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EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS OF SECOND ORDER SEMILINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACE

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Abstract. We consider the Cauchy problem for second order semilinear differential equations in Banach space. Sufficient conditions of local and global existence and uniqueness of mild solutions are presented.

1. Introduction

Let X be a complex Banach space, A a closed densely defined linear operator. We consider the following semilinear differential equation:

$$\frac{d^2x}{dt^2}(t) + Ax(t) = f(t, x(t)), \tag{1}$$

where the function $x(\cdot)$ takes values in X, and f maps some open subset of $\mathrm{R} \times X$ to X. Equations of type (1) are considered in [1], where sufficient conditions of existence and uniqueness of solutions are presented for a broad class of functions f, including discontinuous functions. However, there are examples of functions f for which these results cannot be applied, e.g., $f_1(t,x) = x^3$ and $f_2(t,x) = x \cdot x'$ on $X = L_2(\mathrm{R})$.

In this paper we apply Henry's method [2] to second order semilinear equation (1) and prove several theorems about sufficient conditions of existence and uniqueness of solutions of Cauchy problems for a class of continuous functions f. As shown below, this class includes f_1 and f_2 in $X = L_2(\mathrm{R})$ if $Ax = -x''$ (defined on $x \in X$ such that x'', understood in the sense of distributions, belongs to X).

2. Preliminaries

Let $C(t)$ be an operator cosine function with generator $-A$. Linear operator $-A$ is also a generator of an analytic semigroup $T(t)$. At first let us consider the case when $\sigma(A) \subset \{\lambda | \operatorname{Re}\lambda > 0\}$. For $\alpha > 0$ define

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} T(s)\, ds$$

(see [2, p. 24]).

Operator $A^{-\alpha}$ is bounded in X and has an inverse [2, Theorem 1.4.2, p. 25].

Define $A^\alpha = (A^{-\alpha})^{-1}$. A^α is closed and densely defined. For arbitrary α, β, we have $A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$ on $D(A^\alpha) \cap D(A^\beta) \cap D(A^{\alpha+\beta})$; if $\alpha > \beta$, then $D(A^\alpha) \subset D(A^\beta)$ [2, p. 25–26].

Now consider the case when $\sigma(A) \not\subset \{\lambda | \operatorname{Re}\lambda > 0\}$. Let us denote $\omega = -\inf \operatorname{Re}\sigma(A)$, then for $b > \omega$ we have $\sigma(A + bI) \subset \{\lambda | \operatorname{Re}\lambda > 0\}$. Define: $A_b = A + bI$, $A_b^\alpha = (A_b)^\alpha$,

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$X_b^\alpha = D(A_b^\alpha)$, for $x \in X_b^\alpha$ denote $\|x\|_\alpha = \|A_b^\alpha x\|$. Then

$$A_b^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-sb} T(s)\, ds. \tag{2}$$

Lemma 1. ([2, Theorem 1.4.6, p. 28; Theorem 1.4.8, p. 29]). *The space X_b^α does not depend on the choice of b such that $\sigma(A+bI) \subset \{\lambda | \operatorname{Re}\lambda > 0\}$. $X^\alpha = X_b^\alpha$ is a Banach space in the norm $\|\cdot\|_\alpha$, and for different b the corresponding norms $\|\cdot\|_\alpha$ are equivalent.*

Example 1. Let $X = L_2(\mathrm{R})$, $Ax = -x''$, where $D(A)$ is the set of all $x \in X$ such that x'' (understood in the sense of distributions) belongs to $L_2(\mathrm{R})$; $\alpha = 1/2$. Then $X^{1/2} = H^1(\mathrm{R})$ in the sense that they coincide as subsets of $X = L_2(\mathrm{R})$, and the corresponding norms are equivalent ([2, p. 77]; [3, Theorem V.3, p. 135]).

Lemma 2. *Let $C(t), S(t), T(t)$ be, respectively, cosine function, sine function and a semigroup with the generator $-A$. Then for any $s \geq 0, t \geq 0, \alpha > 0$ and $b > \omega$, the following relations hold:*

$$T(s)C(t) = C(t)T(s), \tag{3}$$

$$A_b^{-\alpha} C(t) = C(t) A_b^{-\alpha}, \quad A_b^{-\alpha} S(t) = S(t) A_b^{-\alpha}, \tag{4}$$

and for $x \in D(A_b^\alpha)$

$$C(t)x \in D(A_b^\alpha), \quad A_b^\alpha C(t)x = C(t)A_b^\alpha x,$$
$$S(t)x \in D(A_b^\alpha), \quad A_b^\alpha S(t)x = S(t)A_b^\alpha x. \tag{5}$$

Proof. Since the semigroup $T(t)$ is analytic,

$$T(s) = \frac{1}{2\pi i} \oint_\Gamma e^{\lambda s} R(\lambda; -A)\, d\lambda,$$

where Γ is a contour in the resolvent set of the operator $-A$ with $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\pi/2; \pi)$. Also we have $R(\lambda; -A)C(t) = C(t)R(\lambda; -A)$. Hence

$$T(s)C(t) = \frac{1}{2\pi i} \oint_\Gamma e^{\lambda s} R(\lambda; -A) C(t)\, d\lambda = \frac{1}{2\pi i} \oint_\Gamma e^{\lambda s} C(t) R(\lambda; -A)\, d\lambda = C(t)T(s)$$

and (3) is proved.

Relations (4) are immediate consequences of (2) and (3).

Relations (5) are easily obtained from (4), as well as the following statement: if bounded linear operators B_1, B_2 commute, B_1^{-1} exists and $x \in D(B_1^{-1})$, then $B_2 B_1^{-1} x = B_1^{-1} B_1 B_2 B_1^{-1} x = B_1^{-1} B_2 B_1 B_1^{-1} x = B_1^{-1} B_2 x$ and $B_2 x \in D(B_1^{-1})$. □

In what follows we assume that the following holds.

Assumption 1. ([4, Assumption 5.1, p. 63]). *Let $b > \omega$. Then $S(t)X \in D(A_b^{1/2})$, and $A_b^{1/2} S(t)$ is a strongly continuous function of the argument t on $-\infty < t < +\infty$.*

Lemma 3. ([4, Lemma 5.2, p. 63; Theorem 5.4, p. 65; eq. 5.12, p. 65]). *If Assumption 1 holds, then $\forall b > \omega \quad \exists C_{1/2} > 0\ \forall t \geq 0$*

$$\left\| A_b^{1/2} S(t) \right\| \leq C_{1/2}(1+t)e^{\omega t}.$$

Assumption 1 holds for any generator of the cosine function in any complex Lebesgue space $L_p(Y, \mu), 1 < p < \infty$ [4, Theorem 6.1, p. 71; Theorem 6.3, p. 73].

Theorem 1. (analogous to [2, Theorem 1.4.3, p. 26]). *Under Assumption 1, for any $\alpha \in \left[0; 1/2\right]$* we have the following:

1) there exists $C_\alpha > 0$ such that for every $t \geq 0$, $\|S(t)\|_\alpha \leq C_\alpha(1+t)e^{\omega t}$;

2) for $\forall x_0 \in D(A_b^\alpha), x_1 \in X$ we have that $\|C(t)x_0 + S(t)x_1 - x_0\|_\alpha \longrightarrow_{t\to 0} 0$.

Proof. To prove the first statement we use Lemma 3

$$\begin{aligned}\|S(t)\|_\alpha &= \|A_b^\alpha S(t)\| \le \left\|A_b^{1/2} S(t) A_b^{-(1/2-\alpha)}\right\| \\ &\le C_{1/2}(1+t)e^{\omega t}\cdot\left\|A_b^{-(1/2-\alpha)}\right\| = C_\alpha(1+t)e^{\omega t}.\end{aligned}$$

The second statement is implied by the following:

$$\begin{aligned}A_b^\alpha\left(C(t)x_0 + S(t)x_1 - x_0\right) &= \left(C(t)-I\right)A_b^\alpha x_0 + A_b^{1/2}S(t)A_b^{-(1/2-\alpha)}x_1 \\ &= (C(t)-I)y_0 + A_b^{1/2}S(t)y_1 \underset{t\to 0}{\longrightarrow} (C(0)-I)y_0 + A_b^{1/2}S(0)y_1 = 0,\end{aligned}$$

because $C(t)$ and $A_b^{1/2}S(t)$ are strongly continuous functions. □

Note, however, that the operators $A_b^\alpha S(t)$, $\alpha > 1/2$ and $A_b^\alpha C(t)$, $\alpha > 0$, can be unbounded.

Example 2. Let $X = L_2(\mathrm{R})$, $Ax = -x''$ as in Example 1. Then $\forall x_0 \in H^1(\mathrm{R}), x_1 \in X$

$$\left(C(t)x_0\right)(s) = \frac{1}{2}\left(x_0(s+t) + x_0(s-t)\right),$$

$$\left(S(t)x_1\right)(s) = \frac{1}{2}\int_{s-t}^{s+t} x_1(\xi)\,d\xi.$$

Take $\alpha = 1/2$, then $\forall t > 0$ the operator $A_b^\alpha C(t)$ is unbounded. Consider, for example,

$$x_n(s) = \begin{cases}\sin n\frac{2\pi s}{t}, & 0 \le s \le t,\\ 0, & \text{otherwise,}\end{cases}$$

then $\{\|x_n\|, n = 0, 1, \ldots\}$ is bounded, but

$$\|A_b^\alpha C(t)x_n\| = \|C(t)x_n\|_\alpha \ge \text{const} \times \|C(t)x_n\|_{H^1} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Take $\alpha = 1$, then $\forall t > 0$ the operator $A_b^\alpha S(t)$ is unbounded,

$$\begin{aligned}\left(AS(t)x_1\right)(s) &= \frac{1}{2}\frac{\partial^2}{\partial s^2}\int_{s-t}^{s+t} x_1(\xi)\,d\xi = \frac{1}{2}\frac{\partial}{\partial s}\left(x_1(s+t) - x_1(s-t)\right) \\ &= \frac{1}{2}\left(x_1'(s+t) - x_1'(s-t)\right).\end{aligned}$$

3. Sufficient conditions of local existence and uniqueness of mild solutions

For functions defined on some interval $[t_0; t_1]$ or semiaxis $[t_0; +\infty)$ and taking values in a Banach space X consider the following semilinear equation:

$$\frac{d^2x}{dt^2}(t) + Ax(t) = f(t, x(t)), \tag{6}$$

where the function f maps some open set $U \subset \mathrm{R} \times X^\alpha$ to X, for fixed $\alpha \in \left[0; 1/2\right]$, and state the following Cauchy problem for it:

$$x(t_0) = x_0 \in D(A_b^\alpha), \quad x'(t_0) = x_1 \in X. \tag{7}$$

A classical solution of the problem (6)–(7) on $[t_0; t_1]$ is a function $x : [t_0; t_1] \to X$ that is twice continuously differentiable, $x(t) \in D(A)$ for all $t \in [t_0; t_1]$, and satisfies (6) and (7).

A mild solution of the problem (6)–(7) on $[t_0; t_1]$ is a continuous function $x : [t_0; t_1] \to X$ that satisfies, on $[t_0; t_1]$, the equation

$$x(t) = C(t-t_0)x_0 + S(t-t_0)x_1 + \int_{t_0}^{t} S(t-\tau)f(\tau, x(\tau))\,d\tau. \tag{8}$$

A classical solution of the problem (6)–(7) is also a mild solution ([1, p. 436]). The converse doesn't always hold, since the mild solution may fail to be twice continuously differentiable.

Theorem 2. *Let $U \subset \mathrm{R} \times X^\alpha$ be an open set and $f : U \to X$ a continuous function that satisfies a local Lipschitz condition, for every point $(t_1, x_1) \in U$ there exists $K > 0$ and a neighborhood $U_1 \subset U$ of the point (t_1, x_1) such that for $x, y \in U_1$ the inequality*

$$\|f(t,x) - f(s,y)\| \le K \left(|t-s| + \|x-y\|_\alpha\right)$$

holds. Then for each pair (t_0, x_0) from U and $x_1 \in X$ there exists $t_1 > t_0$ such that problem (6)–(7) has a unique mild solution on $[t_0; t_1]$ with $x(t_0) = x_0 \in D(A_b^\alpha)$, $x'(t_0) = x_1 \in X$.

(This theorem is analogous to [2, Theorem 3.3.3, p. 54])

Proof. Let $V(\tau,\delta) = \{(t,x) | t \in [t_0; t_0+\tau], \|x - x_0\|_\alpha \le \delta\}$. Choose τ, δ such that $V(\tau,\delta) \subset U$ and for $(t,x), (t,y) \in V(\tau,\delta)$ the following holds: $\|f(t,x) - f(t,y)\| \le K\|x-y\|_\alpha$. Also let $B = \max_{t\in[t_0;t_0+\tau]} \|f(t,x_0)\|$.

Using Theorem 1, choose $t_1 \in (t_0, t_0+\tau]$ such that for $t \in [t_0; t_1]$,

$$\|C(t-t_0)x_0 + S(t-t_0)x_1 - x_0\|_\alpha \le \delta/2$$

and

$$C_\alpha(1+t_1)e^{\omega t_1}(t_1 - t_0)(B + K\delta) \le {}^{\delta}\!/_{2}.$$

Now, define $M = \left\{x \in C([t_0;t_1]; X^\alpha) | \sup_{t_0 \le t \le t_1} \|x(t) - x_0\|_\alpha \le \delta\right\}$ with the usual sup-norm $|||x||| = \sup_{t_0\le t\le t_1} \|x(t)\|_\alpha$. This is a complete metric space.

Consider a map $G : M \to C([t_0;t_1]; X^\alpha)$ defined for $x \in M$ as follows:

$$G(x)(t) = C(t-t_0)x_0 + S(t-t_0)x_1 + \int_{t_0}^{t} S(t-s) f(s, x(s))\, ds.$$

First let us show that G maps M into itself. For any $x \in M$,

$$\begin{aligned}\|G(x)(t) - x_0\|_\alpha &\le \|C(t-t_0)x_0 + S(t-t_0)x_1 - x_0\|_\alpha \\ &\quad + \int_{t_0}^{t} \|S(t-s)\|_\alpha \|f(s,x(s))\|\, ds \\ &\le {}^{\delta}\!/_{2} + (t-t_0)\left(C_\alpha(1+t)e^{\omega t}\right)(B + K\delta) \le {}^{\delta}\!/_{2} + {}^{\delta}\!/_{2} = \delta.\end{aligned}$$

Now, let us show that G is a strict contraction (using Theorem 1), for any $x, y \in M$,

$$\begin{aligned}\|G(x)(t) - G(y)(t)\|_\alpha &\le \int_{t_0}^{t} \|S(t-s)\|_\alpha \|f(s,x(s)) - f(s,y(s))\|\, ds \\ &\le \left(C_\alpha(1+t_1)e^{\omega t_1}(t_1-t_0)K\right)|||x-y||| \le \frac{1}{2}|||x-y|||.\end{aligned}$$

Therefore, $|||G(x) - G(y)||| \le \frac{1}{2}|||x-y|||$.

So, $G : M \to M$ is a strict contraction. By the contraction mapping theorem there exists a unique element $x \in M$ satisfying $G(x)(t) = x(t)$, i.e., relation (8). This element is the sought-for solution. □

Example 3. Consider the following equation:

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial s^2} + x\frac{\partial x}{\partial s}.$$

We rewrite it as

$$\frac{d^2x}{dt^2} + Ax = f(t,x), \tag{9}$$

where $x(t)$ is a function taking values in $X = L_2(\mathrm{R})$; $Ax = -x''$ as in Example 1, $f(t,x) = x \cdot x'$. Let us state the Cauchy problem for it,

$$x(t_0) = x_0 \in X^{1/2}, \quad x'(t_0) = x_1. \tag{10}$$

Now we will prove that the function $f(t,x)$ satisfies the conditions of Theorem 2 with $\alpha = 1/2$.

Lemma 4. *Let $X = L_2(\mathrm{R})$, $Ax = x''$, $\alpha = 1/2$ as in Example 1. Then*

(1) $\exists C_1 > 0 \quad \forall x \in X^{1/2}: \quad \|x\|_{L_\infty} \le C_1 \|x\|_\alpha$;
(2) $\exists C_2 > 0 \quad \forall x \in X^{1/2}: \quad \|x\|_{L_2} \le C_2 \|x\|_\alpha$;
(3) $\exists C_3 > 0 \quad \forall x \in X^{1/2}: \quad \left\|\frac{dx}{ds}\right\|_{L_2} \le C_3 \|x\|_\alpha$.

Proof. (1) As noted in Example 1, $X^{1/2} = H^1(\mathrm{R})$ in the sense that they coincide as subsets of $X = L_2(\mathrm{R})$, and the corresponding norms are equivalent. And $H^1(\mathrm{R})$ is continuously embedded into $L_\infty(\mathrm{R})$ (even into $C(\mathrm{R})$), see [2, p. 9].

(2) Let $x \in X^{1/2}$. Then $\|x\|_{L_2} = \left\|A_b^{-1/2} A_b^{1/2} x\right\|_{L_2} \le \left\|A_b^{-1/2}\right\| \|x\|_\alpha$. We obtain the needed inequality by denoting $C_2 = \left\|A_b^{-1/2}\right\|$.

(3) $\left\|\frac{dx}{ds}\right\|_{L_2} \le \|x\|_{H^1}$, and norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_\alpha$ are equivalent (see part 1 of the proof).

□

Using Lemma 4, for $x, y \in X^{1/2}$ we have

$$\begin{aligned}
\|f(x) - f(y)\|_{L_2} &= \left\|x\frac{dx}{ds} - y\frac{dy}{ds}\right\|_{L_2} \le \left\|x\frac{dx}{ds} - y\frac{dx}{ds}\right\|_{L_2} + \left\|y\frac{dx}{ds} - y\frac{dy}{ds}\right\|_{L_2} \\
&\le \|x - y\|_{L_\infty} \left\|\frac{dx}{ds}\right\|_{L_2} + \|y\|_{L_\infty} \left\|\frac{dx}{ds} - \frac{dy}{ds}\right\|_{L_2} \\
&\le C_1 C_3 \left(\|x - y\|_\alpha \|x\|_\alpha + \|y\|_\alpha \|x - y\|_\alpha\right) \\
&= C_1 C_3 \left(\|x\|_\alpha + \|y\|_\alpha\right) \|x - y\|_\alpha .
\end{aligned}$$

Therefore, by Theorem 2 the problem (9)–(10) has a unique local mild solution.

Theorem 3. *Let $U \subset \mathrm{R} \times X^\alpha$, $f : U \to X$ and $K > 0$ be as in Theorem 2, $x(t)$ a mild solution of the problem (6)–(7) on $[t_0; t_1]$ with $x(t_0) = x_0$, $x'(t_0) = x_1$. Then the following holds.*

(1) *If $x_0 \in D(A_b^\alpha)$, $x_1 \in X$, then $x(t) \in D(A_b^\alpha)$.*
(2) *If $x_0 \in D(A_b^{1/2+\alpha})$ and $x_1 \in D(A_b^\alpha)$, then $x(t)$ is locally Lipschitz as a function $[t_0; t_1] \to X^\alpha$ (and, therefore, as a function $[t_0; t_1] \to X$).*
(3) *If $x_0 \in D(A)$ and $x_1 \in D(A_b^{1/2})$, then $x'(t)$ is locally Lipschitz as a function $[t_0; t_1] \to X$.*

Proof. (1) Take $x_0 \in D(A_b^\alpha)$, $x_1 \in X$. By Lemma 2 for every $t > t_0$ we have that

$$C(t - t_0)x_0 \in D(A_b^\alpha).$$

By assumption 1, $S(t - t_0)x_1 \in D(A_b^{1/2})$, but $D(A_b^{1/2}) \subset D(A_b^\alpha)$. Now,

$$\int_{t_0}^t S(t - s) f(s, x(s))\, ds \in D(A_b^\alpha),$$

since by Assumption 1, we have $S(t - s)f(s, x(s)) \in D(A_b^\alpha)$, and $A_b^\alpha S(t - s) f(s, x(s))$ is a continuous function of the argument s. Hence,

$$x(t) = C(t - t_0)x_0 + S(t - t_0)x_1 + \int_{t_0}^t S(t - s) f(s, x(s))\, ds \in D(A_b^\alpha).$$

(2) Define $g(t) = f(t, x(t))$. Take an arbitrary point $t \in (t_0; t_1)$. For sufficiently small $h > 0$, we have

$$\|g(t+h) - g(t)\| \leq K\left(h + \|x(t+h) - x(t)\|_\alpha\right). \tag{11}$$

Therefore,

$$\begin{aligned}
&\|x(t+h) - x(t)\|_\alpha \\
&\quad\leq \|(C(t-t_0+h) - C(t-t_0))\, A_b^\alpha x_0\| + \|(S(t-t_0+h) - S(t-t_0))\, A_b^\alpha x_1\| \\
&\quad+ \Big\|\int_{t_0}^{t+h} A_b^\alpha S(t+h-\tau) g(\tau)\, d\tau - \int_{t_0}^{t} A_b^\alpha S(t-\tau) g(\tau)\, d\tau\Big\|.
\end{aligned}$$

Recall that $C'(s) = -S(s)A$ and $S'(s) = C(s)$. Now,

$$\begin{aligned}
&\|x(t+h) - x(t)\|_\alpha \\
&\quad\leq h\Big(\sup_{\tau\in[t-t_0;t_1-t_0]} \Big\|A_b^{1/2} S(\tau)\Big\|\Big) \Big\|A_b^{1/2+\alpha} x_0\Big\| + h\Big(\sup_{\tau\in[t-t_0;t_1-t_0]} \|C(\tau)\|\Big) \|A_b^\alpha x_1\| \\
&\quad+ \Big\|\int_{t_0}^{t_0+h} A_b^\alpha S(t+h-\tau) g(\tau)\, d\tau\Big\| + \Big\|\int_{t_0}^{t} A_b^\alpha S(t-\tau)\left(g(\tau+h) - g(\tau)\right)\, d\tau\Big\|.
\end{aligned}$$

Therefore we can choose $K_1, K_2 > 0$, independent of $t, t+h \in (t_0; t_1)$, such that

$$\|x(t+h) - x(t)\|_\alpha \leq hK_1 + K_2 \int_{t_0}^{t} \|g(\tau+h) - g(\tau)\|\, d\tau. \tag{12}$$

Now substitute (12) into (11),

$$\begin{aligned}
\|g(t+h) - g(t)\| &\leq K\left(h + \|x(t+h) - x(t)\|_\alpha\right) \\
&\leq K\Big(h + hK_1 + K_2 \int_{t_0}^{t} \|g(\tau+h) - g(\tau)\|\, d\tau\Big) \\
&= hK\left(1 + K_1\right) + K_2 K \int_{t_0}^{t} \|g(\tau+h) - g(\tau)\|\, d\tau.
\end{aligned}$$

By the Gronwall inequality for $t_0 < t < t+h < t_1$ we have

$$\|g(t+h) - g(t)\| \leq h\left(K\left(1 + K_1\right) e^{K_2 K t}\right), \tag{13}$$

hence

$$\|x(t+h) - x(t)\|_\alpha \leq h\left(K_1 + K_2(t_1 - t_0)(K(1+K_1)e^{K_2 K t})\right),$$

i.e., the functions $g(t)$ and $x(t)$ are locally Lipschitz.

(3) Take $x_0 \in D(A)$. By differentiating (8) we obtain

$$\begin{aligned}
x'(t) &= C'(t-t_0)x_0 + S'(t-t_0)x_1 + \int_{t_0}^{t} S'(t-s) f(s, x(s))\, ds \\
&= -S(t-t_0)Ax_0 + C(t-t_0)x_1 + \int_{t_0}^{t} C(t-s) f(s, x(s))\, ds.
\end{aligned}$$

Arguing as in the case of $\|x(t+h) - x(t)\|_\alpha$ above, we have

$$\begin{aligned}
&\|x'(t+h) - x'(t)\| \\
&\quad\leq \|(S(t-t_0+h) - S(t-t_0))\, Ax_0\| + \|(C(t-t_0+h) - C(t-t_0))\, x_1\| \\
&\quad+ \Big\|\int_{t_0}^{t_0+h} C(t+h-t_0-\tau) g(\tau)\, d\tau\Big\| + \Big\|\int_{t_0}^{t} C(t-\tau)\left(g(\tau+h) - g(\tau)\right)\, d\tau\Big\|,
\end{aligned}$$

and for some $K_3, K_4 > 0$, independent of $t, t+h \in (t_0; t_1)$,

$$\|x'(t+h) - x'(t)\| \leq hK_3 + K_4 \int_{t_0}^{t} \|g(\tau+h) - g(\tau)\|\, d\tau. \tag{14}$$

By substituting (13) into (14) we obtain the assertion of the theorem. □

4. Sufficient conditions for global existence and uniqueness of mild solutions

Theorem 4. *Let the continuous function $f : \mathrm{R}^+ \times X^\alpha \to X$ in equation (6) satisfy a global Lipschitz condition, i.e., there exists $K > 0$ such that for any $t, s \geq t_0$ and $x, y \in X^\alpha$, the following inequality holds:*

$$\|f(t,x) - f(s,y)\| \leq K\left(|t-s| + \|x-y\|_\alpha\right).$$

Then $\forall x_0 \in D(A_b^{1/2})$, $x_1 \in X$ the problem (6)-(7) has a unique mild solution on $[t_0;+\infty)$ with $x(t_0) = x_0$, $x'(t_0) = x_1$.

Note that the theorem requires $x_0 \in D(A_b^{1/2})$ even if $\alpha < 1/2$.

Proof. By Theorem 2, $\exists t_1 > t_0$, problem (6)–(7) has a unique mild solution on $[t_0;t_1]$. Denote by $\tilde{t}_1$ the supremum of t_1 such that a mild solution exists and is unique on $[t_0;t_1]$. Then the solution $x(t)$ exists and is unique on $[t_0;\tilde{t}_1)$.

Assume now that $\tilde{t}_1 < +\infty$. First, we show that the solution is bounded on $[t_0;\tilde{t}_1)$,

$$\|x(t)\|_\alpha \leq \|C(t-t_0)\|\,\|x_0\|_\alpha + \|S(t-t_0)\|_\alpha\,\|x_1\| + \int_{t_0}^{t}\|S(t-\tau)\|_\alpha\,\|f(\tau,x(\tau))\|\,d\tau.$$

The functions $C(t-t_0)$ and $A_b^\alpha S(t-t_0)$ are strongly continuous for every $t \in \mathrm{R}$, so they are bounded on $[t_0;\tilde{t}_1)$. Therefore, $\exists K_1 > 0, K_2 > 0$ such that $\forall t \in [t_0;\tilde{t}_1)$: $\|C(t-t_0)\|\,\|x_0\|_\alpha + \|S(t-t_0)\|_\alpha\,\|x_1\| \leq K_1$, $\|S(t-t_0)\|_\alpha \leq K_2$. Hence,

$$\|x(t)\|_\alpha \leq K_1 + K_2\int_{t_0}^{t}\|f(\tau,x(\tau))\|\,d\tau. \tag{15}$$

Furthermore, for $\tau \in [t_0;\tilde{t}_1)$

$$\begin{aligned}\|f(\tau,x(\tau))\| &\leq \|f(\tau,x(\tau)) - f(t_0,x(t_0))\| + \|f(t_0,x(t_0))\|\\ &\leq K\left((\tau-t_0) + \|x(\tau)-x(t_0)\|_\alpha\right) + \|f(t_0,x(t_0))\|\\ &\leq \left(K\left(\tilde{t}_1 - t_0\right) + K\|x(t_0)\|_\alpha + \|f(t_0,x(t_0))\|\right) + K\|x(\tau)\|_\alpha\\ &= K_3 + K\|x(\tau)\|_\alpha\end{aligned}$$

(where K_3 is independent of τ).

Using (15), we obtain

$$\|x(t)\|_\alpha \leq K_1 + K_2K_3(\tilde{t}_1 - t_0) + K_2K\int_{t_0}^{t}\|x(\tau)\|_\alpha\,d\tau.$$

Therefore, by Gronwall inequality,

$$\|x(t)\|_\alpha \leq \left(K_1 + K_2K_3(\tilde{t}_1 - t_0)\right)e^{K_2Kt}.$$

So $x(\cdot) : [t_0;\tilde{t}_1) \to X^\alpha$ is a bounded continuous function and we can extend it to the point $\tilde{t}_1$, and $x(\tilde{t}_1) \in X^\alpha$.

In a similar way it can be shown that the function $x'(\cdot) : [t_0;\tilde{t}_1) \to X$ is bounded (noting that $(C(t-t_0)x_0)' = -A_b^{1/2}S(t)A_b^{1/2}x_0 - bS(t))$. Hence we have $x'(\tilde{t}_1)$. So we are in a position to apply Theorem 2 with initial time $\tilde{t}_1$ and conditions $x(\tilde{t}_1), x'(\tilde{t}_1)$, which means that we can extend the solution further than $\tilde{t}_1$, which is a contradiction to its maximality. We have therefore proved that $\tilde{t}_1 = +\infty$. □

5. Some particular cases

Theorem 5. *Let $f(t,x) = f_1(t,x,Bx)$, where B is a closed linear operator, relatively bounded with respect to A_b^α ($D(A_b^\alpha) \subset D(B)$ and $\exists C_1, C_2 \geq 0$ $\forall x \in D(A_b^\alpha)$ such that $\|Bx\| \leq C_1\|x\| + C_2\|A_b^\alpha x\|$). Let the continuous function $f_1 : \mathrm{R}^+ \times D(B) \times X \to X$*

satisfy a global Lipschitz condition, i.e., there exists $K > 0$ *such that* $\forall t_1 \geq t_0, t \geq t_0, x_1 \in D(B), x_2 \in D(B), y_1 \in X, y_2 \in X$ *the following inequality holds:*

$$\|f_1(t_1, x_1, y_1) - f_1(t_2, x_2, y_2)\| \leq K\left(|t_1 - t_2| + \|x_1 - x_2\| + \|y_1 - y_2\|\right),$$

Then $\forall x_0 \in D(A_b^{1/2})$, $x_1 \in X$ *the problem (6)-(7) has a unique mild solution on* $[t_0; +\infty)$ *with* $x(t_0) = x_0$, $x'(t_0) = x_1$.

Proof. It suffices to show that $f(t, x)$ satisfies the conditions of Theorem 4.

Choose $t_1 \geq t_0, t \geq t_0, x_1 \in D(B), x_2 \in D(B)$. Then

$$\begin{aligned}
\|f(t_1, x_1) - f(t_2, x_2)\| &= \|f_1(t_1, x_1, Bx_1) - f_1(t_2, x_2, Bx_2)\| \\
&\leq K\left(|t_1 - t_2| + \|x_1 - x_2\| + \|B(x_1 - x_2)\|\right) \\
&\leq K\left(|t_1 - t_2| + (1 + C_1)\|x_1 - x_2\| + C_2\|A_b^{\alpha}(x_1 - x_2)\|\right) \\
&= K\left(|t_1 - t_2| + \left((1 + C_1)\left\|A_b^{-\alpha}\right\| + C_2\right)\|A_b^{\alpha}(x_1 - x_2)\|\right) \\
&= K_1\left(|t_1 - t_2| + \|x_1 - x_2\|_{\alpha}\right).
\end{aligned}$$

□

Let $X = L_2(\mathrm{R})$, $Ax = -x''$, $\alpha = 1/2$, $X^{1/2} = H^1(\mathrm{R})$ as in Example 1. Consider the following Cauchy problem:

$$\frac{d^2x}{dt^2} + Ax = h(x)g(x), \tag{16}$$

$$x(t_0) = x_0, \quad x'(t_0) = x_1. \tag{17}$$

Theorem 6. *Let* $U \subset H^1(\mathrm{R})$ *be an open set and let the functions* $g : U \to L_2(\mathrm{R})$, $h : U \to L_\infty(\mathrm{R})$ *satisfy Lipschitz conditions, namely, if* $x_1 \in U$ *then there exist* $K_g > 0$, $K_h > 0$ *and a neighborhood* $U_1 \subset U$ *of the point* x_1 *such that for* $x, y \in U_1$ *the following inequalities hold:*

$$\begin{aligned}
\|g(x) - g(y)\|_{L_\infty} &\leq K_g\|x - y\|_{H^1}, \quad \|g(x)\|_{L_\infty} \leq K_g\|x\|_{H^1}, \\
\|h(x) - h(y)\|_{L_2} &\leq K_h\|x - y\|_{H^1}, \quad \|h(x)\|_{L_2} \leq K_h\|x\|_{H^1}.
\end{aligned}$$

Then for every $t_0 \in \mathrm{R}$, $x_0 \in U$ *and* $x_1 \in X$ *there exists* $t_1 > t_0$ *such that the problem (16)–(17) has a unique mild solution on* $[t_0; t_1]$.

Proof. To use Theorem 2 it suffices to show that the function $f(x) = g(x)h(x)$ is locally Lipschitz as a function $U \to L_2(\mathrm{R})$.

For any $x, y \in U_1$,

$$\begin{aligned}
\|f(x) - f(y)\|_{L_2} &\leq \|g(x)h(x) - g(x)h(y)\|_{L_2} + \|g(x)h(y) - g(y)h(y)\|_{L_2} \\
&\leq \|g(x)\|_{L_\infty}\|h(x) - h(y)\|_{L_2} + \|g(x) - g(y)\|_{L_\infty}\|h(y)\|_{L_2} \\
&\leq K_gK_h\left(\|x\|_{H^1}\|x - y\|_{H^1} + \|x - y\|_{H^1}\|y\|_{H^1}\right) \leq \\
&\leq CK_gK_h\left(\|x\|_{\alpha} + \|y\|_{\alpha}\right)\|x - y\|_{\alpha},
\end{aligned}$$

(because $\|\cdot\|_{H^1}$ and $\|\cdot\|_{\alpha}$ are equivalent). □

Example 4. As in Example 2, for the equation

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial s^2} + x\frac{\partial x}{\partial s} + x^3 \tag{18}$$

let us state the Cauchy problem (16)–(17) with $h(x) = \frac{dx}{ds} + x^2$, $g(x) = x$. Obviously, the function $g(\cdot)$ satisfies the conditions of Theorem 6. Let us check the conditions of Theorem 6 for $h(\cdot)$. We have

$$\|h(x) - h(y)\|_{L_2} \leq C\left(\|x - y\|_{H^1} + \|x + y\|_{L_\infty}\|x - y\|_{L_2}\right) \leq C_1\|x - y\|_{H^1},$$

and, by analogy,

$$\|h(x)\|_{L_2} \leq C_1\|x\|_{H^1}.$$

So, for given $t_0, x_0 \in H^1(\mathrm{R}), x_1 \in L_2(\mathrm{R})$ there exists a unique local mild solution of the problem (16)–(17) for the equation (18).

6. Sufficient conditions for a mild solution to be classical

Now we consider conditions under which a mild solution of the Cauchy problem (6)–(7) is also a classical solution. Since our case is close to the case when $C(t)$ possesses a group decomposition, we can prove a theorem with conditions similar to [5, Theorem III.1.5].

Theorem 7. *Let $U \subset \mathrm{R} \times X^\alpha$, $f : U \to X$ and $K > 0$ be as in Theorem 2, $x_0 \in D(A)$, $x_1 \in D(A_b^{1/2})$, and $x(t)$ a the mild solution of the problem (6)–(7) on $[t_0; t_1]$ with $x(t_0) = x_0$, $x'(t_0) = x_1$. If for every $(\tau, y) \in U$, $f(\tau, y) \in D(A_b^{1/2})$ and $A_b^{1/2} f(\cdot, \cdot)$ is a continuous function on U, then $x(t)$ is a classical solution.*

Proof. First we note that $C(t)\,x$ is differentiable for any $x \in D(A_b^{1/2})$ and

$$C'(t)\,x = -bS(t) - A_b^{1/2} S(t)\, A_b^{1/2} x = -AS(t)\,x. \tag{19}$$

Then we have

$$x'(t) = -S(t - t_0)\,Ax_0 + C(t - t_0)\,x_1 + \int_{t_0}^{t} C(t - s)\, f(s, x(s))\, ds.$$

Differentiating with respect to t one more time and using (19) we obtain

$$x''(t) = -AC(t - t_0)\,x_0 - AS(t - t_0)\,x_1 + f(t, x(t)) - \int_{t_0}^{t} AS(t - s)\, f(s, x(s))\, ds.$$

Obviously $x''(t)$ is continuous, and

$$x''(t) = -Ax(t) + f(t, x(t)).$$

□

7. Conclusion

In the paper we have constructed sufficient conditions for existence and uniqueness of mild solutions of the Cauchy problem (6)–(7) for continuous functions $f : U \to X$, $U \subset \mathrm{R} \times X^\alpha$, satisfying local Lipschitz condition.

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