

ON J -SELF-ADJOINT EXTENSIONS OF THE PHILLIPS SYMMETRIC OPERATOR

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Dedicated to the blessed memory of I. Gohberg.

ABSTRACT. J -self-adjoint extensions of the Phillips symmetric operator S are studied. The concepts of stable and unstable C -symmetry are introduced in the extension theory framework. The main results are the following: if A is a J -self-adjoint extension of S , then either $\sigma(A) = \mathbb{R}$ or $\sigma(A) = \mathbb{C}$; if A has a real spectrum, then A has a stable C -symmetry and A is similar to a self-adjoint operator; there are no J -self-adjoint extensions of the Phillips operator with unstable C -symmetry.

1. INTRODUCTION

Let \mathfrak{H} be a Hilbert space with inner product (\cdot, \cdot) and with fundamental symmetry J (i.e., $J = J^*$ and $J^2 = I$). The space \mathfrak{H} endowed with the indefinite inner product (indefinite metric) $[x, y]_J := (Jx, y)$, $\forall x, y \in \mathfrak{H}$ is called a Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$.

An operator A in \mathfrak{H} is called J -self-adjoint if A is self-adjoint with respect to the indefinite metric $[\cdot, \cdot]_J$. It is clear that A is J -self-adjoint if and only if

$$(1.1) \quad A^*J = JA.$$

During the past ten years a steady interest in the study of J -self-adjoint operators has been strongly increased by the necessity of mathematically correct and rigorous analysis of pseudo-Hermitian Hamiltonians arising in \mathcal{PT} -symmetric quantum mechanics (PTQM) see e.g. [10]–[19], [32, 35, 38].

In many cases, pseudo-Hermitian Hamiltonians admit the representation $A + V$, where a (fixed) self-adjoint operator A and a non-symmetric potential V satisfy certain (Krein space) symmetry properties which allow one to formalize the expression $A + V$ as a family of J -self-adjoint¹ operators A_ε acting in a Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$. Here $\varepsilon \in \mathbb{C}^m$ is a complex parameter characterizing the potential V .

Let Ξ be the domain of variation of ε . One of important problems for the collection $\{A_\varepsilon\}$, which is directly inspired by PTQM, is the description of quantitative and qualitative changes of spectra $\sigma(A_\varepsilon)$ when ε runs Ξ . Nowadays this topic has been analyzed with a wealth of technical tools (see, e.g., [7, 8, 21, 24, 39]).

In particular, if the potential V is singular, then operators A_ε turn out to be J -self-adjoint extensions of the symmetric operator $S = A \upharpoonright \ker V$ which commutes with J and spectral analysis of A_ε can be carried out by the extension theory methods [2, 3, 4, 22]. Here, the ‘main ingredients’ are: a holomorphic operator function characterizing S (the characteristic function $\Theta(\cdot)$ [26, 28, 37] or the Weyl function $M(\cdot)$ [16, 17, 18]) and the boundary conditions which distinguish A_ε among other J -self-adjoint extensions of S . In

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¹Under a special choice of involution J .

such a setting, the spectral analysis of A_ε is reduced to the routine solution of algebraic equations including $\theta(\cdot)$ and boundary conditions.

In the present paper we are going to study a special case where the characteristic function of a symmetric operator S with finite deficiency indices is equal to zero ($\Theta(\mu) \equiv 0$, $\forall \mu \in \mathbb{C} \setminus \mathbb{R}$).

One of general constructions leading to symmetric operators S with the zero characteristic function is the following: let U be a bilateral shift with a wandering subspace W_0 in \mathfrak{H} (see [20] for the terminology) and let V be its restriction onto $\mathfrak{H} \ominus W_0$, i.e., $V = U \upharpoonright (\mathfrak{H} \ominus W_0)$. Then the operator

$$(1.2) \quad S = i(V + I)(V - I)^{-1}, \quad \mathcal{D}(S) = \mathcal{R}(V - I)$$

is simple² symmetric and its deficiency indices coincide with $\langle \dim W_0, \dim W_0 \rangle$.

In other words, S is the restriction of the Cayley transform of U

$$(1.3) \quad A = i(U + I)(U - I)^{-1}, \quad \mathcal{D}(A) = \mathcal{R}(U - I)$$

onto $\mathcal{D}(S) = \mathcal{R}(V - I)$.

The operator S defined by (1.2), (1.3) was used by Phillips [36] (with $\dim W_0 = 1$) as an example of the symmetric operator, which is invariant with respect to a certain set \mathfrak{U} of unitary operators (\mathfrak{U} -invariant) but it has no \mathfrak{U} -invariant self-adjoint extensions. For this reason, the simple symmetric operator S determined by (1.2) and (1.3) will be referred as the *Phillips symmetric operator*.

Due to specific properties of the Phillips operator (the characteristic function is zero, there are no real points of regular type of S , etc) we obtain an evolution of $\sigma(A_\varepsilon)$ which differs from the matrix models [21, 24, 25] and models based on J -self-adjoint (symmetric) perturbations of the Schrödinger or Dirac operator [5, 15, 32, 39]. For instance, in our case, either the spectrum of an J -self-adjoint extension A_ε of S coincides with real line: $\sigma(A_\varepsilon) = \mathbb{R}$ or with complex plane: $\sigma(A_\varepsilon) = \mathbb{C}$ (Theorem 3.7).

One of the key points in PTQM is the description of a hidden symmetry C which exists for a given pseudo-Hermitian Hamiltonian A in the sector of exact \mathcal{PT} -symmetry [9, 10, 11]. The operator C has some rough analogy with the charge conjugation operator in the quantum field theory [10] and it is determined non-uniquely [13]. The existence of C gives rise to an inner product $(\cdot, \cdot)_C = [C\cdot, \cdot]_J$ and the dynamics generated by A is therefore governed by a unitary time evolution.

For J -self-adjoint extensions $A_\varepsilon \supset S$, where S is an *arbitrary* symmetric operator commuting with J , we introduce the concepts of stable and unstable C -symmetry (Definition 2.11). These concepts are natural in the extension theory framework. Roughly speaking, if A_ε belongs to the sector Σ_J^{st} of stable C -symmetry, then A_ε preserves the property of C -symmetry under small variation of ε .

It follows from the results of [1, 23] that for some types of singular perturbations of the Schrödinger or the Dirac operator, the sector Σ_J^{unst} of unstable C symmetry is not empty and operators A_ε with real spectra and Jordan points correspond to the case where ε belongs to the boundary of Σ_J^{st} .

In the case of the Phillips symmetric operator S , the spectral picture above can be essentially simplified. Precisely, assuming the deficiency indices $\langle 2, 2 \rangle$ of S , we show that $\Sigma_J^{\text{unst}} = \emptyset$ and any J -self-adjoint extension of S with real spectrum is similar to a self-adjoint operator (Theorem 3.9 and Corollary 3.10).

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of a linear operator A , respectively, while $A \upharpoonright \mathcal{D}$ stands for the restriction of

²An operator is called *simple* if its restriction to any nontrivial reducing subspace is not a self-adjoint operator.

A to the set \mathcal{D} . The set of points of regular type of a symmetric operator S is denoted by $\widehat{\rho}(S)$ (i.e., $r \in \widehat{\rho}(S) \iff \|(S - rI)u\| \geq k\|u\|, \forall u \in \mathcal{D}(S), k > 0$).

2. PRELIMINARIES

2.1. Elements of the Krein space theory. Let $(\mathfrak{H}, [\cdot, \cdot]_J)$ be a Krein space with fundamental symmetry J . The corresponding orthoprojectors $P_{\pm} = \frac{1}{2}(I \pm J)$ determine the fundamental decomposition of \mathfrak{H}

$$(2.1) \quad \mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \quad \mathfrak{H}_- = P_- \mathfrak{H}, \quad \mathfrak{H}_+ = P_+ \mathfrak{H}.$$

A subspace \mathfrak{L} of \mathfrak{H} is called *hypermaximal neutral* if

$$\mathfrak{L} = \mathfrak{L}^{[\perp]_J} = \{x \in \mathfrak{H} : [x, y]_J = 0, \forall y \in \mathfrak{L}\}.$$

A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called *uniformly positive (uniformly negative)* if $[x, x]_J \geq a^2\|x\|^2$ ($-[x, x]_J \geq a^2\|x\|^2$) $a \in \mathbb{R}$ for all $x \in \mathfrak{L}$. The subspaces \mathfrak{H}_{\pm} in (2.1) are examples of uniformly positive and uniformly negative subspaces and they possess the property of maximality in the corresponding classes (i.e., \mathfrak{H}_+ (\mathfrak{H}_-) does not belong as a subspace to any uniformly positive (negative) subspace).

Let $\mathfrak{L}_+ (\neq \mathfrak{H}_+)$ be an arbitrary maximal uniformly positive subspace. Then its J -orthogonal complement $\mathfrak{L}_- = \mathfrak{L}_+^{[\perp]_J}$ is a maximal uniformly negative and the direct J -orthogonal sum

$$(2.2) \quad \mathfrak{H} = \mathfrak{L}_+ [+]_J \mathfrak{L}_-$$

gives another (then (2.1)) decomposition of \mathfrak{H} onto its positive \mathfrak{L}_+ and negative \mathfrak{L}_- parts (the brackets $[\cdot]_J$ mean the orthogonality with respect to the indefinite metric).

An arbitrary decomposition of the Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$ onto its positive and negative parts (like (2.2)) is called *canonical*.

The subspaces \mathfrak{L}_{\pm} in (2.2) can be described as

$$\mathfrak{L}_+ = (I + X)\mathfrak{H}_+, \quad \mathfrak{L}_- = (I + X^*)\mathfrak{H}_-,$$

where $X : \mathfrak{H}_+ \rightarrow \mathfrak{H}_-$ is a contraction and $X^* : \mathfrak{H}_- \rightarrow \mathfrak{H}_+$ is the adjoint of X .

The self-adjoint operator $T = XP_+ + X^*P_-$ acting in \mathfrak{H} is called *an operator of transition* from the fundamental decomposition (2.1) to the canonical one (2.2). Obviously, $\mathfrak{L}_+ = (I + T)\mathfrak{H}_+$ and $\mathfrak{L}_- = (I + T)\mathfrak{H}_-$.

Operators of transition admit a simple description. Namely, a self-adjoint operator T in \mathfrak{H} is an operator of transition if and only if $\|T\| < 1$ and $JT = -TJ$.

The set $\{T\}$ of all possible operators of transition is in one-to-one correspondence (via $\mathfrak{L}_{\pm} = (I + T)\mathfrak{H}_{\pm}$) with all possible canonical decompositions (2.2) of the Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$.

The projectors $P_{\mathfrak{L}_{\pm}} : \mathfrak{H} \rightarrow \mathfrak{L}_{\pm}$ onto \mathfrak{L}_{\pm} with respect to the decomposition (2.2) are determined by the formulas

$$P_{\mathfrak{L}_-} = (I - T)^{-1}(P_- - TP_+), \quad P_{\mathfrak{L}_+} = (I - T)^{-1}(P_+ - TP_-).$$

The bounded operator

$$(2.3) \quad C = P_{\mathfrak{L}_+} - P_{\mathfrak{L}_-} = J(I - T)(I + T)^{-1}$$

also describes subspaces \mathfrak{L}_{\pm} in (2.2)

$$(2.4) \quad \mathfrak{L}_+ = \frac{1}{2}(I + C)\mathfrak{H}, \quad \mathfrak{L}_- = \frac{1}{2}(I - C)\mathfrak{H}.$$

The set of operators C determined (2.3) is completely characterized by the conditions

$$(2.5) \quad C^2 = I, \quad JC > 0.$$

2.2. Elements of the Von Neumann extension theory. Let S be a closed symmetric densely defined operator in a Hilbert space \mathfrak{H} with equal (finite or infinite) deficiency indices. Denote by $\mathfrak{N}_i = \mathfrak{H} \ominus \mathcal{R}(S - iI)$ and $\mathfrak{N}_{-i} = \mathfrak{H} \ominus \mathcal{R}(S + iI)$ the defect subspaces of S and consider the Hilbert space $\mathfrak{M} = \mathfrak{N}_{-i} \dot{+} \mathfrak{N}_i$ with the inner product

$$(f, g)_{\mathfrak{M}} = (f_i, g_i) + (f_{-i}, g_{-i}) \quad f = f_i + f_{-i}, \quad g = g_i + g_{-i} \quad \{f_{\pm i}, g_{\pm i}\} \subset \mathfrak{N}_{\pm i}.$$

The operator $Z(f_{-i} + f_i) = f_{-i} - f_i$ is a fundamental symmetry in the Hilbert space \mathfrak{M} and its restriction onto \mathfrak{N}_{-i} and \mathfrak{N}_i coincide, respectively, with I and $-I$.

Let J be a fundamental symmetry in \mathfrak{H} . In what follows we assume that

$$(2.6) \quad SJ = JS.$$

Then the subspaces $\mathfrak{N}_{\pm i}$ reduce J and the restriction $J \upharpoonright \mathfrak{M}$ gives rise to a fundamental symmetry in the Hilbert space \mathfrak{M} . Moreover, according to the properties of Z mentioned above, $JZ = ZJ$. Therefore, JZ is a fundamental symmetry in \mathfrak{M} and sesquilinear form

$$[f, g]_{JZ} = (JZf, g)_{\mathfrak{M}} = (Jf_{-i}, g_{-i}) - (Jf_i, g_i)$$

determines an indefinite metric on \mathfrak{M} .

According to von-Neumann formulas any closed intermediate extension A of S (i.e., $S \subset A \subset S^*$) is uniquely determined by the choice of a subspace $M \subset \mathfrak{M}$. Precisely,

$$(2.7) \quad \mathcal{D}(A) = \mathcal{D}(S) \dot{+} M \quad \text{and} \quad A = S^* \upharpoonright \mathcal{D}(A).$$

We use the notation A_M for J -self-adjoint extensions of S determined by (2.7).

Let A_M and $A_{\widetilde{M}}$ be arbitrary extensions of S that are defined by the subspaces M and \widetilde{M} , respectively. Taking (2.6) and (2.7) into account we derive

$$(2.8) \quad [A_M \psi, \phi]_J - [\psi, A_{\widetilde{M}} \phi]_J = 2i[f, g]_{JZ}$$

for all $\psi = u + f \in \mathcal{D}(A_M)$, $f \in M$, $\phi = v + g \in \mathcal{D}(A_{\widetilde{M}})$, $g \in \widetilde{M}$.

It follows from (1.1) and (2.8) that an extension A_M of S is J -self-adjoint if and only if

$$M = M^{[\perp]_{JZ}} = \{f \in \mathfrak{M} : [f, g]_{JZ} = 0, \forall g \in M\},$$

i.e., if M is a hypermaximal neutral subspace of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_{JZ})$. Formalizing this observation we get the well-known result.

Proposition 2.1. *The correspondence $A \leftrightarrow M$ determined by (2.7) is a bijection between J -self-adjoint (self-adjoint) extensions A of S and hypermaximal neutral subspaces M of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_{JZ})$ (of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_J)$).*

Denote by $\Sigma_J(S)$ the set of J -self-adjoint extensions of S . In general, these extensions may have complex spectra and, moreover, the existence of $A \in \Sigma_J(S)$ with empty resolvent set (i.e., $\sigma(A) = \mathbb{C}$) is also possible. To guarantee nonempty resolvent set for any $A \in \Sigma_J(S)$ we need to impose additional constraints. In this way we recall that a J -self-adjoint operator A is called *definitizable* if the resolvent set of A is nonempty and there exists a polynomial $p(\cdot) \not\equiv 0$ such that $p(A)$ is a nonnegative operator in the Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$.

Proposition 2.2. ([6]). *Let S have finite deficiency indices. Then if there exists a definitizable extension $A \in \Sigma_J(S)$, then an arbitrary operator from $\Sigma_J(S)$ has a nonempty resolvent set and is definitizable.*

2.3. Boundary value spaces technique. Proposition 2.1 provides a description of $\Sigma_J(S)$ in terms of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_Z)$. Another approach which allows one to avoid the use of \mathfrak{M} is based on the concept of *boundary triplets* (or boundary value spaces, see [22] and the references therein).

Definition 2.3. A triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where \mathcal{H} is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(S^*)$ into \mathcal{H} , is called a *boundary triplet of S^** if the abstract Green identity

$$(S^*\psi, \phi) - (\psi, S^*\phi) = (\Gamma_1\psi, \Gamma_0\phi)_{\mathcal{H}} - (\Gamma_0\psi, \Gamma_1\phi)_{\mathcal{H}}, \quad \psi, \phi \in \mathcal{D}(S^*)$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : \mathcal{D}(S^*) \rightarrow \mathcal{H} \oplus \mathcal{H}$ is surjective.

Denote

$$(2.9) \quad \mathfrak{N}_\mu = \mathfrak{H} \ominus \mathcal{R}(S - \mu I) = \ker(S^* - \bar{\mu}I), \quad \mu \in \widehat{\rho}(S).$$

The Weyl function $M(\cdot)$ and the characteristic function $\Theta(\cdot)$ of S associated with a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ are defined as follows [18, 27, 37]:

$$(2.10) \quad \begin{aligned} M(\mu)\Gamma_0 f_{\bar{\mu}} &= \Gamma_1 f_{\bar{\mu}}, \quad \forall f_{\bar{\mu}} \in \mathfrak{N}_{\bar{\mu}}, \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}, \\ \Theta(\mu)(\Gamma_1 + i\Gamma_0)f_{\bar{\mu}} &= (\Gamma_1 - i\Gamma_0)f_{\bar{\mu}}, \quad \forall \mu \in \mathbb{C}_+. \end{aligned}$$

It is clear that $\Theta(\mu) = (M(\mu) - iI)(M(\mu) + iI)^{-1}$, $\forall \mu \in \mathbb{C}_+$.

The Weyl function (or, characteristic function) determines a simple symmetric operator S up to unitary equivalence.

The simplest (*canonical*) boundary triplet can immediately be constructed as a triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1)$, where

$$(2.11) \quad \Gamma_0\psi = f_{-i} + Qf_i, \quad \Gamma_1\psi = if_{-i} - iQf_i, \quad \psi = u + f_{-i} + f_i \in \mathcal{D}(S^*)$$

and Q is an arbitrary unitary mapping $Q : \mathfrak{N}_i \rightarrow \mathfrak{N}_{-i}$.

To underline the dependence of Γ_j on the choice of Q in (2.11), we denote by $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ the corresponding boundary triplet.

If Q commutes with J , then the boundary operators Γ_j defined by (2.11) satisfy the relations

$$(2.12) \quad \Gamma_0 J = J\Gamma_0, \quad \Gamma_1 J = J\Gamma_1.$$

By Proposition 2.1, self-adjoint extensions $A_M \supset S$ commuting with J are described by hypermaximal neutral subspaces

$$(2.13) \quad M_G = \{f_i + Gf_i \mid \forall f_i \in \mathfrak{N}_i\}$$

of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_Z)$ which satisfy the additional relation $JM_G = M_G$. Here $G : \mathfrak{N}_i \rightarrow \mathfrak{N}_{-i}$ are unitary mappings. Obviously, $JM_G = M_G \iff JG = GJ$. The latter gives rise to the existence of boundary triplets $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, -G)$ defined by (2.11) with the additional properties (2.12). We prove the following simple statement:

Proposition 2.4. *Boundary triplets $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ satisfying (2.12) exist if and only if the set of self-adjoint extensions of S commuting with J is non-empty.*

For such type of boundary triplets, Proposition 2.1 can be rewritten as follows:

Proposition 2.5. *Let $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ be a boundary triplet of S^* which satisfies (2.12). Then an arbitrary $A \in \Sigma_J(S)$ with $i \notin \sigma(A)$ coincides with the restriction of S^* onto the domain*

$$(2.14) \quad \mathcal{D}(A) = \{f \in \mathcal{D}(S^*) \mid K(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f\},$$

where K is a J -unitary operator in \mathfrak{N}_{-i} (i.e., $J = K^*JK$).

The correspondence $A = A_K \leftrightarrow K$ determined by (2.14) is a bijection between the set of all J -self-adjoint extensions A_K of S such that $i \notin \sigma(A_K)$ and the set of J -unitary operators in \mathfrak{N}_{-i} . Furthermore,

$$(2.15) \quad A_K^* = A_{(K^*)^{-1}}.$$

Remark 2.6. J -Self-adjoint extensions A_M with $i \in \sigma(A_M)$ are characterized by non-trivial intersections $M \cap \mathfrak{N}_{-i}$ of the corresponding subspaces M in (2.7). In that case, the description (2.14) of $\mathcal{D}(A_M)$ is impossible (since $\ker(\Gamma_1 - i\Gamma_0) = \mathfrak{N}_{-i}$ and $\ker(\Gamma_1 + i\Gamma_0) = \mathfrak{N}_i$ by (2.11)).

2.4. Description of $\Sigma_J(S)$. The case of deficiency indices $\langle 2, 2 \rangle$. We are going to analyze $\Sigma_J(S)$ in more detail for the case where S has deficiency indices $\langle 2, 2 \rangle$. To avoid the study of self-adjoint extensions we assume $J \neq I$. Then, the following subspaces of the Hilbert space \mathfrak{M} :

$$\begin{aligned} \mathfrak{M}_{++} &= (I + Z)(I + J)\mathfrak{M}, & \mathfrak{M}_{--} &= (I - Z)(I - J)\mathfrak{M}, \\ \mathfrak{M}_{+-} &= (I + Z)(I - J)\mathfrak{M}, & \mathfrak{M}_{-+} &= (I - Z)(I + J)\mathfrak{M} \end{aligned}$$

are nontrivial and mutually orthogonal. Therefore, $\dim \mathfrak{M}_{\pm\pm} = 1$ (since $\dim \mathfrak{M} = 4$) and there exists an orthonormal basis $\{e_{\pm\pm}\}$ of the Hilbert space \mathfrak{M} such that

$$\mathfrak{M}_{\pm\pm} = \langle e_{\pm\pm} \rangle, \quad \mathfrak{N}_{-i} = \langle e_{++}, e_{+-} \rangle, \quad \mathfrak{N}_i = \langle e_{-+}, e_{--} \rangle.$$

In that case

$$(2.16) \quad \begin{aligned} Je_{++} &= e_{++}, & Je_{-+} &= e_{-+}, & Je_{+-} &= -e_{+-}, & Je_{--} &= -e_{--}; \\ Ze_{++} &= e_{++}, & Ze_{-+} &= -e_{-+}, & Ze_{+-} &= e_{+-}, & Ze_{--} &= -e_{--}. \end{aligned}$$

Let us consider the boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ defined by (2.11), where a unitary mapping $Q : \mathfrak{N}_i \rightarrow \mathfrak{N}_{-i}$ acts as follows:

$$(2.17) \quad Qe_{-+} = e_{++}, \quad Qe_{--} = e_{+-}.$$

The operator Q commutes with J due to (2.16) and hence, relations (2.12) hold.

Denote by $\mathcal{K} = \|k_{ij}\|$ the matrix representation of a J -unitary operator K in \mathfrak{N}_{-i} with respect to the basis $\{e_{++}, e_{+-}\}$. By (2.16), the restriction of J onto \mathfrak{N}_{-i} can be identified (with respect to the basis $\{e_{++}, e_{+-}\}$) with the matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This means that $\sigma_3 = \overline{\mathcal{K}^t} \sigma_3 \mathcal{K}$ (since K is J -unitary). The simple analysis of the latter relation leads to the following description of \mathcal{K} :

$$(2.18) \quad \mathcal{K} = \mathcal{K}(\zeta, \phi, \omega, \xi) = e^{-i\xi} \begin{pmatrix} -(\cosh \zeta)e^{-i\phi} & (\sinh \zeta)e^{-i\omega} \\ -(\sinh \zeta)e^{i\omega} & (\cosh \zeta)e^{i\phi} \end{pmatrix},$$

where $\zeta \in \mathbb{R}$ and $\xi, \phi, \omega \in [0, 2\pi)$. Using Proposition 2.5, we obtain the following

Proposition 2.7. *The formula*

$$(2.19) \quad S^* \upharpoonright \mathcal{D}(A_M), \quad \mathcal{D}(A_M) = \{f \in \mathcal{D}(S^*) \mid K(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f\},$$

where K is an arbitrary J -unitary operator in \mathfrak{N}_{-i} and the boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ is defined by (2.11) and (2.17) establishes the one to one correspondence between J -self-adjoint extensions $A_M \in \Sigma_J(S)$ with $i \notin \sigma(A_M)$ and matrices $\mathcal{K}(\zeta, \phi, \omega, \xi)$ defined by (2.18).

Remark. It follows from Proposition 2.1 and relations (2.16) that operators $A_M \in \Sigma_J(S)$ with $i \in \sigma(A_M)$ are described by the two-parameter set of hypermaximal neutral subspaces

$$M(k_1, k_2) = \langle e_{++} + e^{ik_1}e_{+-}; e_{--} + e^{ik_2}e_{-+} \rangle, \quad k_1, k_2 \in \mathbb{R}$$

of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_{JZ})$. By virtue of (2.11) and (2.17), the subspaces $M(k_1, k_2)$ can be (formally) described by (2.14) if we put

$$\zeta = \infty, \quad \xi = 0, \quad \omega = \frac{k_2 + k_1}{2}, \quad \phi = \frac{k_2 - k_1}{2}$$

in (2.18) and consider $\cosh \infty = \sinh \infty = \infty$ as a number.

To emphasize the relationship $A_M \leftrightarrow \mathcal{K}$ established in Proposition 2.7, we will use the notation $A_{\mathcal{K}}$ instead of A_M .

Corollary 2.8. *The adjoint operator $A_{\mathcal{K}}^*$ of $A_{\mathcal{K}} \in \Sigma_J(S)$ is defined by $\mathcal{K}(-\zeta, \phi, \xi, \omega)$ i.e.,*

$$(2.20) \quad A_{\mathcal{K}(\zeta, \phi, \omega, \xi)}^* = A_{\mathcal{K}(-\zeta, \phi, \omega, \xi)}.$$

The set of self-adjoint extensions of S commuting with J is described by unitary matrices $\mathcal{K}(0, \phi, \omega, \xi)$.

Proof. The relation (2.20) follows from (2.15) and (2.18).

If a self-adjoint extension $A \supset S$ commutes with J , then A is also J -self-adjoint and $A \equiv A_{\mathcal{K}(\zeta, \phi, \omega, \xi)}$ by Proposition 2.7. Using (2.20) and taking into account (2.18), we get $\zeta = 0$ that completes the proof of Corollary 2.8. \square

2.5. The property of C -symmetry. By analogy with [10] the definition of C -symmetry in the Krein spaces setting can be formalized as follows.

Definition 2.9. *An operator A acting in a Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$ has the property of C -symmetry if there exists a bounded linear operator C in \mathfrak{H} such that: (i) $C^2 = I$; (ii) $JC > 0$; (iii) $AC = CA$.*

By virtue of (2.3) and (2.5) the property of C -symmetry of A means that A can be decomposed

$$(2.21) \quad A = A_+ [+]_J A_-, \quad A_+ = A \upharpoonright_{\mathfrak{L}_+}, \quad A_- = A \upharpoonright_{\mathfrak{L}_-}$$

with respect to the canonical decomposition (2.2) (with subspaces \mathfrak{L}_{\pm} determined by (2.4)).

If a J -self-adjoint operator A possesses the property of C -symmetry, then its counterparts A_{\pm} in (2.21) turn out to be self-adjoint operators in the Hilbert spaces \mathfrak{L}_+ and \mathfrak{L}_- with the inner products $[\cdot, \cdot]_J$ and $-[\cdot, \cdot]_J$, respectively. This simple observation leads to the following statement.

Proposition 2.10. ([1]). *A J -self-adjoint operator A has the property of C -symmetry if and only if A is similar to a self-adjoint operator in \mathfrak{H} . If a J -self-adjoint operator A has the property of C -symmetry then its spectrum is real and the adjoint operator C^* provides the property of C -symmetry for A^* .*

Definition 2.11. *Let $A \in \Sigma_J(S)$ have the property of C -symmetry realized by an operator C . We will say that A belongs to the sector Σ_J^{st} of stable C -symmetry if the operator C commutes with S . Otherwise ($AC = CA$ but $SC \neq CS$), the operator A belongs to the sector Σ_J^{unst} of unstable C -symmetry.*

The next statement immediately follows from Theorem 3.1 in [1].

Proposition 2.12. *Let $A_M \in \Sigma_J(S)$ be defined by (2.7). Then $A_M \in \Sigma_J^{st}$ if and only if $CM = M$, where C realizes the property of C -symmetry for S .*

3. THE PHILLIPS SYMMETRIC OPERATOR

We are going to specify general results of previous section to the case of Phillips symmetric operator S defined by (1.2) and (1.3).

3.1. Preliminaries. The general definition (1.2), (1.3) of S looks rather abstract and, in many cases, it is useful to work with a model realization of S in $\mathfrak{H} = l_2(\mathbb{Z}, N)$ (N is an auxiliary finite-dimensional Hilbert space). In that case

$$(3.1) \quad \begin{aligned} U(\dots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \dots) &= (\dots, x_{-3}, x_{-2}, \underline{x_{-1}}, x_0, x_1, \dots), \\ V(\dots, x_{-2}, x_{-1}, \underline{0}, x_1, x_2, \dots) &= (\dots, x_{-3}, x_{-2}, \underline{x_{-1}}, 0, x_1, \dots), \end{aligned}$$

where $x_j \in N$ and elements at the zero position are underlined.

The self-adjoint operator A takes the form

$$(3.2) \quad \begin{aligned} Af &= i(\dots, x_{-3} + x_{-2}, x_{-2} + x_{-1}, \underline{x_{-1}} + x_0, x_0 + x_1, x_1 + x_2, \dots), \\ f \in \mathcal{D}(A) &\iff f = (\dots, x_{-3} - x_{-2}, x_{-2} - x_{-1}, \underline{x_{-1}} - x_0, x_0 - x_1, x_1 - x_2, \dots), \end{aligned}$$

where $\sum_{i \in \mathbb{Z}} \|x_i\|_N^2 < \infty$ and the symmetric operator S is the restriction of A onto the set

$$(3.3) \quad u \in \mathcal{D}(S) \iff u = (\dots, x_{-3} - x_{-2}, x_{-2} - x_{-1}, \underline{x_{-1}}, -x_1, x_1 - x_2, \dots),$$

which consists of all $u \in \mathcal{D}(A)$ such that $x_0 = 0$.

Recalling (2.9) and using (3.2), (3.3), it is easily to see that (see, e.g., [29])

$$(3.4) \quad \begin{aligned} \mathfrak{N}_i &= \{f_i(x) = (\dots, 0, 0, \underline{x}, 0, 0, \dots) : \forall x \in N\}, \\ \mathfrak{N}_\mu &= \{f_\mu(x) = (\dots, \bar{r}_\mu^2 x, \bar{r}_\mu x, \underline{x}, 0, 0, \dots) : \forall x \in N\}, \quad \mu \in \mathbb{C}_+, \\ \mathfrak{N}_{-i} &= \{f_{-i}(x) = (\dots, 0, 0, \underline{0}, x, 0, \dots) : \forall x \in N\}, \\ \mathfrak{N}_{\bar{\mu}} &= \{f_{\bar{\mu}}(x) = (\dots, 0, 0, \underline{0}, x, r_\mu x, r_\mu^2 x, \dots) : \forall x \in N\}, \end{aligned}$$

where $r_\mu = \frac{\mu - i}{\mu + i}$. Direct calculation with the use of (3.3) and (3.4) gives

$$(3.5) \quad f_\mu((1 - \bar{r}_\mu)x) = u + f_i(x), \quad f_{\bar{\mu}}((1 - r_\mu)x) = v + f_{-i}(x), \quad \forall x \in N,$$

where $u, v \in \mathcal{D}(S)$. Therefore,

$$(3.6) \quad \mathfrak{N}_\mu \subset \mathcal{D}(S) \dot{+} \mathfrak{N}_i, \quad \mathfrak{N}_{\bar{\mu}} \subset \mathcal{D}(S) \dot{+} \mathfrak{N}_{-i}, \quad \forall \mu \in \mathbb{C}_+.$$

Lemma 3.1. *Let $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ be a boundary triplet of the Phillips symmetric operator S (defined by (1.2) and (1.3)). Then the corresponding characteristic function $\Theta(\cdot)$ of S is equal to zero.*

Proof. It is sufficient to verify this statement for the case where S is defined by (3.2) and (3.3). According to (3.6), an arbitrary $f_{\bar{\mu}} \in \mathfrak{N}_{\bar{\mu}}$ has the form $f_{\bar{\mu}} = u + f_{-i}$, where $u \in \mathcal{D}(S)$ and $f_{-i} \in \mathfrak{N}_{-i}$. But then $(\Gamma_1 + i\Gamma_0)f_{\bar{\mu}} = 2if_{-i}$ and $(\Gamma_1 - i\Gamma_0)f_{\bar{\mu}} = 0$ due to (2.11). Therefore, $\Theta(\mu) \equiv 0$ ($\forall \mu \in \mathbb{C}_+$) by (2.10). Lemma 3.1 is proved. \square

Lemma 3.2. *Let S be defined by (3.2) and (3.3) and let J be a fundamental symmetry in $l_2(\mathbb{Z}, N)$. Then J commutes with S if and only if*

$$(3.7) \quad J(\dots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \dots) = (\dots, J_- x_{-2}, J_- x_{-1}, \underline{J_- x_0}, J_+ x_1, J_+ x_2, \dots),$$

where J_\pm are fundamental symmetries in N .

Proof. Let J commute with S . It follows from (2.9) that defect subspaces \mathfrak{N}_μ are invariant with respect J . Taking (3.4) into account we conclude that the restrictions $J_- := J \upharpoonright \mathfrak{N}_i$ and $J_+ := J \upharpoonright \mathfrak{N}_{-i}$ determine two fundamental symmetries J_- and J_+ in N . Further, the equality $JS = SJ$ is equivalent to the relation $JV = VJ$, where V is defined by (3.1). Combining this relation with the first and third relations in (3.4) and taking the definition of J_\pm into account we establish (3.7).

Conversely, if a fundamental symmetry J is defined by (3.7), then relations (3.2) and (3.3) imply that $JS = SJ$. Lemma 3.2 is proved. \square

Lemma 3.3. *Let S be defined by (3.2) and (3.3), let J be a fundamental symmetry in $l_2(\mathbb{Z}, N)$ commuting with S , and let C be a bounded operator in $l_2(\mathbb{Z}, N)$ such that $C^2 = I$ and $JC > 0$. Then C commutes with S if and only if*

$$(3.8) \quad C(\dots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \dots) = (\dots, C_-x_{-2}, C_-x_{-1}, \underline{C_-x_0}, C_+x_1, C_+x_2, \dots),$$

where C_\pm are bounded operators in N such that $C_\pm^2 = I_N$ and $J_\pm C_\pm > 0$ where J_\pm are taken from the formula (3.7).

Proof. By Lemma 3.2, the operator J is defined by (3.7), where J_\pm are fundamental symmetries in N .

Assume that C commutes with S . Then, using (2.6) one gets $SF = FS$, where $F = JC$ is a bounded self-adjoint operator. Hence,

$$SC^* = SFJ = FSJ = FJS = C^*S.$$

The obtained relation $C^*S = SC^*$ and $C^2 = I$ imply that the defect subspaces \mathfrak{N}_μ of S are invariant with respect C . It follows from (3.4) that the restrictions $C_- := C \upharpoonright \mathfrak{N}_i$ and $C_+ := C \upharpoonright \mathfrak{N}_{-i}$ determine bounded operators C_\pm in N such that $C_\pm^2 = I_N$ and $J_\pm C_\pm > 0$. Reasoning by analogy with the proof of Lemma 3.2, we complete the proof. \square

3.2. Description of J -self-adjoint extensions. Using (3.4) we can identify the Hilbert space $\mathfrak{M} = \mathfrak{N}_{-i} \dot{+} \mathfrak{N}_i$ with

$$N \oplus N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in N \right\}.$$

In that case

$$(3.9) \quad Z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad J \upharpoonright \mathfrak{M} = \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix}.$$

Proposition 3.4. *Let S be defined by (3.2) and (3.3). Then the set $\Sigma_J(S)$ of J -self-adjoint extensions of S is non-empty if and only if*

$$(3.10) \quad \dim[(I - J_+)N] = \dim[(I - J_-)N].$$

Proof. By Proposition 2.1, J -self-adjoint extensions of S exist if and only if the Krein space $(\mathfrak{M}, [\cdot, \cdot]_{JZ})$ has hypermaximal neutral subspaces. This is possible only in the case where $\dim[(I + JZ)\mathfrak{M}] = \dim[(I - JZ)\mathfrak{M}]$ or, that is equivalent (see (3.9)),

$$\dim[(I + J_+)N] + \dim[(I - J_-)N] = \dim[(I - J_+)N] + \dim[(I + J_-)N].$$

This identity is equivalent to (3.10) (since $\dim[(I + J_\pm)N] + \dim[(I - J_\pm)N] = \dim N$ and $\dim N < \infty$). Proposition 3.10 is proved. \square

Corollary 3.5. *Let S be defined by (3.2) and (3.3) and let J be a fundamental symmetry commuting with S in $l_2(\mathbb{Z}, N)$. Then self-adjoint extensions of S commuting with J exist if and only if the identity (3.10) holds.*

Proof. If A_M is a self-adjoint extension of S commuting with J , then $A_M \in \Sigma_J(S)$ and relation (3.10) holds due to Proposition 3.4.

Conversely, since $\dim N < \infty$, relation (3.10) is equivalent to the identity

$$\dim[(I + J_+)N] = \dim[(I + J_-)N].$$

This implies the existence of unitary mappings $G : \mathfrak{N}_i \rightarrow \mathfrak{N}_{-i}$ such that $GJ = GJ_- = J_+G = JG$. In that case the hypermaximal neutral subspace M_G of the Krein space $(\mathfrak{M}, [\cdot, \cdot]_Z)$ (defined by (2.13)) satisfies the relation $JM_G = M_G$ and the corresponding self-adjoint extension A_M commutes with J . Corollary 3.5 is proved. \square

Proposition 3.6. *Let S be the Phillips symmetric operator (defined by (1.2) and (1.3)) and let J be a fundamental symmetry commuting with S in \mathfrak{H} . Then boundary triplets $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ of S^* defined by (2.11) and satisfying (2.12) exist if and only if the set $\Sigma_J(S)$ is non-empty.*

Proof. It is sufficient to establish for the Phillips symmetric operator S realized by the formulas (3.2) and (3.3). In that case, by Proposition 3.4 and Corollary 3.5, $\Sigma_J(S) \neq \emptyset \iff$ there exist self-adjoint extensions of S commuting with J . Using now Proposition 2.4 we complete the proof. \square

Theorem 3.7. *Let S be the Phillips symmetric operator, let J be a fundamental symmetry commuting with S in \mathfrak{H} , and let $A_M \in \Sigma_J(S)$. Then the spectrum of A_M either coincides with \mathbb{R} ($\sigma(A_M) = \mathbb{R}$) or covers the whole complex plane ($\sigma(A_M) = \mathbb{C}$) and its non-real part consists of eigenvalues of A_M .*

Proof. Since an arbitrary $A_M \in \Sigma_J(S)$ is a finite rank perturbation of the self-adjoint operator A (see (1.3)), the non-real spectrum of A_M may include complex eigenvalues.

Without loss of generality we can suppose that S is determined by the formulas (3.2) and (3.3). Assume that $\mu_0 \in \mathbb{C}_+$ is an eigenvalue of A_M . Then there exists an element $f_{\bar{\mu}_0} \in \mathfrak{N}_{\bar{\mu}_0} \cap \mathcal{D}(A_M)$ and, according to (3.5),

$$f_{\bar{\mu}_0} = f_{\bar{\mu}_0}(x) = v_0 + f_{-i} \left(\frac{x}{1 - r_{\mu_0}} \right), \quad v_0 \in \mathcal{D}(S)$$

for a certain choice of $x \in N$. Using the second relation in (3.5) for an arbitrary $\mu \in \mathbb{C}_+$, we obtain

$$f_{\bar{\mu}} \left(\frac{1 - r_{\mu} x}{1 - r_{\mu_0}} \right) = v + f_{-i} \left(\frac{x}{1 - r_{\mu_0}} \right), \quad v \in \mathcal{D}(S).$$

Comparing last two relations we arrive at the conclusion that the element

$$f_{\bar{\mu}} \left(\frac{1 - r_{\mu} x}{1 - r_{\mu_0}} \right) = v - v_0 + f_{\bar{\mu}_0}, \quad \mu \in \mathbb{C}_+,$$

belongs to $\mathfrak{N}_{\bar{\mu}} \cap \mathcal{D}(A_M)$. Hence $\mathbb{C}_+ \subset \sigma_p(A_M)$. The relation $\mathbb{C}_- \subset \sigma_p(A_M)$ is established by the same manner. Thus, $\sigma(A_M) = \mathbb{C}$ and $\mathbb{C} \setminus \mathbb{R}$ contains eigenvalues of A_M .

Assume that the spectrum of A_M is real. Since the Phillips symmetric operator has no real points of regular type (see, e.g., [26]), the spectrum of A_M coincides with \mathbb{R} . Theorem 3.7 is proved. \square

Corollary 3.8. *Let S be the Phillips symmetric operator and let J be a fundamental symmetry commuting with S in \mathfrak{H} . Then the set $\Sigma_J(S)$ does not contain definitizable operators.*

Proof. By Proposition (3.6), if $\Sigma_J(S)$ is a non-empty set, then there exists a boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ of S^* which satisfies (2.12). It follows from Proposition 2.5 and Theorem 3.7 that operators $A_M \in \Sigma_J(S)$ with real spectrum are described by the formula (2.14) in terms of the boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$. The rest of operators $A_M \in \Sigma_J(S)$ (which can not be described by (2.14)) have empty resolvent set (due to Remark 2.6 and Theorem 3.7). By Proposition 2.2 this means that $\Sigma_J(S)$ does not contain definitizable operators. Corollary 3.8 is proved. \square

3.3. J -self-adjoint extensions with C -symmetry.

Theorem 3.9. *Let S be the Phillips operator with deficiency indices $\langle 2, 2 \rangle$. Then an arbitrary J -self-adjoint extension $A_M \in \Sigma_J(S)$ has the property of stable C -symmetry ($A_M \in \Sigma_J^{\text{st}}$) if and only if the spectrum of A_M is real.*

Proof. If A_M has C -symmetry, then its spectrum is real (see Proposition 2.10).

Conversely, we assume that $A_M \in \Sigma_J(S)$ has a real spectrum. In that case, by Proposition 2.7, $A_M(= A_{\mathcal{K}})$ is defined by (2.19), where $\mathcal{K} = \mathcal{K}(\zeta, \phi, \omega, \xi)$ has the form (2.18).

Without loss of generality we can assume that S is determined by the formulas (3.2) and (3.3) in the space $l_2(\mathbb{Z}, N)$. Then, by virtue of Proposition 2.12 and Lemma 3.3, the operator $A_M(= A_{\mathcal{K}})$ has a stable C -symmetry if and only if $CM = M$ for at least one of operators C determined by (3.8).

It follows from (2.19) that $M = \{f = f_{-i} + f_i \mid K(\Gamma_1 + i\Gamma_0)f = (\Gamma_1 - i\Gamma_0)f\}$. Employing (2.11) we rewrite the latter relation as follows:

$$M = \{f_{-i} - Q^{-1}Kf_{-i} \mid \forall f_{-i} \in \mathfrak{N}_{-i}\}.$$

The obtained description of M and Lemma 3.3 imply that

$$(3.11) \quad CM = M \iff KC_+ = \widehat{C}_+K \quad (\widehat{C}_+ := QC_-Q^{-1}),$$

where C_+ and \widehat{C}_+ act in $N = \mathfrak{N}_{-i}$ and satisfy the relations $C^2 = I, JC > 0$ ($C \in \{C_+, \widehat{C}_+\}$).

Let

$$C_+ = \|c_{ij}\|$$

be the matrix representation of C_+ with respect to the basis $\{e_{++}, e_{+-}\}$. Then the relations $C_+^2 = I, JC_+ > 0$ take the form

$$(3.12) \quad C_+^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C_+ > 0,$$

where $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix representation of $J \upharpoonright \mathfrak{N}_{-i}$ with respect to $\{e_{++}, e_{+-}\}$ (since (2.16)). A simple analysis of (3.12) leads to the following description of C_+ :

$$(3.13) \quad C_+ = C_{\widetilde{\chi}, \widetilde{\omega}} = \begin{pmatrix} \cosh \widetilde{\chi} & (\sinh \widetilde{\chi})e^{-i\widetilde{\omega}} \\ -(\sinh \widetilde{\chi})e^{i\widetilde{\omega}} & -\cosh \widetilde{\chi} \end{pmatrix}, \quad \widetilde{\chi}, \widetilde{\omega} \in \mathbb{R}.$$

Reasoning by analogy for the matrix representation \widehat{C}_+ of \widehat{C}_+ we get

$$(3.14) \quad \widehat{C}_+ = C_{\widehat{\chi}, \widehat{\omega}} = \begin{pmatrix} \cosh \widehat{\chi} & (\sinh \widehat{\chi})e^{-i\widehat{\omega}} \\ -(\sinh \widehat{\chi})e^{i\widehat{\omega}} & -\cosh \widehat{\chi} \end{pmatrix}, \quad \widehat{\chi}, \widehat{\omega} \in \mathbb{R}.$$

Passing to the matrix representation in (3.11) we conclude that $A_M(= A_{\mathcal{K}})$ has a stable C -symmetry if and only if

$$(3.15) \quad \mathcal{K}(\zeta, \phi, \omega, \xi)C_{\widetilde{\chi}, \widetilde{\omega}} = C_{\widehat{\chi}, \widehat{\omega}}\mathcal{K}(\zeta, \phi, \omega, \xi),$$

where $\mathcal{K}(\zeta, \phi, \omega, \xi)$ is defined by (2.18). A routine analysis of (3.15) with the use of (3.13) and (3.14) shows that (3.15) is equivalent to the system of relations

$$(3.16) \quad \begin{cases} \cosh \widehat{\chi} - \cosh \widetilde{\chi} + \tanh \zeta [e^{i(\widehat{\omega} - \omega - \phi)} \sinh \widehat{\chi} - e^{i(\omega - \widetilde{\omega} - \phi)} \sinh \widetilde{\chi}] = 0, \\ \tanh \zeta [\cosh \widehat{\chi} + \cosh \widetilde{\chi}] + e^{i(\phi + \omega - \widetilde{\omega})} \sinh \widehat{\chi} + e^{-i(\phi - \omega + \widetilde{\omega})} \sinh \widetilde{\chi} = 0. \end{cases}$$

Let us set $\widehat{\chi} = \widetilde{\chi} = \chi$. Then the first relation in (3.16) is satisfied when

$$(3.17) \quad \omega = \frac{\widehat{\omega} + \widetilde{\omega}}{2}$$

and the second one goes over

$$\tanh \zeta + \tanh \chi \cos \left(\phi + \frac{\widetilde{\omega} - \widehat{\omega}}{2} \right) = 0.$$

The latter equation can be solved with respect to χ if and only if

$$(3.18) \quad \left| \tanh \zeta \right| < \left| \cos \left(\phi + \frac{\tilde{\omega} - \hat{\omega}}{2} \right) \right|.$$

Since $\tilde{\omega}, \hat{\omega} \in \mathbb{R}$ are independent variables, conditions (3.17) and (3.18) with fixed ω and ϕ can easily be satisfied by a suitable choice of $\tilde{\omega}$ and $\hat{\omega}$. This means that the system (3.16) has a solution $\tilde{\chi}, \hat{\chi}, \tilde{\omega}, \hat{\omega}$ for any fixed ζ, ϕ, ω, ξ . Therefore, $A_{\mathcal{K}(\zeta, \phi, \omega, \xi)}$ has a stable C -symmetry for any choice of ζ, ϕ, ω , and ξ . Theorem 3.9 is proved. \square

Corollary 3.10. *Let A be J -self-adjoint extension of the Phillips symmetric operator S with deficiency indices $\langle 2, 2 \rangle$. Then A is similar to a self-adjoint operator if and only if the spectrum of A is real.*

Proof. It follows from Proposition 2.10 and Theorem 3.9 \square

3.4. Various realizations of the Phillips operator. It follows from (1.2) and (1.3) that the Phillips symmetric operator S can be obtained as the restriction of a self-adjoint operator A with Lebesgue spectrum onto the domain

$$(3.19) \quad \mathcal{D}(S) = \{f \in \mathcal{D}(A) \mid ((A - iI)f, w) = 0, \forall w \in W_0\},$$

where W_0 is a wandering subspace of the bilateral shift U . In particular, this means that the Phillips symmetric operator S naturally arises in the study of the formal expression $i \frac{d}{dx} + \langle \delta, \cdot \rangle \delta(x)$ and it coincides with the operator

$$S = i \frac{d}{dx}, \quad \mathcal{D}(S) = \{u(\cdot) \in W_2^1(\mathbb{R}) \mid u(0) = 0\}$$

acting in $L_2(\mathbb{R})$. This example illustrates one of possible general approaches to the construction of the Phillips symmetric operator. Indeed, let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and let $S = S_1 \oplus S_2$, where S_1 and S_2 are simple maximal symmetric operators in the Hilbert spaces \mathfrak{H}_1 and \mathfrak{H}_2 with deficiency indices $\langle m, 0 \rangle$ and $\langle 0, m \rangle$ ($m \in \mathbb{N}$), respectively. In that case S is a simple symmetric operator in \mathfrak{H} with deficiency indices $\langle m, m \rangle$ and its characteristic function $\Theta(\cdot)$ associated with an arbitrary boundary triplet $(\mathfrak{N}_{-i}, \Gamma_0, \Gamma_1, Q)$ (see (2.11)) is equal to zero. By Lemma 3.1 this means that S is a Phillips symmetric operator.

Using (1.3) and (3.19) it is easy to calculate the defect subspaces $\mathfrak{N}_{\pm i}$ of S

$$\mathfrak{N}_i = W_0 \quad \text{and} \quad \mathfrak{N}_{-i} = UW_0.$$

According to (1.2) and (1.3), a bilateral shift U and its wandering subspace W_0 are the main ingredients for the determination of the Phillips symmetric operator S . To illustrate this point, we have presented below two mathematical constructions where U appears naturally and W_0 admits a simple description.

3.4.1. Multiresolution approximation of $L_2(\mathbb{R})$. We recall [33, 34] that a multiresolution approximation (MRA) of $L_2(\mathbb{R})$ is a sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L_2(\mathbb{R})$ such that: (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$; (ii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$; (iii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R})$; (iv) $f(\cdot) \in V_j \Leftrightarrow f(2^{-j}\cdot) \in V_0$; (v) there exists a function $\varphi(\cdot) \in V_0$ such that the sequence $\{\varphi(\cdot - k), k \in \mathbb{Z}\}$ is a Riesz basis of V_0 .

Let

$$Uf(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right), \quad \forall f \in L_2(\mathbb{R})$$

be the dilation operator in $\mathfrak{H} = L_2(\mathbb{R})$ and let $\{V_j\}_{j \in \mathbb{Z}}$ be a fixed multiresolution approximation of $L_2(\mathbb{R})$. Then U is a bilateral shift in $L_2(\mathbb{R})$ with a wandering subspace $W_0 = V_1 \ominus V_0$ (due to properties (i)–(iv)).

According to the general results of MRA-based wavelet theory [33, 34] the subspace W_0 is a wavelet subspace and relations (i)–(v) imply the existence of a function (wavelet)

$\psi(\cdot) \in W_0$ such that the sequence $\{\psi(\cdot - k), k \in \mathbb{Z}\}$ forms an orthonormal basis in W_0 . This means that (3.19) can be rewritten as follows:

$$\mathcal{D}(S) = \{f \in \mathcal{D}(A) \mid ((A - iI)f, \psi(\cdot - k)) = 0, \forall k \in \mathbb{Z}\},$$

where the wavelet $\psi(\cdot)$ is directly constructed by the (scaling) function $\varphi(\cdot)$ from condition (v).

3.4.2. *Abstract wave equation.* Let us consider an operator-differential equation

$$(3.20) \quad u_{tt} = -Lu,$$

where L is a positive self-adjoint operator in a Hilbert space H . By H_L we denote the completion of domain of definition $\mathcal{D}(L)$ with respect to the norm $\|u\|_{H_L}^2 := (Lu, u)_H$ and consider the Hilbert space $\mathfrak{H} = H_L \oplus H$ (the energy space). It is convenient to write elements of \mathfrak{H} as column matrices $\begin{pmatrix} u \\ v \end{pmatrix}$, where $u \in H_L$ and $v \in H$. Put $u_t = v$ and rewrite (3.20) as

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = iQ \begin{pmatrix} u \\ v \end{pmatrix}, \quad Q = i \begin{pmatrix} 0 & -I \\ L & 0 \end{pmatrix}.$$

The operator Q with the domain $\mathcal{D}(Q) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid \{u, v\} \subset \mathcal{D}(L) \right\}$ is essentially self-adjoint in \mathfrak{H} . Its closure A is a generator of the group of unitary (in \mathfrak{H}) operators $W_A(t) = e^{iAt}$, which determines solutions of the Cauchy problem for the abstract wave equation (3.20).

The equation (3.20) is said to be *free (unperturbed) wave equation* if there exists a simple maximal symmetric operator B in H such that

$$(3.21) \quad B^2 \subset L \quad \text{and} \quad (Lu, u) = \|B^*u\|_H^2, \quad \forall u \in \mathcal{D}(L).$$

Assume that $\{u_n\}$ belong $\mathcal{D}(B^2)$ and form a Cauchy sequence in H_L . Then $\{Bu_n\}$ is the Cauchy sequence in H (due to the second relation in (3.21)) and hence $\lim_{n \rightarrow \infty} Bu_n = \gamma \in H$. In that case we will say that the sequence $\{u_n\}$ converges to the element x_γ in the space H_L . Obviously the Hilbert space H_L can be identified with the set of elements $\{x_\gamma \mid \forall \gamma \in H\}$ and $(x_\gamma, x_\zeta)_{H_L} = (\gamma, \zeta)_H$.

In what follows, without loss of generality we assume that B has zero defect number in the lower half-plane. Then B admits the representation

$$(3.22) \quad B = T^{-1}i \frac{d}{ds}T, \quad \mathcal{D}(B) = T^{-1}W_2^0(\mathbb{R}_+, \mathcal{N}),$$

where T is an isometric mapping from H onto $L_2(\mathbb{R}_+, \mathcal{N})$, \mathcal{N} is an auxiliary Hilbert space of dimension equal to the nonzero defect number of B and $\mathbb{R}_+ = (0, \infty)$.

Using (3.22), we can define B in various functional spaces getting, as a result, different specific realizations of the free abstract wave equation. In particular, the classical free wave equation $u_{tt}(x, t) = \Delta u(x, t)$ in \mathbb{R}^n (n is odd) can be obtained from (3.22) if we choose \mathcal{N} as the Hilbert space $L_2(S^{n-1})$ of functions square-integrable on the unit sphere S^{n-1} in \mathbb{R}^n and consider the isometric operator $T : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}_+, \mathcal{N})$ defined on the rapidly decreasing smooth functions $u(x) \in \mathcal{S}(\mathbb{R}^n)$ by the formula

$$(Tu)(s, w) = (\partial_s^m Ru)(s, w), \quad m = \frac{(n-1)}{2}, \quad s \geq 0, \quad w \in S^{n-1},$$

where R is the Radon transformation. In that case the Laplace operator $L = -\Delta$ in $L_2(\mathbb{R}^n)$ satisfies condition (3.21) and equation (3.20) takes the form $u_{tt}(x, t) = \Delta u(x, t)$ (see [31] for detail).

Assume that (3.20) is the free wave equation for some choice of B . Then the corresponding generator A is the Cayley transform of a bilateral shift U and a wandering subspace W_0 can be chosen as follows [30]:

$$(3.23) \quad W_0 = \left\{ \begin{pmatrix} x_h \\ -ih \end{pmatrix} \mid \forall h \in \ker(B^* + iI) \right\}.$$

Substituting (3.23) into (3.19) and taking into account that the domain $\mathcal{D}(A)$ can be described explicitly, we find S . For instance, let L be the Friedrichs extension of B^2 . Then $L = B^*B$ and this operator satisfies (3.21). In this case:

$$A \begin{pmatrix} x_\gamma \\ p \end{pmatrix} = -i \begin{pmatrix} x_{Bp} \\ -B^*\gamma \end{pmatrix}, \quad \mathcal{D}(A) = \left\{ \begin{pmatrix} x_\gamma \\ p \end{pmatrix} \mid \forall \gamma \in \mathcal{D}(B^*), \forall p \in \mathcal{D}(B) \right\}$$

and S is the restriction of A onto the set of elements

$$\left\{ \begin{pmatrix} x_\gamma \\ p \end{pmatrix} \mid \gamma \in \mathcal{D}(B^*), p \in \mathcal{D}(B) \right\}$$

such that $((B^* + iI)(p - i\gamma), h) = 0$ for all $h \in \ker(B^* + iI)$.

Corollary 3.11. *Assume that the nonzero defect number of B is 2 and J is a fundamental symmetry in \mathfrak{H} such that $SJ = JS$. Then if $A \in \Sigma_J(S)$ has a real spectrum, then $W_A(t) = e^{iAt}$ is a C_0 -semigroup.*

Proof. Immediately follows from Corollary 3.10. □

4. CONCLUSIONS

In this paper we have studied the collection $\Sigma_J(S)$ of J -self-adjoint extensions of the Phillips symmetric operator S . Our attention to $\Sigma_J(S)$ was inspired by a steady interest in the spectral analysis of new classes of J -self-adjoint operators A_ε with the aim to illustrate quantitative and qualitative changes of spectra $\sigma(A_\varepsilon)$ when parameters ε run the domain of variation Ξ . Due to specific inherent properties of the Phillips operator S (the zero characteristic function, the absence of real points of regular type, etc) we obtained a spectral picture which differs from the matrix models [21, 24, 25] and models based on J -self-adjoint (symmetric) perturbations of the Schrödinger or the Dirac operator [5, 15, 32, 39]. For instance, in our case, either the spectrum of $A \in \Sigma_J(S)$ coincides with real line: $\sigma(A) = \mathbb{R}$ or with complex plane: $\sigma(A) = \mathbb{C}$ (Theorem 3.7).

For operators $A_\varepsilon \in \Sigma_J(S)$ (where S is an arbitrary symmetric operator commuting with J) we have introduced the concepts of stable and unstable C -symmetry (Definition 2.11). These concepts are natural for sets of J -self-adjoint operators appearing in the extension theory framework. Roughly speaking, if A_ε belongs to the sector Σ_J^{st} of stable C -symmetry, then A_ε preserves the property of C -symmetry under small variation of ε .

For singular perturbations of the Schrödinger or the Dirac operator, the corresponding symmetric operator S has real points of regular type. In that case, the sector Σ_J^{unst} of unstable C symmetry is not empty and operators $A_\varepsilon \in \Sigma_J(S)$ with real spectra and Jordan points arise in the case where ε lies on the boundary of Σ_J^{st} [1, 23]. This picture is essentially simplified for the Phillips symmetric operator S since S has no real points of regular type. We have shown that the sector Σ_J^{unst} of unstable C -symmetry is the empty set and there are no J -self-adjoint extensions of A_{sym} with real spectra and Jordan points (this fact follows from Theorem 3.9 and Corollary 3.10). These results have been obtained under the assumption that S has deficiency indices $\langle 2, 2 \rangle$. We believe that they remain true for the general case $\langle n, n \rangle$. However, the corresponding proof requires more cumbersome analysis and the case $\langle n, n \rangle$ will be considered in a forthcoming paper.

An open problem is finding an adequate physical phenomenon for which J -self-adjoint extensions of S can be served as model Hamiltonians. In this way we have just discussed certain representations of S related to abstract wave equation and multiresolution approximation.

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