$(1) \quad \int_{\mathbb{R}} x^n dM(x) = S_n, \quad n \in \mathbb{Z}_+,$$

where  $\{S_n\}_{n=0}^{\infty}$  is a given sequence of Hermitian  $(N \times N)$  complex matrices,  $N \in \mathbb{N}$ .

Sequences  $\{S_n\}_{n=0}^{\infty}$  for which this problem has a solution are called *moment sequences*. This problem was introduced in 1949 by M. G. Krein [1]. He described all solutions in the case when the corresponding J-matrix defines a symmetric operator with maximal deficiency numbers. This result appeared without proof in [2]. Yu. M. Berezansky in 1965 proved the main fact in this theory of M. G. Krein: the convergence of the series from the polynomials of the first kind (even for the operator moment problem) [3, Ch. 7, Section 2]. Using V. P. Potapov's J-theory, in 1983 I. V. Kovalishina described solutions of the matrix Hamburger moment problem in the completely indeterminate case [4] (The completely indeterminate case meant that the limit radii of the matrix Weyl discs had full ranks). Using properties of matrix orthogonal polynomials, in 2001 P. Lopez-Rodriguez obtained a parameterization of solutions in the completely indeterminate case [5] (The completely indeterminate case meant that the corresponding J-matrix generated a symmetric operator with maximal deficiency numbers). In 2004, Yu. M. Dyukarev introduced a notion of an abstract limit interpolation problem and described solutions of the completely indeterminate limit interpolation problem [6]. As one of applications, he obtained a description of solutions of the matrix Hamburger moment problem in the completely indeterminate case (This case meant that the limit radii of the matrix Weyl discs had full ranks).

In the scalar case, a description of all solutions of the moment problem (1) can be found, e.g., in [7], [3] for the nondegenerate case, and in [8] for the degenerate case.

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Recall that the condition of solvability for the matrix Hamburger moment problem is that for arbitrary complex vectors  $\vec{\xi}_j = (\xi_{j,0}, \xi_{j,1}, \dots, \xi_{j,N-1})$ ,  $j = 0, 1, 2, \dots$ , it holds ([1, p. 52]):

$$(2) \quad \sum_{j,k=0}^n \vec{\xi}_k^* S_{j+k} \vec{\xi}_j \geq 0, \quad n = 0, 1, 2, \dots$$

Let us introduce the following matrices

$$(3) \quad \Gamma_n = \begin{pmatrix} S_0 & S_1 & \dots & S_n \\ S_1 & S_2 & \dots & S_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_n & S_{n+1} & \dots & S_{2n} \end{pmatrix}, \quad n \in \mathbb{Z}_+.$$

It is not hard to verify that condition (2) is equivalent to the following inequalities

$$(4) \quad \Gamma_n \geq 0, \quad n \in \mathbb{Z}_+.$$

In 1954, A. V. Shtraus described all generalized resolvents of a densely defined symmetric operator with an arbitrary deficiency index [9]. In 1970, he described all generalized resolvents for an arbitrary, not necessarily densely defined symmetric operator [10]. We shall use these fundamental results to obtain a description of all solutions of the matrix Hamburger moment problem in the case when condition (4) is true.

We shall also study the truncated matrix Hamburger moment problem. The problem is to find a left-continuous non-decreasing matrix function  $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$  on  $\mathbb{R}$ ,  $M(-\infty) = 0$ , such that

$$(5) \quad \int_{\mathbb{R}} x^n dM(x) = S_n, \quad n = 0, 1, \dots, 2d,$$

where  $\{S_n\}_{n=0}^{2d}$  is a given sequence of Hermitian  $(N \times N)$  complex matrices,  $d \in \mathbb{Z}_+$ ,  $N \in \mathbb{N}$ .

The conditions of solvability of the moment problem (5) were given by T. Ando in 1970 [11]. The nondegenerate case of the truncated moment problem (5) is the case when the following condition takes place:

$$(6) \quad \Gamma_d > 0,$$

where  $\Gamma_d$  is defined as in (3). In 1968, V. G. Ershov obtained a description of all solutions of the truncated matrix Hamburger moment problem (5) in the nondegenerate case, using an operator approach [12]. In 1989, H. Dym described all solutions of the moment problem (5) in the nondegenerate case, using the reproducing kernel Hilbert spaces approach [13]. In 1997, V. M. Adamyan and I. M. Tkachenko obtained solutions of the truncated moment problem (5) both in degenerate and nondegenerate cases, using an operator approach [14]. In 1998, G.-N. Chen and Y.-J. Hu obtained solutions of the truncated moment problem (5) both in degenerate and nondegenerate cases, using a generalization of the Schur algorithm and matrix continued fractions [15]. We shall study the moment problem (5) under the following conditions

$$(7) \quad \Gamma_d \geq 0, \quad \text{Ker } \Gamma_{d-1} \subseteq \text{Ker } \widehat{\Gamma}_{d-1},$$

where  $\widehat{\Gamma}_{d-1} = (S_{i+j+2})_{i,j=0}^{d-1}$ . These conditions are necessary and sufficient for the solvability of the moment problem (5). Using A. V. Shtraus's results we describe all solutions of the truncated moment problem (5) (under conditions (7)).

Finally, we consider the scalar truncated Hamburger moment problem with even number of given moments. The problem is to find a left-continuous non-decreasing function

$\sigma(x)$  on  $\mathbb{R}$ ,  $\sigma(-\infty) = 0$ , such that

$$(8) \quad \int_{\mathbb{R}} x^n d\sigma(x) = s_n, \quad n = 0, 1, \dots, 2d + 1,$$

where  $\{s_n\}_{n=0}^{2d+1}$  is a given sequence of real numbers,  $d \in \mathbb{Z}_+$ . Algebraic conditions of solvability of this moment problem were given in [16, Theorem 3.1]. We shall give a simple condition of solvability for the truncated scalar Hamburger moment problem (7).

For additional references on matrix Hamburger moment problems (including truncated) we refer to a historical review in [17].

**Notations.** As usual, we denote by  $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$  the sets of real, complex, positive integer, integer, non-negative integer numbers, respectively. The space of  $n$ -dimensional complex vectors  $a = (a_0, a_1, \dots, a_{n-1})$ , will be denoted by  $\mathbb{C}^n$ ,  $n \in \mathbb{N}$ ;  $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$ . If  $a \in \mathbb{C}^n$  then  $a^*$  means the complex conjugate vector. By  $\mathbb{P}$  we denote a set of all complex polynomials and by  $\mathbb{P}_d$  we mean all complex polynomials with degrees less or equal to  $d$ ,  $d \in \mathbb{Z}_+$ , (including the zero polynomial). Let  $M(x)$  be a left-continuous non-decreasing matrix function  $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$  on  $\mathbb{R}$ ,  $M(-\infty) = 0$ , and  $\tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x)$ ;  $\Psi(x) = (dm_{k,l}/d\tau_M)_{k,l=0}^{N-1}$ . We denote by  $L^2(M)$  a set (of classes of equivalence) of vector functions  $f : \mathbb{R} \rightarrow \mathbb{C}^N$ ,  $f = (f_0, f_1, \dots, f_{N-1})$ , such that (see, e.g., [18])

$$\|f\|_{L^2(M)}^2 := \int_{\mathbb{R}} f(x)\Psi(x)f^*(x)d\tau_M(x) < \infty.$$

The space  $L^2(M)$  is a Hilbert space with the scalar product

$$(f, g)_{L^2(M)} := \int_{\mathbb{R}} f(x)\Psi(x)g^*(x)d\tau_M(x), \quad f, g \in L^2(M).$$

By  $l^2$  we denote a space of infinite complex vectors  $u = (u_0, u_1, \dots)$ , such that  $\|u\|_{l^2}^2 := \sum_{k=0}^{\infty} |u_k|^2 < \infty$ . The space  $l^2$  is a Hilbert space with the scalar product  $(u, v)_{l^2} = \sum_{k=0}^{\infty} u_k \bar{v}_k$ ,  $u, v \in l^2$ . A set of elements  $u = (u_0, u_1, \dots)$  from  $l^2$ , such that all but finite number  $u_k$  are zero will be denoted by  $l_0^2$ . Elements of  $l_0^2$  are called finite vectors.

For a separable Hilbert space  $H$  we denote by  $(\cdot, \cdot)_H$  and  $\|\cdot\|_H$  the scalar product and the norm in  $H$ , respectively. The indices may be omitted in obvious cases.

For a linear operator  $A$  in  $H$  we denote by  $D(A)$  its domain, by  $R(A)$  its range, and by  $A^*$  we denote its adjoint if it exists. If  $A$  is bounded, then  $\|A\|$  stands for its operator norm. By  $\text{Ker } A$  we mean the null subspace of  $A$ . For a set of elements  $\{x_n\}_{n \in A}$  in  $H$ , we denote by  $\text{Lin}\{x_n\}_{n \in A}$  and  $\text{span}\{x_n\}_{n \in A}$  the linear span and the closed linear span (in the norm of  $H$ ), respectively, where  $A$  is an arbitrary set of indices. For a set  $M \subseteq H$  we denote by  $\bar{M}$  the closure of  $M$  with respect to the norm of  $H$ . By  $E_H$  we denote the identity operator in  $H$ , i.e.  $E_H x = x$ ,  $x \in H$ . If  $H_1$  is a subspace of  $H$ , by  $P_{H_1} = P_{H_1}^H$  we denote the operator of the orthogonal projection on  $H_1$  in  $H$ .

## 2. THE MATRIX HAMBURGER MOMENT PROBLEM: SOLVABILITY AND A DESCRIPTION OF SOLUTIONS

Recall that an infinite complex matrix  $K = (K_{n,m})_{n,m=0}^{\infty}$  is called a *positive definite kernel* if

$$(9) \quad \sum_{n,m=0}^{\infty} K_{n,m} \xi_n \bar{\xi}_m \geq 0,$$

for all finite vectors  $(\xi_n)_{n=0}^{\infty}$  of complex numbers, see [3]. In other words,  $K$  is a positive definite kernel if

$$(10) \quad uKu^* = (uK, u)_{l^2} \geq 0, \quad u \in l_0^2,$$

where  $uK$  is defined by the usual matrix multiplication.

We shall use the following important fact (e.g., [19, p. 215]):

**Theorem 1.** *a) Let  $K = (K_{n,m})_{n,m=0}^\infty$  be a positive definite kernel. Then there exist a separable Hilbert space  $H$  with a scalar product  $(\cdot, \cdot)$  and a sequence  $\{x_n\}_{n=0}^\infty$  in  $H$ , such that*

$$(11) \quad K_{n,m} = (x_n, x_m), \quad n, m \in \mathbb{Z}_+,$$

and  $\text{span}\{x_n\}_{n \in \mathbb{Z}_+} = H$ .

*b) Let  $R = (R_{n,m})_{n,m=0}^r \geq 0$  be a positive semi-definite complex  $((r + 1) \times (r + 1))$  matrix,  $r \in \mathbb{Z}_+$ . Then there exist a finite-dimensional Hilbert space  $H_0$  with a scalar product  $(\cdot, \cdot)_0$  and a sequence  $\{y_n\}_{n=0}^r$  in  $H_0$ , such that*

$$(12) \quad R_{n,m} = (y_n, y_m), \quad n, m = 0, 1, \dots, r,$$

and  $\text{span}\{y_n\}_{n=0}^r = H_0$ .

*Proof.* a) Consider an arbitrary infinite-dimensional linear vector space  $V$  (for example a space of complex sequences  $(u_n)_{n \in \mathbb{Z}_+}$ ,  $u_n \in \mathbb{C}$ ). Let  $X = \{x_n\}_{n=0}^\infty$  be an arbitrary infinite sequence of linear independent elements in  $V$ . Let  $L = \text{Lin}\{x_n\}_{n \in \mathbb{Z}_+}$  be the linear span of elements of  $X$ . Introduce the following functional:

$$(13) \quad [x, y] = \sum_{n,m=0}^\infty K_{n,m} a_n \bar{b}_m,$$

for  $x, y \in L$ ,

$$x = \sum_{n=0}^\infty a_n x_n, \quad y = \sum_{m=0}^\infty b_m x_m, \quad a_n, b_m \in \mathbb{C}.$$

The space  $V$  with  $[\cdot, \cdot]$  will be a quasi-Hilbert space. Factorizing and making the completion we obtain the required space  $H$  (see [3, p. 10–11]).

b) In this case we proceed in an analogous manner. □

Consider the matrix Hamburger moment problem (1). If we choose an arbitrary element  $f = (f_0, f_1, \dots, f_{N-1})$ ,  $f_k \in \mathbb{P}$ ,  $k = 0, 1, \dots, N - 1$ , and calculate  $\int_{\mathbb{R}} f dM f^*$ , one can easily deduce the necessity of conditions (2),(4) for the solvability of the moment problem.

On the other hand, suppose that the moment problem (1) is given and condition (4) holds true. Set

$$(14) \quad \Gamma = (S_{k+l})_{k,l=0}^\infty = \begin{pmatrix} S_0 & S_1 & \dots & S_n & \dots \\ S_1 & S_2 & \dots & S_{n+1} & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ S_n & S_{n+1} & \dots & S_{2n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Comparing relations (4) and (10) we conclude that the kernel  $\Gamma = (\Gamma_{n,m})_{n,m=0}^\infty$  is positive definite. Let

$$S_n = (s_n^{k,l})_{k,l=0}^{N-1}, \quad n \in \mathbb{Z}_+.$$

Notice that

$$(15) \quad \Gamma_{rN+j,tN+n} = s_{r+t}^{j,n}, \quad 0 \leq j, n \leq N - 1, \quad r, t \in \mathbb{Z}_+.$$

From (15) it follows that

$$(16) \quad \Gamma_{a+N,b} = \Gamma_{a,b+N}, \quad a, b \in \mathbb{Z}_+.$$

In fact, if  $a = rN + j$ ,  $b = tN + n$ ,  $0 \leq j, n \leq N - 1$ ,  $r, t \in \mathbb{Z}_+$ , we can write

$$\Gamma_{a+N,b} = \Gamma_{(r+1)N+j,tN+n} = s_{r+t+1}^{j,n} = \Gamma_{rN+j,(t+1)N+n} = \Gamma_{a,b+N}.$$

By Theorem 1 there exist a Hilbert space  $H$  and a sequence  $\{x_n\}_{n=0}^\infty$  in  $H$ , such that  $\text{span}\{x_n\}_{n \in \mathbb{Z}_+} = H$ , and

$$(17) \quad (x_n, x_m)_H = \Gamma_{n,m}, \quad n, m \in \mathbb{Z}_+.$$

Set  $L := \text{Lin}\{x_n\}_{n \in \mathbb{Z}_+}$ . Choose an arbitrary  $x \in L$ . Let  $x = \sum_{k=0}^\infty \alpha_k x_k$ ,  $x = \sum_{k=0}^\infty \beta_k x_k$ , where  $\alpha_k, \beta_k \in \mathbb{C}$ , and all but finite number of coefficients  $\alpha_k, \beta_k$  are zero. Using (17), (16) we can write

$$\begin{aligned} \left( \sum_{k=0}^\infty \alpha_k x_{k+N}, x_l \right) &= \sum_{k=0}^\infty \alpha_k (x_{k+N}, x_l) = \sum_{k=0}^\infty \alpha_k \Gamma_{k+N,l} = \sum_{k=0}^\infty \alpha_k \Gamma_{k,l+N} \\ &= \sum_{k=0}^\infty \alpha_k (x_k, x_{l+N}) = \left( \sum_{k=0}^\infty \alpha_k x_k, x_{l+N} \right) = (x, x_{l+N}), \quad l \in \mathbb{Z}_+. \end{aligned}$$

In an analogous manner we obtain that

$$\left( \sum_{k=0}^\infty \beta_k x_{k+N}, x_l \right) = (x, x_{l+N}), \quad l \in \mathbb{Z}_+,$$

and therefore

$$\left( \sum_{k=0}^\infty \alpha_k x_{k+N}, x_l \right) = \left( \sum_{k=0}^\infty \beta_k x_{k+N}, x_l \right), \quad l \in \mathbb{Z}_+.$$

Since  $\bar{L} = H$ , we obtain that

$$(18) \quad \sum_{k=0}^\infty \alpha_k x_{k+N} = \sum_{k=0}^\infty \beta_k x_{k+N}.$$

Set

$$(19) \quad Ax = \sum_{k=0}^\infty \alpha_k x_{k+N}, \quad x \in L, \quad x = \sum_{k=0}^\infty \alpha_k x_k.$$

In particular, we have

$$(20) \quad Ax_k = x_{k+N}, \quad k \in \mathbb{Z}_+.$$

The above considerations show that this definition is correct. Choose arbitrary  $x, y \in L$ ,  $x = \sum_{k=0}^\infty \alpha_k x_k$ ,  $y = \sum_{n=0}^\infty \gamma_n x_n$ , and write

$$\begin{aligned} (Ax, y) &= \left( \sum_{k=0}^\infty \alpha_k x_{k+N}, \sum_{n=0}^\infty \gamma_n x_n \right) = \sum_{k,n=0}^\infty \alpha_k \bar{\gamma}_n (x_{k+N}, x_n) = \sum_{k,n=0}^\infty \alpha_k \bar{\gamma}_n (x_k, x_{n+N}) \\ &= \left( \sum_{k=0}^\infty \alpha_k x_k, \sum_{n=0}^\infty \gamma_n x_{n+N} \right) = (x, Ay). \end{aligned}$$

Thus, the operator  $A$  is a linear symmetric operator in  $H$  with the domain  $D(A) = L$ . Let  $\tilde{A} \supseteq A$  be an arbitrary self-adjoint extension of  $A$  in a Hilbert space  $\tilde{H} \supseteq H$ , and  $\{\tilde{E}_\lambda\}_{\lambda \in \mathbb{R}}$  be its left-continuous orthogonal resolution of unity. Choose an arbitrary  $a \in \mathbb{Z}_+$ ,  $a = rN + j$ ,  $r \in \mathbb{Z}_+$ ,  $0 \leq j \leq N - 1$ . Notice that

$$x_a = x_{rN+j} = Ax_{(r-1)N+j} = \cdots = A^r x_j.$$

Then choose an arbitrary  $b \in \mathbb{Z}_+$ ,  $b = tN + n$ ,  $t \in \mathbb{Z}_+$ ,  $0 \leq n \leq N - 1$ . Using (15) we can write

$$\begin{aligned} s_{r+t}^{j,n} &= \Gamma_{rN+j,tN+n} = (x_{rN+j}, x_{tN+n})_H = (A^r x_j, A^t x_n)_H = (\tilde{A}^r x_j, \tilde{A}^t x_n)_{\tilde{H}} \\ &= \left( \int_{\mathbb{R}} \lambda^r d\tilde{E}_\lambda x_j, \int_{\mathbb{R}} \lambda^t d\tilde{E}_\lambda x_n \right)_{\tilde{H}} = \int_{\mathbb{R}} \lambda^{r+t} d(\tilde{E}_\lambda x_j, x_n)_{\tilde{H}} \\ &= \int_{\mathbb{R}} \lambda^{r+t} d\left( P_{\tilde{H}}^{\tilde{H}} \tilde{E}_\lambda x_j, x_n \right)_H. \end{aligned}$$

From the latter relation we get

$$(21) \quad S_{r+t} = \int_{\mathbb{R}} \lambda^{r+t} d\tilde{M}(\lambda), \quad r, t \in \mathbb{Z}_+,$$

where  $\tilde{M}(\lambda) := \left( \left( P_{\tilde{H}}^{\tilde{H}} \tilde{E}_\lambda x_j, x_n \right)_H \right)_{j,n=0}^{N-1}$ . If we set  $t = 0$  in relation (15), we obtain that the matrix function  $\tilde{M}(\lambda)$  is a solution of the matrix Hamburger moment problem (1) (From the properties of the orthogonal resolution of unity it easily follows that  $\tilde{M}(\lambda)$  is left-continuous non-decreasing and  $\tilde{M}(-\infty) = 0$ ).

Thus, we obtained another proof of the solvability criterion for the matrix Hamburger moment problem (1).

Let  $\hat{A}$  be an arbitrary self-adjoint extension of  $A$  in a Hilbert space  $\hat{H}$ . Let  $R_z(\hat{A})$  be the resolvent of  $\hat{A}$  and  $\{\hat{E}_\lambda\}_{\lambda \in \mathbb{R}}$  be an orthogonal left-continuous resolution of unity of  $\hat{A}$ . Recall that the operator-valued function  $\mathbf{R}_z = P_{\hat{H}}^{\hat{H}} R_z(\hat{A})$  is called a generalized resolvent of  $A$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ . The function  $\mathbf{E}_\lambda = P_{\hat{H}}^{\hat{H}} \hat{E}_\lambda$ ,  $\lambda \in \mathbb{R}$ , is a spectral function of a symmetric operator  $A$ . There exists a one-to-one correspondence between generalized resolvents and spectral functions established by the following relation ([20]):

$$(22) \quad (\mathbf{R}_z f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d(\mathbf{E}_\lambda f, g)_H, \quad f, g \in H, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Formula (21) shows that spectral functions of  $A$  produce solutions of the matrix Hamburger moment problem (1). Can an arbitrary solution of (1) be produced in such a way? Choose an arbitrary solution  $\widehat{M}(x) = (\widehat{m}_{k,l}(x))_{k,l=0}^{N-1}$  of the matrix Hamburger moment problem (1). Consider the space  $L^2(\widehat{M})$  and let  $Q$  be the operator of multiplication by an independent variable in  $L^2(\widehat{M})$ . The operator  $Q$  is self-adjoint and its resolution of unity is (see [18])

$$(23) \quad E_b - E_a = E([a, b]) : h(x) \rightarrow \chi_{[a,b]}(x)h(x),$$

where  $\chi_{[a,b]}(x)$  is the characteristic function of an interval  $[a, b]$ ,  $-\infty \leq a < b \leq +\infty$ . Set  $\vec{e}_k = (e_{k,0}, e_{k,1}, \dots, e_{k,N-1})$ ,  $e_{k,j} = \delta_{k,j}$ ,  $0 \leq j \leq N - 1$ , for  $k = 0, 1, \dots, N - 1$ . A set of (classes of equivalence of) functions  $f \in L^2(\widehat{M})$  such that (the corresponding class includes)  $f = (f_0, f_1, \dots, f_{N-1})$ ,  $f \in \mathbb{P}$ , we denote by  $\mathbb{P}^2(\widehat{M})$  and call a set of vector polynomials in  $L^2(\widehat{M})$ . Set  $L_0^2(\widehat{M}) = \mathbb{P}^2(\widehat{M})$ .

For an arbitrary  $f \in \mathbb{P}^2(\widehat{M})$  there exists a unique representation of the following form:

$$(24) \quad f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \alpha_{k,j} x^j \vec{e}_k, \quad (\alpha_{k,0}, \alpha_{k,1}, \dots) \in l_0^2.$$

Let  $g \in \mathbb{P}^2(\widehat{M})$  have a representation

$$(25) \quad g(x) = \sum_{l=0}^{N-1} \sum_{r=0}^{\infty} \beta_{l,r} x^r \vec{e}_l, \quad (\beta_{l,0}, \beta_{l,1}, \dots) \in l_0^2.$$

We can write

$$(26) \quad \begin{aligned} (f, g)_{L^2(\widehat{M})} &= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \int_{\mathbb{R}} x^{j+r} \vec{e}_k d\widehat{M}(x) \vec{e}_l^* \\ &= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \int_{\mathbb{R}} x^{j+r} d\widehat{m}_{k,l}(x) = \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} s_{j+r}^{k,l}. \end{aligned}$$

On the other hand, we can write

$$(27) \quad \begin{aligned} \left( \sum_{j=0}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=0}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_{rN+l} \right)_H &= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} (x_{jN+k}, x_{rN+l})_H \\ &= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \Gamma_{jN+k, rN+l} = \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} s_{j+r}^{k,l}. \end{aligned}$$

From relations (26), (27) it follows that

$$(28) \quad (f, g)_{L^2(\widehat{M})} = \left( \sum_{j=0}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=0}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_{rN+l} \right)_H.$$

Set

$$(29) \quad Vf = \sum_{j=0}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k},$$

for  $f(x) \in \mathbb{P}^2(\widehat{M})$ ,  $f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \alpha_{k,j} x^j \vec{e}_k$ ,  $(\alpha_{k,0}, \alpha_{k,1}, \dots) \in l_0^2$ .

If  $f, g$  have representations (24), (25), and  $\|f - g\|_{L^2(\widehat{M})} = 0$ , then from (28) it follows that

$$\|Vf - Vg\|_H^2 = (V(f - g), V(f - g))_H = (f - g, f - g)_{L^2(\widehat{M})} = \|f - g\|_{L^2(\widehat{M})}^2 = 0.$$

Thus,  $V$  is a correctly defined operator from  $\mathbb{P}^2(\widehat{M})$  to  $H$ . Relation (28) shows that  $V$  is an isometric transformation from  $\mathbb{P}^2(\widehat{M})$  onto  $L$ . By continuity we extend it to an isometric transformation from  $L_0^2(\widehat{M})$  onto  $H$ . In particular, we note that

$$(30) \quad Vx^j \vec{e}_k = x_{jN+k}, \quad j \in \mathbb{Z}_+, \quad 0 \leq k \leq N-1.$$

Set  $L_1^2(\widehat{M}) := L^2(\widehat{M}) \ominus L_0^2(\widehat{M})$ , and  $U := V \oplus E_{L_1^2(\widehat{M})}$ . The operator  $U$  is an isometric transformation from  $L^2(\widehat{M})$  onto  $H \oplus L_1^2(\widehat{M}) =: \widehat{H}$ . Set

$$\widehat{A} := UQU^{-1}.$$

The operator  $\widehat{A}$  is a self-adjoint operator in  $\widehat{H}$ . Let  $\{\widehat{E}_\lambda\}_{\lambda \in \mathbb{R}}$  be its left-continuous orthogonal resolution of unity. Notice that

$$\begin{aligned} UQU^{-1}x_{jN+k} &= VQV^{-1}x_{jN+k} = VQx^j \vec{e}_k = Vx^{j+1} \vec{e}_k = x_{(j+1)N+k} \\ &= x_{jN+k+N} = Ax_{jN+k}, \quad j \in \mathbb{Z}_+, \quad 0 \leq k \leq N-1. \end{aligned}$$

By linearity we get

$$UQU^{-1}x = Ax, \quad x \in L = D(A),$$

and therefore  $\widehat{A} \supseteq A$ . Choose an arbitrary  $z \in \mathbb{C} \setminus \mathbb{R}$  and write

$$\begin{aligned}
 \int_{\mathbb{R}} \frac{1}{\lambda - z} d(\widehat{E}_\lambda x_k, x_j)_{\widehat{H}} &= \left( \int_{\mathbb{R}} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, x_j \right)_{\widehat{H}} \\
 &= \left( U^{-1} \int_{\mathbb{R}} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, U^{-1} x_j \right)_{L^2(\widehat{M})} \\
 (31) \quad &= \left( \int_{\mathbb{R}} \frac{1}{\lambda - z} dU^{-1} \widehat{E}_\lambda U \vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} = \left( \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_\lambda \vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} \\
 &= \int_{\mathbb{R}} \frac{1}{\lambda - z} d(E_\lambda \vec{e}_k, \vec{e}_j)_{L^2(\widehat{M})}, \quad 0 \leq k, \quad j \leq N - 1.
 \end{aligned}$$

Using (23) we can write

$$(E_\lambda \vec{e}_k, \vec{e}_j)_{L^2(\widehat{M})} = \widehat{m}_{k,j}(\lambda),$$

and therefore

$$(32) \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} d(P_H^{\widehat{H}} \widehat{E}_\lambda x_k, x_j)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\widehat{m}_{k,j}(\lambda), \quad 0 \leq k, \quad j \leq N - 1.$$

By the Stieltjes-Perron inversion formula (see, e.g., [7]) we conclude that

$$(33) \quad \widehat{m}_{k,j}(\lambda) = (P_H^{\widehat{H}} \widehat{E}_\lambda x_k, x_j)_H.$$

Consequently, an answer on the above question is affirmative.

Let us show that the deficiency index of  $A$  is equal to  $(m, n)$ ,  $0 \leq m, n \leq N$ . Choose an arbitrary  $u \in L$ ,  $u = \sum_{k=0}^\infty c_k x_k$ ,  $c_k \in \mathbb{C}$ . Suppose that  $c_k = 0$ ,  $k \geq N + R + 1$ , for some  $R \in \mathbb{Z}_+$ . Consider the following system of linear equations:

$$(34) \quad -z d_k = c_k, \quad k = 0, 1, \dots, N - 1,$$

$$(35) \quad d_{k-N} - z d_k = c_k, \quad k = N, N + 1, N + 2, \dots,$$

where  $\{d_k\}_{k \in \mathbb{Z}_+}$  are unknown complex numbers,  $z \in \mathbb{C} \setminus \mathbb{R}$  is a fixed parameter. Set

$$(36) \quad \begin{aligned} d_k &= 0, \quad k \geq R + 1, \\ d_j &= c_{N+j} + z d_{N+j}, \quad j = R, R - 1, R - 2, \dots, 0. \end{aligned}$$

For such numbers  $\{d_k\}_{k \in \mathbb{Z}_+}$ , all equations in (35) are satisfied. Only equations (34) are not satisfied. Set  $v = \sum_{k=0}^\infty d_k x_k$ ,  $v \in L$ . Notice that

$$(A - zE_H)v = \sum_{k=0}^\infty (d_{k-N} - z d_k)x_k,$$

where  $d_{-1} = d_{-2} = \dots = d_{-N} = 0$ . By the construction of  $d_k$  we have

$$\begin{aligned}
 (37) \quad (A - zE_H)v - u &= \sum_{k=0}^\infty (d_{k-N} - z d_k - c_k)x_k = \sum_{k=0}^{N-1} (-z d_k - c_k)x_k, \\
 u &= (A - zE_H)v + \sum_{k=0}^{N-1} (z d_k + c_k)x_k, \quad u \in L.
 \end{aligned}$$

Set  $H_z := \overline{(A - zE_H)L} = (\overline{A} - zE_H)D(\overline{A})$ , and

$$(38) \quad y_k := x_k - P_{H_z}^H x_k, \quad k = 0, 1, \dots, N - 1.$$

Set  $H_0 := \text{span}\{y_k\}_{k=0}^{N-1}$ . Notice that the dimension of  $H_0$  is less or equal to  $N$ , and  $H_0 \perp H_z$ . From (37) it follows that  $u \in L$  can be represented in the following form:

$$(39) \quad u = u_1 + u_2, \quad u_1 \in H_z, \quad u_2 \in H_0.$$



Therefore we get  $L \subseteq H_z \oplus H_0$ ;  $H \subseteq H_z \oplus H_0$ , and finally  $H = H_z \oplus H_0$ . Thus,  $H_0$  is the corresponding deficiency subspace. So, the deficiency numbers of  $A$  are less or equal to  $N$ .

**Theorem 2.** *Let a matrix Hamburger moment problem (1) be given and condition (4) is true. Let an operator  $A$  be constructed for the moment problem as in (19). All solutions of the moment problem have the following form*

$$(40) \quad M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (\mathbf{E}_\lambda x_k, x_j)_H,$$

where  $\mathbf{E}_\lambda$  is a spectral function of the operator  $A$ . Moreover, the correspondence between all spectral functions of  $A$  and all solutions of the moment problem is one-to-one.

*Proof.* It remains to prove that different spectral functions of the operator  $A$  produce different solutions of the moment problem (1). Suppose to the contrary that two different spectral functions produce the same solution of the moment problem. That means that there exist two self-adjoint extensions  $A_j \supseteq A$ , in Hilbert spaces  $H_j \supseteq H$ , such that

$$(41) \quad P_H^{H_1} E_{1,\lambda} \neq P_H^{H_2} E_{2,\lambda},$$

$$(42) \quad (P_H^{H_1} E_{1,\lambda} x_k, x_j)_H = (P_H^{H_2} E_{2,\lambda} x_k, x_j)_H, \quad 0 \leq k, j \leq N-1, \quad \lambda \in \mathbb{R},$$

where  $\{E_{n,\lambda}\}_{\lambda \in \mathbb{R}}$  are orthogonal left-continuous resolutions of unity of operators  $A_n$ ,  $n = 1, 2$ . Set  $L_N := \text{Lin}\{x_k\}_{k=0, N-1}$ . By linearity we get

$$(43) \quad (P_H^{H_1} E_{1,\lambda} x, y)_H = (P_H^{H_2} E_{2,\lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in \mathbb{R}.$$

Denote by  $R_{n,\lambda}$  the resolvent of  $A_n$ , and set  $\mathbf{R}_{n,\lambda} := P_H^{H_n} R_{n,\lambda}$ ,  $n = 1, 2$ . From (43), (22) it follows that

$$(44) \quad (\mathbf{R}_{1,\lambda} x, y)_H = (\mathbf{R}_{2,\lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Choose an arbitrary  $z \in \mathbb{C} \setminus \mathbb{R}$  and consider the space  $H_z$  defined as above. Since

$$R_{j,z}(A - zE_H)x = (A_j - zE_{H_j})^{-1}(A_j - zE_{H_j})x = x, \quad x \in L = D(A),$$

we get

$$(45) \quad R_{1,z}u = R_{2,z}u \in H, \quad u \in H_z,$$

$$(46) \quad \mathbf{R}_{1,z}u = \mathbf{R}_{2,z}u, \quad u \in H_z, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

We can write

$$(47) \quad (\mathbf{R}_{n,z}x, u)_H = (R_{n,z}x, u)_{H_n} = (x, R_{n,\bar{z}}u)_{H_n} = (x, \mathbf{R}_{n,\bar{z}}u)_H, \\ x \in L_N, \quad u \in H_{\bar{z}}, \quad n = 1, 2,$$

and therefore we get

$$(48) \quad (\mathbf{R}_{1,z}x, u)_H = (\mathbf{R}_{2,z}x, u)_H, \quad x \in L_N, \quad u \in H_{\bar{z}}.$$

By (37) an arbitrary element  $y \in L$  can be represented as  $y = y_{\bar{z}} + y'$ ,  $y_{\bar{z}} \in H_{\bar{z}}$ ,  $y' \in L_N$ . Using (44) and (48) we get

$$(\mathbf{R}_{1,z}x, y)_H = (\mathbf{R}_{1,z}x, y_{\bar{z}} + y')_H = (\mathbf{R}_{2,z}x, y_{\bar{z}} + y')_H = (\mathbf{R}_{2,z}x, y)_H, \quad x \in L_N, \quad y \in L.$$

Since  $\bar{L} = H$ , we obtain

$$(49) \quad \mathbf{R}_{1,z}x = \mathbf{R}_{2,z}x, \quad x \in L_N, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

For an arbitrary  $x \in L$ ,  $x = x_z + x'$ ,  $x_z \in H_z$ ,  $x' \in L_N$ , using relations (46), (49) we obtain

$$(50) \quad \mathbf{R}_{1,z}x = \mathbf{R}_{1,z}(x_z + x') = \mathbf{R}_{2,z}(x_z + x') = \mathbf{R}_{2,z}x, \quad x \in L, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and

$$(51) \quad \mathbf{R}_{1,z}x = \mathbf{R}_{2,z}x, \quad x \in H, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

By (22) that means that the spectral functions coincide and we obtain a contradiction.  $\square$

Recall some known facts from [9] which we shall need here. Let  $B$  be a closed symmetric operator in a Hilbert space  $H$ , with the domain  $D(B)$ ,  $\overline{D(B)} = H$ . Set  $\Delta_B(\lambda) = (B - \lambda E_H)D(B)$ , and  $N_\lambda = N_\lambda(B) = H \ominus \Delta_B(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

Consider an arbitrary bounded linear operator  $C$ , which maps  $N_i$  into  $N_{-i}$ . For

$$(52) \quad g = f + C\psi - \psi, \quad f \in D(B), \quad \psi \in N_i,$$

we set

$$(53) \quad B_C g = Bf + iC\psi + i\psi.$$

Since an intersection of  $D(A)$ ,  $N_i$  and  $N_{-i}$  consists only of the zero element, this definition is correct. Notice that  $B_C$  is a part of the operator  $B^*$ . The operator  $B_C$  is called a *quasiself-adjoint extension of the operator  $B$ , defined by the operator  $C$* .

The following theorem is true, see [9, Theorem 7]:

**Theorem 3.** *Let  $B$  be a closed symmetric operator in a Hilbert space  $H$  with the domain  $D(B)$ ,  $\overline{D(B)} = H$ . All generalized resolvents of the operator  $B$  have the following form:*

$$(54) \quad \mathbf{R}_\lambda = \begin{cases} (B_{F(\lambda)} - \lambda E_H)^{-1}, & \text{Im } \lambda > 0 \\ (B_{F^*(\bar{\lambda})} - \lambda E_H)^{-1}, & \text{Im } \lambda < 0 \end{cases},$$

where  $F(\lambda)$  is an analytic in  $\mathbb{C}_+$  operator-valued function, which values are contractions which map  $N_i(B)$  into  $N_{-i}(B)$  ( $\|F(\lambda)\| \leq 1$ ), and  $B_{F(\lambda)}$  is the quasiself-adjoint extension of  $B$  defined by  $F(\lambda)$ .

On the other hand, for any operator function  $F(\lambda)$  having the above properties there corresponds by relation (54) a generalized resolvent of  $B$ .

By virtue of Theorems 2 and 3 we get a description of all solutions of the matrix Hamburger moment problem (1).

**Theorem 4.** *Let a matrix Hamburger moment problem (1) be given and condition (4) is true. Consider a sequence  $\{x_n\}_{n=0}^\infty$  in a Hilbert space  $H$  such that relation (17) holds and  $\text{span}\{x_n\}_{n \in \mathbb{Z}_+} = H$ . Let  $A$  be a linear operator with  $D(A) = \text{Lin}\{x_n\}_{n \in \mathbb{Z}_+}$ , defined by equalities*

$$Ax_k = x_{k+N}, \quad k \in \mathbb{Z}_+.$$

All solutions of the moment problem have the following form

$$(55) \quad M(x) = (m_{k,j}(x))_{k,j=0}^{N-1},$$

where  $m_{k,j}$  satisfy the following relation

$$(56) \quad \int_{\mathbb{R}} \frac{1}{x - \lambda} dm_{k,j}(x) = ((A_{F(\lambda)} - \lambda E_H)^{-1} x_k, x_j)_H, \quad \lambda \in \mathbb{C}_+,$$

where  $F(\lambda)$  is an analytic in  $\mathbb{C}_+$  operator-valued function, which values are contractions which map  $N_i(\bar{A})$  into  $N_{-i}(\bar{A})$  ( $\|F(\lambda)\| \leq 1$ ), and  $A_{F(\lambda)}$  is the quasiself-adjoint extension of  $\bar{A}$  defined by  $F(\lambda)$ .

On the other hand, to any operator function  $F(\lambda)$  having the above properties there corresponds by relation (56) a solution of the matrix Hamburger moment problem. Moreover, the correspondence between all operator functions having the above properties and all solutions of the moment problem, established by relation (56), is one-to-one.

*Proof.* It remains to check the last statement of the theorem. Note that different functions  $F_1(\lambda)$ ,  $F_2(\lambda)$ , with the above properties generate different generalized resolvents  $\mathbf{R}_1(\lambda)$ ,  $\mathbf{R}_2(\lambda)$  of  $A$  (see [9, Remark 2, p. 85]). Let  $\mathbf{E}_1(\lambda)$ ,  $\mathbf{E}_2(\lambda)$ , be the corresponding spectral functions of  $A$ . Suppose to the contrary that functions  $F_1(\lambda)$ ,  $F_2(\lambda)$ , correspond

to the same solution  $M(x) = (m_{k,j}(x))_{k,j=0}^{N-1}$  of the moment problem. By (56) this means that

$$(57) \quad \int_R \frac{1}{x - \lambda} dm_{k,j}(x) = (\mathbf{R}_1(\lambda)x_k, x_j)_H = (\mathbf{R}_2(\lambda)x_k, x_j)_H.$$

By the Stieltjes-Perron inversion formula we get

$$(58) \quad m_{k,j}(x) = (\mathbf{E}_1(\lambda)x_k, x_j)_H = (\mathbf{E}_2(\lambda)x_k, x_j)_H.$$

We obtain that different spectral functions of  $A$  generate the same solution of the moment problem. This contradicts to Theorem 2.  $\square$

### 3. THE TRUNCATED MATRIX HAMBURGER MOMENT PROBLEM

Let a moment problem (5) be given with  $d \in \mathbb{N}$ , and the first condition in (7) is true. Let  $\Gamma_d = (\gamma_{n,m}^d)_{n,m=0}^{dN+N-1}$ . By Theorem 1 there exist a finite-dimensional Hilbert space  $H$  and a sequence  $\{x_n\}_{n=0}^{dN+N-1}$  in  $H$ , such that

$$(59) \quad \gamma_{n,m}^d = (x_n, x_m), \quad n, m = 0, 1, \dots, dN + N - 1,$$

and  $\text{span}\{x_n\}_{n=0}^{dN+N-1} = H$ . Notice that

$$(60) \quad \gamma_{rN+j,tN+n}^d = s_{r+t}^{j,n}, \quad 0 \leq j, n \leq N - 1, \quad 0 \leq r, t \leq d.$$

From (60) it follows that

$$(61) \quad \gamma_{a+N,b}^d = \gamma_{a,b+N}^d, \quad a = rN + j, \quad b = tN + n, \quad 0 \leq j, n \leq N - 1, \quad 0 \leq r, t \leq d - 1.$$

In fact, we can write

$$\gamma_{a+N,b}^d = \gamma_{(r+1)N+j,tN+n}^d = s_{r+t+1}^{j,n} = \gamma_{rN+j,(t+1)N+n}^d = \gamma_{a,b+N}^d.$$

Denote  $L_a = \text{Lin}\{x_n\}_{n=0}^{dN-1}$ ,  $L = \text{Lin}\{x_n\}_{n=0}^{dN+N-1}$ .

Set

$$(62) \quad Ax = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \quad x \in L_a, \quad x = \sum_{k=0}^{dN-1} \alpha_k x_k.$$

In particular, we have

$$(63) \quad Ax_k = x_{k+N}, \quad 0 \leq k \leq dN - 1.$$

**Proposition 1.** *The operator  $A$  is correctly defined if and only if the second condition in (7) holds.*

*Proof. Necessity.* Since  $A$  is correctly defined, from the equality

$$(64) \quad \sum_{k=0}^{dN-1} \xi_k x_k = 0,$$

with some complex numbers  $\xi_k$ , it should follow the equality

$$(65) \quad \sum_{k=0}^{dN-1} \xi_k x_{k+N} = 0.$$

On the other hand, the equality (64) is equivalent to the equalities

$$(66) \quad \sum_{k=0}^{dN-1} \xi_k (x_k, x_l) = \sum_{k=0}^{dN-1} \xi_k \gamma_{d;k,l} = 0, \quad l = 0, 1, \dots, dN - 1.$$

Analogously, the equality (65) is equivalent to the equalities

$$(67) \quad \sum_{k=0}^{dN-1} \xi_k (x_{k+N}, x_{l+N}) = \sum_{k=0}^{dN-1} \xi_k \gamma_{d;k+N,l+N} = 0, \quad l = 0, 1, \dots, dN - 1.$$

If we shall use the matrix notations, the equality

$$(68) \quad (\xi_0, \xi_1, \dots, \xi_{dN-1})(\gamma_{d;k,l})_{k,l=0}^{dN-1} = 0,$$

implies the equality

$$(69) \quad (\xi_0, \xi_1, \dots, \xi_{dN-1})(\gamma_{d;k+N,l+N})_{k,l=0}^{dN-1} = 0.$$

Thus, the second relation in (7) is true.

*Sufficiency.* If the first relation in (7) is true, then relation (68) implies relation (69). Therefore relation (66) implies relation (67) and, finally, relation (64) implies relation (65). The last property means that the operator  $A$  is defined correctly.  $\square$

We assume that the second condition in (7) is true. Choose arbitrary  $x, y \in L_a$ ,  $x = \sum_{k=0}^{dN-1} \alpha_k x_k$ ,  $y = \sum_{n=0}^{dN-1} \gamma_n x_n$ , and write

$$\begin{aligned} (Ax, y) &= \left( \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \sum_{n=0}^{dN-1} \gamma_n x_n \right) = \sum_{k,n=0}^{dN-1} \alpha_k \overline{\gamma_n} (x_{k+N}, x_n) = \sum_{k,n=0}^{dN-1} \alpha_k \overline{\gamma_n} (x_k, x_{n+N}) \\ &= \left( \sum_{k=0}^{dN-1} \alpha_k x_k, \sum_{n=0}^{dN-1} \gamma_n x_{n+N} \right) = (x, Ay). \end{aligned}$$

Thus, an operator  $A$  is a linear symmetric operator in  $H$  with the domain  $D(A) = L_a$ . It is not necessary that  $A$  is densely defined.

Let  $\tilde{A} \supseteq A$  be an arbitrary self-adjoint extension of  $A$  in a Hilbert space  $\tilde{H} \supseteq H$ , and  $\{\tilde{E}_\lambda\}_{\lambda \in \mathbb{R}}$  be its left-continuous orthogonal resolution of unity. Existence of a self-adjoint extension of a non-densely defined symmetric operator was established by M.A. Krasnoselskiy (e.g. [9]). Choose an arbitrary  $a$ ,  $0 \leq a \leq dN + N - 1$ ,  $a = rN + j$ ,  $0 \leq r \leq d$ ,  $0 \leq j \leq N - 1$ . Notice that

$$x_a = x_{rN+j} = Ax_{(r-1)N+j} = \dots = A^r x_j.$$

Then choose an arbitrary  $b$ ,  $0 \leq b \leq dN + N - 1$ ,  $b = tN + n$ ,  $0 \leq t \leq d$ ,  $0 \leq n \leq N - 1$ . Using (59) we can write

$$\begin{aligned} s_{r+t}^{j,n} &= \gamma_{rN+j,tN+n}^d = (x_{rN+j}, x_{tN+n})_H = (A^r x_j, A^t x_n)_H = (\tilde{A}^r x_j, \tilde{A}^t x_n)_{\tilde{H}} \\ &= \left( \int_{\mathbb{R}} \lambda^r d\tilde{E}_\lambda x_j, \int_{\mathbb{R}} \lambda^t d\tilde{E}_\lambda x_n \right)_{\tilde{H}} = \int_{\mathbb{R}} \lambda^{r+t} d(\tilde{E}_\lambda x_j, x_n)_{\tilde{H}} \\ &= \int_{\mathbb{R}} \lambda^{r+t} d(P_H^{\tilde{H}} \tilde{E}_\lambda x_j, x_n)_H. \end{aligned}$$

From the last relation we obtain

$$(70) \quad S_{r+t} = \int_{\mathbb{R}} \lambda^{r+t} d\tilde{M}(\lambda), \quad 0 \leq r, t \leq d,$$

where  $\tilde{M}(\lambda) := \left( \left( P_H^{\tilde{H}} \tilde{E}_\lambda x_j, x_n \right)_H \right)_{j,n=0}^{N-1}$ . From relation (15) we derive that the matrix function  $\tilde{M}(\lambda)$  is a solution of the matrix Hamburger moment problem (5) (Properties of the orthogonal resolution of unity provide that  $\tilde{M}(\lambda)$  is left-continuous non-decreasing and  $\tilde{M}(-\infty) = 0$ ).

On the other hand, choose an arbitrary solution  $\widehat{M}(x) = (\widehat{m}_{k,l}(x))_{k,l=0}^{N-1}$  of the truncated matrix Hamburger moment problem (5). Consider the space  $L^2(\widehat{M})$  and let  $Q$  be the operator of multiplication by an independent variable in  $L^2(\widehat{M})$ . The operator  $Q$  is self-adjoint and its resolution of unity is given by (23).

Let  $\vec{e}_k$ ,  $k = 0, 1, \dots, N - 1$ , be defined as after (23). A set of (classes of equivalence of) functions  $f \in L^2(\widehat{M})$  such that (the corresponding class includes)  $f = (f_0, f_1, \dots, f_{N-1})$ ,

$f \in \mathbb{P}_d$ , we denote by  $\mathbb{P}_d^2(\widehat{M})$  and call a set of vector polynomials in  $L^2(\widehat{M})$  of degree less or equal to  $d$ . Set  $L_{d,0}^2(\widehat{M}) = \overline{\mathbb{P}_d^2(\widehat{M})}$ .

For an arbitrary  $f \in \mathbb{P}_d^2(\widehat{M})$  there exists a unique representation of the following form:

$$(71) \quad f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^d \alpha_{k,j} x^j \vec{e}_k, \quad \alpha_{k,j} \in \mathbb{C}.$$

Let  $g \in \mathbb{P}_d^2(\widehat{M})$  has a representation

$$g(x) = \sum_{l=0}^{N-1} \sum_{r=0}^d \beta_{l,r} x^r \vec{e}_l, \quad \beta_{l,r} \in \mathbb{C}.$$

As it was done in the case of the full matrix Hamburger moment problem after (23), we obtain that

$$(72) \quad (f, g)_{L^2(\widehat{M})} = \left( \sum_{j=0}^d \sum_{k=0}^{N-1} \alpha_{k,j} x_j x_{N+k}, \sum_{r=0}^d \sum_{l=0}^{N-1} \beta_{l,r} x_r x_{N+l} \right)_H.$$

Set

$$(73) \quad Vf = \sum_{j=0}^d \sum_{k=0}^{N-1} \alpha_{k,j} x_j x_{N+k},$$

for  $f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^d \alpha_{k,j} x^j \vec{e}_k$ ,  $(\alpha_{k,0}, \alpha_{k,1}, \dots) \in l_0^2$ . From relation (72) it easily follows that  $V$  is a correctly defined operator from  $\mathbb{P}_d^2(\widehat{M})$  to  $H$ . Relation (72) shows that  $V$  is an isometric transformation from  $\mathbb{P}_d^2(\widehat{M})$  onto  $L$ . By continuity we extend it to an isometric transformation from  $L_{d,0}^2(\widehat{M})$  onto  $H$ . In particular, we note that

$$(74) \quad Vx^j \vec{e}_k = x_{jN+k}, \quad 0 \leq j \leq d, \quad 0 \leq k \leq N-1.$$

Set  $L_{d,1}^2(\widehat{M}) := L^2(\widehat{M}) \ominus L_{d,0}^2(\widehat{M})$ , and  $U := V \oplus E_{L_{d,1}^2(\widehat{M})}$ . The operator  $U$  is an isometric transformation from  $L^2(\widehat{M})$  onto  $H \oplus L_{d,1}^2(\widehat{M}) =: \widehat{H}$ . Set

$$\widehat{A} := UQU^{-1}.$$

The operator  $\widehat{A}$  is a self-adjoint operator in  $\widehat{H}$ . Let  $\{\widehat{E}_\lambda\}_{\lambda \in \mathbb{R}}$  be its left-continuous orthogonal resolution of unity. Notice that

$$\begin{aligned} UQU^{-1}x_{jN+k} &= VQV^{-1}x_{jN+k} = VQx^j \vec{e}_k = Vx^{j+1} \vec{e}_k = x_{(j+1)N+k} = x_{jN+k+N} \\ &= Ax_{jN+k}, \quad 0 \leq j \leq d-1, \quad 0 \leq k \leq N-1. \end{aligned}$$

By linearity we get

$$UQU^{-1}x = Ax, \quad x \in L_a = D(A),$$

and therefore  $\widehat{A} \supseteq A$ . Choose an arbitrary  $z \in \mathbb{C} \setminus \mathbb{R}$  and write

$$(75) \quad \begin{aligned} \int_{\mathbb{R}} \frac{1}{\lambda - z} d(\widehat{E}_\lambda x_k, x_j)_{\widehat{H}} &= \left( \int_{\mathbb{R}} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, x_j \right)_{\widehat{H}} \\ &= \left( U^{-1} \int_{\mathbb{R}} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, U^{-1} x_j \right)_{L^2(\widehat{M})} \\ &= \left( \int_{\mathbb{R}} \frac{1}{\lambda - z} dU^{-1} \widehat{E}_\lambda U \vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} = \left( \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_\lambda \vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} \\ &= \int_{\mathbb{R}} \frac{1}{\lambda - z} d(E_\lambda \vec{e}_k, \vec{e}_j)_{L^2(\widehat{M})}, \quad 0 \leq k, j \leq N-1. \end{aligned}$$

Using (23) we can write

$$(E_\lambda \vec{e}_k, \vec{e}_j)_{L^2(\widehat{M})} = \widehat{m}_{k,j}(\lambda),$$

and therefore

$$(76) \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} d(P_H^{\widehat{H}} \widehat{E}_\lambda x_k, x_j)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\widehat{m}_{k,j}(\lambda), \quad 0 \leq k, j \leq N - 1.$$

By the Stieltjes-Perron inversion formula we conclude that

$$(77) \quad \widehat{m}_{k,j}(\lambda) = (P_H^{\widehat{H}} \widehat{E}_\lambda x_k, x_j)_H.$$

Consequently, all solutions of the truncated moment problem are generated by spectral functions of  $A$ . For the definitions of a spectral function and a generalized resolvent for a non-densely defined symmetric operator we refer to [9].

Let us show that the deficiency index of  $A$  is equal to  $(m, n)$ ,  $0 \leq m, n \leq N$ . Choose an arbitrary  $u \in L$ ,  $u = \sum_{k=0}^{dN+N-1} c_k x_k$ ,  $c_k \in \mathbb{C}$ . Consider the following system of linear equations:

$$(78) \quad -z d_k = c_k, \quad k = 0, 1, \dots, N - 1,$$

$$(79) \quad d_{k-N} - z d_k = c_k, \quad k = N, N + 1, \dots, dN + N - 1,$$

where  $\{d_k\}_{k=0}^{dN+N-1}$  are unknown complex numbers,  $z \in \mathbb{C} \setminus \mathbb{R}$  is a fixed parameter. Set

$$(80) \quad \begin{aligned} d_k &= 0, & k &= dN, dN + 1, \dots, dN + N - 1, \\ d_{k-N} &= z d_k + c_k, & k &= dN + N - 1, dN + N - 2, \dots, N. \end{aligned}$$

For such numbers  $\{d_k\}_{k \in \mathbb{Z}_+}$ , all equations in (79) are satisfied. Equations (78) are not necessarily satisfied. Set  $v = \sum_{k=0}^{dN+N-1} d_k x_k = \sum_{k=0}^{dN-1} d_k x_k$ . Notice that  $v \in L_a = D(A)$ . We can write

$$(A - zE_H)v = \sum_{k=0}^{dN+N-1} (d_{k-N} - z d_k)x_k,$$

where  $d_{-1} = d_{-2} = \dots = d_{-N} = 0$ . By the construction of  $d_k$  we have

$$(81) \quad \begin{aligned} (A - zE_H)v - u &= \sum_{k=0}^{dN+N-1} (d_{k-N} - z d_k - c_k)x_k = \sum_{k=0}^{N-1} (-z d_k - c_k)x_k, \\ u &= (A - zE_H)v + \sum_{k=0}^{N-1} (z d_k + c_k)x_k, \quad u \in L. \end{aligned}$$

Set  $H_z := \overline{(A - zE_H)L} = (\overline{A} - zE_H)D(\overline{A})$ . Repeating arguments after relation (37) we obtain that the deficiency numbers of  $A$  are less or equal to  $N$ .

**Theorem 5.** *Let a truncated matrix Hamburger moment problem (5) with  $d \in \mathbb{N}$  be given and conditions (7) are true. Let the operator  $A$  be constructed for the moment problem as in (62). All solutions of the moment problem have the following form*

$$(82) \quad M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (\mathbf{E}_\lambda x_k, x_j)_H,$$

where  $\mathbf{E}_\lambda$  is a spectral function of the operator  $A$ . Moreover, the correspondence between all spectral functions of  $A$  and all solutions of the moment problem is one-to-one.

*Proof.* Only the last statement of the theorem was not proved yet. Its proof repeats the corresponding proof of Theorem 2. □

We need some known facts from [10]. Let  $B$  be a closed symmetric operator in a Hilbert space  $H$  with the domain  $D(B)$ , which is not necessarily dense in  $H$ . Set  $\Delta_B(\lambda) = (B - \lambda E_H)D(B)$ , and  $N_\lambda = N_\lambda(B) = H \ominus \Delta_B(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

Define an operator  $X_i: N_i \rightarrow N_{-i}$  in the following way:

$$(83) \quad \varphi = X_i \psi,$$

if  $\psi \in N_i$ ,  $\varphi \in N_{-i}$  and  $\varphi - \psi \in D(B)$ .

The operator  $X_i$  can be defined also in the following way:

$$(84) \quad D(X_i) = P_{N_i}^H(H \ominus \overline{D(B)}),$$

$$(85) \quad X_i P_{N_i}^H h = P_{N_{-i}}^H h, \quad h \in H \ominus \overline{D(B)}.$$

The operator  $X_i$  is called *forbidden with respect to the operator  $B$* .

An operator  $V: N_i \rightarrow N_{-i}$ , is called *admissible with respect to the operator  $B$* , if inclusion  $V\psi - \psi \in D(B)$  is possible only if  $\psi = 0$ . It is equivalent to the condition that the relation  $V\psi = X_z \psi$  is possible only if  $\psi = 0$ .

The formulas

$$(86) \quad D(G) = D(B) \dot{+} (V - E_H)D(V),$$

$$(87) \quad G(f + V\psi - \psi) = Bf + iV\psi + i\psi, \quad f \in D(B), \quad \psi \in D(V),$$

establish a one-to-one correspondence between a set of all admissible with respect to  $B$  isometric operators  $V$ ,  $D(V) \subseteq N_i$ ,  $R(V) \subseteq N_{-i}$ , and a set of all symmetric extensions  $G$  of the operator  $B$ . The operator  $G$  is self-adjoint if and only if  $D(G) = N_i$ ,  $R(G) = N_{-i}$ .

Denote by  $K(B; \mathbb{C}_+; N_i, N_{-i})$  a class of analytic operator-valued functions  $F(\lambda)$  in  $\mathbb{C}_+$ , whose values are contractions which map  $N_i$  into  $N_{-i}$ ,  $\|F(\lambda)\| \leq 1$ .

Set  $\mathbb{C}_+^\varepsilon := \{z \in \mathbb{C}_+ : \varepsilon \leq \arg z \leq \pi - \varepsilon\}$ ,  $0 \leq \varepsilon \leq \frac{\pi}{2}$ . A function  $F \in K(B; \mathbb{C}_+; N_i, N_{-i})$  is called *admissible with respect to the operator  $B$*  if relations

$$(88) \quad \lim_{\lambda \in \mathbb{C}_+^\varepsilon, \lambda \rightarrow \infty} F(\lambda)\psi = X_i \psi,$$

$$(89) \quad \varliminf_{\lambda \in \mathbb{C}_+^\varepsilon, \lambda \rightarrow \infty} (|\lambda|(\|\psi\| - \|F(\lambda)\psi\|)) < +\infty$$

imply  $\psi = 0$ .

The class of all functions from  $K(B; \mathbb{C}_+; N_i, N_{-i})$  which are admissible with respect to  $B$  we denote by  $K_a(B; \mathbb{C}_+; N_i, N_{-i})$ . Notice that in the case  $\overline{D(B)} = H$  we have  $K_a(B; \mathbb{C}_+; N_i, N_{-i}) = K(B; \mathbb{C}_+; N_i, N_{-i})$ .

Let  $F(\lambda) \in K_a(B; \mathbb{C}_+; N_i, N_{-i})$ . In this case the operator  $F(\lambda)$  is admissible with respect to  $B$  [10]. By  $B_{F(\lambda)}$  we mean an operator  $G$  defined as in (86) with  $V = F(\lambda)$ .

The following theorem holds true, see [10, Theorem 12].

**Theorem 6.** *Let  $B$  be a closed symmetric operator in a Hilbert space  $H$  with the domain  $D(B) \subseteq H$ . The formula*

$$(90) \quad \mathbf{R}_\lambda = \begin{cases} (B_{F(\lambda)} - \lambda E_H)^{-1}, & \text{Im } \lambda > 0 \\ (B_{F^*(\bar{\lambda})} - \lambda E_H)^{-1}, & \text{Im } \lambda < 0 \end{cases},$$

*establishes a one-to-one correspondence between the set of all generalized resolvents of  $B$  and the class  $K_a(B; \mathbb{C}_+; N_i, N_{-i})$ .*

Using Theorems 5 and 6 we get a description of all solutions of the truncated matrix Hamburger moment problem.

**Theorem 7.** *Let a truncated matrix Hamburger moment problem (5) with  $d \in \mathbb{N}$  be given and conditions (7) are true. Consider a sequence  $\{x_n\}_{n=0}^{dN+N-1}$  in a Hilbert space  $H$  such that (59) holds and  $\text{span}\{x_n\}_{n=0}^{dN+N-1} = H$ . Let  $A$  be a linear operator with  $D(A) = \text{Lin}\{x_n\}_{n=0}^{dN-1}$ , defined by equalities*

$$Ax_k = x_{k+N}, \quad 0 \leq k \leq dN - 1.$$

All solutions of the truncated moment problem have the following form

$$(91) \quad M(x) = (m_{k,j}(x))_{k,j=0}^{N-1},$$

where  $m_{k,j}$  satisfy the following relation

$$(92) \quad \int_{\mathbb{R}} \frac{1}{x - \lambda} dm_{k,j}(x) = ((A_{F(\lambda)} - \lambda E_H)^{-1} x_k, x_j)_H, \quad \lambda \in \mathbb{C}_+,$$

where  $F(\lambda) \in K_a(A; \mathbb{C}_+; N_i, N_{-i})$ . Here by  $A_{F(\lambda)}$  we mean the operator with the domain

$$D(A_{F(\lambda)}) = D(A) \dot{+} (F(\lambda) - E_H)D(F(\lambda)),$$

and

$$A_{F(\lambda)}(f + F(\lambda)\psi - \psi) = Af + iF(\lambda)\psi + i\psi, \quad f \in D(A), \psi \in D(F(\lambda)).$$

On the other hand, to any operator function  $F(\lambda) \in K_a(A; \mathbb{C}_+; N_i, N_{-i})$  it corresponds by relation (92) a solution of the truncated matrix Hamburger moment problem. Moreover, the correspondence between  $K_a(A; \mathbb{C}_+; N_i, N_{-i})$  and all solutions of the truncated moment problem, established by relation (92), is one-to-one.

*Proof.* To check the last statement of the theorem it is enough to repeat the arguments from the proof of Theorem 4. □

*Remark.* We describe solutions of the moment problem (5) under conditions (7). Observe that these conditions are not only sufficient for the solvability of the moment problem, but they are necessary. In fact, let  $\widehat{M}(x) = (\widehat{m}_{k,l}(x))_{k,l=0}^{N-1}$  be a solution of the truncated matrix Hamburger moment problem (5). The necessity of the first condition in (7) is obvious. By Theorem 1 we can construct a Hilbert space  $H$  and a sequence  $\{x_n\}_{n=0}^{dN+N-1}$  in  $H$ , such that

$$(93) \quad \gamma_{n,m}^d = (x_n, x_m), \quad n, m = 0, 1, \dots, dN + N - 1,$$

and  $\text{span}\{x_n\}_{n=0}^{dN+N-1} = H$ . Repeating the construction and arguments after (70), we shall construct an operator  $\widehat{A} \supseteq A$ . Thus, the operator  $A$  is defined correctly. By Proposition 1 we conclude that the second relation in (7) holds. (Similar reasoning can be found in [21]).

#### 4. SOLVABILITY OF THE SCALAR TRUNCATED MOMENT PROBLEM WITH EVEN NUMBER OF GIVEN MOMENTS

Let a moment problem (8) be given. Set

$$(94) \quad \Gamma_n = \begin{pmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \dots & s_{2n} \end{pmatrix}, \quad n = 0, 1, \dots, d.$$

Let  $\sigma(x)$  be a solution of the moment problem. If we choose an arbitrary polynomial  $p(x) \in \mathbb{P}_d$ , and calculate  $\int_{\mathbb{R}} |p(x)|^2 d\sigma(x) \geq 0$ , we can easily see that

$$(95) \quad \Gamma_d \geq 0,$$



and therefore all matrices  $\Gamma_n$ ,  $0 \leq n \leq d$ , are real positive semi-definite. Thus, condition (95) is necessary for the solvability of the moment problem.

Suppose now that a moment problem (8) is given and condition (95) is true. If  $\Gamma_0 = s_0 = 0$ , then there exists a unique solution  $\sigma(x) = 0$ , if  $s_1 = s_2 = \dots = s_{2d+1} = 0$ , or there are no solutions in the opposite case.

Assume that  $\Gamma_0 = s_0 > 0$ . Set

$$r := \max\{n : 0 \leq n \leq d, \det \Gamma_n > 0\}, \quad 1 \leq r \leq d.$$

a) *Case*  $r = d$ . In this case  $\Gamma_d > 0$ . We can define a real number  $s_{2d+2}$  such that

$$(96) \quad \det \Gamma_{d+1} > 0, \quad \Gamma_{d+1} := \begin{pmatrix} s_0 & s_1 & \dots & s_{d+1} \\ s_1 & s_2 & \dots & s_{d+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{d+1} & s_{d+2} & \dots & s_{2d+2} \end{pmatrix}.$$

To show that, expand the latter determinant by the elements of the last row and choose  $s_{2d+2}$  sufficiently large. Thus, in this case, by results of V.G. Ershov and H. Dym (see the Introduction) and also by results in [7] on the truncated Hamburger moment problem, it follows that the moment problem (8) has a solution.

b) *Case*  $r < d$ . In this case we have

$$(97) \quad \Gamma_r > 0, \quad \det \Gamma_{r+1} = 0.$$

Let  $\vec{c} = (c_0, c_1, \dots, c_{r+1})$  be a non-zero real vector such that

$$(98) \quad \Gamma_{r+1} \vec{c}^* = 0, \quad c_{r+1} = 1.$$

Consider a non-zero real polynomial  $p(x) = \sum_{k=0}^{r+1} c_k x^k$ , of degree exactly  $r + 1$ .

If there exists a solution  $\sigma(x)$ , then

$$(99) \quad \int_{\mathbb{R}} p^2(x) d\sigma(x) = \sum_{k,n=0}^{r+1} c_k c_n s_{k+n} = 0.$$

This implies that  $\sigma(x)$  has points of increase only in zeros of  $p(x)$ , which we shall denote by  $x_0, x_1, \dots, x_r$ . Roots of the polynomial  $p(x)$  in this case are real and distinct (or we could replace  $p(x)$  by a polynomial of a less degree such that (99) held, this contradicts (97)). Thus,  $\sigma(x)$  is a piecewise constant function,  $\sigma(-\infty) = 0$ , with jumps in a real distinct points  $\{x_k\}_{k=0}^r$ . Denote the jump of  $\sigma$  at  $x_k$  by  $\mu_k$ ,  $0 \leq k \leq r$ . The moment equalities (8) are equivalent to

$$(100) \quad \sum_{k=0}^r x_k^n \mu_k = s_n, \quad n = 0, 1, \dots, r,$$

$$(101) \quad \sum_{k=0}^r x_k^n \mu_k = s_n, \quad n = r + 1, r + 2, \dots, 2r + 1.$$

The linear system of equations (100) has a non-zero Vandermonde's determinant, and has a unique solution. This solution should satisfy relations (101).

**Theorem 8.** *Let a truncated Hamburger moment problem (8) be given. It has a solution if and only if*

a)  $s_k = 0$ ,  $k = 0, 1, \dots, 2d + 1$ ; or

b)  $\Gamma_d > 0$ ; or

c)  $1 \leq r < d$ , where  $r := \max\{n : 0 \leq n \leq d, \det \Gamma_n > 0\}$ ; the polynomial  $p(x) = \sum_{k=0}^{r+1} c_k x^k$ , where  $c_k$  are complex numbers satisfying (98), has real distinct zeros  $\{x_k\}_{k=0}^r$ ; and the unique solution of linear system (100) consists of non-negative numbers  $\mu_k \geq 0$ ,  $k = 0, 1, \dots, r$ , which satisfy relations (101).

*In cases a) and c) the solution is unique.*

*Proof.* The necessity follows from the above considerations. The sufficiency of condition b) was shown. The sufficiency of conditions a) and c) is obvious.  $\square$

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