DIRECT AND INVERSE PROBLEMS FOR GENERALIZED PICK MATRIX

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This paper is dedicated to memory of Israel Gohberg.

ABSTRACT. All matrix modifications of classical Nevanlinna-Pick interpolation problem with a finite number of nonreal nodes which can be investigated by V. P. Potapov method are described.

1. INTRODUCTION

In the present paper some aspects of matrix generalizations of the classical Nevanlinna-Pick interpolation problem are considered in the class of Nevanlinna functions.

1.1. A function $w(z)$ is said to be a Nevanlinna $\mathcal{R}$-*function* if it is holomorphic in the open upper half-plane $\mathbf{C}_+$ and maps $\mathbf{C}_+$ into the closed upper half-plane, i.e., $\operatorname{Im} w(z) \geq 0$ for $z \in \mathbf{C}_+$. It is known (see for example [1]) that an $\mathcal{R}$-function admits the integral representation

$$(1.1) \qquad w(z) = \alpha + \beta z + \int_{-\infty}^{\infty} ((t-z)^{-1} - t(1+t^2)^{-1}) \, d\sigma(t),$$

where α and β are real numbers, $\beta \geq 0$, $d\sigma(t) \geq 0$, $\int_{-\infty}^{\infty}(1+t^2)^{-1}d\sigma(t) < \infty$. We denote by $\mathcal{R}$ the set of all $\mathcal{R}$-functions.

The classical Nevanlinna-Pick problem is as follows. Given two sequences of complex numbers $\{z_k\}_1^n$, *nodes of interpolation*, and $\{w_k\}_1^n$, *interpolation data*, such that all z_k are pairwise distinct and lie in $\mathbf{C}_+$, find conditions under which there exists a function $w \in \mathcal{R}$ such that $w(z_k) = w_k$, $k = 1, 2, \ldots, n$. In the case when solutions exist it is necessary to describe all such functions. It is known (see for example [1]), that this problem is solvable if and only if *the Pick matrix*

$$W = \left(\frac{w_j - \overline{w_k}}{z_j - \overline{z_k}} \right)_{j,k=1}^n$$

is Hermite-nonnegative.

In the case where $z_1 = z_2 = \cdots = z_n(= z_0)$, we have one node of multiplicity n and along with $w_0 = w(z_0)$ the values of the derivatives $w^{(k)}(z_0)$, $k = 1, 2, \ldots, n-1$, should be included in the data. Then the interpolation problem can be stated as follows. Find all $\mathcal{R}$-functions $w(z)$ whose expansion in a neighborhood of z_0 is of the form

$$w(z) = w_0 + w_1(z - z_0) + \cdots + w_{n-1}(z - z_0)^{n-1} + o((z - z_0)^{n-1}) \quad (z \to z_0).$$

Considering this problem in the class of holomorphic functions mapping the open unit disk into the closed unit disk or into the closed right half-plane with $z_0 = 0$ we arrive at the classical Schur problem or the Carathéodory problem, respectively.

1.2. A direct generalization of the Nevanlinna-Pick problem for the matrix case differs from the scalar version in the following: here w_k are quadratic matrices of order m, i.e., $w_k \in \mathbf{C}^{m \times m}$ and the solution is to be found in the class $\mathcal{R}_m$ of matrix-valued functions $w(z)$ which satisfy the conditions

1) $w(z)$ take on values in $\mathbf{C}^{m \times m}$,
2) $w(z)$ are holomorphic in $\mathbf{C}_+$,
3) $\mathrm{Im}\,w(z)(= (w(z) - w(z)^*)/2i) \geq 0$ for $z \in \mathbf{C}_+$ (in the Hermite sense),
4) $w(z)$ are defined in the lower half-plane $\mathbf{C}_-$ by the formula

(1.2) $$w(z) = w(\overline{z})^*.$$

V. P. Potapov proposed an elegant method for investigation of the matrix Nevanlinna-Pick problem [3] based on "J-theory" developed by him [4]. Later on, this method has been used by him, his colleagues, and followers for treating the matrix Schur and Carathéodory problems, the infinite matrix moment problem, the problem of continuation of Hermite-positive matrix-valued functions (see [5]) and more complicated problems [6].

Here all possible matrix generalizations of Nevanlinna-Pick interpolation problems in the class $\mathcal{R}_m$ with a finite number of nonreal nodes of interpolation solvable by Potapov method are described.

Let us first describe a simple cases of matrix modifications of the Nevanlinna-Pick problem.

In the simplest left-sided tangential problem, pairwise different nodes of interpolation $\{z_k\}_1^n \subset \mathbf{C}_+$ are given together with two sequences of m-dimensional vector-rows $\{b_k\}_1^n$ and $\{c_k\}_1^n$. Conditions are to be found under which there exists $w \in \mathcal{R}_m$ such that

(1.3) $$b_k w(z_k) = c_k, \quad k = 1, 2, \ldots, n.$$

The formulation of the simplest right-sided Nevanlinna-Pick problem is similar: given pairwise different nodes of interpolation, $\{\zeta_k\}_1^n \subset \mathbf{C}_+$, together with two sequences of m-dimensional vector-columns $\{\beta_k\}_1^n$ and $\{\gamma_k\}_1^n$, find all $w \in \mathcal{R}_m$ for which

$$w(\zeta_k)\beta_k = \gamma_k, \quad k = 1, 2, \ldots, n.$$

Let us state the simplest bitangential Nevanlinna-Pick problem. Given two sequences $\{z_k\}_1^{n_1} \subset \mathbf{C}_+$ and $\{\zeta_k\}_1^{n_2} \subset \mathbf{C}_+$ of nodes of interpolation, two sequences of m-dimensional vector-rows $\{b_k\}_1^{n_1}$ and $\{c_k\}_1^{n_1}$ and two sequences of m-dimensional vector-columns $\{\beta_k\}_1^{n_2}$ and $\{\gamma_k\}_1^{n_2}$. If the sequences $\{z_k\}_1^{n_1}$ and $\{\zeta_k\}_1^{n_2}$ do not intersect then the task is to find $w \in \mathcal{R}_m$ for which

$$b_k w(z_k) = c_k, \quad k = 1, 2, \ldots, n_1, \quad w(\zeta_j)\beta_j = \gamma_j, \quad j = 1, 2, \ldots, n_2.$$

If, for example, $z_k = \zeta_j$ (denote these coinciding nodes by η_{kj}) then, in addition, $d_1 = b_k w'(\eta_{kj})\beta_j$ must be given. Thus if we set $d_0 = b_k w(\eta_{kj})\beta_j(= c_k\beta_j = b_k\gamma_j)$ then the additional condition can be represented in the form

$$b_k w(z)\beta_j = d_0 + d_1(z - \eta_{kj}) + o(z - \eta_{kj}), \quad z \to \eta_{kj}.$$

At first, one-sided tangential problem (for the Schur matrix-valued functions) was stated by M. G. Krein and was solved by I. P. Fedchina [14] under his guidance. The author stated more general one-sided tangential interpolation problems for Nevanlinna, Schur, and Caratheodory pairs of matrix-valued functions [12]. The two-sided problem was stated and solved by J. Ball, I. Gohberg, L. Rodman [15] in development of methods of [12].

2. MAIN RELATIONS

2.1. Let us deduce the main relations using the simplest left-sided tangential problem as an example. Let pairwise different points z_k, $k = 1, 2, \ldots, n$, in $\mathbf{C}_+$ be given together with two sequences of m-dimensional rows $\{b_k\}_1^n$ and $\{c_k\}_1^n$. It is necessary to find all $w \in \mathcal{R}_m$ for which

$$(2.1) \qquad b_k w(z_k) = c_k, \quad k = 1, 2, \ldots, n.$$

Let us find a necessary condition for solvability of the problem under consideration. The Pick matrix

$$U = \left(\frac{w(z_j) - w(z_k)^*}{z_j - \overline{z_k}} \right)_{j,k=1}^n$$

is positive semidefinite. Let a matrix $T \in \mathbf{C}^{n \times nm}$ be block diagonal were the k-th diagonal block is $b_k \in \mathbf{C}^{1 \times m}$. The matrix

$$W = TUT^* = \left(\frac{c_j b_k{}^* - b_j c_k{}^*}{z_j - \overline{z_k}} \right)_{j,k=1}^n$$

is positive semidefinite too. So a necessary condition for solvability of the above mentioned problem is $W \geq 0$. This condition is also sufficient (see Sec. 7).

2.2. Using integral representation (1.1) one can rewrite (2.1) as

$$(2.2) \qquad c_k = b_k \alpha + z_k b_k \beta + \int_{-\infty}^{\infty} ((t - z_k)^{-1} - t(1 + t^2)^{-1}) b_k \, d\sigma(t).$$

Introducing the matrices

$$A = \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

we rewrite (2.2) as follows:

$$(2.3) \qquad C = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI_n - A)^{-1} - t(1 + t^2)^{-1} I_n) \, d\sigma(t),$$

where I_n is the identity matrix of order n. We shall omit the index n if there are no doubts about the order. The symbol 0 stands for the zero matrix of appropriate size.

The conditions of many interpolation problems can be given in the form of (2.3) with an appropriate choice of matrices A, B, C.

2.3. There exists a simple connection between the matrices A, B, C and W. The identities

$$z_j \frac{c_j b_k{}^* - b_j c_k{}^*}{z_j - \overline{z_k}} - \frac{c_j b_k{}^* - b_j c_k{}^*}{z_j - \overline{z_k}} \overline{z_k} = c_j b_k{}^* - b_j c_k{}^*$$

represented in the matrix form are

$$(2.4) \qquad AW - WA^* = CB^* - BC^*.$$

Such matrix identities (in some other form) were used by V. P. Potapov, his students, and followers as a technical tool for factorization of certain matrices while solving matrix versions of classical interpolation problems. It should be mentioned that if the spectra of the matrices A and A^* do not intersect then relations (2.4) uniquely determine the matrix W (see, e.g. [7] or [8]), i.e., (2.4) permits to find W using the interpolation data. For example, for the simplest left-sided tangential problem rewriting (2.4) element-wise we obtain

$$z_j w_{jk} - \overline{z_k} w_{jk} = c_j b_k{}^* - b_j c_k{}^*,$$

which implies

$$W = \left(\frac{c_j b_k{}^* - b_j c_k{}^*}{z_j - \overline{z_k}} \right)^n_{j,k=1}.$$

2.4. Finally, let us obtain an integral representation for the matrix W in the considered case. Such representations are important for this paper.

Let us calculate

$$w_{jk} = \frac{c_j b_k{}^* - b_j c_k{}^*}{z_j - \overline{z_k}} = \frac{b_j w(z_j) b_k{}^* - b_j w(z_k)^* b_k{}^*}{z_j - \overline{z_k}}$$

using (2.2),

$$\begin{aligned}
w_{jk} &= \frac{1}{z_j - \overline{z_k}} \big(b_j (\alpha + \beta z_j + \int_{-\infty}^{\infty} ((t - z_j)^{-1} - t(1 + t^2)^{-1}) \, d\sigma(t)) b_k{}^* \\
&\quad - b_j (\alpha + \beta \overline{z_k} + \int_{-\infty}^{\infty} ((t - \overline{z_k})^{-1} - t(1 + t^2)^{-1}) \, d\sigma(t)) b_k{}^*) \\
&= b_j \beta b_k^* + \int_{-\infty}^{\infty} (t - z_j)^{-1} b_j \, d\sigma(t) b_k^* (t - \overline{z_k})^{-1}.
\end{aligned}$$

This implies

$$(2.5) \qquad W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}.$$

2.5. Using integral representation (1.1) for solutions and choosing appropriate matrices A, B and C, formulae (2.3) and (2.5) can be obtained for other solvable interpolation problems in $\mathcal{R}_m$. These matrices A, B and C and *the generalized Pick matrix W* in the considered problem satisfy relation (2.4). By a generalized Pick matrix we mean here a positive semidefinite matrix, which is a necessary and sufficient condition for solvability of the corresponding interpolation problem.

In the present paper we solve the inverse problem: given the *main matrix identity* (2.4) where $A \in \mathbf{C}^{N \times N}$, $B \in \mathbf{C}^{N \times m}$, $C \in \mathbf{C}^{N \times m}$, $(m \le N)$, and $W = W^* \ge 0$, $W \in \mathbf{C}^{N \times N}$ and the spectrum of A being non-real. The question is: does an interpolation problem exist for which W is a generalized Pick matrix? We give an affirmative answer to this question under the essential condition that rank $B = m$ (this condition is fulfilled for all direct matrix generalizations of classical interpolation problems in $\mathcal{R}_m$) and describe the corresponding interpolation problem. Therefore, forestalling we will call by a generalized Pick matrix the matrix W involved in (2.4).

2.6. The scheme of our investigation is as follows. We prove that from (2.4) and $W \ge 0$ directly follows the representation

$$(2.6) \qquad W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1},$$

where $\beta = \beta^* \in \mathbf{C}^{m \times m}$, $\beta \ge 0$, $d\sigma(t) \ge 0$, $\int_{-\infty}^{\infty} (1 + t^2)^{-1} d\sigma(t) < \infty$. For rank$B = m$, this representation implies

$$C = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI - A)^{-1} - t(1 + t^2)^{-1} I) B \, d\sigma(t),$$

where $\alpha = \alpha^* \in \mathbf{C}^{m \times m}$. Using the parameters α, β, $d\sigma(t)$ it is possible to introduce an $\mathcal{R}_m$-function by

$$w(z) = \alpha + \beta z + \int_{-\infty}^{\infty} ((t - z)^{-1} - t(1 + t^2)^{-1}) \, d\sigma(t).$$

This function will be called *associated with the quadruple* (A, B, C, W). It is shown in this paper that to every such a quadruple there corresponds an interpolation problem the set of solutions of which coincides with the set of the associated functions and that W is a generalized Pick matrix for this problem.

This approach is related to that of T. S. Ivanchenko and L. A. Sakhnovich [9] (see also [10]). However, these authors consider A, B, C and W as operators acting on a Hilbert space with restrictive conditions. Associated functions they called solutions of an interpolation problem not formulating this problem except for some particular cases. They also do not prove the integral representation of the nonnegative W and just assume its existence.

In this paper, also a simple algorithm is given to calculate the elements of the generalized Pick matrix using the interpolation data.

3. Block structure of the generalized Pick matrix

3.1. Definition. A quadruple $A \in \mathbf{C}^{N \times N}$, $B \in \mathbf{C}^{N \times m}$, $C \in \mathbf{C}^{N \times m}$, $W = W^* \in \mathbf{C}^{N \times N}$ (the pair A, B, respectively) is said to be *similar* to a quadruple $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{W}$, (the pair $\tilde{A}$, $\tilde{B}$ respectively) if there exists a non-singular matrix $T \in \mathbf{C}^{N \times N}$ such that

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = TC, \quad \tilde{W} = TWT^*$$

($\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, respectively).

It is easy to see that (2.4) is invariant under the similarity transformation, i.e., $\tilde{A}\tilde{W} - \tilde{W}\tilde{A}^* = \tilde{C}\tilde{B}^* - \tilde{B}\tilde{C}^*$ and under such a transformation formulae (2.3) and (2.5) are also invariant while α, β and $d\sigma(t)$ do not change. Thus the same set of associated $\mathcal{R}_m$-functions corresponds to similar quadruples if $W \geq 0$. The spectra of A and $\tilde{A}$ coincide, and $\tilde{W} \geq 0$ if and only if $W \geq 0$.

To investigate interpolation properties of $\mathcal{R}_m$-functions associated with (A, B, C, W) it is possible, due to the above mentioned, to assume that the matrix A is reduced to a lower Jordan form, $A = \operatorname{diag}(A_1, A_2, \ldots, A_\nu)$, where A_j is a lower Jordan cell the order of which is denoted by $n_j + 1$ (it is convenient for us to index rows and columns starting with 0),

$$A_j = \begin{pmatrix} z_j & 0 & 0 & \ldots & 0 & 0 \\ 1 & z_j & 0 & \ldots & 0 & 0 \\ 0 & 1 & z_j & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & z_j \end{pmatrix}.$$

Let us split B, C and W into blocks corresponding to the block structure of A,

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_\nu \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ \vdots \\ C_\nu \end{pmatrix}, \quad W = (W_{jk})_{j,k=1}^\nu,$$

where

$$B_j = \begin{pmatrix} b_0^j \\ \vdots \\ b_{n_j}^j \end{pmatrix} \in \mathbf{C}^{(n_j+1) \times m}, \quad C_j = \begin{pmatrix} c_0^j \\ \vdots \\ c_{n_j}^j \end{pmatrix} \in \mathbf{C}^{(n_j+1) \times m}, \quad W_{jk} \in \mathbf{C}^{(n_j+1) \times (n_k+1)}.$$

Evidently,

(3.1) $$A_j W_{jk} - W_{jk} A_k^* = C_j B_k^* - B_j C_k^*.$$

To simplify the notations, the elements $w_{pq}^{(jk)}$ of the block W_{jk} are denoted by v_{pq}. Comparing elements in both sides of (3.1) we arrive at

$$(3.2) \qquad v_{p-1,q} - v_{p,q-1} + (z_j - \overline{z_k})v_{pq} = c_p^j(b_q^k)^* - b_p^j(c_q^k)^*,$$
$$p = 0, 1, \ldots, n_j, \quad q = 0, 1, \ldots, n_k, \quad v_{-1,q} = 0, \quad v_{p,-1} = 0.$$

3.2. First assume that $z_j \neq \overline{z}_k$. Setting $p = 0$ and choosing $q = 0, 1, \ldots, n_k$ successively we find, from the relation

$$-v_{0,q-1} + (z_j - \overline{z_k})v_{0q} = c_0^j(b_p^k)^* - b_0^j(c_p^j)^*,$$

the elements of the first row ($p = 0$) of the block W_{jk}. In the same way setting $q = 0$ and $p = 0, 1, \ldots, n_j$ we find, from the relation

$$v_{p-1,0} + (z_j - \overline{z_k})v_{p0} = c_p^j(b_0^k)^* - b_p^j(c_0^k)^*,$$

the elements of the first column ($q = 0$). After this using (3.2) we can find all the rest of elements of the block W_{jk}. Thus, for $z_j \neq \overline{z_k}$, all the elements of the block W_{jk} are uniquely determined from elements of the matrices B_j, C_j, B_k^* and C_k^*. Notice that the condition $z_j \neq \overline{z_k}$ is fulfilled if all the eigenvalues of the matrix A lie in the upper (or in the lower) half-plane.

3.3. Let us consider now the case of $z_j = \overline{z}_k$. Then (3.2) can be reduced to

$$(3.3) \qquad v_{p-1,q} - v_{p,q-1} = c_p^j(b_q^k)^* - b_p^j(c_q^k)^*.$$

Without loss of generality we can assume that if $n_j \neq n_k$ then $n_j > n_k$. Otherwise, we can consider $W_{kj} = W_{jk}^*$ instead of W_{jk}. For $q = 0$ using (3.3) we obtain

$$v_{p-1,0} = c_p^j(b_0^k)^* - b_p^j(c_0^k)^*$$

for elements in the first column. Setting $p = 0, 1, \ldots, n_j$ we find all elements of this column except for the last one, $v_{n_j,0}$. Now (3.3) allow to find successively the elements $v_{p-1,1}, v_{p-2,2}, \ldots$ (the sum of indices is equal p) using $v_{p,0}$. The last element in this series is $v_{0,p}$ if $p \leq n_k$ and v_{p-n_k,n_k} if $p > n_k$. All the rest of the elements can be expressed in the same way in terms of elements of the last row, i.e., through $v_{n_j,0}, v_{n_j,1}, \ldots, v_{n_j,n_k}$. These elements of the last row can not be determined from (3.3) because they do not depend on elements of the matrices B_j, C_j, B_k^* and C_k^*.

4. INTEGRAL REPRESENTATION OF NON-NEGATIVE GENERALIZED PICK MATRIX

4.1. Let matrices $A \in \mathbf{C}^{N \times N}$, $B \in \mathbf{C}^{N \times m}$, $C \in \mathbf{C}^{N \times m}$, $(m < N)$ and $W \in \mathbf{C}^{N \times N}$, $W = W^*$ be given and satisfy the main matrix identity

$$(4.1) \qquad AW - WA^* = CB^* - BC^*.$$

It is supposed that the spectrum of A is non-real. The subsequent considerations are based on the following two theorems.

Theorem 1. *The following assertions are equivalent*:
  1) $W \geq 0$.
  2) *The representation*

$$(4.2) \qquad W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}$$

is valid where $\beta \in \mathbf{C}^{m \times m}$ is a Hermite-semidefinite matrix and $d\sigma(t)$ is a matrix-valued measure taking values in $\mathbf{C}^{m \times m}$ for which the integral $\int_{-\infty}^{\infty} (1 + t^2)^{-1} d\sigma(t)$ converges.
  In the case where $\det W = 0$, additionally $\operatorname{rank} B = m$ is required.

Theorem 2. 1) *If representation (4.2) is true and* $\operatorname{rank} B = m$, *then there exists a matrix* $\alpha = \alpha^* \in \mathbf{C}^{m \times m}$ *such that*

$$(4.3) \qquad C = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI_n - A)^{-1} - t(1+t^2)^{-1}I_n)\,d\sigma(t).$$

2)*If the spectra of the matrices* A *and* A^* *do not intersect then (4.3) implies (4.2).*

The parameters α, β, and $d\sigma(t)$ appeared in (4.3) allow to introduce the $\mathcal{R}_m$-functions

$$w(z) = \alpha + \beta z + \int_{-\infty}^{\infty} ((t-z)^{-1} - t(1+t^2)^{-1})\,d\sigma(t),$$

which we call *associated with the quadruple* (A, B, C, W). Their role was already mentioned in Sec. 2.

5. Proof of Theorem 1

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Let us start with the case of $W > 0$. While proving we will construct one of the representations of the matrix W.

5.1. Introduce the matrix

$$(5.1) \qquad V = AW + BC^*.$$

The main matrix identity $AW - WA^* = CB^* - BC^*$ implies that $V^* = V$.
We consider the linear pencil

$$(5.2) \qquad V - zW.$$

A number λ is said to be an eigenvalue of the pencil if there exists a vector-column $g \neq 0$, called an eigenvector, such that $(V - \lambda W)g = 0$. It is easy to see that the eigenvalues λ_j of the pencil coincide with the eigenvalues of the Hermitian matrix $W^{-1/2}VW^{-1/2}$ and, therefore, are real. The eigenvectors g_j of the pencil are related to the eigenvectors f_j of this matrix by the formula $g_j = W^{-1/2}f_j$. Since the vectors $\{f_j\}_1^N$ constitute a complete orthogonal system which can be considered normalized (i.e. $f_j^* f_k = \delta_{jk}$, where δ_{jk} is the Kronecker symbol), the eigenvectors of the pencil possess the following property:

$$(5.3) \qquad g_j^* W g_k = \delta_{jk}.$$

Using this property it is easy to verify that

$$(5.4) \qquad W = \sum_{j=1}^{N} W g_j g_j^* W.$$

Indeed, denote the right-hand side of (5.4) by U. Using (5.3) we obtain

$$U g_k = \sum_{j=1}^{N} W g_j \delta_{jk} = W g_k, \quad k = 1, 2, \ldots, N,$$

therefore $U = W$.

Using Definition (5.1) let us rewrite $(V - \lambda_j W)g_j = 0$ in the form

$$(A - \lambda_j I)W g_j + BC^* g_j = 0,$$

what implies that

$$W g_j = (\lambda_j I - A)^{-1} BC^* g_j, \quad j = 1, 2, \ldots, N.$$

Therefore (5.4) can be presented as

$$W = \sum_{j=1}^{N} (\lambda_j I - A)^{-1} BC^* g_j g_j^* CB^* (\lambda_j I - A^*)^{-1}.$$

Uniting the eigenvalues which are equal to each other in the groups (the group may consist of one eigenvalue), let us number these groups and let us denote the common value of the eigenvalues from the k-th group by t_k. Summing the items which correspond to these eigenvalues let us put

$$\rho_k = \sum_{\lambda_j = t_k} C^* g_j g_j{}^* C \geq 0 \quad (\rho_k \in \mathbf{C}^{m \times m})$$

and so

$$W = \sum_{k=1}^{M} (t_k I - A)^{-1} B \rho_k B^* (t_k I - A^*)^{-1},$$

where M is the number of the groups. Thus,

$$W = \int_{-\infty}^{\infty} (tI - A)^{-1} B d\sigma(t) B^* (tI - A^*)^{-1},$$

where $\sigma(t)$ is a piece-wise constant matrix-valued function with jumps ρ_k at the points t_k. Theorem is proved for the case of $W > 0$.

5.2. We start to consider the case of $W \geq 0$, $\det W = 0$, we assume temporarily that $V \neq 0$ and the pair of matrices (A, B) is controllable.

The initial definition of controllability in the control theory involves terms of this theory. We will just mention two criteria of controllability (see, e.g. [11]). The first criterion can easily be checked, the second is used in the proof of Theorem 1.

Proposition 1. *A pair of matrices $A \in \mathbf{C}^{N \times N}$ and $B \in \mathbf{C}^{N \times m}$ is controllable if and only if*

$$\operatorname{rank}\left(B, AB, A^2 B, \ldots, A^{N-1} B\right) = N.$$

Proposition 2. *A pair of matrices $A \in \mathbf{C}^{N \times N}$ and $B \in \mathbf{C}^{N \times m}$ is controllable if and only if for $f \in \mathbf{C}^{N \times 1}$ the identity $f^*(tI - A)^{-1} B \equiv 0$ is true only for $f = 0$.*

Let us show that for any controllable pair (A, B) there exists a positive matrix W_0 and a matrix $C_0 \in C^{N \times m}$ which satisfy the relation

(5.5) $AW_0 - W_0 A^* = C_0 B^* - B C_0^*.$

We set $P(t) = \det(tI - A)$,

$$C_0 = \int_{-\infty}^{\infty} (tI - A)^{-1} |P(t)|^2 B e^{-t^2} dt$$

(the integral converges, since $(tI - A)^{-1} P(t)$ is a polynomial matrix) and

$$W_0 = \int_{-\infty}^{\infty} (tI - A)^{-1} B B^* (tI - A^*)^{-1} |P(t)|^2 e^{-t^2} dt \ (\geq 0).$$

Let us check that $W_0 > 0$. For an arbitrary $f \in \mathbf{C}^{N \times 1}$ we have

$$f^* W_0 f = \int_{-\infty}^{\infty} |Q(t)|^2 |P(t)|^2 e^{-t^2} dt,$$

where $Q(t) = f^*(tI - A)^{-1}B$. The identity $f^*W_0f = 0$ means that $Q(t) \equiv 0$, which, according to Proposition 2, is possible only if $f = 0$. It only remains to prove (5.5),

$$C_0B^* - BC_0^*$$

$$= \int_{-\infty}^{\infty} (tI - A)^{-1}(BB^*(tI - A^*) - (tI - A)BB^*)(tI - A^*)^{-1}|P(t)|^2e^{-t^2}\,dt$$

$$= \int_{-\infty}^{\infty} (tI - A)^{-1}(ABB^* - BB^*A^*)(tI - A^*)^{-1}|P(t)|^2e^{-t^2}\,dt$$

$$= AW_0 - W_0A^*.$$

Set $W(\epsilon) = W + \epsilon W_0, (\epsilon > 0), C(\epsilon) = C + \epsilon C_0, V(\epsilon) = AW(\epsilon) + BC(\epsilon)$. It it easy to see that the matrix $W(\epsilon)$ is positive and

$$AW(\epsilon) - W(\epsilon)A^* = C(\epsilon)B^* - BC(\epsilon)^*.$$

Let us denote by $\lambda_k(\epsilon)$ the eigenvalues and by $g_k(\epsilon)$ the corresponding eigenvectors of the pencil $V(\epsilon) - zW(\epsilon)$, normalized by the condition $g_j^*(\epsilon)W(\epsilon)g_k(\epsilon) = \delta_{jk}, \quad j, k = 1, 2, \ldots, N$.

As $\epsilon \to +0$ we have $W(\epsilon) \to W$, $V(\epsilon) \to V$, and $W(\epsilon)$ decreases if ϵ decreases. A part of the eigenvalues $\lambda_k(\epsilon)$ of the pencil $V(\epsilon) - zW(\epsilon)$ disappear at infinity when $\epsilon \to +0$, while the rest of them converge to the corresponding eigenvalues of the pencil $V - zW$. In the representation

$$W(\epsilon) = \sum_{j=1}^{N} W(\epsilon)g_j(\epsilon)g_j^*(\epsilon)W(\epsilon)$$

each of the summands is less than $W(\epsilon_0)$ for $0 < \epsilon < \epsilon_0$ and therefore is bounded. Using the fact that the trace of the product of two matrices do not depend on the order of the factors we conclude that for $0 < \epsilon < \epsilon_0$ the vectors $W(\epsilon)g_k(\epsilon)$ are bounded,

$$\|W(\epsilon)g_k(\epsilon)\|^2 = (g_k(\epsilon))^*W(\epsilon)^2g_k(\epsilon)$$

$$= \mathrm{trace}((g_k(\epsilon)^*W(\epsilon))(W(\epsilon)g_k(\epsilon)))$$

$$= \mathrm{trace}(W(\epsilon)g_k(\epsilon)g_k(\epsilon)^*W(\epsilon)) < \mathrm{trace}W(\epsilon) < \mathrm{trace}W(\epsilon_0).$$

Therefore, for a certain sequence $\epsilon_n \to 0$ there exist the limits

$$f_k = \lim_{n\to\infty} W(\epsilon_n)g_k(\epsilon_n).$$

Let $\lambda_k(\epsilon_n)$ converge to a finite number λ_k as $n \to \infty$. Then, due to rank $B = m$, the relation

(5.6) $$(\lambda_k(\epsilon_n)I - A)W(\epsilon_n)g_k(\epsilon_n) = B(C(\epsilon_n))^*g_k(\epsilon_n)$$

implies existence of the limits

$$h_k = \lim_{n\to\infty} (C(\epsilon_n))^*g_k(\epsilon_n)$$

and, therefore,

$$(\lambda_kI - A)f_k = Bh_k.$$

Let now $\lambda_k(\epsilon_n) \to \infty$ as $\epsilon_n \to +0$. Let us rewrite (5.6) in the form

$$(I - \lambda_k^{-1}(\epsilon_n)A)W(\epsilon_n)g_k(\epsilon_n) = B\lambda_k^{-1}(\epsilon_n)(C(\epsilon_n))^*g_k(\epsilon_n).$$

The limit in the left-hand side is equal to f_k, thus, due to rank $B = m$, there exists

$$h_k = \lim_{n\to\infty} \lambda_k(\epsilon_n)^{-1}(C(\epsilon_n))^*g_k(\epsilon_n)$$

and, therefore, $f_k = Bh_k$. Now the summands in the representation

$$W = \sum_{k=1}^{N} f_k f_k^*$$

can be split into two groups. The first group corresponds to those of the eigenvalues $\lambda_k(\epsilon_n)$ which tend to infinity as $\epsilon \to +0$. Let us index them with the numbers $k = 1, 2, \ldots, n_1$. The limit of their sum can be presented in the form $B\beta B^*$ where $\beta = \sum_{k=1}^{n_1} h_k h_k^*$.

The sum of the rest of the summands, as in Subsection 4.1, can be presented in the form

$$\int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}.$$

Therefore,

$$W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}.$$

5.3. Let us dispose of the requirement of controllability of the pair (A, B). We will use the following assertion (see, e.g. [11]).

Proposition 3. *A non-controllable pair (A, B) is similar to the pair $(\tilde{A}, \tilde{B})$, where*

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}$$

and the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

Let us transform similarly the quadruple (A, B, C, W) with $W \geq 0$ into the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{W})$

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ \tilde{W}_{21} & \tilde{W}_{22} \end{pmatrix} \geq 0.$$

Comparing the right-lower blocks in the identity $\tilde{A}\tilde{W} - \tilde{W}\tilde{A}^* = \tilde{C}\tilde{B}^* - \tilde{B}\tilde{C}^*$ we obtain the equation $\tilde{A}_{22}\tilde{W}_{22} - \tilde{W}_{22}\tilde{A}_{22}^* = 0$. Without loss of generality one can assume that $\tilde{A}_{22} = \mathrm{diag}(G_1, \ldots, G_\mu)$, where G_j are lower Jordan cells. Let U_{jj} be the corresponding diagonal block of the matrix $\tilde{W}_{22}$. Since $G_j U_{jj} - U_{jj} G_j^* = 0$ and the spectra of the blocks G_j and G_j^* do not intersect, we arrive at $U_{jj} = 0$. Thus, all the diagonal blocks of $\tilde{W}_{22}$ are zero and due to $\tilde{W}_{22} \geq 0$ we conclude that $\tilde{W}_{22} = 0$. This implies $\tilde{W}_{12} = 0$ and $\tilde{W}_{21} = 0$. Consequently, $\tilde{W} = \begin{pmatrix} \tilde{W}_{11} & 0 \\ 0 & 0 \end{pmatrix}$. Since the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable and in the case where $\det\tilde{W}_{11} = 0$ we have $\mathrm{rank}\,\tilde{B}_1 = \mathrm{rank}\,B = m$, the representation

$$\tilde{W}_{11} = \tilde{B}_1 \beta \tilde{B}_1^* + \int_{-\infty}^{\infty} (tI - \tilde{A}_{11})^{-1} \tilde{B}_1 d\sigma(t) \tilde{B}_1^* (tI - \tilde{A}_{11}^*)^{-1}$$

holds. Taking into account the identity

$$(tI - \tilde{A})^{-1} = \begin{pmatrix} (tI - \tilde{A}_{11})^{-1} & -(tI - \tilde{A}_{11})^{-1}\tilde{A}_{12}(tI - \tilde{A}_{22})^{-1} \\ 0 & (tI - \tilde{A}_{22})^{-1} \end{pmatrix}$$

we obtain

$$\tilde{B}\beta\tilde{B}^* + \int_{-\infty}^{\infty} (tI - \tilde{A})^{-1}\tilde{B}\,d\sigma(t)\tilde{B}^*(tI - A^*)^{-1}$$

$$= \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} \beta(\tilde{B}_1^*, 0) + \int_{-\infty}^{\infty} \begin{pmatrix} (tI - \tilde{A}_{11})^{-1}\tilde{B}_1 \\ 0 \end{pmatrix} d\sigma(t)(\tilde{B}_1^*(tI - \tilde{A}_{11}^*)^{-1}, 0)$$

$$= \begin{pmatrix} \tilde{W}_{11} & 0 \\ 0 & 0 \end{pmatrix} = \tilde{W}.$$

The inverse similarity transformation leads to the integral representation of the matrix W while $V \neq 0$.

5.4. Let us consider the case where $V = 0$, i.e., in accordance with definition (5.1) the case where

(5.7) $$AW + BC^* = 0.$$

In this case Theorem 1 remains true but the previous proof is not valid.

Let the matrix W have ν nonzero (positive) eigenvalues and let Λ be the diagonal $\nu \times \nu$-matrix with these eigenvalues as diagonal elements. Then denote by G the $N \times \nu$-matrix the columns of which are the corresponding eigenvectors of W normalized by 1. Then $WG = G\Lambda$ and $W = G\Lambda G^*$. Setting $F = G\Lambda^{1/2}$ we immediately obtain $W = FF^*$. Multiplying the both sides of (5.7) by G from the right we arrive at

$$-AG\Lambda = BC^*G,$$

what implies that

$$(0I - A)F = BH,$$

where

$$H = C^*G\Lambda^{-1} \in \mathbf{C}^{m\times\nu}.$$

From this it follows that

$$W = (0I - A)^{-1}BHH^*B^*(0I - A^*)^{-1} = \int_{-\infty}^{\infty} (tI - A)^{-1}B\,d\sigma(t)B^*(tI - A^*)^{-1},$$

where the matrix $\sigma(t)$ is constant on the intervals $(-\infty, 0)$ and $(0, \infty)$ and has a jump $HH^* \in \mathbf{C}^{m\times m}$ at $t = 0$. Theorem 1 is proved.

6. Proof of Theorem 2

1) Let

(6.1) $$W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1}B\,d\sigma(t)B^*(tI - A^*)^{-1}.$$

Set

(6.2) $$\tilde{C} = AB\beta + \int_{-\infty}^{\infty} ((tI - A)^{-1} - t(1 + t^2)^{-1}I)B\,d\sigma(t)$$

and check the identity

(6.3) $$AW - WA^* = \tilde{C}B^* - B\tilde{C}^*.$$

Indeed, denoting the summands in (6.1) by W_1 and W_2, the summands in (6.2) by $\tilde{C}_1$ and $\tilde{C}_2$ we find

$$\tilde{C}_1 B^* - B\tilde{C}_1^* = AB\beta B^* - B\beta B^* A^* = AW_1 - W_1 A^*$$

and

$$\tilde{C}_2 B^* - B \tilde{C}_2^*$$

$$= \int_{-\infty}^{\infty} (tI - A)^{-1} (B \, d\sigma(t) B^* (tI - A^*) - (tI - A) B \, d\sigma(t) B^*)(tI - A^*)^{-1}$$

$$= A \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}$$

$$- \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1} A^*$$

$$= A W_2 - W_2 A^*.$$

This implies (6.3).

Since, together with (6.3), the main matrix identity $AW - WA^* = CB^* - BC^*$ is true, we have

$$\tilde{C} B^* - B \tilde{C}^* = C B^* - B C^*$$

or

(6.4) $$(C - \tilde{C}) B^* = B(C - \tilde{C}).$$

Since rank $B = m$, there exists a nonsingular matrix T for which $TB = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$. Set $T(C - \tilde{C}) = \begin{pmatrix} \alpha \\ a \end{pmatrix}$. Multiplying the both sides of (6.4) by T from the left and by T^* from the right we obtain $\begin{pmatrix} \alpha \\ a \end{pmatrix} (I_m \ 0) = \begin{pmatrix} I_m \\ 0 \end{pmatrix} (\alpha^* \ a^*)$, which implies $\alpha = \alpha^*$ and $a = 0$. Consequently, $T(C - \tilde{C}) = \begin{pmatrix} I_m \\ 0 \end{pmatrix} \alpha = TB\alpha$, i.e., $C = B\alpha + \tilde{C}$. The first assertion of the theorem is proved.

2) Let $AW - WA^* = CB^* - BC^*$,

$$C = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI - A)^{-1} - t(1 + t^2)^{-1} I) B \, d\sigma(t)$$

and the spectra of the matrices A and A^* do not intersect. Introduce the matrix

$$\tilde{W} = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}.$$

It is easy to check that $A\tilde{W} - \tilde{W}A^* = CB^* - BC^*$ and, since this equation possesses a unique solution with respect to the unknown W, we conclude that $W = \tilde{W}$. Theorem 2 is proved.

7. INTERPOLATION CHARACTERISTICS FOR ASSOCIATED $\mathcal{R}_m$-FUNCTIONS

7.1. Let all the eigenvalues of the matrix A be non-real, let z_j be one of them with the corresponding lower Jordan cell A_j. Let also

$$B_j = \begin{pmatrix} b_0^j \\ b_1^j \\ \vdots \\ b_{n_j}^j \end{pmatrix}, \quad C_j = \begin{pmatrix} c_0^j \\ c_1^j \\ \vdots \\ c_{n_j}^j \end{pmatrix}$$

and W_{jj} be the corresponding blocks of B, C and $W \geq 0$, rank $B = m$. According to Theorem 1, the following representation is true:

$$W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1} B d\sigma(t) B^* (tI - A^*)^{-1},$$

which, due to Theorem 2, implies that

$$C = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI - A)^{-1} - t(1+t^2)^{-1}I) B \, d\sigma(t),$$

which in turn implies

(7.1) $$C_j = B_j\alpha + A_j B_j\beta + \int_{-\infty}^{\infty} (tI_{n_j+1} - A_j)^{-1} - t(1+t^2)^{-1} I_{n_j+1}) B \, d\sigma(t),$$

$j = 1, 2, \ldots, \nu$. From (7.1) we conclude that the $\mathcal{R}_m$-function

$$w(z) = \alpha + \beta z + \int_{-\infty}^{\infty} ((t-z)^{-1} - t(1+t^2)^{-1}) \, d\sigma(t)$$

associated with the quadruple (A, B, C, W) is also associated with the quadruples (A_j, B_j, C_j, W_{jj}), $j = 1, 2, \ldots, \nu$.

Taking into account that

$$(tI_{n_j+1} - A_j)^{-1} - t(1+t^2)^{-1} I_{n_j+1}$$

$$= \begin{pmatrix} \frac{1}{t-z_j} - \frac{t}{1+t^2} & 0 & \cdots & 0 & 0 \\ \frac{1}{(t-z_j)^2} & \frac{1}{t-z_j} - \frac{t}{1+t^2} & \ddots & 0 & 0 \\ \frac{1}{(t-z_j)^3} & \frac{1}{(t-z_j)^2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{(t-z_j)^{(n_j+1)}} & \frac{1}{(t-z_j)^{n_j}} & \cdots & \frac{1}{(t-z_j)^2} & \frac{1}{t-z_j} - \frac{t}{1+t^2} \end{pmatrix}$$

we obtain from (7.1) that

(7.2) $$\begin{cases} c_0^j = b_0^j w(z_j), \\ c_1^j = b_0^j \frac{w'(z_j)}{1!} + b_1^j w(z_j), \\ \qquad \cdots \cdots \cdots \cdots \cdots \cdots \\ c_{n_j}^j = b_0^j \frac{w^{(n_j)}(z_j)}{n_j!} + b_1^j \frac{w^{(n_j-1)}(z_j)}{(n_j-1)!} + \cdots + b_{n_j}^j w(z_j). \end{cases}$$

Introducing the polynomials

$$b_j(z) = \sum_{k=0}^{n_j} b_k^j (z - z_j)^k, \quad c_j(z) = \sum_{k=0}^{n_j} c_k^j (z - z_j)^k$$

rewrite relations (7.2) in an equivalent form,

(7.3) $$b_j(z)w(z) = c_j(z) + o((z - z_j)^{n_j}), \quad z \to z_j.$$

We arrive at the so-called left-sided tangential problem at the node z_j of multiplicity $n_j + 1$. If the node z_j lies in the lower half-plane $\mathbf{C}_-$, one can consider the node $\zeta_j = \overline{z}_j \in \mathbf{C}_+$ with the help of the transformation

$$(b_j(\overline{z})w(\overline{z}))^* = (c_j(\overline{z}) + o((\overline{z} - z_j)^{n_j}))^*,$$

i.e., to pass to a right-sided tangential problem at the node ζ_j of multiplicity $n_j + 1$,

$$w(z)\beta_j(z) = \gamma_j(z) + o((z - \zeta_j)^{n_j}),$$

where $\beta_j(z) = \sum_{k=0}^{n_j} (b_k^j)^*(z - \zeta_j)^k$, $\gamma_j(z) = \sum_{k=0}^{n_j} (c_k^j)^*(z - \zeta_j)^k$. However, in the sequel we will not use this transformation. Thus, all the one-sided tangential problems will be left-sided.

If $z_j \neq \overline{z}_k$ for each $k = 1, 2, \ldots, \nu$ then we have already used in problem (7.3) all the data contained in B_j, C_j and the blocks W_{jk} can be uniquely determined from the equations $A_j W_{jk} - W_{jk} A_j^* = C_j B_k^* - B_j C_k^*$ and they do not contain new interpolation data corresponding to the node z_j.

Let us notice that all the steps of the above proof are invertible, therefore, (7.3) and (7.1) are equivalent.

7.2. Now we consider the case where $z_j = \overline{z}_k$ for some k.

We keep the assumption that the matrix A is reduced to its lower Jordan form, $A = \operatorname{diag}(A_1, A_2, \ldots, A_\nu)$, where A_j is the lower Jordan cell corresponding to the eigenvalue z_j. If $z_j = \overline{z}_k$, then according to what was proved in Sec. 3, the last row $(v_{n_j,0}, v_{n_j,1}, \ldots, v_{n_j,n_k})$ of the block W_{jk} $(n_j \geq n_k)$ does not depend on the matrices A_j, B_j, C_j, B_k, C_k and the elements of this last row must determine additional interpolation conditions for the associated $\mathcal{R}_m$-functions.

Let

$$(7.4) \qquad w(z) = \sum_{p=0}^{\infty} w_p^j(z - z_j)^p$$

be the expansion of the associated $\mathcal{R}_m$-function and

$$(7.5) \qquad b_j(z)w(z) = \sum_{p=0}^{n_j} c_p^j(z - z_j)^p + \sum_{p=0}^{\infty} c_p^j(z - z_j)^p.$$

The right-hand side of (7.5) is split into two groups of summands in order to underline that the coefficients c_p^j in the first group are common for all associated $\mathcal{R}_m$-functions while it is not true for the coefficients in the second group. Taking into account (7.4) and (7.5) we can write

$$(7.6) \qquad c_p^j = \sum_{q=0}^{p} b_q^j w_{p-q}^j, \quad p = 0, 1 \ldots, \quad b_q^j = 0 \quad \text{if} \quad q > n_j.$$

Let us express elements of the last row $(v_{n_j,0}, v_{n_j,1}, \ldots, v_{n_j,n_k})$ of the block W_{jk} in terms of the coefficients of decomposition (7.5) and the coefficients of the polynomial

$$(b(\overline{z}))^* = \sum_{p=0}^{n_k} (b_p^k)^*(z - \overline{z}_k)^p = \sum_{p=0}^{n_k} (b_p^k)^*(z - z_j)^p$$

using the representation

$$(7.7) \qquad W_{jk} = B_j \beta B_k^* + \int_{-\infty}^{\infty} (tI_{n_j+1} - A_j)^{-1} B_j d\sigma(t) B_k^* (tI_{n_k+1} - A_k^*)^{-1}.$$

Since the last row of the matrix $(tI_{n_j+1} - A_j)^{-1} B_j$ has the form

$$\sum_{p=0}^{n_j} \frac{b_p^j}{(t - z_j)^{n_j+1-p}}$$

and the r-th column of the matrix $B_k^* (tI_{n_k+1} - A_k^*)^{-1}$ is

$$\sum_{q=0}^{n_k} \frac{(b_q^j)^*}{(t - z_j)^{n_k+2+r-p}},$$

taking into account that

$$\int\limits_{-\infty}^{\infty} \frac{1}{(t-z)^2}\,d\sigma(t) = w'(z_j) - \beta = w_1^j - \beta,$$

$$\int\limits_{-\infty}^{\infty} \frac{1}{(t-z)^{p+1}}\,d\sigma(t) = w^{(p)}(z_j)/p! = w_p^j, \quad p = 2, 3\ldots,$$

we obtain from (7.7) the following relations:

$$v_{n_j,r} = \left(\sum_{p=0}^{n_j} b_p^j w_{n_j+r+1-p}^j\right)(b_0^k)^* + \left(\sum_{p=0}^{n_j} b_p^j w_{n_j+r-p}^j\right)(b_1^k)^* + \cdots + \left(\sum_{p=0}^{n_j} b_p^j w_{n_j+1-p}^j\right)(b_r^k)^*.$$

Using (7.6) we can write

(7.8) $$v_{n_j,r} = \sum_{p=0}^{r} c_{n_j+1+r-p}^j (b_p^k)^*, \quad r = 0, 1, \ldots, n_k.$$

Let us consider the product

$$b_j(z)w(z)(b_k(\bar{z}))^* = \left(\sum_{p=0}^{\infty} c_p^j(z-z_j)^p\right)\sum_{q=0}^{n_k}(b_q^k)^*(z-z_j)^q = \sum_{p=0}^{\infty} d_p^{(j,k)}(z-z_j)^p,$$

where

(7.9) $$d_p^{(j,k)} = \sum_{q=0}^{p} c_{p-q}^j (b_q^k)^*, \quad p = 0, 1, \ldots, \quad b_q^k = 0 \quad \text{if} \quad q > n_k.$$

Equality (7.9) shows that for $p \le n_j$ the coefficients $d_p^{(j,k)}$ are uniquely defined by the interpolation data which are contained in B_j, B_k, C_j, C_k. Then, for $r = 0, 1, \ldots, n_k$ we have

(7.10)
$$\begin{aligned}
d_{n_j+1+r}^{(j,k)} &= \sum_{q=0}^{n_j+1+r} c_{n_j+1+r-q}^j (b_q^k)^* = \sum_{q=0}^{n_k} c_{n_j+1+r-q}^j (b_q^k)^* \\
&= \sum_{q=0}^{r} c_{n_j+1+r-q}^j (b_q^k)^* + \sum_{q=r+1}^{n_k} c_{n_j+1+r-q}^j (b_q^k)^*.
\end{aligned}$$

With account of (7.8) we obtain

$$d_{n_j+1+r}^{(j,k)} = v_{n_j,r} + \sum_{q=r+1}^{n_k} c_{n_j+1+r-q}^j (b_q^k)^*, \quad r = 0, 1, \ldots, n_k,$$

which shows that the coefficients $d_{n_j+1}^{(j,k)}$, $d_{n_j+2}^{(j,k)}$, $\ldots$, $d_{n_j+n_k+1}^{(j,k)}$ are uniquely determined by the last row of the block W_{jk} and the interpolation data contained in B_j, B_k, C_j, C_k. Consequently, additional interpolation conditions for $z_j = \bar{z}_k$ are of the form

(7.11) $$b_j(z)w(z)(b_k(\bar{z}))^* = d_{j,k}(z) + o((z-z_j)^{n_j+n_k+1}), \quad z \to z_j,$$

where

$$d_{j,k}(z) = \sum_{p=0}^{n_j+n_k+1} d_p^{(j,k)}(z-z_j)^p.$$

7.3. Let us show that the inverse statement is also true, i.e., every $\mathcal{R}_m$-function

(7.12) $$w(z) = \alpha + \beta z + \int_{-\infty}^{\infty} ((t-z)^{-1} - t(1+t^2)^{-1})\,d\sigma(t),$$

which is a solution of the above stated interpolation problem, is also associated with a properly constructed quadruple (A, B, C, W).

Let ν non-real nodes of interpolation $\{z_j\}_1^\nu$ be given. To each node we attribute its multiplicity $n_j + 1$ and the pair of polynomials

$$b_j(z) = \sum_{p=0}^{n_j} b_p^j (z - z_j)^p, \quad c_j(z) = \sum_{p=0}^{n_j} c_p^j (z - z_j)^p$$

with the coefficients from $\mathbf{C}^{1 \times m}$. Moreover, if $z_j = \bar{z}_k$ let the scalar polynomial

$$d_{j,k}(z) = \sum_{p=0}^{n_j + n_k + 1} d_p^{(j,k)} (z - z_j)^p$$

be given. Without loss of generality it is possible to assume that $n_j \geq n_k$, otherwise we can reach the desired inequalities by changing the enumeration of the nodes. Suppose that the $\mathcal{R}_m$-function given by (7.10) satisfy the conditions

$$(7.13) \qquad b_j(z)w(z) = c_j(z) + o((z - z_j)^{n_j}), \quad z \to z_j, \quad j = 1, 2, \ldots, \nu$$

and, in the case where $z_j = \bar{z}_k$, it satisfies also the conditions

$$b_j(z)w(z)(b_k(\bar{z}))^* = d_{j,k}(z) + o((z - z_j)^{n_j + n_k + 1}), \quad z \to z_j.$$

Let us construct the quadruple (A, B, C, W).

We denote by A_j, $j = 1, 2, \ldots, \nu$ the lower Jordan cell of order $n_j + 1$ with z_j on the main diagonal, by A the block-diagonal matrix $A = \mathrm{diag}(A_1, A_2, \ldots, A_{nu})$, and let $N = \sum_{j=1}^{\nu}(n_j + 1)$. Then we set

$$B_j = \begin{pmatrix} b_0^j \\ \vdots \\ b_{n_j} \end{pmatrix} \in \mathbf{C}^{(n_j+1) \times m}, \qquad C_j = \begin{pmatrix} c_0^j \\ \vdots \\ c_{n_j} \end{pmatrix} \in \mathbf{C}^{(n_j+1) \times m},$$

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_\nu \end{pmatrix} \in \mathbf{C}^{N \times m}, \qquad C = \begin{pmatrix} C_1 \\ \vdots \\ C_\nu \end{pmatrix} \in \mathbf{C}^{N \times m}.$$

If $z_j \neq \bar{z}_k$ then using the algorithm of Sec. 3.2 it is possible to construct the block $W_{jk} \in \mathbf{C}^{(n_j+1) \times (n_k+1)}$. According to the way of constructing the relation

$$(7.14) \qquad\qquad A_j W_{jk} - W_{jk} A_k^* = C_j B_k^* - B_j C_k^*$$

holds true automatically. If $z_j \neq \bar{z}_k$ for all $j, k = 1, 2, \ldots, \nu$, i.e., the spectra of the matrices A and A^* do not intersect, then (7.14) implies that the matrix W satisfies the relation

$$AW - WA^* = CB^* - BC^*.$$

As it was shown in Sec. 7.1, relations (7.13) are equivalent to representations (7.1) for $j = 1, 2, \ldots, \nu$ and, consequently, to the representation

$$C = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI - A)^{-1} - t(1 + t^2)^{-1}I)B \, d\sigma(t),$$

which, according to Theorem 2, in the case under consideration is equivalent to

$$W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}.$$

If $z_j = \bar{z}_k$ for some values of j and k then, in order to construct the block $W_{j,k}$, it is necessary for the last row $(v_{n_j,0}, v_{n_j,1}, \ldots, v_{n_j,n_k})$ to be given. Setting

$$(7.15) \qquad v_{n_j,r} = d^{(j,k)}_{n_j+1+r} - \sum_{q=r+1}^{n_k} c^j_{n_j+1+r-q}(b^j_{q,k})^*$$

we construct such blocks and using them compose the matrix $W = (W_{j,k})^\nu_{j,k=1}$ that satisfies

$$AW - WA^* = CB^* - BC^*$$

by the construction. Moreover, let us introduce the matrix

$$\tilde{W} = B\beta B^* + \int\limits_{-\infty}^{\infty} (tI - A)^{-1} B \, d\sigma(t) B^* (tI - A^*)^{-1}$$

and prove that $W = \tilde{W}$. If $z_j \neq \bar{z}_k$ then, as it is shown above, $W_{jk} = \tilde{W}_{jk}$. If $z_j = \bar{z}_k$ then according to (7.8) the elements of the last row, $\tilde{v}_{n_j,0}, \tilde{v}_{n_j,1}, \ldots, \tilde{v}_{n_j,n_k}$, of the block $\tilde{W}_{jk}$ are of the form

$$\tilde{v}_{n_j,r} = \sum_{p=0}^{r} c^j_{n_j+1+r-p}(b^k_p)^*, \quad r = 0, 1, \ldots, n_k.$$

On the other hand, formula (7.10) obtained directly from the expansion

$$b_j(z)w(z)(b_k(z))^* = \sum_{p=0}^{\infty} d^{(j,k)}_p (z - z_j)^p$$

with account of (7.15) leads to

$$v_{n_j,r} = \sum_{p=0}^{r} c^j_{n_j+1+r-p}(b^k_p)^*, \quad r = 0, 1, \ldots, n_k,$$

which implies that $\tilde{v}_{n_j.r} = v_{n_j.r}$. Consequently, $W_{jk} = \tilde{W}_{jk}$ and $W = \tilde{W}$. Q.E.D.

Simultaneously we conclude that for solvability of the inverse problems stated in Sec. 7.3 it is necessary and sufficient that the generalized Pick matrix W be Hermite non-negative.

8. Main matrix inequality

V. P. Potapov's method is based on the so-called main matrix inequality (MMI) with respect to the matrix-valued function $w(z)$. The set of solutions of the MMI coincides with the set of solutions of the interpolation problem we consider. Here in the framework of the proposed approach the MMI will be obtained the set of solutions of which coincides with the set of associated matrix-valued functions and, consequently, with the set of solutions of the corresponding interpolation problem.

8.1. Let a quadruple (A, B, C, W) satisfy the relation $AW - WA^* = CB^* - BC^*$ and $W \geq 0$.

For $z \in \mathbf{C}_+$ we introduce the matrices

$$\mathbf{A}(z) = \begin{pmatrix} A & 0 \\ 0 & zI_m \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B \\ I_m \end{pmatrix}, \quad \mathbf{C}(\mathbf{z}) = \begin{pmatrix} C \\ v(z) \end{pmatrix},$$

$$\mathbf{W}(z) = \begin{pmatrix} W & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{pmatrix}$$

and choose $W_{12}(z), W_{21}(z), W_{22}(z)$ in order that the relations

(8.1) $$\mathbf{A}(z)\mathbf{W}(z) - \mathbf{W}(z)(\mathbf{A}(z))^* = \mathbf{C}(z)\mathbf{B}^* - \mathbf{B}(\mathbf{C}(z))^*$$

be satisfied. Comparing the blocks in the left-hand and the right-hand sides we obtain (if $\bar{z}$ does not belong to the spectrum of A) that

$$\mathbf{W}(z) = \begin{pmatrix} W & (A - \bar{z}I)^{-1}(C - B(v(z))^*) \\ (C^* - v(z)B^*)(A^* - zI)^{-1} & \frac{v(z) - (v(z))^*}{z - \bar{z}} \end{pmatrix}.$$

8.2. Theorem 3. *An $\mathcal{R}_m$-function $w(z)$ is associated with the quadruple (A, B, C, W) if and only if it is a solution of the MMI*

(8.2) $$\begin{pmatrix} W & (A - \bar{z}I)^{-1}(C - B(w(z))^*) \\ (C^* - w(z)B^*)(A^* - zI)^{-1} & \frac{w(z) - (w(z))^*}{z - \bar{z}} \end{pmatrix} \geq 0.$$

for $\operatorname{Im} z > 0$.

Proof. Let a point $\bar{z}_0$ do not belong to the spectrum of A. If $\mathbf{W}(z_0) \geq 0$ then taking into account that rank $\mathbf{B} = m$ we arrive at the following representation:

(8.3) $$\mathbf{W}(z_0) = \mathbf{B}\beta\mathbf{B}^* + \int_{-\infty}^{\infty} (tI - \mathbf{A}(z_0))^{-1}\mathbf{B}\, d\sigma(t)\mathbf{B}^*(tI - (\mathbf{A}(z_0))^*)^{-1}$$

and, consequently,

(8.4) $$\mathbf{C}(z_0) = \mathbf{B}\alpha + \mathbf{A}(z_0)\mathbf{B}\beta + \int_{-\infty}^{\infty} ((tI - \mathbf{A}(z_0))^{-1} - t(1 + t^2)^{-1}I)\mathbf{B}\, d\sigma(t)$$

is valid. Representation (8.3) implies

(8.5) $$W = B\beta B^* + \int_{-\infty}^{\infty} (tI - A)^{-1}B\, d\sigma(t)B^*(tI - A^*)^{-1}$$

while (8.4) implies

$$C = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI - A)^{-1} - t(1 + t^2)^{-1}I)B\, d\sigma(t)$$

and

$$v(z_0) = \alpha + \beta z_0 + \int_{-\infty}^{\infty} ((t - z_0)^{-1} - t(1 + t^2)^{-1})\, d\sigma(t).$$

Thus, the $\mathcal{R}_m$-function

$$w(z) = \alpha + \beta z + \int_{-\infty}^{\infty} ((t - z)^{-1} - t(1 + t^2)^{-1})\, d\sigma(t)$$

is associated with the quadruple (A, B, C, W) and $v(z_0)$ is its value at z_0.

Conversely, let $w(z)$ be an $\mathcal{R}_m$-function,

$$w(z) = \alpha + \beta z + \int_{-\infty}^{\infty} ((t - z)^{-1} - t(1 + t^2)^{-1})\, d\sigma(t)$$

associated with the quadruple (A, B, C, W) with $W \geq 0$. Then

(8.6) $$W_{22}(z) = \frac{w(z) - (w(z))^*}{z - \bar{z}} = \beta + \int_{-\infty}^{\infty} \frac{d\sigma(t)}{|t - z|^2},$$

,

$$C - B(w(z))^* = B\alpha + AB\beta + \int_{-\infty}^{\infty} ((tI - A)^{-1} - t(1 + t^2)^{-1}I)B \, d\sigma(t)$$

$$- \left(B\alpha + B\beta\bar{z} + \int_{-\infty}^{\infty} ((t - \bar{z})^{-1} - t(1 + t^2)^{-1})B \, d\sigma(t) \right)$$

$$= (A - \bar{z}I)^{-1} \left(B\beta + \int_{-\infty}^{\infty} (tI - A)^{-1}(t - \bar{z})^{-1}B \, d\sigma(t) \right)$$

and

(8.7) $$W_{12}(z) = B\beta + \int_{-\infty}^{\infty} (tI - A)^{-1}(t - \bar{z})^{-1}B \, d\sigma(t).$$

This formula allows to define the value of $W_{12}(z)$ for those z for which $\bar{z}$ belong to the spectrum of A. Taking into account (8.6), (8.7) and that $W_{21}(z) = (W_{12}(z))^*$ we obtain

$$\mathbf{W}(z) = \begin{pmatrix} B \\ I_m \end{pmatrix} \beta \begin{pmatrix} B \\ I_m \end{pmatrix}^*$$

$$+ \int_{-\infty}^{\infty} \begin{pmatrix} tI - A & 0 \\ 0 & (t - z)I_m \end{pmatrix}^{-1} \begin{pmatrix} B \\ I_m \end{pmatrix} d\sigma(t) \begin{pmatrix} B \\ I_m \end{pmatrix}^* \begin{pmatrix} tI - A^* & o \\ 0 & (t - \bar{z})I_m \end{pmatrix}^{-1} \geq 0.$$

$\square$

9. Proposition on nonnegative block matrix

Proposition 4. (See, e.g. [13], p. 223–224). *Let a Hermitian matrix $H \in \mathbf{C}^{M \times M}$ be split into the blocks*

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \quad H_{11} \in \mathbf{C}^{N \times N}.$$

The inequality $H \geq 0$ is equivalent to the following three conditions:
  1) The matrix H_{11} is positive semidefinite.
  2) The equation

(9.1) $$H_{11}X = H_{12}$$

is solvable.
  3) Each solution X of equation (9.1) satisfies the inequality

(9.2) $$lH_{22} - X^* H_{11}X \geq 0.$$

Corollary. *If there exists H_{11}^{-1} then the inequality $H \geq 0$ is equivalent to the following two conditions:*
  1) $H_{11} \geq 0$,
  2) $H_{22} - H_{21}H_{11}^{-1}H_{12} \geq 0$.

10. Parametrization of the set of associated $\mathcal{R}_m$-functions in the case of $W > 0$

10.1. Let us introduce the matrix

$$J = \begin{pmatrix} 0 & -iI_m \\ iI_m & 0 \end{pmatrix} \in \mathbf{C}^{2m \times 2m}.$$

It is easy to see that $J^* = J = J^{-1}$ and that a holomorphic in $\mathbf{C}_+$ function $w(z)$ of order m belongs to $\mathcal{R}_m$ if and only if

(10.1) $$(w(z), I_m)J(w(z), I_m)^* \geq 0, \quad z \in \mathbf{C}_+.$$

Multiplying (10.1) from the left by a meromorphic in $\mathbf{C}_+$ matrix-valued function $Q(z)$, which takes values in $\mathbf{C}^{m \times m}$, and from the right by $Q^*(z)$ we arrive at the inequality

$$(10.2) \qquad (P(z), Q(z))J(P(z), Q(z))^* \geq 0, \quad z \in \mathbf{C}_+.$$

Definition. A pair $(P(z), Q(z))$ of $\mathbf{C}^{m \times m}$-valued functions, which are meromorphic in $\mathbf{C}_+$, is said to be an $\mathcal{R}_m$-*pair* if (10.2) is valid for the values of $z \in \mathbf{C}_+$ for which both $P(z)$ and $Q(z)$ are defined.

Definition. Pairs $(P(z), Q(z))$ and $(\tilde{P}(z), \tilde{Q}(z))$ are said to be *equivalent* if there exists an invertible $\mathbf{C}^{m \times m}$-matrix-valued function $u(z)$ meromorphic in $\mathbf{C}_+$ for which $(\tilde{P}(z), \tilde{Q}(z)) = u(z)(P(z), Q(z))$.

Evidently, if $(P(z), Q(z))$ is an $\mathcal{R}_m$-pair, then such are all the pairs equivalent to it. If the matrix-valued function $Q(z)$ involved in an $\mathcal{R}_m$-pair is invertible then this pair is equivalent to the pair $(w(z), I_m)$ where $w(z) = (Q(z))^{-1}P(z) \in \mathcal{R}_m$ (all the singular points of $(Q(z))^{-1}P(z)$ in $\mathbf{C}_+$ are removable).

Definition. A pair $(P(z), Q(z))$ is said to be *non-degenerate* if for $f \in C^{m \times 1}$ the equalities $f^*P(z) = 0$, $f^*Q(z) = 0$ imply $f = 0$.

Instead of the MMI for matrix-valued functions $w(z)$, we will consider the MMI for non-degenerate pairs of matrix-valued functions $(P(z), Q(z))$,

$$(10.3) \qquad \begin{pmatrix} W & (A - \overline{z}I)^{-1}(C(Q(z))^* - B(P(z))^*) \\ (Q(z)C^* - P(z)B^*)(A^* - zI)^{-1} & \frac{P(z)(Q(z))^* - Q(z)(P(z))^*}{z - \overline{z}} \end{pmatrix} \geq 0.$$

If the matrix-valued function $Q(z)$ is invertible then multiplying (10.3) by $T(z) = \begin{pmatrix} I_N & 0 \\ 0 & (Q(z))^{-1} \end{pmatrix}$ from the left and by $(T(z))^*$ from the right we obtain the MMI for $w(z)$.

Lemma. *If* $\operatorname{rank} B = m$ *and the pair* $(P(z), Q(z))$ *satisfying (10.3) is non-degenerate then the matrix-valued function* $Q(z)$ *is invertible.*

Proof. Let $f^*Q(z) = 0$ $(f \in \mathbf{C}^{m \times 1})$ for some z in the domains of $P(z)$, $Q(z)$ and $(A - \overline{z}I)^{-1}$. Multiplying the both sides of (10.3) on the right by $T = \begin{pmatrix} g & 0 \\ 0 & f \end{pmatrix}$, where g is an arbitrary vector from $\mathbf{C}^{N \times 1}$, and on the left by T^* we obtain the inequality

$$\begin{pmatrix} g^*Wg & -g^*(A - \overline{z}I)^{-1}B(P(z))^*f \\ -f^*P(z)B^*(A^* - zI)^{-1}g & 0 \end{pmatrix} \geq 0,$$

which implies $g^*(A - \overline{z}I)^{-1}B(P(z))^*f = 0$. Since g is an arbitrary vector, we have $B(P(z))^*f = 0$ and since $\operatorname{rank} B = m$ we conclude that $f^*P(z) = 0$, i.e., $f = 0$. Therefore, $Q(z)^{-1}$ exists. $\qquad \square$

Note that if we omit the requirement $\operatorname{rank} B = m$ then the formulations of the interpolation problems must be more complicated [12].

10.2. To simplify notations we denote $K = (C, B)$, $R(z) = (P(z), Q(z))$. Then the main matrix identity takes the form

$$(10.4) \qquad AW - WA^* = iKJK^*$$

and the MMI for $R(z)$ is

$$(10.5) \qquad \begin{pmatrix} W & i(A - \overline{z}I)^{-1}KJ(R(z))^* \\ -iR(z)JK^*(A^* - zI)^{-1} & \frac{iR(z)J(R(z))^*}{z - \overline{z}} \end{pmatrix} \geq 0.$$

Since $W > 0$, inequality (10.5) according to Proposition 4 is equivalent to

$$R(z)\{J - \frac{z - \bar{z}}{i} JK^*(A^* - zI)^{-1}W^{-1}(A - \bar{z}I)^{-1}KJ\}(R(z))^* \geq 0.$$

It turns out that the matrix in parentheses can be factorized as it was done by V. P. Potapov when he was considering a matrix version of Nevanlinna-Pick problem

(10.6) $$J - \frac{z - \bar{z}}{i} JK^*(A^* - zI)^{-1}W^{-1}(A - \bar{z}I)^{-1}KJ = \mathcal{A}(z)J(\mathcal{A}(z))^*.$$

Moreover the following factorization holds true:

(10.7) $$J - \frac{z - \bar{\zeta}}{i} JK^*(A^* - zI)^{-1}W^{-1}(A - \bar{\zeta}I)^{-1}KJ = \mathcal{A}(z)J(\mathcal{A}(\zeta))^*.$$

Assuming this factorization to hold we can find the matrix-valued function $\mathcal{A}(z)$ by fixing its value at some point ζ_0. It is convenient to set $\mathcal{A}(\infty) = I_{2m}$. From (10.7) at $\zeta = \infty$ we obtain

(10.8) $$\mathcal{A}(z) = I_{2m} + iJK^*(A^* - zI)^{-1}W^{-1}K.$$

Direct calculations with a use of main identity (10.4) and the equalities

$$(A - \bar{\zeta}I)^{-1}A = I + \bar{\zeta}(A - \bar{\zeta}I)^{-1}, \quad A^*(A^* - zI)^{-1} = I + z(A^* - zI)^{-1}$$

confirm validity of factorization (10.7).

Note that for $\zeta = \bar{z}$ equality (10.7) becomes $\mathcal{A}(z)J(\mathcal{A}(\bar{z}))^* = J$, what implies

$$\mathcal{A}(z)J(\mathcal{A}(\bar{z}))^*J = I_{2m}.$$

Therefore, for $\mathcal{B}(z) = (\mathcal{A}(z))^{-1}$ we obtain

(10.9) $$\mathcal{B}(z) = J(\mathcal{A}(\bar{z}))^*J = I_{2m} - iJK^*W^{-1}(A - zI)^{-1}K.$$

If we split $\mathcal{B}(z)$ into blocks $b_{jk}(z)$ $(j, k = 1, 2)$ of order m,

$$\mathcal{B}(z) = \left(\begin{array}{cc} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{array} \right),$$

then

$$b_{11}(z) = I_m - B^*W^{-1}(A - zI)^{-1}C, \qquad b_{12}(z) = -B^*W^{-1}(A - zI)^{-1}B,$$
$$b_{21}(z) = C^*W^{-1}(A - zI)^{-1}C, \qquad b_{22}(z`) = I_m + C^*W^{-1}(A - zI)^{-1}B.$$

10.3. Now the MMI for the pairs $(P(z), Q(z))$ looks as follows:

(10.10) $$(P(z), Q(z))\mathcal{A}(z)J(\mathcal{A}(z))^*(P(z), Q(z))^* \geq 0.$$

Let us put

(10.11) $$(P(z), Q(z))\mathcal{A}(z) = (p(z), q(z)).$$

It is evident that the matrix-valued functions $p(z)$ and $q(z)$ are meromorphic in $\mathbf{C}_+$, the pair $(p(z), q(z))$ is an $\mathcal{R}_m$-pair and is non-degenerate if and only if such is $(P(z), Q(z))$. Conversely, let $(p(z), q(z))$ be an arbitrary non-degenerate $\mathcal{R}_m$-pair. Let us put

$$(P(z), Q(z)) = (p(z), q(z))\mathcal{B}(z).$$

Then (10.11) and (10.10) are true, so $(P(z), Q(z))$ meets the MMI and is non-degenerate.

If the pairs $(P_1(z), Q_1(z))$ and $(P_2(z), Q_2(z))$ have invertible entries $Q_1(z)$ and $Q_2(z)$ such that $(Q_1(z))^{-1}P_1(z) = (Q_2(z))^{-1}P_2(z)(= w(z))$ then these pairs are equivalent because each of them is equivalent to $(w(z), I_m)$. In this case the corresponding pairs $(p_1(z), q_1(z))$, $(p_2(z), q_2(z))$ are also equivalent.

These considerations lead to the following theorem.

Theorem 4. *Let*

$$AW - WA^* = CB^* - BC^*, \quad W > 0, \quad and \quad \operatorname{rank} B = m.$$

Then the formula

$$w(z) = (p(z)b_{12}(z) + q(z)b_{22}(z))^{-1}(p(z)b_{11}(z) + q(z)b_{21}(z))$$

gives a one-to-one correspondence between the set of associated with (A, B, C, W) $\mathcal{R}_m$-functions $w(z)$ and the set of the equivalence of arbitrary non-degenerate $\mathcal{R}_m$-pairs $(p(z), q(z))$.

11. PARAMETRIZATION OF THE SET OF ASSOCIATED $\mathcal{R}_m$-FUNCTIONS IN THE CASE $W \geq 0$, $\operatorname{rank} W = r < N$

11.1. Without loss of generality we can assume that the matrix W has the form

$$(11.1) \qquad W = \begin{pmatrix} W_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad W_1 > 0, \quad W_1 \in \mathbf{C}^{r \times r}.$$

Then we split the rest of the matrices into corresponding blocks,
$$(11.2)$$
$$A = \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix}, \ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \ B = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad A_1 \in \mathbf{C}^{r \times r}, \ B_1, C_1 \in \mathbf{C}^{r \times m}.$$

Comparing block-wise parts of the main matrix identity we obtain

$$(11.3) \qquad \begin{aligned} A_1 W_1 - W_1 A_1^* &= C_1 B_1^* - B_1 C_1^*, & -W_1 A_{21}^* &= C_1 B_2^* - B_1 C_2^*, \\ A_{21} W_1 &= C_2 B_1^* - B_2 C_1^*, & C_2 B_2^* - B_2 C_2^* &= 0. \end{aligned}$$

According to Proposition 4, MMI (9.3) is valid if and only if
1) $W \geq 0$ (this is assumed).
2) The equation

$$(11.4) \qquad WX = (A - \bar{z}I)^{-1}(C(Q(z))^* - B(P(z))^*)$$

is solvable.
3) For each such solution X the following inequality is valid:

$$(11.5) \qquad \frac{P(z)(Q(z))^* - Q(z)(P(z))^*}{z - \bar{z}} - X^* W X \geq 0.$$

Splitting $X \in \mathbf{C}^{N \times m}$ into the blocks $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$,

$X_1 \in \mathbf{C}^{r \times m}$, we rewrite (11.4) as

$$\begin{pmatrix} A_1 - \bar{z}I & A_{12} \\ A_{21} & A_2 - \bar{z}I \end{pmatrix} \begin{pmatrix} W_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(Q(Z))^* - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(P(Z))^*,$$

what implies

$$(11.6) \qquad (A_1 - \bar{z}I)W_1 X_1 = C_1(Q(z))^* - B_1(P(z))^*,$$

$$(11.7) \qquad A_{21} W_1 X_1 = C_2(Q(z))^* - B_2(P(z))^*.$$

Therefore,

$$(11.8) \qquad X_1 = W^{-1}(A_1 - \bar{z}I)^{-1}(C_1(Q(z))^* - B_1(P(z))^*),$$

where $P(z)$ and $Q(z)$ must satisfy (11.7), which using (11.8) and (11.3) can be written in the form

$$(11.9) \qquad \begin{aligned} (C_2 B_1^* - B_2 C_1^*)W^{-1}&(A_1 - \bar{z}I)^{-1}(C_1(Q(z))^* - B_1(P(z))^*) \\ &= (C_2(Q(z))^* - B_2(P(z))^*), \end{aligned}$$

X_2 is an arbitrary matrix from $C^{(N-r)\times m}$.

11.2. Inequality (11.5) with the found values of X takes the form
(11.10)
$$\frac{P(z)(Q(z))^* - Q(z)(P(z)^*)}{z - \overline{z}}$$
$$- (Q(z)C_1^* - P(z)B_1^*)(A_1^* - zI)^{-1}W_1^{-1}(A_1 - \overline{z}I)^{-1}(C_1(Q(z))^* - B_1(P(z))^*) \geq 0.$$

This is the MMI for the quadruple (A_1, B_1, C_1, W_1), since
$$\frac{P(z)(Q(z))^* - Q(z)(P(z)^*)}{z - \overline{z}} = \frac{iR(z)^{\prime}J(R(z))^*}{z - \overline{z}},$$

$$Q(z)C_1^* - P(z)B_1^* = -iR(z)JK_1^*, \quad C_1(Q(z))^* - B_1(P(z))^* = iK_1J(R(z))^*,$$

where $R(z) = (P(z), Q(z))$, $K_1 = (C_1, B_1)$ (see (10.6)). So solutions of (11.10) can be parametrized by means of the formula
$$(P(z), Q(z)) = (p(z), q(z))\mathcal{B}_1(z),$$

where $(p(z), q(z))$ is an arbitrary non-degenerate $\mathcal{R}_m$-pair, $\mathcal{B}_1(z)$ is the matrix obtained by changing K for $K_1 = (C_1, B_1)$, W for W_1, A for A_1 in (10.9)
(11.11)
$$\mathcal{B}_1(z) = I_{2m} - iJK_1^*W_1^{-1}(A_1 - zI)^{-1}K_1.$$

11.3. Let us clarify what conditions on the $\mathcal{R}_m$-pair $(p(z), q(z))$ are imposed by condition (11.9). Condition (11.9) can be written as follows:
$$(-B_2, C_2)\begin{pmatrix} -C_1^* \\ B_1^* \end{pmatrix} W_1^{-1}(A_1 - \overline{z}I)^{-1}(-B_1, C_1)\begin{pmatrix} (P(z))^* \\ (Q(z))^* \end{pmatrix} = (-B_2, C_2)\begin{pmatrix} (P(z))^* \\ (Q(z))^* \end{pmatrix}$$

or
$$(P(z), Q(z))\left(I_{2m} - \begin{pmatrix} -B_1^* \\ C_1^* \end{pmatrix}(A_1^* - zI)^{-1}W_1^{-1}(C_1, B_1)\right)\begin{pmatrix} -B_2^* \\ C_2^* \end{pmatrix} = 0.$$

It should be noted (see (10.10)) that
$$I_{2m} - \begin{pmatrix} -B_1^* \\ C_1 \end{pmatrix}(A_1^* - zI)^{-1}W_1^{-1}(C_1, B_1)$$
$$= I_{2m} + iJK_1^*(A_1^* - zI)^{-1}W_1^{-1}K_1 = \mathcal{A}_1(z) = (\mathcal{B}_1(z))^{-1}$$

therefore, condition (11.9) is equivalent to the condition
(11.12)
$$p(z)B_2^* = q(z)C_2^*.$$

Let us show that the set of non-degenerate $\mathcal{R}_m$-pairs satisfying (11.12) is not empty. Equation $C_2B_2^* - B_2C_2^* = 0$ (see (11.3)) is equivalent to
$$(B_2 + iC_2)(B_2 + iC_2)^* = (B_2 - iC_2)(B_2 - iC_2)^*.$$

Therefore, there exists a unitary matrix $u \in C^{m\times m}$ such that $B_2 + iC_2 = (B_2 - iC_2)u$ and, consequently,
$$p_0B_2^* = q_0C_2^*,$$

where $p_0 = i(I - u)$, $q_0 == I + u$. It is easy to see that the pair (p_0, q_0) is non-degenerate and is an $\mathcal{R}_m$-pair, because $p_0q_0^* - q_0p_0^* = 0$.

Now we state the final result of this section in terms of (11.2) and (11.10).

Theorem 5. *Let* $AW - WA^* = CB^* - BC^*$, $W \geq 0$, *then the formula*
$$w(z) = (p(z)b_{12}^{(1)}(z) + q(z)b_{22}^{(1)}(z))^{-1}(p(z)b_{11}^{(1)}(z) + q(z)b_{21}^{(1)}(z))$$

establishes a one-to-one correspondence between the set of $\mathcal{R}_m$-functions associated with (A, B, C, W) and the set of those classes of equivalence of non-degenerate $\mathcal{R}_m$-pairs

$(p(z), q(z))$ *which satisfy the condition* $p(z)B_2^* = q(z)C_2^*$. *Here* $b_{jk}^{(1)}(z)$ *are m order blocks of the matrix* $\mathcal{B}_1(z)$.

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