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G-FRAMES AND OPERATOR VALUED-FRAMES IN HILBERT C*-MODULES

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Abstract. g-frames and fusion frames in Hilbert C*-modules have been defined by the second author and B. Khosravi in [15] and operator-valued frames in Hilbert C*-modules have been defined by Kaftal et al in [11]. We show that every operator-valued frame is a g-frame, we also show that in Hilbert C*-modules tensor product of orthonormal basis is an orthonormal basis and tensor product of g-frames is a g-frame, we get some relations between their g-frame operators, and we study tensor product of operator-valued frames in Hilbert C*-modules.

1. Introduction

Frames were first introduced in 1946 by Gabor [8], reintroduced in 1986 by Daubechies, Grossman and Meyer [4], and popularized from then on. Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing, image processing, data compression, sampling theory and many other fields.

Later the notion of frames in C*-algebras and frames in Hilbert C*-modules were introduced and some of their properties were investigated [6, 7, 11, 14, 15]. The second author and B. Khosravi in [15] introduced g-frames and fusion frames in Hilbert C*-modules. Since tensor product is useful in the approximation of multi-variate functions of combinations of univariate ones, in this paper, we study the g-frames in tensor product of Hilbert C*-modules and we generalize the techniques of [11], [13–16] to C*-modules. In section 2 we briefly recall the definitions and basic properties of Hilbert C*-modules. In section 3, we investigate tensor product of Hilbert C*-modules, which is introduced in [11] and we show that tensor product of orthonormal basis is an orthonormal basis. We also show that tensor product of g-frames for Hilbert C*-modules H and F, present a g-frame for $H \otimes F$, and tensor product of their g-frame operators is the g-frame operator of their tensor product g-frame. Finally, we recall the definitions and some properties of operator-valued frames in Hilbert C*-modules which were introduced in [11] and we show that every operator-valued frame is a g-frame. We study tensor product of operator-valued frames, tensor product of frame transforms for Hilbert C*-modules.

Throughout this paper, $\mathbb{N}$ and $\mathbb{C}$ will denote the set of natural numbers and the set of complex numbers, respectively. I, J and I_j's will be countable index sets. A and B will be unital C*-algebras.

2. Preliminaries

In this section we briefly recall the definitions and basic properties of Hilbert C*-modules and g-frames in Hilbert C*-modules. For more information about g-frames in Hilbert spaces and Hilbert C*-modules we refer to [15], [18].

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For a C*-algebra if $a \in A$ is positive, we write $a \geqslant 0$ and A^+ denotes the set of positive elements of A. If $0 \leqslant a$ and $a \leqslant b$ then we have $a^{\frac{1}{2}} \leqslant b^{\frac{1}{2}}$ and if $-b \leqslant a \leqslant b$ then $\| a \| \leqslant \| b \|$ for $a, b \in A$.

Definition 2.1. Let A be a unital C*-algebra and H be a left A-module, such that the linear structures of A and H are compatible. H is a pre-Hilbert A-module if H is equipped with an A-valued inner product $\langle .,. \rangle : H \times H \longrightarrow A$ that posses the following properties:

(i) $\langle x, x\rangle \geq 0$ for all $x \in H$ and $\langle x, x\rangle = o$ if and only if $x = 0$;
(ii) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ for all $a \in A$ and $x, y, z \in H$;
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in H$.

For $x \in H$, we define $\| x \| = \| \langle x, x\rangle \|^{\frac{1}{2}}$, if H is complete with $\| . \|$, it is called a Hilbert A-module or a Hilbert C*-module over A. For every a in C*-algebra A,we have $| a | = (a^*a)^{\frac{1}{2}}$ and the A-valued norm on H is defined by $| x | = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in H$. Throughout this paper V is a Hilbert A-module, W is a Hilbert B-module, and $\{V_i\}_{i\in I}$ is a family of closed submodules of V and $\{W_j\}_{j\in J}$ is a family of closed submodules of W. Now we recall some definitions, see [15], [18]. If H and F are Hilbert A-modules,then $L(H, F)$ is the collection of all bounded adjointable linear operators from H to F and we abbreviate $L(H, H)$ by $L(H)$, an operator $T \in L(H, F)$ is a unitary if $TT^* = I_F$, and $T^*T = I_H$.

Example 2.2. (a) Let $\{(H_i, \langle \dot{;} \rangle_i : i \in I)\}$ be a sequence of Hilbert spaces. Then $(\bigoplus H_i)_{l_2} = \{(x_i)_{i\in I} : x_i \in H_i, ||(x_i)||_2^2 = \sum_i \langle x_i, x_i\rangle_i < \infty\}$ with pointwise operations and inner product defined by $\langle (x_i), (y_i)\rangle = \sum_i \langle x_i, y_i\rangle_i$ is a Hilbert space. If $H_i = H$ for each $i \in I$, then we denote $(\bigoplus_{i\in I}(H_i))_{l_2}$ by $l^2(H, I)$. So for any sequence $\{H_i : i \in I\}$ of Hilbert spaces, there exists a Hilbert space $H = (\bigoplus_{i\in I} H_i)_{l_2}$ which contains all of the H_i's.

(b) Let $\{V_i : i \in I$ be a sequence of Hilbert A-modules and $(\bigoplus_{i\in I} V_i)_{l_2} = \{x = (x_i) : x_i \in V_i \textit{ and } \sum_i \langle x_i, x_i\rangle \textit{ is norm convergent in } A\}$. Then $(\bigoplus_{i\in I} V_i)_{l_2}$ is a Hilbert A-module with A-valued inner product $\langle x, y\rangle = \sum_{i\in I} \langle x_i, y_i\rangle$, where $x = (x_i)_{i\in I}$ $y = (y_i)_{i\in I}$, pointwise operations and the norm defined by $\| a \| = \| \langle a, a\rangle \|^{\frac{1}{2}}$.

Definition 2.3. (See [18]). Let H be a Hilbert space, and $\{V_i\}_{i\in I}$ is a family of Hilbert spaces. $B(H, V_i)$ is the collection of all adjointable bounded linear operators from H into V_i. A family $\{\Lambda_i \in B(H, V_i)\}_{i\in I}$ is said to be a g-frame for H with respect to $\{V_i\}_{i\in I}$, if there are real constants $0 < A \leqslant B < \infty$, such that for all $x \in H$,

$$A \| x \|^2 \leq \sum \| \Lambda_i x \|^2 \leq B \| x \|^2 . \tag{1}$$

The optimal constants (i.e. maximal for A and minimal for B) are called g-frame bounds. The g-frame $\{\Lambda_i : i \in I\}$ is said to be a tight g-frame if $A = B$, and said to be a Parseval g-frame if $A = B = 1$. The family $\{\Lambda_i : i \in I\}$ is said to be a g-Bessel sequence for H with respect to $\{V_i\}_{i\in I}$, if the right-hand side inequality of (1) holds.

In [18] it is shown that frames, pseudoframes, oblique frames, outer frames and frames of subspaces (fusion frames) are a class of g-frames, see examples in [18]. Also in [18] the g-frame operator S_Λ is defined by $S_\Lambda f = \sum_i \Lambda_i^*\Lambda_i f$ for each $f \in H$ and in [16] the synthesis operator $T_\Lambda : (\bigoplus V_i)_{l_2} \to H$, $T_\Lambda((y_i)_i) = \sum_i \Lambda_i^*(y_i)$ and the analysis operator $T_\Lambda^* : H \to (\bigoplus V_i)_{l_2}$, $T^*(x) = (\Lambda_i x)_{i\in I}$ are defined for every g-Bessel sequence $\{\Lambda_i \in B(H, V_i) : i \in I\}$ and some of their properties were investigated. To generalize this notion to the situation of Hilbert C*-modules first we recall some definitions. Let A be a unital C^*-algebra and H be a finitely or countably generated Hilbert A-module. A sequence $\{f_i : i \in I\}$ is a frame for H if there exist real constants $0 < A \leq B < \infty$ such that for every $f \in H$,

$$A\langle f, f\rangle \leq \sum_{i\in I}\langle f, f_i\rangle\langle f_i, f\rangle \leq B\langle f, f\rangle. \tag{2}$$

A and B are called bounds of the frame, if $A = B = \lambda$, the frame is called λ-tight, if $A = B = 1$, the frame is a Parseval frame, and if only the right hand inequality is required, it is called a Bessel sequence, The frame is standard if the series in (2) converges in norm in A.

Definition 2.4. (See [15]). A sequence $\{\Lambda_i \in L(H, V_i) : i \in I\}$ is a g-frame in H with respect to $\{V_i : i \in I\}$ if there exist constants $A, B > 0$ such that for every $x \in H$,

$$A\langle x, x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq B\langle x, x\rangle. \tag{3}$$

As usual A and B are g-frame bounds of $\{\Lambda_i : i \in I\}$. If $A = B = \lambda$, the g-frame is called λ-*tight* and if $A = B = 1$, it is called a *Parseval* g-frame. The g-frame is *standard* if for every $x \in H$, the sum in (3) converges in norm.

In this paper we consider standard g-frames. If $\{\Lambda_i : i \in I\}$ is a standard g-frame in a finitely or countably generated Hilbert A-module, then we can define the frame transform θ, the synthesis operator θ^* and g-frame operator S as follows:

$$\theta : H \to (\bigoplus V_i)_{l_2}\ , \qquad \theta(x) = (\Lambda_i x)_{i\in I}$$

$$\theta^* : (\bigoplus V_i)_{l_2} \to H\ , \qquad \theta^*(y) = \sum_i \Lambda_i^*(y_i),$$

for all $y = (y_i)$ in $(\bigoplus V_i)_{l_2}$ and $S = \theta^*\theta : H \to H$ is given by $S(x) = \sum_i \Lambda_i^*\Lambda_i(x)$, for each $x \in H$. We know that $||\theta^*|| \leq \sqrt{D}$, and $\theta : H \to \theta(H)$ is invertible and $||\theta^{-1}|| \leq \frac{1}{\sqrt{C}}$. Moreover S is positive, self adjoint and invertible with $\| S \| \leq D$ and $\| S^{-1} \| \leq \frac{1}{C}$, see [15].

3. Frames in Hilbert C*-modules

Suppose that A, B are C*-algebras and we take $A \otimes B$ as the completion of $A \otimes_{\text{alg}} B$ with the spatial norm see [10]. $A \otimes B$ is the spatial tensor product of A and B, also suppose that H is a Hilbert A-module and F is a Hilbert B-module. We want to define $H \otimes F$ as a Hilbert $(A \otimes B)$-module. Start by forming the algebraic tensor product $H \otimes_{\text{alg}} F$ of the vector spaces H, F (over $\mathbb{C}$). This is a left module over $(A \otimes_{\text{alg}} B)$ (the module action being given by $(a \otimes b)(x \otimes y) = ax \otimes by \quad (a \in A, b \in B, x \in H, y \in F)$). For $(x_1, x_2 \in H, y_1, y_2 \in F)$ we define $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle \otimes \langle y_1, y_2\rangle$. We also know that for $z = \sum_{i=1}^n x_i \otimes y_i$ in $H \otimes_{\text{alg}} F$ we have

$$\langle z, z\rangle = \sum_{i,j}\langle x_i, x_j\rangle \otimes \langle y_i, y_j\rangle \geq 0$$

and

$$\langle z, z\rangle = 0 \quad iff \quad z = 0.$$

This extends by linearity to an $(A \otimes_{\text{alg}} B)$-valued sesquilinear form on $H \otimes_{\text{alg}} F$, which makes $H \otimes_{\text{alg}} F$ into a semi-inner-product module over the pre-C*-algebra $(A \otimes_{\text{alg}} B)$. The semi-inner-product on $H \otimes_{\text{alg}} F$ is actually an inner product, see [17]. Then $H \otimes_{\text{alg}} F$ is an inner-product module over the pre-C*-algebra $(A \otimes_{\text{alg}} B)$,and we can perform the double completion discussed in chapter 1 of [17] to conclude that the completion $H \otimes F$ of $H \otimes_{\text{alg}} F$ is a Hilbert $(A \otimes B)$-module. We call $H \otimes F$ the exterior tensor product of H and F. With H, F as above, we wish to investigate the adjointable operators on $H \otimes F$. Suppose that $S \in L(H), T \in L(F)$. Define a linear operator $S \otimes T$ on $H \otimes F$ by $S \otimes T(x \otimes y) = Sx \otimes Ty \quad (x \in H, y \in F)$. It is a routine verification that $S^* \otimes T^*$ is an adjoint for $S \otimes T$, so in fact $S \otimes T \in L(H \otimes F)$. For more details see [10], [17]. We note that if $a \in A^+$ and $b \in B^+$, then $a \otimes b \in (A \otimes B)^+$. Plainly if a, b are Hermitian elements of A and $a \geq b$, then for every positive element x of B, we have $a \otimes x \geq b \otimes x$.

Definition 3.1. Let H be a Hilbert C*-module over an arbitrary C*-algebra A. An element $v \in H$ is said to be a basic vector if $e = \langle v, v\rangle$ is a minimal projection in A, in the sense that $eAe = \mathbb{C}e$. A system $(v_\lambda)_{\lambda\in\Lambda}$ in H is orthonormal if each v_λ is a basic vector and $\langle v_\lambda, v_\mu\rangle = 0$ for all $\lambda \neq \mu$. An orthonormal system $(v_\lambda)_{\lambda\in\Lambda}$ in H is said to be an orthonormal basis for H if it generates a dense submodule of H.

If we consider C^*-algebra of compact operators $K(H)$ on some Hilbert space H, then every Hilbert $K(H)$-module V possesses an orthonormal basis and all closed submodules of Hilbert $K(H)$-modules are orthogonally complemented. Furthermore, if $e \in K(H)$ is an orthogonal one-dimensional projection, then $Ve := \{xe : x \in V\}$ is a Hilbert space with respect to inner product $(x, y) = tr(\langle y, x\rangle)$ where "tr" means the trace. It is easy to see that $\langle x, y\rangle = (y, x)e$ for all $x \in Ve$, $y \in Ve$ see [1], [2].

Theorem 3.2. *Let H, F be Hilbert C^*-modules over a C^*-algebra A, let $\{x_i\}_{i\in I}$ be an orthonormal basis for H, and $\{y_j\}_{j\in J}$ be an orthonormal basis for F. Then $\{x_i \otimes y_j\}_{i\in I, j\in J}$ is an orthonormal basis for $H \otimes F$ on $A \otimes A$.*

Proof. We have $\langle x_i, x_i\rangle = e_i$ for all $i \in I$, $e_iAe_i = \mathbb{C}e_i$, $\langle x_i, x_l\rangle = 0$ if $i \neq l$, and $(x_i)_{i\in I}$ generate a dense submodule of H. Also, $\langle y_j, y_j\rangle = e_j$ for all $j \in J$, $e_jAe_j = \mathbb{C}e_j$, $\langle y_j, y_k\rangle = 0$ if $j \neq k$, and $(y_j)_{j\in J}$ generate a dense submodule of F. Now we have

$$\langle x_i \otimes y_j, x_i \otimes y_j\rangle = \langle x_i, x_i\rangle \otimes \langle y_j, y_j\rangle = e_i \otimes e_j = e_i \otimes e_j.$$

So $x_i \otimes y_j$ is a minimal projection in $A \otimes A$. We show that the system $\{x_i \otimes y_j\}_{i\in I, j\in J}$ is orthonormal in $H \otimes F$.

$$\langle x_i \otimes y_j, x_l \otimes y_k\rangle = \langle x_i, x_l\rangle \otimes \langle y_j, y_k\rangle = \delta_{il} \otimes \delta_{jk} = 0$$

if $i \neq l$ or $j \neq k$.

Now we show that the system $\{x_i \otimes y_j\}_{i\in I, j\in J}$ is an orthonormal basis for $H \otimes F$.

Let $x \otimes y \in H \otimes F$ $(x \in H, y \in F)$. So $x = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} a_i x_i$,

$$y = \lim_{m\longrightarrow\infty} \sum_{j=1}^{m} b_j y_j$$

for some $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$.

Then we have

$$\begin{aligned} x \otimes y &= \lim_{n\to\infty} \sum_{i=1}^{n} a_i x_i \otimes \lim_{m\to\infty} \sum_{j=1}^{m} b_j y_j \\ &= \lim_{n\to\infty, m\to\infty} \sum_{i=1, j=1}^{n,m} (a_i \otimes b_j)(x_i \otimes y_j). \end{aligned}$$

So $\{x_i \otimes y_j\}_{i\in I, j\in J}$ generate a dense submodule of $H \otimes F$. □

Our next result is about frame of submodules, so we recall its definition and the definition of fusion frames.

Definition 3.3. (See [1]). Let V be a countably generated Hilbert $K(H)$-module and $I \subseteq \mathbb{N}$ finite or countable. Let $\{\lambda_i : i \in I\}$ be a family of weights. A family of closed submodules $\{W_i : i \in I\}$ of V is a frame of submodules for V with respect to $\{\lambda_i : i \in I\}$ if there exist constants $C, D > 0$ such that for every $x \in V$,

$$C\langle x, x\rangle \leq \sum_i \lambda_i^2 \langle \pi_i(x), \pi_i(x)\rangle \leq D\langle x, x\rangle, \tag{4}$$

where $\pi_i \in B(V)$, denotes the orthogonal projection onto W_i, and the sum in the middle of (4) is in norm. Its frame operator is $S_{W,\lambda} = \sum \lambda_i^2 \pi_i(x)$.

Definition 3.4. (See [15]). Let A be a unital C^*-algebra. X be a Hilbert A-module, and let $\{v_i : i \in I\}$ be a family of weights in A, i.e. each v_i is a positive invertible element from the center of A and let $\{M_i : i \in I\}$ be a family of orthogonally complemented submodules of X. Then $\{(M_i, v_i) : i \in I\}$ is a *fusion frame* if there exist real constants $0 < C \leq D < \infty$ such that

$$C\langle x, x\rangle \leq \sum_i v_i^2 \langle \pi_i(x), \pi_i(x)\rangle \leq D\langle x, x\rangle, \quad \text{for} \quad x \in X.$$

Hence every frame of submodules $\{W_i : i \in I\}$ with respect to weights $\{\lambda_i : i \in I\}$ is a fusion frame $\{(W_i, \lambda_i) : i \in I\}$, where the weights are real numbers, see [1], [15].

We note that every frame of submodule $\{W_i : i \in I\}$ with respect to weights $\{\lambda_i : i \in I\}$ is a fusion frame $\{(W_i, \lambda_i) : i \in I\}$, where the weights are real numbers, see [1], [15].

Lemma 3.5. *Let V and W be Hilbert $K(H)$-modules, let $\{(V_i, \lambda_i) : i \in I\}$ be a standard frame of submodules for V with bounds C, D, frame operator $S_{V,\lambda}$, and let $\{(W_j, \mu_j) : j \in J\}$ be a standard frame of submodules for W with bounds C', D', frame operator $S_{W,\mu}$. Let $e, e' \in K(H)$ be orthogonal one-dimensional projections. Then $\{(V_i \otimes W_j)(e \otimes e') : i \in I, j \in J\}$ is a frame of subspaces for $(V \otimes W)(e \otimes e')$ with respect to $\{\lambda_i \mu_j : i \in I, j \in J\}$ with frame operator*

$$(S_{V,\lambda} \otimes S_{W,\mu})_{|(V\otimes W)(e\otimes e')} = S_{(V\otimes W)(e\otimes e'),\lambda\mu} = S_{Ve,\lambda} \otimes S_{We',\mu}.$$

Proof. By [15], we have $\{((V_i \otimes W_j), (\lambda_i \otimes \mu_j)) : i \in I, j \in J\}$ is a fusion frame for $V \otimes W$ with frame bounds CC', DD' and frame operator $S_{V,\lambda} \otimes S_{W,\mu}$. We know that $e \otimes e'$ is an orthogonal projection and $(V \otimes W)(e \otimes e')$ is a Hilbert space. Then by [1], $\{(V_i \otimes W_j) : i \in I, j \in J\}$ is a frame of submodules with respect to weights $\{\lambda_i \mu_j : i \in I, j \in J\}$, for $V \otimes W$ if and only if $\{(V_i \otimes W_j)(e \otimes e') : i \in I, j \in J\}$ is a frame of subspaces with respect to $\lambda\mu = \{\lambda_i \mu_j : i \in I, j \in J\}$, for $(V \otimes W)(e \otimes e')$ with frame bounds CC', DD' and frame operator $(S_{(V\otimes W),(\lambda\mu)})_{|(V\otimes W)(e\otimes e')} = S_{(V\otimes W)(e\otimes e'),(\lambda\mu)}$. But $S_{(V\otimes W)(e\otimes e'),(\lambda\mu)} = S_{Ve,\lambda} \otimes S_{We',\mu}$ and

$$(S_{V,\lambda})_{|Ve} \otimes (S_{W,\mu})_{|We'} = (S_{V\otimes W,\lambda\mu})_{|(V\otimes W)(e\otimes e')}$$

which completes the proof. □

Lemma 3.6. *Let V, V_i be Hilbert $K(H)$-modules and let $\{\Lambda_i \in L(V, V_i) : i \in I\}$ be a standard g-frame for V with respect to $\{V_i : i \in I\}$ and $e \in K(H)$ be an orthogonal projection of rank 1. Then $\{\Lambda_i \in L(Ve, V_i e) : i \in I\}$ is a g-frame for Ve.*

Proof. Suppose that $\{\Lambda_i \in L(V, V_i) : i \in I\}$ is a standard g-frame for V with frame bounds A, B, it means that

$$A\langle x, x\rangle \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle \leq B\langle x, x\rangle,$$

for all $x \in V$. Since $\langle xe, ye\rangle = (ye, xe)e$ for all $xe, ye \in Ve$, by choosing xe instead of x in the above inequalities, we get

$$A(xe, xe) \leq \sum_{i\in I} \langle \Lambda_i xe, \Lambda_i xe\rangle \leq B(xe, xe), \quad \text{for all} \quad x \in V,$$

which implies that $A(x, x) \leq \sum_{i\in I} (\Lambda_i x, \Lambda_i x) \leq B(x, x)$, for all $x \in Ve$. Therefore $\{\Lambda_i : i \in I\}$ is a g-frame for the Hilbert space Ve with respect to $\{V_i e : i \in I\}$. □

3.1. **G-frames in Hilbert C^*-modules.** We know that every finitely or countably generated Hilbert C*-module over a σ-unital C*-algebra has a standard Parseval frame, see [7]. By using this fact we have the following characterization of g-frames, see [18, Theorem 3.1].

Theorem 3.7. *Let H be a Hilbert A-module and F be a Hilbert B-module. Let $\{\Lambda_i\}_{i\in I}$, $\{\Gamma_j\}_{j\in J}$ be standard g-frames in Hilbert C^*-modules H, F with respect to $\{V_i\}_{i\in I}$, $\{W_j\}_{j\in J}$, respectively. If S, S' and S'' are the g-frame operators of $\{\Lambda_i\}_{i\in I}, \{\Gamma_j\}_{j\in J}$ and $\{\Lambda_i\otimes\Gamma_j\}_{i\in I, j\in J}$, respectively, then $S''=S\otimes S'$.*

Proof. Since S is A-linear and S' is B-linear, and they are bounded, then $S\otimes S'$ is $A\otimes B$-linear. For every $x\in H$ and $y\in F$, we have $Sx=\sum_{i\in I}\Lambda_i^*\Lambda_i x$ and $S'y=\sum_{j\in J}\Gamma_j^*\Gamma_j y$. Therefore

$$\begin{aligned}(S\otimes S')(x\otimes y) &= Sx\otimes S'y=\sum_{i\in I}\Lambda_i^*\Lambda_i x\otimes\sum_{j\in J}\Gamma_j^*\Gamma_j y\\ &=\sum_{i,j}\Lambda_i^*\Lambda_i x\otimes\Gamma_j^*\Gamma_j y=\sum_{i,j}(\Lambda_i^*\otimes\Gamma_j^*)(\Lambda_i x\otimes\Gamma_j y)\\ &=\sum_{i,j}(\Lambda_i^*\otimes\Gamma_j^*)(\Lambda_i\otimes\Gamma_j)(x\otimes y)=\sum_{i,j}(\Lambda_i\otimes\Gamma_j)^*(\Lambda_i\otimes\Gamma_j)(x\otimes y).\end{aligned}$$

Now by the uniqueness of g-frame operator, the last expression is equal to $S''(x\otimes y)$. Consequently we have $(S\otimes S')(x\otimes y)=S''(x\otimes y)$. From this equality it follows that for all $z=\sum_{k=1}^{k=n}x_k\otimes y_k$ in $H\otimes_{\text{alg}}F$, $(S\otimes S')z=S''z$. Hence the above relation holds for all z in $H\otimes F$. So $S''=S\otimes S'$, which is a bounded $A\otimes B$-linear, self-adjoint, positive and invertible operator on $H\otimes F$. We also have $\| S''\|=\| S\otimes S'\|\leq\| S\|\| S'\|$. □

For the g-frame operator we have the following result.

Theorem 3.8. *If $Q\in L(H)$ is an invertible A-linear map and $\{\Lambda_i\}_{i\in I}$ is a g-frame for $H\otimes F$ with respect to $\{V_i\}_{i\in I}$ with g-frame operator S, then $\{\Lambda_i(Q^*\otimes I)\}_{i\in I}$ is a g-frame for $H\otimes F$ with respect to $\{V_i\}_{i\in I}$ with g-frame operator $(Q\otimes I)S(Q^*\otimes I)$.*

Proof. Since $Q\in L(H), Q\otimes I\in L(H\otimes F)$ with inverse $Q^{-1}\otimes I$. It is obvious that $Q\otimes I$ is $A\otimes B$-linear, adjointable, with adjoint $Q^*\otimes I$. An easy calculation shows that for every elementary tensor $x\otimes y$,

$$\begin{aligned}\| (Q\otimes I)(x\otimes y)\|^2 &=\| Q(x)\otimes y\|^2=\| Q(x)\|^2\| y\|^2\\ &\leq\| Q\|^2\| x\|^2\| y\|^2=\| Q\|^2\| x\otimes y\|^2 .\end{aligned}$$

So $Q\otimes I$ is bounded, and therefore it can be extended to $H\otimes F$. Similarly for $Q^*\otimes I$. Hence $Q\otimes I$ is $A\otimes B$-linear, adjointable with adjoint $Q^*\otimes I$, and as we mentioned in the proof of [15, Theorem 3.2] Q^* is invertible and bounded. Hence for every $T\in H\otimes F$, we have

$$\| (Q^*)^{-1}\|^{-1}| T|\leq|(Q^*\otimes I)T|\leq\| Q\|| T| .$$

Hence $Q\otimes I\in L(H\otimes F)$. Now by Theorem 3.2 in [15], we have the result. □

As an application of g-frames we can introduce atomic resolution of bounded A-linear operators, see [14], [15], [18].

Let H be a Hilbert A-module and $\{\Lambda_i\in L(H,V_i): i\in I\}$ be a g-frame for H with canonical dual g-frame $\{\tilde{\Lambda}_i=\Lambda_i S^{-1}: i\in I\}$. Then for every $f\in H$, $f=\sum\Lambda_i^*\tilde{\Lambda}_i f=\sum\tilde{\Lambda}_i^*\Lambda f$. If T is a bounded A-linear map on H, then

$$T=\sum T\Lambda_i^*\tilde{\Lambda}_i=\sum T\tilde{\Lambda}_i^*\Lambda_i=\sum\Lambda_i^*\tilde{\Lambda}_i T=\sum\tilde{\Lambda}_i\Lambda_i T, \tag{5}$$

where the convergence is in strong *-topology. (5) is called atomic resolution of operator T. By using the same proof of Theorem 5.2 in [15] or Proposition 4.2 in [14] we have the following result:

Proposition 3.9. *Let A, B be C^*-algebras, H be a Hilbert A-module and K be a Hilbert B-module. If $T = \sum T_iT = \sum TT_i$ and $S = \sum S_iS = \sum SS_i$ are atomic resolutions of $T \in L(H)$ and $S \in L(K)$, respectively, where $(T_i) \subseteq L(H)$, $(S_i) \subseteq L(K)$, then*

$$T \otimes S = \sum_i (T_i \otimes S_i)(T \otimes S) = \sum_i (T \otimes S)(T_i \otimes S_i).$$

3.2. **operator-valued frames in Hilbert C*-modules.** Let A be a C*-algebra and E be a Hilbert A-module. For each pair of elements x and y in E, a bounded rank-one operator is defined by $\theta_{x,y}(z) = x\langle y, z\rangle$ for all $z \in E$. The closed linear span of all rank-one operators is denoted by $K(E)$. When $A = \mathbb{C}$, $K(E)$ coincides with the ideal $K = K(E)$ of all compact operators on E. We know that for a Hilbert A-module E, $L(E)$ is multiplier algebra of K(E), and $K(E)$ is always a closed ideal of $L(E)$, see [17].

Definition 3.10. We say that $T_\lambda \to T$ in strict topology of $L(E)$ if $\| (T_\lambda - T)S \| \to 0$ and $\| S(T_\lambda - T) \| \to 0$ (for all $S \in K(E)$), where the convergence is in the $L(E)$ norm. We say that $T_\lambda \to T$ in strong*-operator topology if $\| (T_\lambda - T)\xi \| \to 0$ and $\| (T_\lambda - T)^*\xi \| \to 0$ for $\xi \in H$.

We will use the following elementary properties :$T_\lambda \to T$ strictly if and only if $T_\lambda^* \to T^*$ strictly, and either of these convergences implies $BT_\lambda \to BT$ and $T_\lambda B \to TB$ strictly for all $B \in L(E)$. Also, if $T_\lambda \to T$ strictly and $S_\lambda \to S$ strictly, then $T_\lambda S_\lambda \to TS$ strictly. To avoid unnecessary complications, from now on, we assume that A is a σ-unital C*-algebra.

Definition 3.11. Let A be a σ-unital C*-algebra, E be a Hilbert A-module and J be a countable index set. Let E_0 be a projection in $M(\mathrm{K}(E)) = L(E)$. Denote by H_E the submodule E_0E and identify $L(E, H_E)$ with $E_0M(K(E))$. A collection $\{A_j\}_{j\in J} \subseteq L(E, H_E)$ for $j \in J$ is called an operator-valued frame on E with range in H_E if the sum $\sum_j A_j^*A_j$ converges in strict topology (σ *-strong**-topology) to a bounded invertible operator on E, denoted by D. $\{A_j\}_{j\in J}$ is called a tight operator-valued frame (resp., a Parseval operator-valued frame) if $D = \lambda I$ for a positive number λ (resp., $D = I$).

Lemma 3.12. *Every operator-valued frame $\{A_i\}$ with associated operator D is a g-frame with g-frame operator $S = D$.*

Proof. Since σ *-strong**-topology is stronger than *strong**-topology [3], by using part (iii) of Remark 2.3 in [11] we conclude that for every $x \in H$, $\sum_j A_j^*A_jx$ is convergent to $D(x)$, and

$$\frac{1}{\| D^{-1} \|}\langle x, x\rangle \leq \langle \sum_j A_j^*A_jx, x\rangle = \sum_j \langle A_jx, A_jx\rangle \leq \| D \| \langle x, x\rangle.$$

Therefore $\{A_i\}$ is a g-frame with g-frame operator $S = D$. □

By frame, we will mean an operator-valued frame on a Hilbert C*-module.

Remark 3.13. *By Lemma 3.2 of [11] if $\{A_j\}_{j\in J}$ is a frame in $L(E, H_E)$, then $\{B_j = A_jD^{-1}\}_{j\in J}$ is a frame in $L(E, H_E)$ and $\{A_jD^{\frac{-1}{2}}\}_{j\in J}$ is a Parseval frame.*

Next we show that tensor product of operator-valued frame is an operator-valued frame.

Theorem 3.14. *Let E,F be Hilbert A-modules, and let $E_0 \in M(K(E))$,$F_0 \in M(K(F))$ be projections. We put $H_E = E_0E$ and $H_F = F_0F$. Let $\{A_j\}_{j\in J}$ be a frame in $L(E, H_E)$, and let $\{B_i\}_{i\in I}$ be a frame in $L(F, H_F)$. Then $\{A_j \otimes B_i\}_{j\in J,i\in I}$ is a frame in $L(E \otimes F, H_E \otimes H_F)$. In particular, if $\{A_j\}_{j\in J}$ and $\{B_i\}_{i\in I}$ are tight or Parseval frames, then so is $\{A_j \otimes B_i\}_{j\in J,i\in I}$.*

Proof. There is a bounded and invertible operator $D_E \in L(E)$ such that $\sum_j A_j^*A_j$ converges in strict topology to D_E, and there is a bounded and invertible operator $D_F \in L(F)$ such that $\sum_i B_i^*B_i$ converges in strict topology to D_F. We have $K(E \otimes F) = K(E) \otimes K(F)$. Let $x_1, x_2 \in E$, $y_1, y_2 \in F$. Then it is enough to show that $\| (\sum_{i,j}(A_j \otimes B_i)^*(A_j \otimes B_i) - D_E \otimes D_F)\theta_{x_1\otimes y_1,x_2\otimes y_2} \| \to 0$ and $\| \theta_{x_1\otimes y_1,x_2\otimes y_2}(\sum_{i,j}(A_j \otimes B_i)^*(A_j \otimes B_i) - D_E \otimes D_F) \| \to 0$. We have $\theta_{x_1\otimes y_1,x_2\otimes y_2} = \theta_{x_1,x_2} \otimes \theta_{y_1,y_2}$, and therefore

$$
\begin{aligned}
&\| \big(\sum_{i,j}(A_j \otimes B_i)^*(A_j \otimes B_i) - D_E \otimes D_F\big)\theta_{x_1\otimes y_1,x_2\otimes y_2} \| \\
&\quad =\| \big(\sum_j A_j^*A_j \otimes \big(\sum_i B_i^*B_i - D_F\big)\big)\theta_{x_1,x_2} \otimes \theta_{y_1,y_2} \\
&\quad + \big(\big(\sum_j A_j^*A_j - D_E\big) \otimes D_F\big)\theta_{x_1,x_2} \otimes \theta_{y_1,y_2} \| \\
&\quad =\| \sum_j A_j^*A_j\theta_{x_1,x_2} \otimes \big(\sum_i B_i^*B_i - D_F\big)\theta_{y_1,y_2} \\
&\quad + \big(\sum_j A_j^*A_j - D_E\big)\theta_{x_1,x_2} \otimes D_F\theta_{y_1,y_2} \| \\
&\quad \le\| \sum_j A_j^*A_j\theta_{x_1,x_2} \otimes \big(\sum_i B_i^*B_i - D_F\big)\theta_{y_1,y_2} \| \\
&\quad + \| \big(\sum_j A_j^*A_j - D_E\big)\theta_{x_1,x_2} \otimes D_F\theta_{y_1,y_2} \| \\
&\quad \le\| \sum_j A_j^*A_j\theta_{x_1,x_2} \|\| \big(\sum_i B_i^*B_i - D_F\big)\theta_{y_1,y_2} \| \\
&\quad + \| \big(\sum_j A_j^*A_j - D_E\big)\theta_{x_1,x_2} \|\| D_F\theta_{y_1,y_2} \| \to 0.
\end{aligned}
$$

A similar argument shows that

$$\| \theta_{x_1\otimes y_1,x_2\otimes y_2}\big(\sum_{i,j}(A_j \otimes B_i)^*(A_j \otimes B_i) - D_E \otimes D_F\big) \| \to 0.$$

Since the linear span of $\theta_{x_1\otimes y_1,x_2\otimes y_2}$ $(x_1, x_2 \in E, y_1, y_2 \in F)$ is dense in $K(E \otimes F)$, it follows that $\| \big(\sum_{i,j}(A_j \otimes B_i)^*(A_j \otimes B_i) - D_E \otimes D_F\big)S \| \to 0$ $\forall S \in K(E \otimes F)$. □

First we recall some definitions from [11].

Definition 3.15. Assume that $\{A_j\}_{j\in J}$ is a frame in $L(E, H_E)$ for the Hilbert C*-module E. Decompose the identity of $M(K(E)) = L(E)$, into a strictly converging sum of mutually orthogonal projections$\{E_j\}_{j\in J}$ in $M(K(E))$ with $E_j \sim E_{00} \ge E_0$. Let $\{L_j\}_{j\in J}$ be partial isometries in $M(K(E))$ such that $L_jL_j^* = E_j$ and $L_j^*L_j = E_{00}$. Define the frame transform θ_A of the frame $\{A_j\}_{j\in J}$ as θ_A=$\sum_j L_jA_j : E \to E$, and the range projection of θ_A, $P_A = \theta_AD_A^{-1}\theta_A^*$ is called the frame projection.

Theorem 3.16. *Let $\{A_j\}_{j\in J}$ be a frame in $L(E, E_0E)$ with frame transform θ_A, frame projection P_A, and $\{B_i\}_{i\in I}$ be a frame in $L(F, F_0F)$ with frame transform θ_B, frame* *projection P_B. Then $\theta_A \otimes \theta_B$ is the frame transform of the frame $\{A_j \otimes B_i\}_{j\in J, i\in I}$ and $P_{A\otimes B} = P_A \otimes P_B$.*

Proof. There are partial isometries $\{L_j\}_{j\in J} \subseteq M(K(E))$, and $\{K_i\}_{i\in I} \subseteq M(K(F))$ such that $E_j = L_j L_j^*$, $E_0 \leq E_{00} = L_j^* L_j$, $I_E = \sum_j E_j$, $\theta_A = \sum_j L_j A_j$ and $F_i = K_i K_i^*$, $F_0 \leq F_{00} = K_i^* K_i$, $I_F = \sum_i F_i$, $\theta_B = \sum_i K_i B_i$. We have $E_0 \leq E_{00}$, $F_0 \leq F_{00}$. It means $E_0 E_{00} = E_{00} E_0 = E_0$ and $F_0 F_{00} = F_{00} F_0 = F_0$. So $(E_{00} \otimes F_{00})(E_0 \otimes F_0) = E_{00} E_0 \otimes F_{00} F_0 = E_0 \otimes F_0 = (E_0 \otimes F_0)(E_{00} \otimes F_{00})$.

Consequently $(E_{00} \otimes F_{00}) \geq (E_0 \otimes F_0)$, and it is obvious that $\{E_j \otimes F_i\}_{j\in J, i\in I}$ are projections in $B(E \otimes F)$. But

$$(L_j \otimes K_i)(L_j^* \otimes K_i^*) = L_j L_j^* \otimes K_i K_i^* = E_j \otimes F_i = (L_j \otimes K_i)(L_j \otimes K_i)^*,$$
$$(L_j^* \otimes K_i^*)(L_j \otimes K_i) = L_j^* L_j \otimes K_i^* K_i = (E_{00} \otimes F_{00}) = (L_j \otimes K_i)^*(L_j \otimes K_i)$$

and $\sum_{j,i} E_j \otimes F_i = \sum_j E_j \otimes \sum_i F_i = I_E \otimes I_F$. Then

$$\theta_A \otimes \theta_B = \sum_j L_j A_j \otimes \sum_i K_i B_i = \sum_{j,i} L_j A_j \otimes K_i B_i$$
$$= \sum_{j,i} (L_j \otimes K_i)(A_j \otimes B_i) = \theta_{A\otimes B}.$$

Also we have

$$P_{A\otimes B} = \theta_{A\otimes B} D_{A\otimes B}^{-1} \theta_{A\otimes B}^* = (\theta_A \otimes \theta_B)(D_A \otimes D_B)^{-1}(\theta_A \otimes \theta_B)^*$$
$$= \theta_A D_A^{-1} \theta_A^* \otimes \theta_B D_B^{-1} \theta_B^* = P_A \otimes P_B.$$

□

Definition 3.17. Two frames $\{A_j\}_{j\in J}$, $\{B_j\}_{j\in J}$ in $L(E, E_0 E)$ are said to be right-similar if there exists an invertible element $T \in M(K(E))$ such that $B_j = A_j T$ for all $j \in J$.

Lemma 3.18. *If $\{A_j\}_{j\in J}$, $\{B_j\}_{j\in J}$ are right-similar in $L(E, H_E)$ and $\{C_i\}_{i\in I}$, $\{D_i\}_{i\in I}$ are right-similar in $L(F, H_F)$, then $\{A_j \otimes C_i\}_{j,i}$ and $\{B_j \otimes D_i\}_{j,i}$ are right-similar in $L(E \otimes F, H_E \otimes H_F)$.*

Proof. Let T_1 be an invertible element of $M(K(E))$ such that $B_j = A_j T_1$ for all $j \in J$ and T_2 be an invertible element of $M(K(F))$ such that $D_i = C_i T_2$ for all $i \in I$. Then we have $(T_1 \otimes T_2)^{-1} = T_1^{-1} \otimes T_2^{-1}$, and $M(K(E)) \otimes M(K(F)) \subseteqq M(K(E \otimes F))$. But $T_1 \otimes T_2 \in M(K(E)) \otimes M(K(F)$. Hence $T_1 \otimes T_2$ is an invertible operator in $M(K(E \otimes F))$ such that $B_j \otimes D_i = A_j T_1 \otimes C_i T_2 = (A_j \otimes C_i)(T_1 \otimes T_2)$. Consequently $\{A_j \otimes C_i\}_{j,i}$ and $\{B_j \otimes D_i\}_{j,i}$ are right-similar in $L(E \otimes F, H_E \otimes H_F)$. □

By using Theorem 3.3 in [11] and the above lemma, we have the following result.

Theorem 3.19. *Let $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$ be frames in $L(E, H_E)$ and $\{C_i\}$, $\{D_i\}$ be frames in $L(F, H_F)$. Then the following are equivalent:*

- (i) *$\{A_j \otimes C_i\}_{j,i}$ and $\{B_j \otimes D_i\}_{j,i}$ are right similar in $L(E \otimes F, H_E \otimes H_F)$,*
- (ii) *$\{A_i\}$, $\{B_i\}$ are right similar in $L(E, H_E)$ and $\{C_j\}$, $\{D_j\}$ are right-similar in $L(F, H_F)$,*
- (iii) *$P_{A\otimes C} = P_{B\otimes D}$,*
- (iv) *$P_A = P_B$ and $P_C = P_D$.*

Proof. By Theorem 3.3 in [11], (i) is equivalent to (iii) and by Theorem 3.16, (iii) is equivalent to (iv). Also by Theorem 3.3 in [11], (ii) is equivalent to (iv), and by the above lemma, (ii) implies (i). So we have the result. □

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