Methods of Functional Analysis and Topology
Vol. 16 (2010), no. 4, pp. 349–358

SPECTRAL PROBLEM FOR FIGURE-OF-EIGHT GRAPH OF STIELTJES STRINGS

O. MARTINYUK AND V. PIVOVARCHIK

To the memory of Israel Gohberg.

Abstract. We describe the spectrum of the problem generated by the Stieltjes string recurrence relations on a figure-of-eight graph. The continuity and the force balance conditions are imposed at the vertex of the graph. It is shown that the eigenvalues of such (main) problem are interlaced with the elements of the union of sets of eigenvalues of the Dirichlet problems generated by the parts of the string which correspond to the loops of the figure-of-eight graph. Also the eigenvalues of the main problem are interlaced with the elements of the union of sets of eigenvalues of the periodic problems generated by the same parts of the string.

In 1950 M. Krein and F. Gantmakher [1, pp. 332–349] found a mechanical interpretation for the classical results of Stieltjes on the development into continued fractions for functions of a certain class. These functions occur in the theory of small transversal vibrations of strings which are zero density threads bearing point masses. After M. Krein and F. Gantmakher these strings were called *Stieltjes*. Similar development into continued fractions were used earlier by W. Cauer in the theory of synthesis of electrical circuits. He associated the synthesis of reactive two-port having given entrance resistance with the continued fraction coefficients of which are capacities and inductances [2] (see also [3], pp. 498–508).

In Section 1 of this paper, we consider a periodic problem generated by the equation of the Stieltjes string transversal vibrations. This problem occurs in the following situation. A stretched Stieltjes sting is girdling a hard cylinder as a loop. It is assumed that there is no friction between the string and the cylinder. The string can vibrate in the direction parallel to the axis of the cylinder. One meets another physical interpretation of the periodic problem when considers a closed transmission line containing capacities in the longitudinal and inductances in the transversal elements of the circle or vice versa depending on the pole of input resistance [3, pp. 498–508]. The spectrum of such a problem consists of normal (i.e. isolated Fredholm) eigenvalues (see [4, p. 23] for the definition). As an auxiliary result it is proved that these eigenvalues are interlaced in a non-strict sense with the eigenvalues of the Dirichlet problem generated by the same string.

In Section 2 a figure-of-eight graph composed by two Stieltjes strings is considered. At the vertex, continuity conditions are imposed together with the Kirchhoff condition which, in the mechanical interpretation of the problem, describes the balance of forces. It is shown that the relation

$$\varphi_N = \varphi_N^I \varphi_D^{II} + \varphi_N^{II} \varphi_D^I$$

2000 *Mathematics Subject Classification.* Primary 34K29; Secondary 34K10, 39A70.

Key words and phrases. Characteristic polynomial, figure-of-eight graph, Stieltjes string, eigenvalue, multiplicity of eigenvalue.

The second author was supported by Swiss National Science Foundation, grant IZ73Z0–128135.

obtained in [5] for trees remains true in the case of figure-of-eight graph. Here the upper indices correspond to subgraphs into which the vertex divides the graph. In the case of figure-of-eight graph the subgraphs are the loops. The lower indices correspond to the Dirichlet (D) and to the generalized Neumann (N), i.e., the Kirchhoff and continuity conditions. It is proved that the eigenvalues of the problem on the graph, i.e., the set of zeros of the function $\varphi_N(\lambda^2)$ are interlaced in a non-strict sense with the union of the spectra of the Dirichlet problems generated by the Stieltjes strings of the loops as well as with the union of the spectra of the periodic problems generated by the strings of the loops.

It should be mentioned that the spectral problem on a figure-of-eight graph generated by the Sturm-Liouville equation was considered in [6].

1. Let us consider a Stieltjes string of length l, i.e., a massless elastic thread bearing n point masses. Let the ends of the thread be joined at a point composing a cycle. Denote by $m_k,\ k = 1, 2, \dots, n$, the values of the point masses and by $l_k,\ k = 0, 1, \dots, n$, the subintervals into which the masses and the point A break the string such that l_k lies to the left from m_{k+1} while l_{k+1} lies to the right.

Small transversal vibrations of such a string can be described by transversal displacements of the masses $V_k(t)$ where t denotes the time.

The equation of transversal motion for the mass m_k is

$$\frac{V_k(t) - V_{k-1}(t)}{l_{k-1}} + \frac{V_k(t) - V_{k+1}(t)}{l_k} + m_k V_k''(t) = 0 \quad (k = 1, 2, \dots, n). \tag{1}$$

Continuity condition at A gives

$$V_0(t) = V_{n+1}(t). \tag{2}$$

The periodic problem we obtain if we add to (1) and (2) the equation of balance of forces at A (the Kirchhoff condition),

$$\frac{V_1(t) - V_0(t)}{l_0} = \frac{V_{n+1}(t) - V_n(t)}{l_n}. \tag{3}$$

Substituting $V_k(t) = U_k e^{i\lambda t}$ into (1)–(3) we obtain the following spectral problem:

$$\frac{U_k - U_{k-1}}{l_{k-1}} + \frac{U_k - U_{k+1}}{l_k} - m_k \lambda^2 U_k = 0 \quad (k = 1, 2, \dots, n), \tag{4}$$

$$U_0 = U_{n+1}, \tag{5}$$

$$\frac{U_1 - U_0}{l_0} = \frac{U_{n+1} - U_n}{l_n}, \tag{6}$$

where λ is the spectral parameter and U_k is the vibration amplitude of k-th mass.

With the notations,

$$Y = \{U_0, U_1, U_2, \dots, U_{n+1}\}^T,$$

$$M = \operatorname{diag}\{m_1, m_2, \dots, m_n, 0, 0\},$$

$$K = \begin{pmatrix} -\frac{1}{l_0} & \frac{1}{l_1} + \frac{1}{l_0} & -\frac{1}{l_1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\frac{1}{l_1} & \frac{1}{l_2} + \frac{1}{l_1} & -\frac{1}{l_2} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{l_2} & \frac{1}{l_3} + \frac{1}{l_2} & -\frac{1}{l_3} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\frac{1}{l_{n-1}} & \frac{1}{l_n} + \frac{1}{l_{n-1}} & -\frac{1}{l_n} \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \\ -\frac{1}{l_0} & \frac{1}{l_0} & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{l_n} & -\frac{1}{l_n} \end{pmatrix}$$

problem (4)–(6) can be rewritten as an eigenvalue problem generated by the operator pencil $K - \lambda^2 M$.

According to [1] we look for the solution $\{U_0, U_1, \dots, U_n, U_{n+1}\}$ of problem (4)–(6) in the form

$$U_k = R_{2k-2}(\lambda^2)a + Q_{2k-2}(\lambda^2)b, \tag{7}$$

where $R_{2k-2}(\lambda^2)$ and $Q_{2k-2}(\lambda^2)$ are the polynomials defined by the recurrence relations

$$\begin{aligned} R_{2k}(\lambda^2) &= l_k R_{2k-1}(\lambda^2) + R_{2k-2}(\lambda^2), \\ R_{2k-1}(\lambda^2) &= R_{2k-3}(\lambda^2) - m_k \lambda^2 R_{2k-2}(\lambda^2), \end{aligned} \tag{8}$$

$$\begin{aligned} Q_{2k}(\lambda^2) &= l_k Q_{2k-1}(\lambda^2) + Q_{2k-2}(\lambda^2), \\ Q_{2k-1}(\lambda^2) &= Q_{2k-3}(\lambda^2) - m_k \lambda^2 Q_{2k-2}(\lambda^2), \end{aligned} \tag{9}$$

with the initial conditions

$$R_0(\lambda^2) = 1, \quad R_{-1}(\lambda^2) = \frac{1}{l_0}, \tag{10}$$

$$Q_0(\lambda^2) = 1, \quad Q_{-1}(\lambda^2) = 0. \tag{11}$$

Let us prove an analogue of the Lagrange identity [7],

$$R_{2k-1}(\lambda^2)Q_{2k}(\lambda^2) - R_{2k}(\lambda^2)Q_{2k-1}(\lambda^2) = \frac{1}{l_0}. \tag{12}$$

Using (8)–(9) we get

$$\begin{aligned} &R_{2k-1}(\lambda^2)Q_{2k}(\lambda^2) - R_{2k}(\lambda^2)Q_{2k-1}(\lambda^2) \\ &\quad = l_k Q_{2k-1}(\lambda^2)R_{2k-3}(\lambda^2) + R_{2k-3}(\lambda^2)Q_{2k-2}(\lambda^2) - l_k m_k \lambda^2 R_{2k-2}(\lambda^2)Q_{2k-1}(\lambda^2) \\ &\quad - l_k R_{2k-1}(\lambda^2)Q_{2k-3}(\lambda^2) - R_{2k-2}(\lambda^2)Q_{2k-3}(\lambda^2) + l_k m_k \lambda^2 R_{2k-1}(\lambda^2)Q_{2k-2}(\lambda^2). \end{aligned}$$

Substituting

$$R_{2k-3}(\lambda^2) = R_{2k-1}(\lambda^2) + m_k \lambda^2 R_{2k-2}(\lambda^2)$$

we obtain

$$\begin{aligned} &R_{2k-1}(\lambda^2)Q_{2k}(\lambda^2) - R_{2k}(\lambda^2)Q_{2k-1}(\lambda^2) \\ &\quad = l_k R_{2k-1}(\lambda^2)(Q_{2k-1}(\lambda^2) - Q_{2k-3}(\lambda^2)) + R_{2k-3}(\lambda^2)Q_{2k-2}(\lambda^2) \\ &\quad - R_{2k-2}(\lambda^2)Q_{2k-3}(\lambda^2) + l_k m_k \lambda^2 R_{2k-1}(\lambda^2)Q_{2k-2}(\lambda^2) \\ &\quad = R_{2k-3}(\lambda^2)Q_{2k-2}(\lambda^2) - R_{2k-2}(\lambda^2)Q_{2k-3}(\lambda^2). \end{aligned}$$

Repeating this procedure and using (10) and (11) we arrive at

$$R_{2k-1}(\lambda^2)Q_{2k}(\lambda^2) - R_{2k}(\lambda^2)Q_{2k-1}(\lambda^2) = R_{-1}(\lambda^2)Q_0(\lambda^2) - R_0(\lambda^2)Q_{-1}(\lambda^2) = \frac{1}{l_0},$$

what is to be proved.

Due to (8)–(11) the vector $(U_0, U_1, U_2, \dots, U_n, U_{n+1})$, where U_k is given by (7), satisfies (4).

Equation (5) with account of (8)–(11) implies

$$R_{2n}(\lambda^2)a + \big(Q_{2n}(\lambda^2) - 1\big)\, b = 0. \tag{13}$$

From (6) we obtain

$$\left(\frac{1}{l_0} - R_{2n-1}(\lambda^2)\right) a - Q_{2n-1}(\lambda^2)b = 0. \tag{14}$$

The system of linear algebraic equations (13)–(14) with respect to the unknowns a and b possesses a nontrivial solution if and only if its determinant

$$\varphi(\lambda^2) = \begin{vmatrix} R_{2n}(\lambda^2) & Q_{2n}(\lambda^2) - 1 \\ \frac{1}{l_0} - R_{2n-1}(\lambda^2) & -Q_{2n-1}(\lambda^2) \end{vmatrix} \tag{15}$$

is zero.

Taking into account (12) we rewrite (15) in the form

$$\varphi(\lambda^2) \stackrel{def}{=} \frac{Q_{2n}(\lambda^2) + l_0 R_{2n-1}(\lambda^2) - 2}{l_0}. \tag{16}$$

As an auxiliary one we choose the Dirichlet problem generated by equations (4) and the boundary conditions

$$U_0 = 0,$$
$$U_{n+1} = 0.$$

It is clear that $R_{2n}(\lambda^2)$ is the characteristic polynomial of this problem called Dirichlet-Dirichlet.

Definition 1. A function $f(z)$ is said to belong to the Nevanlinna class if

1) it is analytic in the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$;
2) $f(\overline{z}) = \overline{f(z)}$ $(\operatorname{Im} z \neq 0)$;
3) $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z \neq 0$.

The class of Nevanlinna functions will be denoted by $\mathcal{N}$.

Definition 2. The function $f(z)$ analytic in $C\backslash[0, +\infty)$ is said to be an S-function if

1) $f(z) \in \mathcal{N}$,
2) $f(z) \geq 0$ for all $z < 0$.

We denote the class of S-functions by $\mathcal{S}$.

Where it is convenient we will use another parameter $z = \lambda^2$.

Theorem 1.

$$\frac{R_{2n}(z)}{\varphi(z)} \in S.$$

Proof. Consider the system of equations (4)

$$\frac{U_1 - U_2}{l_1} + \frac{U_1 - U_0}{l_0} - m_1 z U_1 = 0,$$
$$\frac{U_2 - U_3}{l_2} + \frac{U_2 - U_1}{l_1} - m_2 z U_2 = 0,$$
$$\dots$$
$$\frac{U_k - U_{k+1}}{l_k} + \frac{U_k - U_{k-1}}{l_{k-1}} - m_k z U_k = 0,$$
$$\dots$$
$$\frac{U_n - U_{n+1}}{l_n} + \frac{U_n - U_{n-1}}{l_{n-1}} - m_n z U_n = 0.$$

We multiply the first equation by $\overline{U}_1$ the second one by $\overline{U}_2$ and so on and the last one by $\overline{U}_n$ and add the obtained equations. Then we arrive at

$$\frac{U_1 - U_0}{l_0}\overline{U}_1 + \frac{U_n - U_{n+1}}{l_n}\overline{U}_n = z \sum_{k=1}^{n} m_k |U_k|^2 . \tag{17}$$

Now we take the imaginary part of (17)

$$\operatorname{Im}\left(\frac{U_n - U_{n+1}}{l_n}\overline{U}_n + \frac{U_1 - U_0}{l_0}\overline{U}_1\right) = \operatorname{Im} z \sum_{k=1}^{n} m_k \left|U_k\right|^2 . \tag{18}$$

Obviously,

$$\begin{aligned}\operatorname{Im}\left(\frac{U_n - U_{n+1}}{l_n}\overline{U}_n\right) &= \operatorname{Im}\left(\frac{U_n - U_{n+1}}{l_n}(\overline{U}_n - \overline{U}_{n+1} + \overline{U}_{n+1})\right)\\ &= \operatorname{Im}\left(\frac{U_n - U_{n+1}}{l_n}\overline{U}_{n+1}\right)\end{aligned} \tag{19}$$

and

$$\operatorname{Im}\left(\frac{U_1 - U_0}{l_0}\overline{U}_1\right) = \operatorname{Im}\left(\frac{U_1 - U_0}{l_0}\overline{U}_0\right). \tag{20}$$

Due to (19), (20) and (5), equation (18) can be reduced to

$$\operatorname{Im}\left(\overline{U}_{n+1}\left(\frac{U_n - U_{n+1}}{l_n} + \frac{U_1 - U_0}{l_0}\right)\right) = \operatorname{Im} z \sum_{k=1}^{n} m_k \left|U_k\right|^2 . \tag{21}$$

If we multiply this equation by $\left|U_{n+1}\right|^{-2}$, then we obtain

$$-\operatorname{Im} \frac{\dfrac{U_1 - U_0}{l_0} - \dfrac{U_{n+1} - U_n}{l_n}}{U_{n+1}} = \frac{\operatorname{Im} z}{\left|U_{n+1}\right|^2} \sum_{k=1}^{n} m_k \left|U_k\right|^2 \tag{22}$$

or

$$-\operatorname{Im}\left(\frac{U_{n+1}}{\dfrac{U_1 - U_0}{l_0} - \dfrac{U_{n+1} - U_n}{l_n}}\right)^{-1} = \frac{\operatorname{Im} z}{\left|U_{n+1}\right|^2} \sum_{k=1}^{n} m_k \left|U_k\right|^2 . \tag{23}$$

Hence,

$$-\operatorname{Im} z \operatorname{Im}\left(\frac{U_{n+1}}{\dfrac{U_1 - U_0}{l_0} - \dfrac{U_{n+1} - U_n}{l_n}}\right)^{-1} > 0$$

and, consequently,

$$\operatorname{Im} z \operatorname{Im}\left(\frac{U_{n+1}}{\dfrac{U_1 - U_0}{l_0} - \dfrac{U_{n+1} - U_n}{l_n}}\right) > 0.$$

The set of zeros of the denominator in the left-hand side of the above inequality coincides with the set of zeros of the function $\varphi(z)$ while zeros of the numerator coincide with the zeros of $R_{2n}(z)$.

Therefore, $\dfrac{R_{2n}(z)}{\varphi(z)}$ is a Nevanlinna function.

Using equations (8)–(11) we conclude that for $z < 0$ the following inequalities are true:

$$Q_{2k}(z) \geq 1, \quad l_0 R_{2k-1}(z) \geq 1 \quad (k = 1, 2, \ldots, n). \tag{24}$$

Thus,

$$\frac{R_{2n}(z)}{\varphi(z)} \in S. \tag{25}$$

Theorem is proved. □

Let us prove that the maximal possible multiplicity of zeros of $\varphi(z)$ is 2. It can happen that for some value of z all elements of the matrix of system (13)–(14) are equal to 0, i.e.,

$$R_{2n}(z) = Q_{2n}(z) - 1 = \frac{1}{l_0} - R_{2n-1}(z) = -Q_{2n-1}(z) = 0.$$

In this case system of equations (13)–(14) possesses two linearly independent solutions, i.e., the multiplicity of the corresponding zero of $\varphi(z)$ is 2. Zeros of $R_{2n}(z)$ are simple (see [1]) and interlaced with zeros of $\varphi(z)$ and (25) would be impossible if the multiplicity of a zero of $\varphi(z)$ exceeded 2.

2. In this section we consider a figure-of-eight metric graph which consists of two loops of lengths L_1 and L_2 joined at the vertex. The first loop is a Stieltjes string bearing point masses m_k, $k = 1, 2, \ldots, n_1$, which together with the vertex break the string into subintervals $l_k > 0$, $k = 0, 1, \ldots, n_1$ ($\sum_{k=0}^{n_1} l_k = L_1$). The amplitudes of vibrations of these masses can be described by (4) with $n = n_1$. The second loop bears point masses $\widetilde{m}_k$, $k = 1, 2, \ldots, n_2$, which together with the vertex break the string into subintervals $\tilde{l}_k > 0$, $k = 0, 1, \ldots, n_2$ ($\sum_{k=0}^{n_2} \tilde{l}_k = L_2$). Small transversal vibrations of the second string can be described by the amplitude vector $\{\widetilde{U}_0, \widetilde{U}_1, \ldots, \widetilde{U}_{n+1}\}$, where $\widetilde{U}_k$ satisfy the equations

$$\frac{\widetilde{U}_k - \widetilde{U}_{k-1}}{\tilde{l}_{k-1}} + \frac{\widetilde{U}_k - \widetilde{U}_{k+1}}{\tilde{l}_k} + \widetilde{m}_k \lambda^2 \widetilde{U}_k = 0 \quad (k = 1, 2, \ldots, n_2). \tag{26}$$

The continuity conditions at the vertex are

$$U_0 = U_{n+1} = \widetilde{U}_0 = \widetilde{U}_{n+1}, \tag{27}$$

and the Kirchhoff condition gives

$$\frac{U_1 - U_0}{l_0} + \frac{\widetilde{U}_1 - \widetilde{U}_0}{\tilde{l}_0} = \frac{U_{n+1} - U_n}{l_n} + \frac{\widetilde{U}_{n+1} - \widetilde{U}_n}{\tilde{l}_n}. \tag{28}$$

Let us introduce the notations

$$X = \{U_0, U_1, U_2, \ldots, U_{n+1}\}^T,$$
$$\widetilde{X} = \{\widetilde{U}_0, \widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_{n+1}\}^T,$$
$$M = \mathrm{diag}\{m_1, m_2, \ldots, m_n, 0\},$$
$$\widetilde{M} = \mathrm{diag}\{\widetilde{m}_1, \widetilde{m}_2, \ldots, \widetilde{m}_n, 0\},$$

$$A = \begin{pmatrix} -\frac{1}{l_0} & \frac{1}{l_1} + \frac{1}{l_0} & -\frac{1}{l_1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -\frac{1}{l_1} & \frac{1}{l_2} + \frac{1}{l_1} & -\frac{1}{l_2} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{l_2} & \frac{1}{l_3} + \frac{1}{l_2} & -\frac{1}{l_3} & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & -\frac{1}{l_{n-1}} & \frac{1}{l_n} + \frac{1}{l_{n-1}} & -\frac{1}{l_n} \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 \end{pmatrix}.$$

By $\widetilde{A}$ we denote the matrix obtained from A by change l_k for $\tilde{l}_k$ in each element. Then we rewrite problem (4), (26)–(28) in a matrix form,

$$\begin{bmatrix} A & & & & & 0 & & & & \\ 0 & & & & & \widetilde{A} & & & & \\ 1 \ldots\ldots\ldots\ldots\ldots 0 & & & & & -1 \ldots\ldots\ldots\ldots\ldots 0 & & & & \\ -\frac{1}{l_0} \ \ \frac{1}{l_0} \ \ 0\ldots0 \ \ \frac{1}{l_n} \ \ -\frac{1}{l_n} & & & & & -\frac{1}{\tilde{l}_0} \ \ \frac{1}{\tilde{l}_0} \ \ 0\ldots0 \ \ \frac{1}{\tilde{l}_n} \ \ -\frac{1}{\tilde{l}_n} & & & & \end{bmatrix} \begin{bmatrix} X \\ \widetilde{X} \\ 0 \\ 0 \end{bmatrix} = \lambda^2 \begin{bmatrix} M \\ \widetilde{M} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} X \\ \widetilde{X} \\ 0 \\ 0 \end{bmatrix}.$$

We look for the solution $(U_0, U_1, U_2, \dots, U_n, U_{n+1}, \widetilde{U}_0, \widetilde{U}_1, \widetilde{U}_2, \dots, \widetilde{U}_n, \widetilde{U}_{n+1})$ of the system composed by equations (4) with $n = n_1$ and by (26)–(28) in the form

$$U_k = R_{2k-2}(\lambda^2)a + Q_{2k-2}(\lambda^2)b, \tag{29}$$

$$\widetilde{U}_k = \widetilde{R}_{2k-2}(\lambda^2)\widetilde{a} + \widetilde{Q}_{2k-2}(\lambda^2)\widetilde{b}, \tag{30}$$

where $R_{2k-2}(\lambda^2)$ and $Q_{2k-2}(\lambda^2)$ are the polynomials defined by (8)–(11). Analogous relations with $\widetilde{m}_k$ and $\widetilde{l}_k$ instead of $\widetilde{m}_k$ and $\widetilde{l}_k$ are true for $\widetilde{R}_{2k-2}(\lambda^2)$ and $\widetilde{Q}_{2k-2}(\lambda^2)$.

The solution $(U_0, U_1, U_2, \dots, U_n, U_{n+1}, \widetilde{U}_1, \widetilde{U}_2, \dots, \widetilde{U}_n, \widetilde{U}_{n+1})$ where U_k and $\widetilde{U}_k$ are given by (29), (30) satisfies (4) with $n = n_1$ and (26). With account of (8)–(11) we obtain from (27) and (28) that

$$b = \widetilde{b}, \tag{31}$$

$$R_{2n_1}(\lambda^2)a + \left(Q_{2n_1}(\lambda^2) - 1\right) b = 0, \tag{32}$$

$$\widetilde{R}_{2n_2}(\lambda^2)\widetilde{a} + \left(\widetilde{Q}_{2n_2}(\lambda^2) - 1\right) \widetilde{b} = 0, \tag{33}$$

$$\left(R_{2n_1-1}(\lambda^2) - \frac{1}{l_0}\right) a + \left(\widetilde{R}_{2n_2-1}(\lambda^2) - \frac{1}{\widetilde{l}_0}\right) \widetilde{a} + Q_{2n_1-1}(\lambda^2)b + \widetilde{Q}_{2n_2-1}(\lambda^2)\widetilde{b} = 0. \tag{34}$$

Equations (31)–(34) complete a homogeneous system with respect to the unknowns $a, \widetilde{a}, b, \widetilde{b}$. This system possesses a nontrivial solution if and only if its determinant

$$\begin{aligned}\varphi(\lambda^2) \stackrel{\text{def}}{=} {} & R_{2n_1}(\lambda^2) \left(\frac{1}{\widetilde{l}_0} \widetilde{Q}_{2n_2}(\lambda^2) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(\lambda^2) \right) \\ & + \widetilde{R}_{2n_2}(\lambda^2) \left(\frac{1}{l_0} Q_{2n_1}(\lambda^2) - \frac{2}{l_0} + R_{2n_1-1}(\lambda^2) \right)\end{aligned} \tag{35}$$

is equal to 0.

Equation (35) can be interpreted as follows. The conditions of continuity at an interior vertex of the graph together with the Kircchoff condition can be considered as generalized Neumann conditions. If these conditions are imposed at a pendant vertex of the graph, then the continuity condition is fulfilled automatically and the Kirchhoff condition coincides with the classical Neumann condition. Therefore, $\varphi(\lambda^2)$ in (35) is nothing but the characteristic polynomial of the Neumann boundary value problem on the figure-of-eight graph (with the Neumann condition at the vertex). The polynomials

$$\frac{1}{l_0} Q_{2n_1}(\lambda^2) - \frac{2}{l_0} + R_{2n_1-1}(\lambda^2)$$

and

$$\frac{1}{\widetilde{l}_0} \widetilde{Q}_{2n_2}(\lambda^2) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(\lambda^2),$$

which are characteristic functions of the periodic problems on the loops can be interpreted as characteristic polynomials for the Neumann eigenvalue problems on the loops. The characteristic polynomials of the Dirichlet-Dirichlet problems on the loops are $R_{2n_1}(\lambda^2)$ and $\tilde{R}_{2n_2}(\lambda^2)$.

Thus, the relation

$$\varphi_N(\lambda^2) = \varphi_N^I(\lambda^2)\varphi_D^{II}(\lambda^2) + \varphi_N^{II}(\lambda^2)\varphi_D^I(\lambda^2), \tag{36}$$

obtained in [5] for the Sturm-Liouville problems on trees appears to be valid in our case too.

Theorem 2. *Let $\{\lambda_k\}_{k=-(n_1+n_2), k\neq 0}^{n_1+n_2}$, $\lambda_{-k} = -\lambda_k$, be a set of eigenvalues of the boundary value problem composed by equations (4) with $n = n_1$ and by equations (26)-(28); let*

$\left\{\nu_k^{(1)}\right\}_{k=-n_1,k\neq 0}^{n_1}$, $\nu_{-k}^{(1)}=-\nu_k^{(1)}$ *be a set of eigenvalues of the Dirichlet-Dirichlet problem generated by the string of the first loop on the interval of length* L_1 *and* $\left\{\nu_k^{(2)}\right\}_{k=-n_2,k\neq 0}^{n_2}$ *be the set of eigenvalues of the Dirichlet-Dirichlet eigenvalue problem generated by the string of the second loop on the interval of length* L_2. *Denote by*

$$\{\xi_k\}_{k=-(n_1+n_2),k\neq 0}^{n_1+n_2}=\left\{\nu_k^{(1)}\right\}_{k=-n_1,k\neq 0}^{n_1}\bigcup\left\{\nu_k^{(2)}\right\}_{k=-n_2,k\neq 0}^{n_2}.$$

Then $\{\lambda_k\}_{k=-(n_1+n_2),k\neq 0}^{n_1+n_2}$ *are interlaced with* $\{\xi_k\}_{k=-(n_1+n_2),k\neq 0}^{n_1+n_2}$

$$0=\lambda_1^2<\xi_1^2\leq\cdots\leq\lambda_{n_1+n_2}^2\leq\xi_{n_1+n_2}^2.$$

Proof. According to the proof of Theorem 1, equation (18) is fulfilled for the first loop as well as the equation

$$\operatorname{Im}\left(\overline{\widetilde{U}}_{n+1}\left(\frac{\widetilde{U}_n-\widetilde{U}_{n+1}}{\widetilde{l}_n}+\frac{\widetilde{U}_1-\widetilde{U}_0}{\widetilde{l}_0}\right)\right)=\operatorname{Im}z\left(\sum_{k=1}^{n}\widetilde{m}_k\left|\widetilde{U}_k\right|^2\right) \tag{37}$$

is satisfied for the second loop. Adding (18) to (37) we obtain

$$\begin{aligned}&\operatorname{Im}\left(\overline{U}_{n+1}\left(\frac{U_n-U_{n+1}}{l_n}+\frac{U_1-U_0}{l_0}\right)\right)+\operatorname{Im}\left(\overline{\widetilde{U}}_{n+1}\left(\frac{\widetilde{U}_n-\widetilde{U}_{n+1}}{\widetilde{l}_n}+\frac{\widetilde{U}_1-\widetilde{U}_0}{\widetilde{l}_0}\right)\right)\\&=\operatorname{Im}z\left(\sum_{k=1}^{n}m_k\left|U_k\right|^2+\sum_{k=1}^{n}\widetilde{m}_k\left|\widetilde{U}_k\right|^2\right).\end{aligned} \tag{38}$$

This implies

$$\begin{aligned}&-\operatorname{Im}\left(\frac{1}{U_{n+1}}\left(\frac{U_1-U_0}{l_0}+\frac{\widetilde{U}_1-\widetilde{U}_0}{\widetilde{l}_0}-\frac{U_{n+1}-U_n}{l_n}-\frac{\widetilde{U}_{n+1}-\widetilde{U}_n}{\widetilde{l}_n}\right)\right)\\&=\operatorname{Im}z\frac{\left(\sum_{k=1}^{n}m_k\left|U_k\right|^2+\sum_{k=1}^{n}\widetilde{m}_k\left|\widetilde{U}_k\right|^2\right)}{|U_{n+1}|^2}.\end{aligned} \tag{39}$$

The right-hand side of (39) is positive for $\operatorname{Im}z>0$ and negative for $\operatorname{Im}z<0$, therefore,

$$-\operatorname{Im}z\operatorname{Im}\left(\frac{U_{n+1}}{\frac{U_1-U_0}{l_0}+\frac{\widetilde{U}_1-\widetilde{U}_0}{\widetilde{l}_0}-\frac{U_{n+1}-U_n}{l_n}-\frac{\widetilde{U}_{n+1}-\widetilde{U}_n}{\widetilde{l}_n}}\right)^{-1}>0.$$

Thus we obtain

$$\operatorname{Im}z\operatorname{Im}\left(\frac{U_{n+1}}{\frac{U_1-U_0}{l_0}+\frac{\widetilde{U}_1-\widetilde{U}_0}{\widetilde{l}_0}-\frac{U_{n+1}-U_n}{l_n}-\frac{\widetilde{U}_{n+1}-\widetilde{U}_n}{\widetilde{l}_n}}\right)>0.$$

The set of zeros of the denominator

$$\frac{U_1-U_0}{l_0}+\frac{\widetilde{U}_1-\widetilde{U}_0}{\widetilde{l}_0}-\frac{U_{n+1}-U_n}{l_n}-\frac{\widetilde{U}_{n+1}-\widetilde{U}_n}{\widetilde{l}_n}$$

coincides with the set $\{\lambda_k^2\}_{k=1}^{n_1+n_2}$ of zeros of $\varphi(z)$ while the set of zeros of the numerator coincides with the union of the sets $\{(\nu_k^{(1)})^2\}_{k=1}^{n_1}$ and $\{(\nu_k^{(2)})^2\}_{k=1}^{n_2}$ where $\left\{\nu_k^{(1)}\right\}_{k=-n_1,k\neq 0}^{n_1}$,

$\nu_{-k}^{(1)} = -\nu_k^{(1)}$, is the spectrum of the problem generated by equations (4), (5) together with the equation $U_{n+1} = 0$, while $\left\{\nu_k^{(2)}\right\}_{k=-n_1,k\neq 0}^{n_2}$, $\nu_{-k}^{(2)} = -\nu_k^{(2)}$ is the spectrum of the problem generated by equations (26) and the equations $\widetilde{U}_0 = \widetilde{U}_{n+1} = 0$.

Thus, $\dfrac{R_{2n_1}(z)\widetilde{R}_{2n_2}(z)}{\varphi(z)} \in N$.

Obviously, similar to (24), the inequalities

$$\widetilde{Q}_{2k}(z) \geq 1, \quad \widetilde{l}_0\widetilde{R}_{2k-1}(z) \geq 1 \quad (k = (1, 2, \ldots, n_2)$$

are true for $z < 0$. Hence, after reduction of the fraction, $\dfrac{R_{2n_1}(z)\widetilde{R}_{2n_2}(z)}{\varphi(z)} \in S$. Due to the property of S-functions we conclude that its poles are interlaced with its zeros.

To finish the proof it is enough to show that $\lambda_1 = 0$. This is true because it is clear that all the eigenvalues are nonnegative and it is easy to check that the vector with

$$U_0 = U_1 = \cdots = U_{n+1} = \widetilde{U}_0 = \widetilde{U}_1 = \cdots = \widetilde{U}_{n+1} = \text{const} \neq 0$$

is the eigenvector corresponding to $\lambda_1 = 0$. □

Let $\varphi(z)$ be the characteristic polynomial (35) of problem (26)–(28) with $z = \lambda^2$. We know that all the zeros of $R_{2n_1}(z)$ and $\widetilde{R}_{2n_2}(z)$ are simple and we have showed at the end of Section 1 that the maximal multiplicity of zeros of the polynomials $\frac{1}{\widetilde{l}_0}\widetilde{Q}_{2n_2}(z) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(z)$ and $\frac{1}{l_0}Q_{2n_1}(z) - \frac{2}{l_0} + R_{2n_1-1}(z)$ can be 2. Moreover, it is possible that for some value of z the equations

$$\frac{1}{\widetilde{l}_0}\widetilde{Q}_{2n_2}(z) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(z) = R_{2n_1}(z) = \frac{1}{l_0}Q_{2n_1}(z) - \frac{2}{l_0} + R_{2n_1-1}(z) = R_{2n_2}(z) = 0$$

are satisfied. Thus it follows from (35) that the multiplicity of a zero of $\varphi(z)$ can be equal and does not exceed 3.

Theorem 3. *Let $\{\lambda_k\}_{k=-(n_1+n_2),k\neq 0}^{n_1+n_2}$, $\lambda_{-k} = -\lambda_k$, be a set of eigenvalues of the problem which consists of equations (4), with $n = n_1$ and by equations (26)–(28), let $\left\{\mu_k^{(1)}\right\}_{k=-n_1,k\neq 0}^{n_1}$, $\mu_{-k}^{(1)} = -\mu_k^{(1)}$, be a set of eigenvalues of the periodic problem generated by the first string on the loop of the length L_1 and let $\left\{\mu_k^{(2)}\right\}_{k=-n_2,k\neq 0}^{n_2}$, $\mu_{-k}^{(2)} = -\mu_k^{(2)}$, be a set of eigenvalues of the periodic problem generated by the second string on the loop of the length L_2. Denote*

$$\{\zeta_k\}_{k=-(n_1+n_2),k\neq 0}^{n_1+n_2} = \left\{\mu_k^{(1)}\right\}_{k=-n_1,k\neq 0}^{n_1} \bigcup \left\{\mu_k^{(2)}\right\}_{k=-n_2,k\neq 0}^{n_2}.$$

Then $\{\lambda_k\}_{k=-(n_1+n_2),k\neq 0}^{n_1+n_2}$ is interlaced with $\{\zeta_k\}_{k=-(n_1+n_2),k\neq 0}^{n_1+n_2}$ in the following sense:

$$0 = \zeta_1^2 = \lambda_1^2 = \zeta_2^2 \leq \lambda_2^2 \leq \cdots \leq \zeta_{n_1+n_2}^2 \leq \lambda_{n_1+n_2}^2,$$

where $0 = \zeta_1^2 = \lambda_1^2 = \zeta_2^2 = \mu_1^{(1)} = \mu_1^{(2)}$.

Proof. With account of (35) we evaluate

$$\frac{\varphi(z)}{\left(\frac{1}{l_0}Q_{2n_1}(z) - \frac{2}{l_0} + R_{2n_1-1}(z)\right)\left(\frac{1}{\widetilde{l}_0}\widetilde{Q}_{2n_2}(z) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(z)\right)}$$
$$= \frac{R_{2n_1}(z)\left(\frac{1}{\widetilde{l}_0}\widetilde{Q}_{2n_2}(z) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(z)\right) + \widetilde{R}_{2n_2}(z)\left(\frac{1}{\widetilde{l}_0}Q_{2n_1}(z) - \frac{2}{l_0} + R_{2n_1-1}(z)\right)}{\left(\frac{1}{l_0}Q_{2n_1}(z) - \frac{2}{l_0} + R_{2n_1-1}(z)\right)\left(\frac{1}{\widetilde{l}_0}\widetilde{Q}_{2n_2}(z) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(z)\right)}$$

$$= \frac{R_{2n_1}(z)}{\frac{1}{l_0}Q_{2n_1}(z) - \frac{2}{l_0} + R_{2n_1-1}(z)} + \frac{\widetilde{R}_{2n_2}(z)}{\frac{1}{\widetilde{l}_0}\widetilde{Q}_{2n_1}(z) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(z)}.$$

According to Theorem 1 the both rational functions in the last expression are S-functions. Therefore,

$$\frac{\varphi(z)}{\left(\frac{1}{l_0}Q_{2n_1}(z) - \frac{2}{l_0} + R_{2n_1-1}(z)\right)\left(\frac{1}{\widetilde{l}_0}\widetilde{Q}_{2n_2}(z) - \frac{2}{\widetilde{l}_0} + \widetilde{R}_{2n_2-1}(z)\right)} \in S,$$

and the sets $\left\{\zeta_k^2\right\}_{k=1}^{n_1+n_2}$ and $\left\{\lambda_k^2\right\}_{k=1}^{n_1+n_2}$ are interlaced. Problem (4)–(6) has the solution $U_0 = U_1 = ... = U_{n+1} = const \neq 0$ for $\lambda = 0$. The same is true for the periodic problem on the second loop. Hence, $\zeta_1^2 = \zeta_2^2 = 0$. Theorem is proved. □

Analogues of Theorems 2, 3 for eigenvalue problems generated by Sturm-Liouville equations on trees were obtained in [8].

References

1. F. R. Gantmakher, M. G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, Gostechizdat, Moscow—Leningrad, 1950. (Russian); German transl. Akademie Verlag, Berlin, 1960.
2. W. Cauer, *Die Verwirklichung von Wechselstromwiderstandem Vorgeschriebner Frequenzabhangigkeit*, Arch. Electrotech **17** (1926), 355–388.
3. G. V. Zeveke, P. A. Ionkin, A. V. Netushil, S. B. Strakhov, *Foundations of Cirquits Theory*, Energia, Moscow, 1975. (Russian)
4. I. C. Gohberg, M. G. Krein, *Introduction to the Theory of Linear Non-Selfadjoint Operators in Hilbert Space*, Nauka, Moscow, 1965. (Russian)
5. C.-K. Law, V. Pivovarchik, *Characteristic functions of quantum graphs*, J. of Phys. A: Math. Theor. **42** (2009), no. 3, 035302, 12 pp.
6. A. M. Gomilko, V. Pivovarchik, *The universe Sturm-Liouville problem on a figure-eight-type graph*, Ukrain. Mat. Zh. **60** (2008), no. 9, 1168–1188. (Russian); English transl. Ukrainian Math. J. **60** (2008), no. 9, 1360–1385.
7. I. S. Kac, M. G. Krein, *On the Spectral Functions of the String*, Supplement II in the Russian translation of the book by F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Mir, Moscow, 1968, 648–737; English transl. Amer. Math. Soc. Transl., Ser. 2, **103** (1974), no. 9, 19–102.
8. V. N. Pivovarchik, *Existence of a tree of Stieltjes strings corresponding to two given spectra*, J. of Phys. A: Math. Theor. **42** (2009), no. 37, 375213, 16 pp.

South-Ukrainian National Pedagogical University, 26 Staroportofrankivs'ka, Odessa, 65023, Ukraine
E-mail address: martynyuk_olga@mail.ru

South-Ukrainian National Pedagogical University, 26 Staroportofrankivs'ka, Odessa, 65023, Ukraine
E-mail address: v.pivovarchik@paco.net

Received 23/02/2010; Revised 28/05/2010