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# SINGULARLY PERTURBED NORMAL OPERATORS

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*This paper is dedicated to the memory of Israel Gohberg.*

Abstract. We give an effective description of finite rank singular perturbations of a normal operator by using the concepts we introduce of an admissible subspace and corresponding admissible operators. We give a description of rank one singular perturbations in terms of a scale of Hilbert spaces, which is constructed from the unperturbed operator.

Let $\mathcal{H}$ be a separable Hilbert space. Denote by $A$ a linear operator with a domain $\mathfrak{D}(A)$ dense in $\mathcal{H}$. A closed operator $A$ is normal if $A^*A = AA^*$. In such a case, $\mathfrak{D}(A^*) = \mathfrak{D}(A)$ and $\|A^*x\| = \|Ax\|$ [1], which is equivalent to the equality

$$(Ax, Ay) = (A^*x, A^*y), \quad x, y \in \mathfrak{D}(A) = \mathfrak{D}(A^*), \tag{1}$$

since the bilinear forms $(Ax, Ay)$ and $(A^*x, A^*y)$ are uniquely determined by the corresponding quadratic forms. Self-adjoint operators and unitary operators are examples of a normal operator. If $L$ is any self-adjoint operator, then the operator $A = aL + bI$ is normal for arbitrary complex numbers $a$ and $b$, and we will call such a normal operator a linear function of the self-adjoint operator. An analogue of a symmetric operator is a formally normal operator $N$ with dense domain $\mathfrak{D}(N)$ in $\mathcal{H}$. The operator $N$ is called formally normal if $\mathfrak{D}(N^*) \supset \mathfrak{D}(N)$ and $\|N^*x\| = \|Nx\|$, $x \in \mathfrak{D}(N)$. If the formally normal operator has a normal extension, then we call such an operator prenormal. Comparing with a complete theory of self-adjoint extensions of a symmetric operator, the theory of normal extensions of a formally normal operator has been worked out to a lesser degree of completion [2]–[6]. In this work we use the approach from [7] and investigate normal operators $\tilde{A}$ that are singularly perturbed with respect to a given normal operator $A$ in the sense that the set $\mathfrak{D} \subset \mathfrak{D}(\tilde{A}) \cap \mathfrak{D}(A)$ on which the operator $\tilde{A}$ coincides with $A$ is dense in $\mathcal{H}$ and $\tilde{A} \neq A$. In such a case, the operators $\tilde{A}$ and $A$ have the common prenormal part $N = \tilde{A} \restriction_{\mathfrak{D}} = A \restriction_{\mathfrak{D}} = \tilde{A} \wedge A$, and the operators $\tilde{A}$ and $A$ are different normal extensions of the operator $N$. If $\mathfrak{D}(A) = \mathfrak{D} \dot{+} \mathfrak{N}$, where $\mathfrak{N}$ is an $n$-dimensional subspace of $\mathcal{H}$, then we say that the operator $\tilde{A}$ is a singular perturbation of rank $n$ with respect to $A$.

Let us note that it is interesting not only to describe all finite rank singular perturbations of a normal operator but also to find efficient conditions that would imply that no normal singular perturbation exists. For example, a formally normal operator $N$ whose real and imaginary parts are symmetric operators with deficiency indices equal to $(0, 0)$ or $(1, 1)$ can be connected with the complex moment problem [8]. If a normal operator $A$ is an extension of an operator $N$, then its spectral measure gives an integral representations of the moment sequence. Conditions under which the normal operator $A$ does not have a singular perturbation of rank $n \leq 2$ are sufficient for uniqueness of the

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normal extension of $N$ and, consequently, for uniqueness of the integral representation of the moment sequence. Let us also make a remark on an interesting application to integration of nonlinear evolution equations by the method of the inverse spectral problem of a normal block Jacobi type matrix [9], in addition to the complex moment problem [10]–[11].

$\mathbf{1^0}$. Suppose the normal operator $A$ has at least one regular point $z$, i.e., the operator $(A-zI)^{-1}$ exists and is bounded. Since the operator $A-zI$ is also normal, without loss of generality we suppose everywhere in the sequel that *the non-perturbed operator $A$ has bounded inverse*. It gives a possibility to assume, in the representation $\mathfrak{D}(A)=\mathfrak{D}\dot{+}\mathfrak{R}$, that $A\mathfrak{D}\perp A\mathfrak{R}$, and put $\mathfrak{R}=A^{-1}(\mathcal{H}\ominus A\mathfrak{D})$, and due to (1) to also have $A^*\mathfrak{D}\perp A^*\mathfrak{R}$.

**Lemma 1.** *Let a normal operator $\tilde{A}$ be a finite rank singular perturbation of a normal operator $A$. Then 1) $A\mathfrak{R}\cap\mathfrak{D}(A)=\{0\}$; 2) $\tilde{A}^*\restriction_{\mathfrak{D}}=A^*\restriction_{\mathfrak{D}}$; 3) There exist unique linear operators $T$ and $T_*$ in the subspace $\mathfrak{R}$ such that*

$$\mathfrak{D}(\tilde{A})=\mathfrak{D}\dot{+}(A^*+T)\mathfrak{R}=\mathfrak{D}\dot{+}(A+T_*)\mathfrak{R}, \tag{2}$$

*and the operators $\tilde{A}$ and $\tilde{A}^*$ act as follows:*

$$\tilde{A}(x_0+(A^*+T)r)=Ax_0+ATr,\quad \tilde{A}^*(x_0+(A+T_*)r)=A^*x_0+A^*T_*r, \tag{3}$$

*where $x_0\in\mathfrak{D}$ and $r\in\mathfrak{R}$. The operators $T$ and $T_*$ satisfy the relations $AT_*A^{-1}=(ATA^{-1})^*$, $A^*T_*(A^*)^{-1}=(A^*T(A^*)^{-1})^*$ on the subspaces $\mathfrak{N}=A\mathfrak{R}$ and $\mathfrak{M}=A^*\mathfrak{R}$, respectively.*

*Proof.* The conditions 1) and 2) follow from (1) and from density of the set $\mathfrak{D}$ in $\mathcal{H}$. If for some vector we had $r\in\mathfrak{R}$, $r\neq 0$, $Ar\in\mathfrak{D}(A)=\mathfrak{D}(A^*)$, then we would obtain that the vector $A^*(Ar)\neq 0$ would be orthogonal to $\mathfrak{D}$, $(\varphi,A^*(Ar))=(A\varphi,Ar)=0$. But this is impossible, since the set $\mathfrak{D}$ is dense in $\mathcal{H}$.

Since $\mathfrak{D}(\tilde{A}^*)=\mathfrak{D}(A)\supset\mathfrak{D}$ and $\tilde{A}\restriction_{\mathfrak{D}}=A\restriction_{\mathfrak{D}}$, for an arbitrary $\varphi,\psi\in\mathfrak{D}$, $0=(\varphi,\tilde{A}\psi-A\psi)=(\tilde{A}^*\varphi-A^*\varphi,\psi)$, and using density of $\mathfrak{D}$ in $\mathcal{H}$ we have $\tilde{A}^*\varphi=A^*\varphi$.

Let $\tilde{x}\in\mathfrak{D}(\tilde{A})$ and $x=A^{-1}\tilde{A}\tilde{x}=x_0+r$, where $x_0\in\mathfrak{D}$ and $r\in\mathfrak{R}$. Since $\tilde{A}\tilde{x}-Ax=0$, for arbitrary $\varphi\in\mathfrak{D}$, $0=(\tilde{A}\tilde{x}-Ax,\varphi)=(\tilde{x}-x,A^*\varphi)$. Hence, $\tilde{x}-x\perp A^*\mathfrak{D}$ and, consequently, there exists an element $\rho\in\mathfrak{R}$ such that $\tilde{x}=x+A^*\rho$. Taking into account that $x=x_0+r$ and using the element $\tilde{x}\in\mathfrak{D}(\tilde{A})$ we construct two elements $r,\rho\in\mathfrak{R}$ connected via the linear operator $T$ in $\mathfrak{R}$, $r=T\rho$.

The action of the operator $\tilde{A}$ on the elements $\tilde{x}=x_0+Tr+A^*r$ is determined by the action of the operator $A$ on the element $x=x_0+Tr$, which gives (3) for the operator $\tilde{A}$. Analogously we prove (2)–(3) for the operator $\tilde{A}^*$. Since $(\tilde{A}x,y)=(x,\tilde{A}^*y)$, using (2) we put $x=Tr+A^*r$, $y=T_*\rho+A\rho$, where $r,\rho\in\mathfrak{R}$, and taking into account (3) we obtain $(ATr,A\rho)=(A^*r,A^*T_*\rho)$. Then using (1) we obtain the relations for operators $T_*$ and $T$. $\square$

**Lemma 2.** *Let a normal operator $\tilde{A}$ be a singular perturbation of rank one with respect to the normal operator $A$. Then there exist real numbers $\theta$ and $\xi$, $-\frac{\pi}{2}<\theta\leq\frac{\pi}{2}$, $\xi\in\mathbb{R}^1$, such that vectors $r\in\mathfrak{R}$ have the property*

$$e^{-i\theta}Ar-e^{i\theta}A^*r=2i\xi r. \tag{4}$$

*Proof.* The subspace $\mathfrak{R}$ is one dimensional and the operators $T$ and $T_*$ from (2) are operators of multiplication by the numbers $t$ and $\bar{t}$, respectively. Due to (2) we can represent the vector $\tilde{x}=tr+A^*r\in\mathfrak{D}(\tilde{A})$, $r\in\mathfrak{R}$, in the form $\tilde{x}=x_0+\bar{t}pr+pAr$, where $x_0\in\mathfrak{D}$ and $p$ is a complex number. Thus $\tilde{A}\tilde{x}=tAr$, $\tilde{A}^*\tilde{x}=A^*x_0+\bar{t}pA^*r$. Hence, $0=(\tilde{A}x_0,\tilde{A}\tilde{x})=(\tilde{A}^*x_0,\tilde{A}^*\tilde{x})=(\tilde{A}^*x_0,\tilde{A}^*x_0)$, i.e, $x_0=0$. From the another side, $(\tilde{A}\tilde{x},\tilde{A}\tilde{x})=(\tilde{A}^*\tilde{x},\tilde{A}^*\tilde{x})$, which gives $p=e^{-2i\theta}$, where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2}]$. The equality $\tilde{x}=tr+A^*r=e^{-2i\theta}(\bar{t}r+Ar)$ gives (4), where $\xi=\frac{1}{2i}(te^{i\theta}-\bar{t}e^{-i\theta})$. $\square$

**Definition 1.** *A vector $r \in \mathfrak{D}(A)$ such that $Ar \notin \mathfrak{D}(A)$ is called admissible for the operator $A$ with characteristics $\theta$, $\xi$ if the equality (4) is true, i.e., $r$ is an eigenvector of the operator $Im\,(e^{-i\theta}A)$ with the eigenvalue $\xi$.*

**Theorem 1.** *For the set $\mathcal{P}_1(A)$ of all normal operators $\tilde{A} \neq A$ that are rank one singular perturbations of the normal operator $A$ to be nonempty, it is necessary and sufficient that there existed an admissible vector $r$ for the operator $A$.*

*If $r$ is an admissible vector with characteristics $\theta$, $\xi$, and $\tau$ is an arbitrary real number, then the operator $\tilde{A}_{r,\tau}$ that has the domain*

$$\mathfrak{D}(\tilde{A}_{r,\tau}) = \mathfrak{D}\dot{+}\{e^{-i\theta}(\tau + i\xi)\rho + A^*\rho\}, \quad \mathfrak{D} = A^{-1}(H \ominus \{Ar\}), \quad \rho = ar, \tag{5}$$

*and acts by*

$$\tilde{A}_{r,\tau}(x_0 + e^{-i\theta}(\tau + i\xi)\rho + A^*\rho) = Ax_0 + e^{-i\theta}(\tau + i\xi)A\rho, \quad x_0 \in \mathfrak{D}, \tag{6}$$

*belongs to the set $\mathcal{P}_1(A)$.*

*Each operator $\tilde{A} \in \mathcal{P}_1(A)$ admits the representation (5), (6). Such a representation defines a one-to-one correspondence between the set $\mathcal{P}_1(A)$ and the set of pairs $\{\mathfrak{R}, \tau\}$, where $\mathfrak{R}$ is the one-dimensional subspace spanned by the admissible vector and $\tau$ is a self-adjoint operator in $\mathfrak{R}$.*

*The operator $\tilde{A}_{r,\tau}$ has bounded inverse iff $\tau + i\xi \neq 0$, and*

$$\tilde{A}^{-1}_{r,\tau} = A^{-1} + e^{i\theta}(\tau + i\xi)^{-1}(\cdot, Ar)A^*r. \tag{7}$$

*Proof.* The necessity follows from Lemma 2. By a direct calculation we obtain that the operator $\tilde{A}_{r,\tau}$ defined in (5), (6) is normal, $\tilde{A}_{r,\tau} \in \mathcal{P}_1(A)$, and representation (7) takes place. □

Theorem 1 has the following equivalent form.

**Theorem 2.** *The set $\mathcal{P}_1(A)$ is nonempty iff the normal operator $A$ has a cyclic invariant subspace $\tilde{\mathcal{H}}$, where the operator $A$ is a linear function of an unbounded self-adjoint operator $L$ in $\tilde{\mathcal{H}}$.*

*Proof.* If $\mathcal{P}_1(A) \neq \varnothing$, then there exists an admissible vector $r$ with the characteristics $\theta$, $\xi$. Let $\tilde{H} = H(r)$ be the cyclic invariant subspace of the operator $A$ generated by the vector $r$ [1]. Since $\tilde{H}$ reduces the operator $A$ and $Ar \notin \mathfrak{D}(A)$, the operator $L = \frac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*)$ is an unbounded self-adjoint operator on $\tilde{H}$, and the operator $A \restriction_{\tilde{H}} = e^{i\theta}(L + i\xi I)$ is a linear function of $L$.

Sufficiency follows from the fact that the unbounded self-adjoint operator $L$ has singular perturbations of an arbitrary rank [7]. □

**Corollary 1.** *If the intersection of an arbitrary line on the complex plane and the spectrum of the operator $A$ is a bounded or an empty set, then the normal operator $A$ does not have a singular perturbation of the rank one, that is, $\mathcal{P}_1(A) = \varnothing$.*

**$2^0$.** We give a description of operators $\tilde{A} \in \mathcal{P}_1(A)$ in terms of a scale of Hilbert spaces, $\mathcal{H}_s \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-s}$, $s > 0$, constructed from the positive self-adjoint operator $|A| = (AA^*)^{1/2} = (A^*A)^{1/2}$ [12]. In this construction, we set $\mathcal{H}_s = \mathfrak{D}(|A|^{s/2})$, $\|u\|_s = \||A|^{s/2}u\|$, and $\mathcal{H}_{-s}$ is the closure of $\mathcal{H}_0$ in the norm $\|u\|_{-s} = \||A|^{-s/2}u\|$. Let us remark that $\mathcal{H}_2 = \mathfrak{D}(A) = \mathfrak{D}(A^*)$, and the operators $A$ and $A^*$ have extensions by continuity to $\mathbf{A}$ $\mathbf{A}^*$, being isometric operators acting from all $\mathcal{H} = \mathcal{H}_0$ into all $\mathcal{H}_{-2}$. The scalar product $(\cdot, \cdot)$ in $\mathcal{H}$ can be extended by continuity to a pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{H}_s$ and $\mathcal{H}_{-s}$. If $\mathbf{B}$ is an operator acting from $\mathcal{H}$ into $\mathcal{H}_{-2}$, then its restriction $B$ to the space $\mathcal{H}$ is defined on the set $\mathfrak{D}(B) = \{u \in \mathfrak{D}(\mathbf{B}),\ \mathbf{B}u \in \mathcal{H}\}$: $B = \mathbf{B} \restriction_{\mathcal{H}} = \mathbf{B} \restriction_{\mathfrak{D}(B)}$. If $\mathbf{B} \neq B$, then we call the operator $\mathbf{B}$ singular on $\mathcal{H}$.

For each element $\psi \in \mathcal{H}_{-2} \setminus \mathcal{H}$ and a complex number $z \neq 0$, we can construct a one-dimensional singular operator $\mathbf{V} = z\langle \cdot, \psi\rangle\psi$, acting from $\mathcal{H}_2$ into $\mathcal{H}_{-2}$. This operator is defined on all $\mathcal{H}_2$.

If $\psi \in \mathcal{H}_{-1}$, then we continue the functional $\langle \cdot, \psi\rangle$ to the whole $\mathcal{H}_1$ and, in such a case, the operator $\mathbf{V}$ is defined on the elements $\mathbf{A}^{-1}\psi$, $(\mathbf{A}^*)^{-1}\psi$ and, hence, it is defined on all elements $\varphi + c_1\mathbf{A}^{-1}\psi + c_2(\mathbf{A}^*)^{-1}\psi$, where $\varphi \in \mathcal{H}_2$, $c_1, c_2 \in \mathbb{C}$.

Let $\psi \in \mathcal{H}_{-2}$, $\|\psi\|_{-2} = 1$, $\psi = (\mathbf{A}\mathbf{A}^*)r$, where $r$ is an admissible vector with characteristics $\theta, \xi$ of the operator $A$, according to Definition 1. In this case, to extend the operator $\mathbf{V}$ to all elements of the form $\varphi + c_1\mathbf{A}^{-1}\psi + c_2(\mathbf{A}^*)^{-1}\psi$, it is sufficient to extend the functional $\langle \cdot, \psi\rangle$ to the elements $\omega = \frac{1}{2}(e^{i\theta}\mathbf{A}^{-1}\psi + e^{-i\theta}(\mathbf{A}^*)^{-1}\psi)$ using the real number $\gamma$, $\langle \omega, \psi\rangle_\gamma = \gamma$. Then the extended operator $\mathbf{V}_\gamma$ is defined on the elements $\mathbf{A}^{-1}\psi$ and $(\mathbf{A}^*)^{-1}\psi$ and has the value

$$\mathbf{V}_\gamma(\mathbf{A}^{-1}\psi) = ze^{-i\theta}(\gamma - i\xi)\psi, \quad \mathbf{V}_\gamma((\mathbf{A}^*)^{-1}\psi) = ze^{i\theta}(\gamma + i\xi)\psi.$$

Such an extension of the operator $\mathbf{V}$ will be called a regularisation and the real number $\gamma$ is called a parameter of the regularisation. If $\psi \in \mathcal{H}_{-1}$, the parameter of the regularisation $\gamma = \langle \omega, \psi\rangle$ is defined by continuity. If $\psi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, we can choose the parameter of regularisation to be arbitrary or in such a way that the operator $\mathbf{V}$ would have some symmetry properties [13].

**Definition 2.** *The sum of a normal operator $A$ and a one-dimensions regularised singular operator $\mathbf{V}_\gamma$, in the space $\mathcal{H}$, we understand an operator $\tilde{A}$ that is the restriction of the operator $\mathbf{A} + \mathbf{V}_\gamma$, $\tilde{A} = (\mathbf{A} + \mathbf{V}_\gamma) \restriction_{\mathcal{H}} = A + z\langle \cdot, \psi\rangle_\gamma\psi$.*

**Theorem 3.** *Let $\psi \in \mathcal{H}_{-2}$, $\|\psi\|_{-2} = 1$, $\psi = (\mathbf{A}\mathbf{A}^*)r$, and $r$ be an admissible vector of the operator $A$ with characteristics $\theta, \xi$. Let a real number $\gamma$ be the parameter of regularisation of the one-dimensional operator constructed from the vector $\psi$, and suppose that a real number $\lambda \neq 0$. Then*

$$\tilde{A}_{\psi,\lambda} = A + e^{i\theta}\lambda\langle \cdot, \psi\rangle_\gamma\psi \in \mathcal{P}_1(A). \tag{8}$$

*If $\tau = -\lambda^{-1} - \gamma$, then the operator $\tilde{A}_{\psi,\lambda}$ coincides with the operator $\tilde{A}_{r,\tau}$ defined in Theorem 1.*

*Proof.* The domain of the operator $\tilde{A}_{\psi,\lambda}$ consist of vectors of the form $\tilde{x} = x_0 + c_1 r + c_2(\mathbf{A})^{-1}\psi$ such that $\tilde{A}_{\psi,\lambda}\tilde{x} \in \mathcal{H}$. It gives the condition $c_1 = e^{-i\theta}(-\lambda^{-1} - \gamma + i\xi)c_2$. The comparison the operator $\tilde{A}_{\psi,\lambda}$ with the operator $\tilde{A}_{r,\tau}$ from the Theorem 1 shows that for $\tau = -\lambda^{-1} - \gamma$ the operator $\tilde{A}_{r,\tau}$ is equals to the operator $\tilde{A}_{\psi,\lambda}$. What is more with this connection if $\tau + \gamma = 0$, then $\lambda = \infty$, and the operator $\tilde{A}_{\psi,\infty}$ in (8) we understand as $\tilde{A}_{r,\tau}$, with $\tau = -\gamma$ [14]. Let us remark that

$$\tilde{A}^*_{\psi,\lambda} = A^* + e^{-i\theta}\lambda\langle \cdot, \psi\rangle_\gamma\psi.$$

□

**3⁰**. Let us consider the singular perturbation of an arbitrary finite rank. If the normal operator $\tilde{A}$ is a singular rank $n$ perturbation of the normal operator $A$, then, due to the equality $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(\tilde{A}^*)$ and Lemma 1, we conclude that $\mathfrak{R} \dot{+} A\mathfrak{R} = \mathfrak{R} \dot{+} A^*\mathfrak{R}$. The last equality takes place if and only if there exist linear operators $P$ and $Q$ on the $n$-dimensional subspace $\mathfrak{R}$ such that the operator $P$ has an inverse, and

$$Ar = A^*Pr + Qr, \quad r \in \mathfrak{R}. \tag{9}$$

**Definition 3.** *The finite dimensional subspace $\mathfrak{R} \subset \mathfrak{D}(A)$, $A\mathfrak{R} \cap \mathfrak{D}(A) = \{0\}$, is called $(P,Q)$-admissible for the normal operator $A$ if there exist linear operators $P$, $P^{-1}$ and $Q$ in $\mathfrak{R}$ such that the equality (9) holds true.*

From the Lemma 1, we have that the absence of an $n$-dimensional admissible subspace of the normal operator $A$ is a sufficient condition for the absence of the normal singular perturbation of rank $n$.

Let us consider an example. Let $p_{n+1}(t) = (t-z_1)(t-z_2)\cdots(t-z_{n+1})$ be a polynomial of the degree $n+1$, where we choose the numbers $z_j$ so that all the numbers $\alpha_j = p_{n+1}(\bar{z}_j)$, $j = 1, 2, \ldots, n+1$, are distinct. Then the operator $A$ of multiplication by $p_{n+1}(t)$ is normal in the space $L_2([1,\infty))$ and does not have a singular normal perturbation of the rank less than $n+1$. Indeed, if there existed a singular perturbation of the rank $m \leq n$, the $m$-dimensional subspace $\mathfrak{N}$ would be admissible for the operator $A$. And due to the equality (9), we would immediately have that $\det[p_{n+1}(t)I - \bar{p}_{n+1}(t)P - Q] \equiv 0$. Putting $t = \bar{z}_j$, $j = 1, 2, \ldots, n+1$, we would obtain the identity $\det[\alpha_j I - Q] = 0$, i.e., the operator $Q$ would have $n+1$ different eigenvalues in the $m$-dimensional subspace $\mathfrak{N}$, which is impossible.

The proposed example has a generalization. The range of values of the constructed polynomial $p_{n+1}(t)$, $\quad t \in \mathbb{R}$ is an algebraic curve $\gamma_{n+1}$ in the complex plane. *If the spectrum of the normal operator $A$ belongs to the algebraic curve $\gamma_{n+1}$, then the operator $A$ does not have a normal singular perturbation of the rank less than $n+1$.*

In particular, the last observation gives the following. *If a Borel measure $\mu$ on the complex plane $\mathbb{C}$ is supported on an algebraic curve $\gamma_{n+1}$, $(n \geq 2)$, and all the complex moments $c_{k,m} = \int z^k \bar{z}^m d\mu$ are finite, then such an integral representation is unique for the sequence $c_{k,m}$, $(k, m = 0, 1, 2, \ldots)$.*

The operator $T$ in Lemma 1 is connected with characteristics $(P, Q)$ of the admissible subspace $\mathfrak{N}$. For a description of this connection, it is convenient to translate the operators $T, P, Q$ acting on the subspace $\mathfrak{N}$ to act on the subspace $\mathfrak{M} = A\mathfrak{N}$ by using the identities $\hat{T} = ATA^{-1}$, $\hat{P} = APA^{-1}$ and $\hat{Q} = AQA^{-1}$. Then from (2) and (9), we obtain

$$\hat{T}\hat{P} - \hat{T}^* = \hat{Q}. \tag{10}$$

And from the condition that $\|\tilde{A}\psi\| = \|\tilde{A}^*\psi\|$ and from (3) we get

$$\|\hat{T}^* n\| = \|\hat{T}\hat{P} n\|, \quad n \in \mathfrak{M} = A\mathfrak{N}. \tag{11}$$

**Definition 4.** *The operator $T$ acting on a $(P, Q)$-admissible subspace $\mathfrak{N}$ of the normal operator $A$ is called admissible on $\mathfrak{N}$ if the identities (10) and (11) hold.*

**Theorem 4.** *Relations (2), (3) establish a bijection between the set of all singular perturbations of finite rank of the normal operator $A$ and the set of the pairs $\{\mathfrak{N}, T\}$, where $\mathfrak{N}$ is an admissible subspace of the operator $A$ and $T$ is an admissible operator on $\mathfrak{N}$.*

*Proof.* Each normal operator $\tilde{A}$, which is a singular perturbation of finite rank of a normal operator $A$, uniquely generates, due the Lemma 1, a $(P, Q)$-admissible subspace $\mathfrak{N}$ and an admissible operator $T$ on $\mathfrak{N}$.

The converse is proved immediately using the construction of operators $\tilde{A}$ and $\tilde{A}^*$ given in (2) and (3). $\square$

Let us remark that the subspace $\mathfrak{N}$ is admissible for a strictly positive self-adjoint operator $A$ if and only if $\mathfrak{N} = A^{-1}\mathfrak{M}$, where the finite dimensional subspace $\mathfrak{M}$ satisfies the condition $\mathfrak{M} \cap \mathfrak{D}(A) = \{0\}$. In such a case, the operator $T$ in $\mathfrak{N}$ is admissible if and only if the operator $\hat{T} = ATA^{-1}$ is self-adjoint on $\mathfrak{M}$ [7].

The description of a singularly perturbed normal operator, proposed in this paper, is more effective than the description given in [2]–[6], where one tries to describe all normal extensions of a formally normal operator.

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## References


1. A. I. Plesner, *Spectral Theory of Linear Operators*, Nauka, Moscow, 1965.
2. Y. Kilpi, *Uber lineare normale Transformationen im Hilbertschen Raum*, Ann. Acad. Sci. Fennicae, Math. Ser. A-I, 1953, no. 154.
3. E. A. Coddington, *Normal extensions of formally normal operators*, Pacific J. Math. **10** (1960), no. 4, 1203–1209.
4. G. Biriuk, E. A. Coddington, *Normal extensions of unbounded formally normal operators*, J. Math. Mech. **13** (1964), no. 4, 617–638.
5. K. Schmüdgen, *A formally normal operator having no normal extension*, Proc. Amer. Math. Soc. **95** (1985), no. 3, 503–504.
6. J. Stochel, F. H. Szafraniec, *On normal extensions of unbounded operators.* I, J. Operator Theory **14** (1985), 31–55.
7. S. Albeverio, S. Kuzhel, L. Nizhnik, *On the perturbation theory of self-adjoint operators*, Tokyo Journal of Mathematics **31** (2008), no. 2, 273–292.
8. Y. Kilpi, *Uber das komplexe Momentenproblem*, Ann. Acad. Sci. Fennicae, Math. Ser. A-I, 1957, no. 236, 1–32.
9. Yu. M. Berezansky and A. A. Mokhon'ko, *Integration of some differential-difference nonlinear equations using the spectral theory of normal block Jacobi matrices*, Funct. Anal. Appl. **42** (2008), no. 1, 1–21.
10. Yu. M. Berezansky, M. E. Dudkin, *The complex moment problem and direct and inverse spectral problems for the block Jacobi type bounded normal matrices*, Methods Funct. Anal. Topology **12** (2006), no. 1, 1–31.
11. Yu. M. Berezansky, M. E. Dudkin, *On the complex moment problem*, Math. Nachr. **280** (2007), no. 1–2, 60–73.
12. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
13. S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators. Solvable Schrödinger Type Operators*, Cambridge University Press, Cambridge, 2000.
14. A. Kiselev, B. Simon, *Rank one perturbations with infinitesimal coupling*, J. Funct. Anal. **130** (1995), 345–356.



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