Methods of Functional Analysis and Topology
Vol. 16 (2010), no. 4, pp. 298–303

SINGULARLY PERTURBED NORMAL OPERATORS

M. E. DUDKIN AND L. P. NIZHNIK

This paper is dedicated to the memory of Israel Gohberg.

Abstract. We give an effective description of finite rank singular perturbations of a normal operator by using the concepts we introduce of an admissible subspace and corresponding admissible operators. We give a description of rank one singular perturbations in terms of a scale of Hilbert spaces, which is constructed from the unperturbed operator.

Let $\mathcal{H}$ be a separable Hilbert space. Denote by A a linear operator with a domain $\mathfrak{D}(A)$ dense in $\mathcal{H}$. A closed operator A is normal if $A^*A = AA^*$. In such a case, $\mathfrak{D}(A^*) = \mathfrak{D}(A)$ and $\|A^*x\| = \|Ax\|$ [1], which is equivalent to the equality

$$(Ax, Ay) = (A^*x, A^*y), \quad x, y \in \mathfrak{D}(A) = \mathfrak{D}(A^*), \tag{1}$$

since the bilinear forms (Ax, Ay) and (A^*x, A^*y) are uniquely determined by the corresponding quadratic forms. Self-adjoint operators and unitary operators are examples of a normal operator. If L is any self-adjoint operator, then the operator $A = aL + bI$ is normal for arbitrary complex numbers a and b, and we will call such a normal operator a linear function of the self-adjoint operator. An analogue of a symmetric operator is a formally normal operator N with dense domain $\mathfrak{D}(N)$ in $\mathcal{H}$. The operator N is called formally normal if $\mathfrak{D}(N^*) \supset \mathfrak{D}(N)$ and $\|N^*x\| = \|Nx\|$, $x \in \mathfrak{D}(N)$. If the formally normal operator has a normal extension, then we call such an operator prenormal. Comparing with a complete theory of self-adjoint extensions of a symmetric operator, the theory of normal extensions of a formally normal operator has been worked out to a lesser degree of completion [2]–[6]. In this work we use the approach from [7] and investigate normal operators $\tilde{A}$ that are singularly perturbed with respect to a given normal operator A in the sense that the set $\mathfrak{D} \subset \mathfrak{D}(\tilde{A}) \cap \mathfrak{D}(A)$ on which the operator $\tilde{A}$ coincides with A is dense in $\mathcal{H}$ and $\tilde{A} \neq A$. In such a case, the operators $\tilde{A}$ and A have the common prenormal part $N = \tilde{A} \restriction_{\mathfrak{D}} = A \restriction_{\mathfrak{D}} = \tilde{A} \wedge A$, and the operators $\tilde{A}$ and A are different normal extensions of the operator N. If $\mathfrak{D}(A) = \mathfrak{D} \dot{+} \mathfrak{R}$, where $\mathfrak{R}$ is an n-dimensional subspace of $\mathcal{H}$, then we say that the operator $\tilde{A}$ is a singular perturbation of rank n with respect to A.

Let us note that it is interesting not only to describe all finite rank singular perturbations of a normal operator but also to find efficient conditions that would imply that no normal singular perturbation exists. For example, a formally normal operator N whose real and imaginary parts are symmetric operators with deficiency indices equal to $(0, 0)$ or $(1, 1)$ can be connected with the complex moment problem [8]. If a normal operator A is an extension of an operator N, then its spectral measure gives an integral representations of the moment sequence. Conditions under which the normal operator A does not have a singular perturbation of rank $n \leq 2$ are sufficient for uniqueness of the

2000 *Mathematics Subject Classification.* 47A10, 47A20, 47B15.

Key words and phrases. Normal operator, singular perturbation, perturbation of the finite rank, formally normal operator, the extension of the operator.

This work is partially supported by FRSF of Ukraine, project F29.1/010.

normal extension of N and, consequently, for uniqueness of the integral representation of the moment sequence. Let us also make a remark on an interesting application to integration of nonlinear evolution equations by the method of the inverse spectral problem of a normal block Jacobi type matrix [9], in addition to the complex moment problem [10]–[11].

1⁰. Suppose the normal operator A has at least one regular point z, i.e., the operator $(A-zI)^{-1}$ exists and is bounded. Since the operator $A-zI$ is also normal, without loss of generality we suppose everywhere in the sequel that *the non-perturbed operator A has bounded inverse*. It gives a possibility to assume, in the representation $\mathfrak{D}(A)=\mathfrak{D}\dot{+}\mathfrak{R}$, that $A\mathfrak{D}\perp A\mathfrak{R}$, and put $\mathfrak{R}=A^{-1}(\mathcal{H}\ominus A\mathfrak{D})$, and due to (1) to also have $A^*\mathfrak{D}\perp A^*\mathfrak{R}$.

Lemma 1. *Let a normal operator $\tilde{A}$ be a finite rank singular perturbation of a normal operator A. Then 1) $A\mathfrak{R}\cap\mathfrak{D}(A)=\{0\}$; 2) $\tilde{A}^*\restriction_{\mathfrak{D}}=A^*\restriction_{\mathfrak{D}}$; 3) There exist unique linear operators T and T_* in the subspace $\mathfrak{R}$ such that*

$$\mathfrak{D}(\tilde{A})=\mathfrak{D}\dot{+}(A^*+T)\mathfrak{R}=\mathfrak{D}\dot{+}(A+T_*)\mathfrak{R}, \tag{2}$$

and the operators $\tilde{A}$ and $\tilde{A}^*$ act as follows:

$$\tilde{A}(x_0+(A^*+T)r)=Ax_0+ATr,\quad \tilde{A}^*(x_0+(A+T_*)r)=A^*x_0+A^*T_*r, \tag{3}$$

where $x_0\in\mathfrak{D}$ and $r\in\mathfrak{R}$. The operators T and T_* satisfy the relations $AT_*A^{-1}=(ATA^{-1})^*$, $A^*T_*(A^*)^{-1}=(A^*T(A^*)^{-1})^*$ on the subspaces $\mathfrak{N}=A\mathfrak{R}$ and $\mathfrak{M}=A^*\mathfrak{R}$, respectively.

Proof. The conditions 1) and 2) follow from (1) and from density of the set $\mathfrak{D}$ in $\mathcal{H}$. If for some vector we had $r\in\mathfrak{R}$, $r\neq 0$, $Ar\in\mathfrak{D}(A)=\mathfrak{D}(A^*)$, then we would obtain that the vector $A^*(Ar)\neq 0$ would be orthogonal to $\mathfrak{D}$, $(\varphi,A^*(Ar))=(A\varphi,Ar)=0$. But this is impossible, since the set $\mathfrak{D}$ is dense in $\mathcal{H}$.

Since $\mathfrak{D}(\tilde{A}^*)=\mathfrak{D}(A)\supset\mathfrak{D}$ and $\tilde{A}\restriction_{\mathfrak{D}}=A\restriction_{\mathfrak{D}}$, for an arbitrary $\varphi,\psi\in\mathfrak{D}$, $0=(\varphi,\tilde{A}\psi-A\psi)=(\tilde{A}^*\varphi-A^*\varphi,\psi)$, and using density of $\mathfrak{D}$ in $\mathcal{H}$ we have $\tilde{A}^*\varphi=A^*\varphi$.

Let $\tilde{x}\in\mathfrak{D}(\tilde{A})$ and $x=A^{-1}\tilde{A}\tilde{x}=x_0+r$, where $x_0\in\mathfrak{D}$ and $r\in\mathfrak{R}$. Since $\tilde{A}\tilde{x}-Ax=0$, for arbitrary $\varphi\in\mathfrak{D}$, $0=(\tilde{A}\tilde{x}-Ax,\varphi)=(\tilde{x}-x,A^*\varphi)$. Hence, $\tilde{x}-x\perp A^*\mathfrak{D}$ and, consequently, there exists an element $\rho\in\mathfrak{R}$ such that $\tilde{x}=x+A^*\rho$. Taking into account that $x=x_0+r$ and using the element $\tilde{x}\in\mathfrak{D}(\tilde{A})$ we construct two elements $r,\rho\in\mathfrak{R}$ connected via the linear operator T in $\mathfrak{R}$, $r=T\rho$.

The action of the operator $\tilde{A}$ on the elements $\tilde{x}=x_0+Tr+A^*r$ is determined by the action of the operator A on the element $x=x_0+Tr$, which gives (3) for the operator $\tilde{A}$. Analogously we prove (2)–(3) for the operator $\tilde{A}^*$. Since $(\tilde{A}x,y)=(x,\tilde{A}^*y)$, using (2) we put $x=Tr+A^*r$, $y=T_*\rho+A\rho$, where $r,\rho\in\mathfrak{R}$, and taking into account (3) we obtain $(ATr,A\rho)=(A^*r,A^*T_*\rho)$. Then using (1) we obtain the relations for operators T_* and T. □

Lemma 2. *Let a normal operator $\tilde{A}$ be a singular perturbation of rank one with respect to the normal operator A. Then there exist real numbers θ and ξ, $-\frac{\pi}{2}<\theta\leq\frac{\pi}{2}$, $\xi\in\mathbb{R}^1$, such that vectors $r\in\mathfrak{R}$ have the property*

$$e^{-i\theta}Ar-e^{i\theta}A^*r=2i\xi r. \tag{4}$$

Proof. The subspace $\mathfrak{R}$ is one dimensional and the operators T and T_* from (2) are operators of multiplication by the numbers t and $\bar{t}$, respectively. Due to (2) we can represent the vector $\tilde{x}=tr+A^*r\in\mathfrak{D}(\tilde{A})$, $r\in\mathfrak{R}$, in the form $\tilde{x}=x_0+\bar{t}pr+pAr$, where $x_0\in\mathfrak{D}$ and p is a complex number. Thus $\tilde{A}\tilde{x}=tAr$, $\tilde{A}^*\tilde{x}=A^*x_0+\bar{t}pA^*r$. Hence, $0=(\tilde{A}x_0,\tilde{A}\tilde{x})=(\tilde{A}^*x_0,\tilde{A}^*\tilde{x})=(\tilde{A}^*x_0,\tilde{A}^*x_0)$, i.e, $x_0=0$. From the another side, $(\tilde{A}\tilde{x},\tilde{A}\tilde{x})=(\tilde{A}^*\tilde{x},\tilde{A}^*\tilde{x})$, which gives $p=e^{-2i\theta}$, where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2}]$. The equality $\tilde{x}=tr+A^*r=e^{-2i\theta}(\bar{t}r+Ar)$ gives (4), where $\xi=\frac{1}{2i}(te^{i\theta}-\bar{t}e^{-i\theta})$. □

Definition 1. *A vector $r \in \mathfrak{D}(A)$ such that $Ar \notin \mathfrak{D}(A)$ is called admissible for the operator A with characteristics θ, ξ if the equality (4) is true, i.e., r is an eigenvector of the operator $Im\,(e^{-i\theta}A)$ with the eigenvalue ξ.*

Theorem 1. *For the set $\mathcal{P}_1(A)$ of all normal operators $\tilde{A} \neq A$ that are rank one singular perturbations of the normal operator A to be nonempty, it is necessary and sufficient that there existed an admissible vector r for the operator A.*

If r is an admissible vector with characteristics θ, ξ, and τ is an arbitrary real number, then the operator $\tilde{A}_{r,\tau}$ that has the domain

$$\mathfrak{D}(\tilde{A}_{r,\tau}) = \mathfrak{D} \dot{+} \{e^{-i\theta}(\tau + i\xi)\rho + A^*\rho\}, \quad \mathfrak{D} = A^{-1}(H \ominus \{Ar\}), \quad \rho = ar, \tag{5}$$

and acts by

$$\tilde{A}_{r,\tau}(x_0 + e^{-i\theta}(\tau + i\xi)\rho + A^*\rho) = Ax_0 + e^{-i\theta}(\tau + i\xi)A\rho, \quad x_0 \in \mathfrak{D}, \tag{6}$$

belongs to the set $\mathcal{P}_1(A)$.

Each operator $\tilde{A} \in \mathcal{P}_1(A)$ admits the representation (5), (6). Such a representation defines a one-to-one correspondence between the set $\mathcal{P}_1(A)$ and the set of pairs $\{\mathfrak{R}, \tau\}$, where $\mathfrak{R}$ is the one-dimensional subspace spanned by the admissible vector and τ is a self-adjoint operator in $\mathfrak{R}$.

The operator $\tilde{A}_{r,\tau}$ has bounded inverse iff $\tau + i\xi \neq 0$, and

$$\tilde{A}_{r,\tau}^{-1} = A^{-1} + e^{i\theta}(\tau + i\xi)^{-1}(\cdot, Ar)A^*r. \tag{7}$$

Proof. The necessity follows from Lemma 2. By a direct calculation we obtain that the operator $\tilde{A}_{r,\tau}$ defined in (5), (6) is normal, $\tilde{A}_{r,\tau} \in \mathcal{P}_1(A)$, and representation (7) takes place. $\square$

Theorem 1 has the following equivalent form.

Theorem 2. *The set $\mathcal{P}_1(A)$ is nonempty iff the normal operator A has a cyclic invariant subspace $\tilde{\mathcal{H}}$, where the operator A is a linear function of an unbounded self-adjoint operator L in $\tilde{\mathcal{H}}$.*

Proof. If $\mathcal{P}_1(A) \neq \varnothing$, then there exists an admissible vector r with the characteristics θ, ξ. Let $\tilde{H} = H(r)$ be the cyclic invariant subspace of the operator A generated by the vector r [1]. Since $\tilde{H}$ reduces the operator A and $Ar \notin \mathfrak{D}(A)$, the operator $L = \frac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*)$ is an unbounded self-adjoint operator on $\tilde{H}$, and the operator $A \restriction_{\tilde{H}} = e^{i\theta}(L + i\xi I)$ is a linear function of L.

Sufficiency follows from the fact that the unbounded self-adjoint operator L has singular perturbations of an arbitrary rank [7]. $\square$

Corollary 1. *If the intersection of an arbitrary line on the complex plane and the spectrum of the operator A is a bounded or an empty set, then the normal operator A does not have a singular perturbation of the rank one, that is, $\mathcal{P}_1(A) = \varnothing$.*

$\mathbf{2^0}$. We give a description of operators $\tilde{A} \in \mathcal{P}_1(A)$ in terms of a scale of Hilbert spaces, $\mathcal{H}_s \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-s}$, $s > 0$, constructed from the positive self-adjoint operator $|A| = (AA^*)^{1/2} = (A^*A)^{1/2}$ [12]. In this construction, we set $\mathcal{H}_s = \mathfrak{D}(|A|^{s/2})$, $\|u\|_s = \||A|^{s/2}u\|$, and $\mathcal{H}_{-s}$ is the closure of $\mathcal{H}_0$ in the norm $\|u\|_{-s} = \||A|^{-s/2}u\|$. Let us remark that $\mathcal{H}_2 = \mathfrak{D}(A) = \mathfrak{D}(A^*)$, and the operators A and A^* have extensions by continuity to $\mathbf{A}$ $\mathbf{A}^*$, being isometric operators acting from all $\mathcal{H} = \mathcal{H}_0$ into all $\mathcal{H}_{-2}$. The scalar product $(\cdot, \cdot)$ in $\mathcal{H}$ can be extended by continuity to a pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{H}_s$ and $\mathcal{H}_{-s}$. If $\mathbf{B}$ is an operator acting from $\mathcal{H}$ into $\mathcal{H}_{-2}$, then its restriction B to the space $\mathcal{H}$ is defined on the set $\mathfrak{D}(B) = \{u \in \mathfrak{D}(\mathbf{B}),\ \mathbf{B}u \in \mathcal{H}\}$: $B = \mathbf{B} \restriction_{\mathcal{H}} = \mathbf{B} \restriction_{\mathfrak{D}(B)}$. If $\mathbf{B} \neq B$, then we call the operator $\mathbf{B}$ singular on $\mathcal{H}$.

For each element $\psi \in \mathcal{H}_{-2} \setminus \mathcal{H}$ and a complex number $z \neq 0$, we can construct a one-dimensional singular operator $\mathbf{V} = z\langle \cdot, \psi\rangle\psi$, acting from $\mathcal{H}_2$ into $\mathcal{H}_{-2}$. This operator is defined on all $\mathcal{H}_2$.

If $\psi \in \mathcal{H}_{-1}$, then we continue the functional $\langle \cdot, \psi\rangle$ to the whole $\mathcal{H}_1$ and, in such a case, the operator $\mathbf{V}$ is defined on the elements $\mathbf{A}^{-1}\psi$, $(\mathbf{A}^*)^{-1}\psi$ and, hence, it is defined on all elements $\varphi + c_1\mathbf{A}^{-1}\psi + c_2(\mathbf{A}^*)^{-1}\psi$, where $\varphi \in \mathcal{H}_2$, $c_1, c_2 \in \mathbb{C}$.

Let $\psi \in \mathcal{H}_{-2}$, $\|\psi\|_{-2} = 1$, $\psi = (\mathbf{A}\mathbf{A}^*)r$, where r is an admissible vector with characteristics θ, ξ of the operator A, according to Definition 1. In this case, to extend the operator $\mathbf{V}$ to all elements of the form $\varphi + c_1\mathbf{A}^{-1}\psi + c_2(\mathbf{A}^*)^{-1}\psi$, it is sufficient to extend the functional $\langle \cdot, \psi\rangle$ to the elements $\omega = \frac{1}{2}(e^{i\theta}\mathbf{A}^{-1}\psi + e^{-i\theta}(\mathbf{A}^*)^{-1}\psi)$ using the real number γ, $\langle \omega, \psi\rangle_\gamma = \gamma$. Then the extended operator $\mathbf{V}_\gamma$ is defined on the elements $\mathbf{A}^{-1}\psi$ and $(\mathbf{A}^*)^{-1}\psi$ and has the value

$$\mathbf{V}_\gamma(\mathbf{A}^{-1}\psi) = ze^{-i\theta}(\gamma - i\xi)\psi, \quad \mathbf{V}_\gamma((\mathbf{A}^*)^{-1}\psi) = ze^{i\theta}(\gamma + i\xi)\psi.$$

Such an extension of the operator $\mathbf{V}$ will be called a regularisation and the real number γ is called a parameter of the regularisation. If $\psi \in \mathcal{H}_{-1}$, the parameter of the regularisation $\gamma = \langle \omega, \psi\rangle$ is defined by continuity. If $\psi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, we can choose the parameter of regularisation to be arbitrary or in such a way that the operator $\mathbf{V}$ would have some symmetry properties [13].

Definition 2. *The sum of a normal operator A and a one-dimensions regularised singular operator $\mathbf{V}_\gamma$, in the space $\mathcal{H}$, we understand an operator $\tilde{A}$ that is the restriction of the operator $\mathbf{A} + \mathbf{V}_\gamma$, $\tilde{A} = (\mathbf{A} + \mathbf{V}_\gamma) \restriction_{\mathcal{H}} = A + z\langle \cdot, \psi\rangle_\gamma\psi$.*

Theorem 3. *Let $\psi \in \mathcal{H}_{-2}$, $\|\psi\|_{-2} = 1$, $\psi = (\mathbf{A}\mathbf{A}^*)r$, and r be an admissible vector of the operator A with characteristics θ, ξ. Let a real number γ be the parameter of regularisation of the one-dimensional operator constructed from the vector ψ, and suppose that a real number $\lambda \neq 0$. Then*

$$\tilde{A}_{\psi,\lambda} = A + e^{i\theta}\lambda\langle \cdot, \psi\rangle_\gamma\psi \in \mathcal{P}_1(A). \tag{8}$$

If $\tau = -\lambda^{-1} - \gamma$, then the operator $\tilde{A}_{\psi,\lambda}$ coincides with the operator $\tilde{A}_{r,\tau}$ defined in Theorem 1.

Proof. The domain of the operator $\tilde{A}_{\psi,\lambda}$ consist of vectors of the form $\tilde{x} = x_0 + c_1 r + c_2(\mathbf{A})^{-1}\psi$ such that $\tilde{A}_{\psi,\lambda}\tilde{x} \in \mathcal{H}$. It gives the condition $c_1 = e^{-i\theta}(-\lambda^{-1} - \gamma + i\xi)c_2$. The comparison the operator $\tilde{A}_{\psi,\lambda}$ with the operator $\tilde{A}_{r,\tau}$ from the Theorem 1 shows that for $\tau = -\lambda^{-1} - \gamma$ the operator $\tilde{A}_{r,\tau}$ is equals to the operator $\tilde{A}_{\psi,\lambda}$. What is more with this connection if $\tau + \gamma = 0$, then $\lambda = \infty$, and the operator $\tilde{A}_{\psi,\infty}$ in (8) we understand as $\tilde{A}_{r,\tau}$, with $\tau = -\gamma$ [14]. Let us remark that

$$\tilde{A}^*_{\psi,\lambda} = A^* + e^{-i\theta}\lambda\langle \cdot, \psi\rangle_\gamma\psi.$$

□

3^0. Let us consider the singular perturbation of an arbitrary finite rank. If the normal operator $\tilde{A}$ is a singular rank n perturbation of the normal operator A, then, due to the equality $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(\tilde{A}^*)$ and Lemma 1, we conclude that $\mathfrak{R} \dot{+} A\mathfrak{R} = \mathfrak{R} \dot{+} A^*\mathfrak{R}$. The last equality takes place if and only if there exist linear operators P and Q on the n-dimensional subspace $\mathfrak{R}$ such that the operator P has an inverse, and

$$Ar = A^*Pr + Qr, \quad r \in \mathfrak{R}. \tag{9}$$

Definition 3. *The finite dimensional subspace $\mathfrak{R} \subset \mathfrak{D}(A)$, $A\mathfrak{R} \cap \mathfrak{D}(A) = \{0\}$, is called (P, Q)-admissible for the normal operator A if there exist linear operators P, P^{-1} and Q in $\mathfrak{R}$ such that the equality (9) holds true.*

From the Lemma 1, we have that the absence of an n-dimensional admissible subspace of the normal operator A is a sufficient condition for the absence of the normal singular perturbation of rank n.

Let us consider an example. Let $p_{n+1}(t) = (t-z_1)(t-z_2)\cdots(t-z_{n+1})$ be a polynomial of the degree $n+1$, where we choose the numbers z_j so that all the numbers $\alpha_j = p_{n+1}(\bar{z}_j)$, $j = 1, 2, \ldots, n+1$, are distinct. Then the operator A of multiplication by $p_{n+1}(t)$ is normal in the space $L_2([1, \infty))$ and does not have a singular normal perturbation of the rank less than $n+1$. Indeed, if there existed a singular perturbation of the rank $m \leq n$, the m-dimensional subspace $\mathfrak{R}$ would be admissible for the operator A. And due to the equality (9), we would immediately have that $\det[p_{n+1}(t)I - \bar{p}_{n+1}(t)P - Q] \equiv 0$. Putting $t = \bar{z}_j$, $j = 1, 2, \ldots, n+1$, we would obtain the identity $\det[\alpha_j I - Q] = 0$, i.e., the operator Q would have $n+1$ different eigenvalues in the m-dimensional subspace $\mathfrak{R}$, which is impossible.

The proposed example has a generalization. The range of values of the constructed polynomial $p_{n+1}(t)$, $t \in \mathbb{R}$ is an algebraic curve γ_{n+1} in the complex plane. *If the spectrum of the normal operator A belongs to the algebraic curve γ_{n+1}, then the operator A does not have a normal singular perturbation of the rank less than $n+1$.*

In particular, the last observation gives the following. *If a Borel measure μ on the complex plane $\mathbb{C}$ is supported on an algebraic curve γ_{n+1}, $(n \geq 2)$, and all the complex moments $c_{k,m} = \int z^k \bar{z}^m d\mu$ are finite, then such an integral representation is unique for the sequence $c_{k,m}$, $(k, m = 0, 1, 2, \ldots)$.*

The operator T in Lemma 1 is connected with characteristics (P, Q) of the admissible subspace $\mathfrak{R}$. For a description of this connection, it is convenient to translate the operators T, P, Q acting on the subspace $\mathfrak{R}$ to act on the subspace $\mathfrak{N} = A\mathfrak{R}$ by using the identities $\hat{T} = ATA^{-1}$, $\hat{P} = APA^{-1}$ and $\hat{Q} = AQA^{-1}$. Then from (2) and (9), we obtain

$$\hat{T}\hat{P} - \hat{T}^* = \hat{Q}. \tag{10}$$

And from the condition that $\|\tilde{A}\psi\| = \|\tilde{A}^*\psi\|$ and from (3) we get

$$\|\hat{T}^* n\| = \|\hat{T}\hat{P}n\|, \quad n \in \mathfrak{N} = A\mathfrak{R}. \tag{11}$$

Definition 4. *The operator T acting on a (P, Q)-admissible subspace $\mathfrak{R}$ of the normal operator A is called admissible on $\mathfrak{R}$ if the identities (10) and (11) hold.*

Theorem 4. *Relations (2), (3) establish a bijection between the set of all singular perturbations of finite rank of the normal operator A and the set of the pairs $\{\mathfrak{R}, T\}$, where $\mathfrak{R}$ is an admissible subspace of the operator A and T is an admissible operator on $\mathfrak{R}$.*

Proof. Each normal operator $\tilde{A}$, which is a singular perturbation of finite rank of a normal operator A, uniquely generates, due the Lemma 1, a (P, Q)-admissible subspace $\mathfrak{R}$ and an admissible operator T on $\mathfrak{R}$.

The converse is proved immediately using the construction of operators $\tilde{A}$ and $\tilde{A}^*$ given in (2) and (3). □

Let us remark that the subspace $\mathfrak{R}$ is admissible for a strictly positive self-adjoint operator A if and only if $\mathfrak{R} = A^{-1}\mathfrak{N}$, where the finite dimensional subspace $\mathfrak{N}$ satisfies the condition $\mathfrak{N} \cap \mathfrak{D}(A) = \{0\}$. In such a case, the operator T in $\mathfrak{R}$ is admissible if and only if the operator $\hat{T} = ATA^{-1}$ is self-adjoint on $\mathfrak{N}$ [7].

The description of a singularly perturbed normal operator, proposed in this paper, is more effective than the description given in [2]–[6], where one tries to describe all normal extensions of a formally normal operator.

Acknowledgments. The authors express their sincere gratitude to Prof. Yu. M. Berezansky for helpful discussions and suggestions.

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National Technical University of Ukraine (KPI), 37 Peremogy Av., Kyiv, 03056, Ukraine
E-mail address: `dudkin@imath.kiev.ua`

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: `nizhnik@imath.kiev.ua`

Received 02/12/2009