

THE FADDEEV EQUATION AND ESSENTIAL SPECTRUM OF A HAMILTONIAN IN FOCK SPACE

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ABSTRACT. A Hamiltonian (model operator) H associated to a quantum system describing three particles in interaction, without conservation of the number of particles, is considered. The Faddeev type system of equations for eigenvectors of H is constructed. The essential spectrum of H is described by the spectrum of the channel operator.

1. INTRODUCTION

The essential spectrum of the systems with a fixed number of particles has been studied in many articles, see for example, for the continuous case [16, 19] and for the lattice case [6, 11, 12].

In quantum field theory, condensed matter physics and the theory of chemical reactions there naturally occur quantum systems with non conserved number of particles. Often, the number of particles can be arbitrary large as in cases involving photons (see e.g. [3]), in other cases, such as scattering of spin waves on defects, scattering massive particles and chemical reactions, there are only participants at any given time, though their number can change.

Recall that the study of systems describing N ($1 \leq N < \infty$) particles in interaction, without conservation of the number of the particles, is reduced to the investigation of the spectral properties of self-adjoint operators, acting in the *cut subspace* $\mathcal{H}^{(N)}$ of Fock space, consisting of $n \leq N$ particles [4, 5, 9, 10, 17, 20].

The perturbation problem of an operator (the Friedrichs model), with point and continuous spectrum (which acts in $\mathcal{H}^{(2)}$) has played a considerable role in the study of spectral problems connected to the quantum theory of fields [4].

A two level atom coupled to the radiation field was considered in [5] and using a Mourre type estimate, a complete spectral characterization of the spin boson Hamiltonian was studied for a sufficiently small but nonzero coupling. In [17] the quantum systems with non conserved but finite number of particles was considered, and for such systems, geometric and commutator techniques were developed, which were used to find the location of the spectrum, to prove the absence of singular continuous spectrum and identify accumulation points of the discrete spectrum.

In the present paper we consider the model operator H associated to a system describing three particles in interaction without conservation of the number of particles, acting in $\mathcal{H}^{(3)}$, which is a lattice analog of the spin-boson Hamiltonian [5, 9]. Note that this operator was not studied before and can be considered as a generalization of the above models studied in [1, 2, 7, 8, 13, 14, 15, 18].

2000 *Mathematics Subject Classification.* Primary 81Q10; Secondary 35P20, 47N50.

Key words and phrases. Model operator, Fock space, without conservation of the number of particles, channel operator, Hilbert-Schmidt class, trace class, Faddeev type system of equations, essential spectrum.

The Faddeev equation and the location of the essential spectrum for the similar to H model operators acting in symmetric and non symmetric Fock spaces have been studied in [15, 18] in the case when the operators V_i , $i = 1, 2$ (defined below) are partial integral operators generated kernels. But the techniques developed in those papers are not applicable to the more general case considered in the present paper.

We obtain the following results:

- (i) The Faddeev equation for the eigenvectors of H is constructed.
- (ii) We describe the location of the essential spectrum of the operator H in terms of the spectrum of the channel operator \hat{H} .

The paper is organized as follows. In Section 2, the model operator H is introduced and the main results are stated. In Section 3 the spectrum of \hat{H} is described in terms of the spectrum of a family of generalized Friedrichs models, and some auxiliary statements that play an important role in the proof of the main results of the paper are proved. In Section 4 we obtain an analogue of the Faddeev type system of integral equations for the eigenfunctions of H (Theorem 2.1) and prove that the essential spectrum of H coincides with the spectrum of the channel operator \hat{H} (Theorem 2.3). In section 5 we give an example of calculation of the essential spectrum of H , which shows the efficiency of the proposed method of calculation of the essential spectrum.

Throughout this paper we adopt the following convention. Denote by \mathbf{T}^ν the ν -dimensional torus, the cube $(-\pi, \pi]^\nu$ with appropriately identified sides. The torus \mathbf{T}^ν will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the ν -dimensional space \mathbf{R}^ν modulo $(2\pi\mathbf{Z})^\nu$, where \mathbf{Z} is the one-dimensional lattice.

2. THE MODEL OPERATOR AND MAIN RESULTS

Let us introduce some notations used in this work. Let \mathbf{C} be the field of complex numbers, $L_2(\mathbf{T}^\nu)$ the Hilbert space of square integrable (complex) functions defined on \mathbf{T}^ν and $L_2^s((\mathbf{T}^\nu)^2)$ the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbf{T}^\nu)^2$.

Denote by \mathcal{H} the direct sum of spaces $\mathcal{H}_0 = \mathbf{C}$, $\mathcal{H}_1 = L_2(\mathbf{T}^\nu)$ and $\mathcal{H}_2 = L_2^s((\mathbf{T}^\nu)^2)$, that is, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

The Hilbert space \mathcal{H} is called the "three-particle cut subspace" of the Fock space.

Let H_{ij} be annihilation (creation) operators [4] defined in the Fock space for $i < j$ ($i > j$). We note that in physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

In this paper we consider the case where the number of annihilations and creations of the particles of the considering system equal to 1. It means that $H_{ij} \equiv 0$ for all $|i - j| > 1$. So, a model operator H associated to a system describing three particles in interaction, without conservation of the number of particles, acts in the Hilbert space \mathcal{H} as a matrix operator

$$H = \begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{10} & H_{11} & H_{12} \\ 0 & H_{21} & H_{22} \end{pmatrix}.$$

Let its components $H_{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i$, $i, j = 0, 1, 2$, be defined by the rule

$$(H_{00}f_0)_0 = w_0f_0, \quad (H_{01}f_1)_0 = \int_{\mathbf{T}^\nu} v_0(s)f_1(s) ds, \quad (H_{10}f_0)_1(p) = v_0(p)f_0,$$

$$(H_{11}f_1)_1(p) = w_1(p)f_1(p), \quad (H_{12}f_2)_1(p) = \int_{\mathbf{T}^\nu} v_1(s)f_2(p, s) ds,$$

$$\begin{aligned} (H_{21}f_1)_2(p, q) &= \frac{1}{2}(v_1(p)f_1(q) + v_1(q)f_1(p)), \\ H_{22} &= H_{22}^0 - V_1 - V_2, \quad (H_{22}^0f_2)_2(p, q) = w_2(p, q)f_2(p, q), \\ (V_1f_2)_2(p, q) &= \int_{\mathbf{T}^\nu} v_2(p, s)f_2(s, q) ds, \quad (V_2f_2)_2(p, q) = \int_{\mathbf{T}^\nu} v_2(s, q)f_2(p, s) ds. \end{aligned}$$

Here $f_i \in \mathcal{H}_i$, $i = 0, 1, 2$, w_0 is a real number, $w_1(\cdot)$, $v_i(\cdot)$, $i = 0, 1$, are real-valued continuous functions on \mathbf{T}^ν , $w_2(\cdot, \cdot)$ and $v_2(\cdot, \cdot)$ are real-valued continuous symmetric functions on $(\mathbf{T}^\nu)^2$.

Under these assumptions the operator H is bounded and self-adjoint in \mathcal{H} .
Set

$$\overline{\mathcal{H}}_0 = \mathcal{H}_0, \quad \overline{\mathcal{H}}_1 = \mathcal{H}_1, \quad \overline{\mathcal{H}}_2 = L_2((\mathbf{T}^\nu)^2) \quad \text{and} \quad \mathcal{H}^{(n,m)} = \bigoplus_{i=n}^m \overline{\mathcal{H}}_i, \quad 0 \leq n < m \leq 2,$$

where $L_2((\mathbf{T}^\nu)^2)$ is the Hilbert space of square integrable (complex) functions on $(\mathbf{T}^\nu)^2$.

Throughout the paper we additionally assume that the operators V_i , $i = 1, 2$, acting in the Hilbert space $\overline{\mathcal{H}}_2$ are positive and use this fact without comments. Denote by \tilde{V}_i , $i = 1, 2$, the positive square root of the operators V_i , $i = 1, 2$. Then the operators \tilde{V}_i , $i = 1, 2$, have form (see Lemma 3.2)

$$(2.1) \quad \begin{aligned} (\tilde{V}_1f_2)(p, q) &= \int_{\mathbf{T}^\nu} \tilde{v}_2(p, s)f_2(s, q) ds, \quad (\tilde{V}_2f_2)(p, q) = \int_{\mathbf{T}^\nu} \tilde{v}_2(q, s)f_2(p, s) ds, \\ f_2 &\in \overline{\mathcal{H}}_2. \end{aligned}$$

Here the kernel of \tilde{V}_i , $i = 1, 2$, is formally denoted by $\tilde{v}_2(\cdot, \cdot)$.

To formulate our main results we introduce the channel operator \hat{H} acting in $\mathcal{H}^{(1,2)}$ by the following rule:

$$\hat{H} = \begin{pmatrix} H_{11} & \frac{1}{\sqrt{2}} H_{12} \\ \frac{1}{\sqrt{2}} H_{21}^{(1)} & H_{22}^0 - V_2 \end{pmatrix}$$

with

$$(H_{21}^{(1)}f_1)(p, q) = v_1(q)f_1(p), \quad f_1 \in \overline{\mathcal{H}}_1.$$

It is easy to show that the operator \hat{H} is bounded and self-adjoint in $\mathcal{H}^{(1,2)}$.

Let

$$m = \min_{p, q \in \mathbf{T}^\nu} w_2(p, q), \quad M = \max_{p, q \in \mathbf{T}^\nu} w_2(p, q).$$

For each $z \in \mathbf{C} \setminus [m; M]$ we define the operator matrices $A(z)$ and $K(z)$ acting in the Hilbert space $\mathcal{H}^{(0,2)}$ by

$$A(z) = \begin{pmatrix} A_{00}(z) & 0 & 0 \\ 0 & A_{11}(z) & A_{12}(z) \\ 0 & A_{21}(z) & A_{22}(z) \end{pmatrix}, \quad K(z) = \begin{pmatrix} K_{00}(z) & K_{01}(z) & 0 \\ K_{10}(z) & K_{11}(z) & K_{12}(z) \\ 0 & K_{21}(z) & K_{22}(z) \end{pmatrix},$$

where the operators $A_{ij}(z) : \overline{\mathcal{H}}_j \rightarrow \overline{\mathcal{H}}_i$, $i, j = 0, 1, 2$, are defined as

$$(A_{00}(z)g_0)_0 = g_0, \quad (A_{11}(z)g_1)_1(p) = \left(w_1(p) - z - \frac{1}{2} \int_{\mathbf{T}^\nu} \frac{v_1^2(s) ds}{w_2(p, s) - z} \right) g_1(p),$$

$$(A_{12}(z)g_2)_1(p) = \int_{\mathbf{T}^\nu} \frac{v_1(s)}{w_2(p, s) - z} \int_{\mathbf{T}^\nu} \tilde{v}_2(t, s)g_2(p, t) dt ds,$$

$$(A_{21}(z)g_1)_2(p, q) = \frac{1}{2} \int_{\mathbf{T}^\nu} \frac{\tilde{v}_2(s, q)v_1(s)}{w_2(p, s) - z} ds g_1(p),$$

$$(A_{22}(z)g_2)(p, q) = g_2(p, q) - (\tilde{V}_2R_{22}^0(z)\tilde{V}_2g_2)(p, q)$$

and the operators $K_{ij}(z) : \overline{\mathcal{H}}_j \rightarrow \overline{\mathcal{H}}_i$, $i, j = 0, 1, 2$, are defined as

$$(K_{00}(z)g_0)_0 = (w_0 - z + 1)g_0, \quad K_{01}(z) \equiv H_{01}, \quad K_{10}(z) \equiv -H_{10},$$

$$\begin{aligned}
(K_{11}(z)g_1)_1(p) &= \frac{v_1(p)}{2} \int_{\mathbf{T}^\nu} \frac{v_1(s)g_1(s) ds}{w_2(p, s) - z}, \\
(K_{12}(z)g_2)_1(p) &= \int_{\mathbf{T}^\nu} \frac{v_1(s)}{w_2(p, s) - z} \int_{\mathbf{T}^\nu} \tilde{v}_2(p, t)g_2(t, s) dt ds, \\
(K_{21}(z)g_1)_2(p, q) &= -\frac{v_1(p)}{2} \int_{\mathbf{T}^\nu} \frac{\tilde{v}_2(s, q)g_1(s)}{w_2(p, s) - z} ds, \\
(K_{22}(z)g_2)_2(p, q) &= (\tilde{V}_2 R_{22}^0(z) \tilde{V}_1 g_2)(p, q),
\end{aligned}$$

where $g_i \in \overline{\mathcal{H}}_i$, $i = 0, 1, 2$, and $R_{22}^0(z) = (H_{22}^0 - z)^{-1}$ is the resolvent of the operator H_{22}^0 .

We note that for each $z \in \mathbf{C} \setminus [m; M]$ the operators $K_{ij}(z)$, $i, j = 0, 1, 2$, belong to the Hilbert-Schmidt class and therefore $K(z)$ is a compact operator.

Let $\sigma(\hat{H})$ be the spectrum of \hat{H} . Since for any fixed $z \in \mathbf{C} \setminus \sigma(\hat{H})$ the operator $A(z)$ is bounded and invertible (see Lemma 3.6), for such z we can define the operator $T(z) = A^{-1}(z)K(z)$.

Now we give the main results of the paper.

The following theorem establishes a connection between eigenvalues of H and $T(z)$.

Theorem 2.1. *A number $z \in \mathbf{C} \setminus \sigma(\hat{H})$ is an eigenvalue of the operator H if and only if the number $\lambda = 1$ is an eigenvalue of the operator $T(z)$.*

Remark 2.2. We point out that the equation $T(z)g = g$ is an analogue of the Faddeev type system of integral equations for eigenvectors of the operator H and it plays a crucial role in our analysis of the spectrum of H .

The following theorem describes the essential spectrum of the operator H .

Theorem 2.3. *The essential spectrum of H coincides with the spectrum of \hat{H} .*

Since the channel operator \hat{H} has a more simple structure than H , Theorem 2.3 plays an important role in the next investigations of the spectrum of H . We note that by Lemma 3.4 (see Section 4) Theorem 2.3 describes the location of the essential spectrum of H in terms of the spectrum of the channel operator \hat{H} , where two-particle and three-particle branches of this spectrum are separated.

3. SOME AUXILIARY STATEMENTS

In this section we describe the spectrum of the channel operator \hat{H} . Using the decomposition into direct operator integrals (see [16]) we reduce the study of the spectral properties of the operator \hat{H} to an investigation of the spectral properties of the family of operators $h(p)$, $p \in \mathbf{T}^\nu$, defined below. We also give some auxiliary statements that allow us to prove the main results of the paper.

Let the operator v act in \mathcal{H}_1 as

$$(vf)(p) = \int_{\mathbf{T}^\nu} v_2(p, s)f(s) ds, \quad f \in \mathcal{H}_1.$$

Lemma 3.1. *The operator v is positive and its positive square root $\tilde{v} \equiv v^{\frac{1}{2}}$ has the form*

$$(3.1) \quad (\tilde{v}f)(q) = \int_{\mathbf{T}^\nu} \tilde{v}_2(q, s)f(s) ds, \quad f \in \mathcal{H}_1.$$

Moreover, the function $\tilde{v}_2(\cdot, \cdot)$ is square integrable on $(\mathbf{T}^\nu)^2$.

Proof. Since $v_2(\cdot, \cdot)$ is a continuous function on $(\mathbf{T}^\nu)^2$, we have

$$\int_{\mathbf{T}^\nu} |v_2(s, s)| ds < \infty.$$

The function $v(\cdot, \cdot)$ is symmetric and hence the last inequality means that the operator v belongs to the trace class. From the positivity of V_i , $i = 1, 2$, it follows that the operator v is also positive. Therefore, every nontrivial eigenvalue λ_k of v is positive. By the Hilbert-Schmidt theorem we have

$$v = \sum_k \lambda_k (\varphi_k, \cdot) \varphi_k$$

with $\sum_k \lambda_k < \infty$, where φ_k is the eigenfunction of the operator v corresponding to the eigenvalue λ_k . Then

$$\tilde{v} = \sum_k \sqrt{\lambda_k} (\varphi_k, \cdot) \varphi_k.$$

Taking into account that $\sum_k \lambda_k < \infty$ we obtain that \tilde{v} is a Hilbert-Schmidt operator. Therefore the kernel $\tilde{v}_2(\cdot, \cdot)$ of the integral operator \tilde{v} is a square integrable function. \square

Let I_i , $i = 0, 1, 2$, be an identity operator in \mathcal{H}_i , $i = 0, 1, 2$.

Lemma 3.2. *The positive square root of V_i , $i = 1, 2$, has form (2.1).*

Proof. The operators V_i , $i = 1, 2$, can be decomposed as

$$V_1 = v \otimes I_1, \quad V_2 = I_1 \otimes v.$$

By Lemma 3.1 the operator v is positive and its positive square root has form (3.1). Now it is easy to check that $\tilde{V}_1 = \tilde{v} \otimes I_1$ and $\tilde{V}_2 = I_1 \otimes \tilde{v}$. \square

We now study the operator \hat{H} that commutes with any multiplication operator U_α by a bounded function $\alpha(\cdot)$ on \mathbf{T}^ν ,

$$U_\alpha \begin{pmatrix} g_1(p) \\ g_2(p, q) \end{pmatrix} = \begin{pmatrix} \alpha(p)g_1(p) \\ \alpha(p)g_2(p, q) \end{pmatrix}, \quad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}^{(1,2)}.$$

Therefore the decomposition of the space $\mathcal{H}^{(1,2)}$ into the direct integral (see XIII.16 in [16])

$$\mathcal{H}^{(1,2)} = \int_{\mathbf{T}^\nu} \oplus \mathcal{H}^{(0,1)} dp$$

yields a decomposition into the direct integral

$$(3.2) \quad \hat{H} = \int_{\mathbf{T}^\nu} \oplus h(p) dp,$$

where the family of the bounded and self-adjoint operators $h(p)$, $p \in \mathbf{T}^\nu$, acts in $\mathcal{H}^{(0,1)}$ as

$$(3.3) \quad h(p) = \begin{pmatrix} h_{00}(p) & h_{01} \\ h_{10} & h_{11}^0(p) - v \end{pmatrix}$$

with the entries

$$(h_{00}(p)f_0)_0 = w_1(p)f_0, \quad (h_{01}f_1)_0 = \frac{1}{\sqrt{2}} \int_{\mathbf{T}^\nu} v_1(s)f_1(s) ds,$$

$$(h_{10}f_0)_1(q) = \frac{1}{\sqrt{2}}v_1(q)f_0, \quad (h_{11}^0(p)f_1)_1(q) = w_2(p, q)f_1(q).$$

Let the operator $h_0(p)$, $p \in \mathbf{T}^\nu$, act in $\mathcal{H}^{(0,1)}$ as

$$h_0(p) = \begin{pmatrix} 0 & 0 \\ 0 & h_{11}^0(p) \end{pmatrix}, \quad p \in \mathbf{T}^\nu.$$

The perturbation $h(p) - h_0(p)$, $p \in \mathbf{T}^\nu$, of the operator $h_0(p)$, $p \in \mathbf{T}^\nu$, is a compact operator. Therefore in accordance with the invariance of the essential spectrum

under compact perturbations, the essential spectrum $\sigma_{\text{ess}}(h(p))$ of $h(p)$, $p \in \mathbf{T}^\nu$, fills the following interval of the real axis:

$$\sigma_{\text{ess}}(h(p)) = [m(p); M(p)],$$

where the numbers $m(p)$ and $M(p)$ are defined by

$$m(p) = \min_{q \in \mathbf{T}^\nu} w_2(p, q), \quad M(p) = \max_{q \in \mathbf{T}^\nu} w_2(p, q).$$

For any fixed $p \in \mathbf{T}^\nu$ we define the matrix operator $A_0(p; z)$ acting in $\mathcal{H}^{(0,1)}$ by

$$(3.4) \quad A_0(p; z) = \begin{pmatrix} A_{00}(p; z) & A_{01}(p; z) \\ A_{10}(p; z) & A_{11}(p; z) \end{pmatrix}, \quad z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p)),$$

where

$$\begin{aligned} (A_{00}(p; z)g_0)_0 &= \left(w_1(p) - z - \frac{1}{2} \int_{\mathbf{T}^\nu} \frac{v_1^2(s) ds}{w_2(p, s) - z} - 1 \right) g_0, \\ (A_{01}(p; z)g_1)_0 &= \int_{\mathbf{T}^\nu} \frac{v_1(s)}{w_2(p, s) - z} \int_{\mathbf{T}^\nu} \tilde{v}_2(t, s) g_1(t) dt ds, \\ (A_{10}(p; z)g_0)_1(q) &= \frac{1}{2} \int_{\mathbf{T}^\nu} \frac{\tilde{v}_2(s, q) v_1(s)}{w_2(p, s) - z} ds g_0, \\ (A_{11}(p; z)g_1)_1(q) &= - \int_{\mathbf{T}^\nu} \frac{\tilde{v}_2(q, t)}{w_2(p, t) - z} \int_{\mathbf{T}^\nu} \tilde{v}_2(t, s) g_1(s) ds dt. \end{aligned}$$

We note that for any fixed $p \in \mathbf{T}^\nu$ and $z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p))$ the operator $A_0(p; z)$ belongs to the trace class. Therefore (see [16]) the determinant $\det[E + A_0(p; z)]$ of the operator $E + A_0(p; z)$ is well defined, where $E = \text{diag}\{I_0, I_1\}$.

The following lemma establishes a connection between the eigenvalues of $h(p)$, $p \in \mathbf{T}^\nu$, and the zeroes of the function $\det[E + A_0(p; \cdot)]$, $p \in \mathbf{T}^\nu$.

Lemma 3.3. *A number $z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p))$ is an eigenvalue of the operator $h(p)$, $p \in \mathbf{T}^\nu$, if and only if $\det[E + A_0(p; z)] = 0$.*

Proof. Let the number $z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p))$ be an eigenvalue of the operator $h(p)$, $p \in \mathbf{T}^\nu$, and $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$ be the corresponding eigenvector, i.e., the equation $h(p)f = zf$ or the system of equations

$$(3.5) \quad \begin{aligned} (w_1(p) - z)f_0 + \frac{1}{\sqrt{2}} \int_{\mathbf{T}^\nu} v_1(s) f_1(s) ds &= 0, \\ \frac{1}{\sqrt{2}} v_1(q) f_0 + (w_2(p, q) - z)f_1(q) - \int_{\mathbf{T}^\nu} v_2(q, s) f_1(s) ds &= 0 \end{aligned}$$

has a nontrivial solution $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$.

Since $z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p))$ from the second equation of system (3.5) we find

$$(3.6) \quad f_1(q) = \frac{(\tilde{v}f_1)(q)}{w_2(p, q) - z} - \frac{1}{\sqrt{2}} \frac{v_1(q) f_0}{w_2(p, q) - z},$$

where the operator \tilde{v} is defined by (3.1) and

$$(3.7) \quad \tilde{f}_1(q) = (\tilde{v}f_1)(q).$$

Substituting the expression (3.6) for f_1 into the first equation of system (3.5) and the equality (3.7), we get that the system of equations (3.5) has a nontrivial solution if and only if the system of equations

$$\begin{aligned} \left(w_1(p) - z - \frac{1}{2} \int_{\mathbf{T}^\nu} \frac{v_1^2(s) ds}{w_2(p, s) - z} \right) f_0 + \int_{\mathbf{T}^\nu} \frac{v_1(s)}{w_2(p, s) - z} \int_{\mathbf{T}^\nu} \tilde{v}_2(t, s) \tilde{f}_1(t) dt ds &= 0, \\ \frac{1}{\sqrt{2}} \int_{\mathbf{T}^\nu} \frac{\tilde{v}_2(s, q) v_1(s)}{w_2(p, s) - z} ds f_0 + \tilde{f}_1(q) - \int_{\mathbf{T}^\nu} \frac{\tilde{v}_2(q, t)}{w_2(p, t) - z} \int_{\mathbf{T}^\nu} \tilde{v}_2(t, s) \tilde{f}_1(s) ds dt &= 0 \end{aligned}$$

or the equation

$$E\Phi + A_0(p; z)\Phi = 0, \quad \Phi = (f_0, \tilde{f}_1) \in \mathcal{H}^{(0,1)}$$

has a nontrivial solution, i.e., $\det[E + A_0(p; z)] = 0$. \square

By Lemma 3.3 the number z belongs to the discrete spectrum of $h(p)$ if and only if $\det[E + A_0(p; z)] = 0$. The following equality immediately follows:

$$(3.8) \quad \sigma_{\text{disc}}(h(p)) = \{z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p)) : \det[E + A_0(p; z)] = 0\}, \quad p \in \mathbf{T}^\nu.$$

Lemma 3.4. *For the spectrum $\sigma(\hat{H})$ of \hat{H} the equality*

$$\sigma(\hat{H}) = \bigcup_{p \in \mathbf{T}^\nu} \sigma_{\text{disc}}(h(p)) \cup [m; M]$$

holds.

Proof. The assertion of this lemma follows from the representation (3.2), the equalities

$$\sigma(h(p)) = [m(p); M(p)] \cup \sigma_{\text{disc}}(h(p)), \quad \bigcup_{p \in \mathbf{T}^\nu} [m(p); M(p)] = [m; M],$$

and the theorem on the spectrum of decomposable operators (see [16]). \square

Now we introduce the new subsets of the essential spectrum of H .

Definition 3.5. The sets $\sigma_{\text{two}}(H) = \bigcup_{p \in \mathbf{T}^\nu} \sigma_{\text{disc}}(h(p))$ and $\sigma_{\text{three}}(H) = [m; M]$ are called two-particle and three-particle branches of the essential spectrum of H , respectively.

Lemma 3.6. *The operator $A(z)$, $z \in \mathbf{C} \setminus \sigma_{\text{three}}(H)$, is bounded and invertible if and only if $z \in \mathbf{C} \setminus \sigma(\hat{H})$.*

Proof. Let us introduce the operator matrix $A_0(z)$ acting in $\mathcal{H}^{(1,2)}$ as

$$A_0(z) = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix}.$$

By the definition of $A(z)$ and $A_0(z)$ we have that the operator $A(z)$ is invertible if and only if the operator $A_0(z)$ is invertible.

In analogy with the operator \hat{H} one can give the decomposition

$$(3.9) \quad A_0(z) = \int_{\mathbf{T}^\nu} \oplus [E + A_0(p; z)] dp,$$

where the operator $A_0(p; z)$ is defined by (3.4).

By Lemmas 3.3 and 3.4 for any fixed $p \in \mathbf{T}^\nu$ and $z \in \mathbf{C} \setminus \sigma(\hat{H})$ we have $\det[E + A_0(p; z)] \neq 0$. Therefore, for any fixed $z \in \mathbf{C} \setminus \sigma(\hat{H})$ the operator $A_0(z)$ is invertible. The converse trivially follows from the decomposition (3.9). \square

4. PROOF OF THE MAIN RESULTS

In this section we prove Theorems 2.1 and 2.3.

Proof of Theorem 2.1. Let $z \in \mathbf{C} \setminus \sigma(\hat{H})$ be an eigenvalue of the operator H and $f = (f_0, f_1, f_2) \in \mathcal{H}$ be the corresponding eigenvector, that is, the equation $Hf = zf$ or the system of equations

$$(4.1) \quad \begin{aligned} & ((H_{00} - zI_0)f_0)_0 + (H_{01}f_1)_0 = 0, \\ & (H_{10}f_0)_1(p) + ((H_{11} - zI_1)f_1)_1(p) + (H_{12}f_2)_1(p) = 0, \\ & (H_{21}f_1)_2(p, q) + ((H_{22} - zI_2)f_2)_2(p, q) = 0 \end{aligned}$$

have a nontrivial solution $f = (f_0, f_1, f_2) \in \mathcal{H}$. Since $z \notin \sigma_{\text{three}}(H)$, from the third equation of the system (4.1) for f_2 , we have

$$(4.2) \quad f_2(p, q) = (R_{22}^0(z)V_1 f_2)(p, q) + (R_{22}^0(z)V_2 f_2)(p, q) - (R_{22}^0(z)H_{21} f_1)(p, q).$$

Let

$$(4.3) \quad \tilde{f}_2(p, q) = (\tilde{V}_2 f_2)(p, q).$$

Then

$$(\tilde{V}_2 \tilde{f}_2)(p, q) = (\tilde{V}_1 \tilde{f}_2)(q, p).$$

Therefore the equality (4.2) has form

$$(4.4) \quad f_2(p, q) = (R_{22}^0(z)\tilde{V}_1 \tilde{f}_2)(p, q) + (R_{22}^0(z)\tilde{V}_2 \tilde{f}_2)(p, q) - (R_{22}^0(z)H_{21} f_1)(p, q).$$

Substituting the expression (4.4) for f_2 into the system of equations (4.1) and the equality (4.3) we obtain that the system of equations

$$f_0 = (w_0 - z + 1)f_0 + \int_{\mathbf{T}^\nu} v_0(s)f_1(s) ds,$$

(4.5)

$$\begin{aligned} & \left(w_1(p) - z - \frac{1}{2} \int_{\mathbf{T}^\nu} \frac{v_1^2(s) ds}{w_2(p, s) - z} \right) f_1(p) + \int_{\mathbf{T}^\nu} \frac{v_1(s)}{w_2(p, s) - z} \int \tilde{v}_2(t, s) \tilde{f}_2(p, t) dt ds \\ &= -v_0(p)f_0 + \int_{\mathbf{T}^\nu} \frac{v_1(s)}{w_2(p, s) - z} \int \tilde{v}_2(p, t) \tilde{f}_2(t, s) dt ds + \frac{v_1(p)}{2} \int_{\mathbf{T}^\nu} \frac{v_1(s)f_1(s) ds}{w_2(p, s) - z}, \\ & \frac{1}{2} \int \frac{\tilde{v}_2(s, q)v_1(s)}{w_2(p, s) - z} ds f_1(p) + \tilde{f}_2(p, q) - (\tilde{V}_2 R_{22}^0(z)\tilde{V}_2 \tilde{f}_2)(p, q) \\ &= (\tilde{V}_2 R_{22}^0(z)\tilde{V}_1 \tilde{f}_2)(p, q) - \frac{v_1(p)}{2} \int \frac{\tilde{v}_2(s, q)f_1(s)}{w_2(p, s) - z} ds \end{aligned}$$

has a nontrivial solution if and only if the system of equations (4.1) has a nontrivial solution.

The system of equations (4.5) can be written in the following form:

$$A(z)\Phi = K(z)\Phi, \quad \Phi = (f_0, f_1, \tilde{f}_2) \in \mathcal{H}^{(0,2)}.$$

By Lemma 3.6 for each $z \in \mathbf{C} \setminus \sigma(\hat{H})$ the operator $A(z)$ is invertible and hence the equation $\Phi = A^{-1}(z)K(z)\Phi$ or $\Phi = T(z)\Phi$ has a nontrivial solution if and only if the system of equations (4.5) has a nontrivial solution. \square

Now applying the Weyl criterion and Theorem 2.1 we prove Theorem 2.3.

Proof of Theorem 2.3. The inclusion $\sigma_{\text{three}}(H) \subset \sigma_{\text{ess}}(H)$ can be proved quite similarly to the corresponding inclusion of [7].

We prove that $\sigma_{\text{two}}(H) \subset \sigma_{\text{ess}}(H)$. Let $z_0 \in \sigma_{\text{two}}(H)$ be an arbitrary point.

Two cases are possible,

$z_0 \in \sigma_{\text{three}}(H)$ or $z_0 \notin \sigma_{\text{three}}(H)$.

If $z_0 \in \sigma_{\text{three}}(H)$, then $z_0 \in \sigma_{\text{ess}}(H)$. Let $z_0 \notin \sigma_{\text{three}}(H)$, but $z_0 \in \sigma_{\text{two}}(H)$. Then by Lemma 3.6 the operator $A(z_0)$ isn't invertible. It means that there exists an orthonormal system $\Phi^{(n)} = (0, f_1^{(n)}, \tilde{f}_2^{(n)})$ such that $\|A(z_0)\Phi^{(n)}\|_{\mathcal{H}^{(0,2)}} \rightarrow 0$ as $n \rightarrow +\infty$.

We choose a sequence of orthogonal vector-functions $\{f^{(n)}\}$ as

$$f^{(n)} = \begin{pmatrix} 0 \\ f_1^{(n)}(p) \\ f_2^{(n)}(p, q) \end{pmatrix},$$

where

$$f_2^{(n)}(p, q) = (R_{22}^0(z_0)\tilde{V}_1 \tilde{f}_2^{(n)})(p, q) + (R_{22}^0(z_0)\tilde{V}_2 \tilde{f}_2^{(n)})(p, q) - (R_{22}^0(z_0)H_{21} f_1^{(n)})(p, q).$$

We consider $(H - z_0)f^{(n)}$ and estimate its norm as

$$\begin{aligned} \|(H - z_0)f^{(n)}\|_{\mathcal{H}}^2 &= \|(H - z_0)f^{(n)} + \tilde{f}_2^{(n)} - \tilde{V}_2 f_2^{(n)} - (\tilde{f}_2^{(n)} - \tilde{V}_2 f_2^{(n)})\|_{\mathcal{H}}^2 \\ &\leq \|(A(z_0) - K(z_0))\Phi^{(n)}\|_{\mathcal{H}^{(0,2)}}^2 + \|\tilde{f}_2^{(n)} - \tilde{V}_2 f_2^{(n)}\|_{\overline{\mathcal{H}}_2}^2. \end{aligned}$$

Let

$$(A(z_0) - K(z_0))\Phi^{(n)} = \begin{pmatrix} ((A(z_0) - K(z_0))\Phi^{(n)})_0 \\ ((A(z_0) - K(z_0))\Phi^{(n)})_1 \\ ((A(z_0) - K(z_0))\Phi^{(n)})_2 \end{pmatrix}.$$

Since the operator $K(z_0)$ is compact, we have $\|K(z_0)\Phi^{(n)}\|_{\mathcal{H}^{(0,2)}} \rightarrow 0$ as $n \rightarrow +\infty$. Therefore, from $\|A(z_0)\Phi^{(n)}\|_{\mathcal{H}^{(0,2)}} \rightarrow 0$ as $n \rightarrow +\infty$ it follows that

$$\begin{aligned} (4.6) \quad &\|(A(z_0) - K(z_0))\Phi^{(n)}\|_{\mathcal{H}^{(0,2)}}^2 \\ &= \|((A(z_0) - K(z_0))\Phi^{(n)})_0\|_{\overline{\mathcal{H}}_0}^2 + \|((A(z_0) - K(z_0))\Phi^{(n)})_1\|_{\overline{\mathcal{H}}_1}^2 \\ &\quad + \|((A(z_0) - K(z_0))\Phi^{(n)})_2\|_{\overline{\mathcal{H}}_2}^2 \leq \|A(z_0)\Phi^{(n)}\|_{\mathcal{H}^{(0,2)}}^2 + \|K(z_0)\Phi^{(n)}\|_{\mathcal{H}^{(0,2)}}^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$. It follows that $\|((A(z_0) - K(z_0))\Phi^{(n)})_i\|_{\overline{\mathcal{H}}_i} \rightarrow 0$, $i = 0, 1, 2$, as $n \rightarrow \infty$. Therefore from the equality

$$\|((A(z_0) - K(z_0))\Phi^{(n)})_2\|_{\overline{\mathcal{H}}_2} = \|\tilde{f}_2^{(n)} - \tilde{V}_2 f_2^{(n)}\|_{\overline{\mathcal{H}}_2}$$

and the relation (4.6) we have that $\|(H - z_0)f^{(n)}\|_{\mathcal{H}} \rightarrow 0$ as $n \rightarrow +\infty$. This implies that $z_0 \in \sigma_{\text{ess}}(H)$. Since the point $z_0 \in \sigma_{\text{two}}(\hat{H})$ is arbitrary, it follows that $\sigma_{\text{two}}(H) \subset \sigma_{\text{ess}}(H)$.

Now we prove the inclusion $\sigma_{\text{ess}}(H) \subset \sigma(\hat{H})$. Since for each $z \in \mathbf{C} \setminus \sigma(\hat{H})$ the operator $K(z)$ is compact and $A^{-1}(z)$ is bounded, we have that $f(z) = A^{-1}(z)K(z)$ is a compact-valued analytic function in $\mathbf{C} \setminus \sigma(\hat{H})$. From the self-adjointness of H and Theorem 2.1 it follows that the operator $(\mathbf{I} - f(z))^{-1}$ exists for all $\text{Im } z \neq 0$, where \mathbf{I} is the identical operator in $\mathcal{H}^{(0,2)}$. In accordance with the analytic Fredholm theorem, we conclude that the set

$$\sigma(H) \setminus \sigma(\hat{H}) = \{z : \det(\mathbf{I} - f(z)) = 0\}$$

is discrete. Thus $\sigma(H) \setminus \sigma(\hat{H}) \subset \sigma_{\text{disc}}(H) = \sigma(H) \setminus \sigma_{\text{ess}}(H)$. Therefore the inclusion $\sigma_{\text{ess}}(H) \subset \sigma(\hat{H})$ holds. \square

5. EXAMPLE

In this section we consider the case $\nu = 3$ and calculate the essential spectrum of the operator H in the case where w_0 is an arbitrary real number, $w_1(\cdot)$, $v_i(\cdot)$, $i = 0, 1$, are arbitrary real-valued continuous functions on \mathbf{T}^3 , $w_2(\cdot, \cdot)$ is an arbitrary real-valued continuous symmetric function on $(\mathbf{T}^3)^2$ and the function $v_2(\cdot, \cdot)$ has the form

$$(5.1) \quad v_2(p, q) = \sum_{i=1}^3 \cos(p_i - q_i), \quad p = (p_1, p_2, p_3), \quad q = (q_1, q_2, q_3) \in \mathbf{T}^3.$$

In this case for the kernel $\tilde{v}_2(\cdot, \cdot)$ of the integral operator \tilde{v} defined by (3.1) the equality $\tilde{v}_2(p, q) = (4\pi^3)^{-1}v_2(p, q)$ holds.

Additionally we also assume that the function $v_1(\cdot)$ is even on \mathbf{T}^3 and the function $w_2(\cdot, \cdot)$ is even in each coordinate on \mathbf{T}^1 , for example,

$$(5.2) \quad \begin{aligned} v_1(p) &= \sum_{i=1}^3 \cos p_i, \quad p = (p_1, p_2, p_3) \in \mathbf{T}^3, \\ w_2(p, q) &= \sum_{i=1}^3 (2 - \cos p_i - \cos q_i), \quad p = (p_1, p_2, p_3), \quad q = (q_1, q_2, q_3) \in \mathbf{T}^3. \end{aligned}$$

From Theorem 2.3 and Lemma 3.4 it follows that in the study of the essential spectrum of H the crucial role is played by the discrete spectrum of $h(p)$ defined by (3.3).

By the equality (3.8) for the study $\sigma_{\text{disc}}(h(p))$ we construct the determinant $\det[E + A_0(p; z)]$.

Let the number $z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p))$ be an eigenvalue of the operator $h(p)$, $p \in \mathbf{T}^3$ and $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$ be the corresponding eigenvector, i.e., the equation $h(p)f = zf$ or the system of equations

$$(5.3) \quad \begin{aligned} (w_1(p) - z)f_0 + \frac{1}{\sqrt{2}} \int_{\mathbf{T}^3} v_1(s) f_1(s) ds &= 0, \\ \frac{1}{\sqrt{2}} v_1(q) f_0 + (w_2(p, q) - z) f_1(q) - \int_{\mathbf{T}^3} \sum_{i=1}^3 \cos(s_i - q_i) f_1(s) ds &= 0 \end{aligned}$$

has a nontrivial solution $f = (f_0, f_1) \in \mathcal{H}^{(0,1)}$.

Denote

$$(5.4) \quad d_i = \int_{\mathbf{T}^3} \cos s_i f_1(s) ds, \quad e_i = \int_{\mathbf{T}^3} \sin s_i f_1(s) ds.$$

Since $z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p))$ from the second equation of system (5.3) we find

$$(5.5) \quad f_1(q) = \frac{\sum_{i=1}^3 (d_i \cos q_i + e_i \sin q_i)}{w_2(p, q) - z} - \frac{1}{\sqrt{2}} \frac{v_1(q) f_0}{w_2(p, q) - z}.$$

For any $p \in \mathbf{T}^3$ we define the following continuous functions in $\mathbf{C} \setminus [m(p), M(p)]$ by

$$\begin{aligned} a_{ij}(p; z) &= - \int_{\mathbf{T}^3} \frac{\cos s_i \cos s_j ds}{w_2(p, s) - z}, \quad i, j = 1, 2, 3, \\ b_i(p; z) &= \int_{\mathbf{T}^3} \frac{\sin^2 s_i ds}{w_2(p, s) - z}, \quad c_i(p; z) = \frac{1}{\sqrt{2}} \int_{\mathbf{T}^3} \frac{\cos s_i v_1(s) ds}{w_2(p, s) - z}, \quad i = 1, 2, 3, \\ D_0(p; z) &= w_1(p) - z - \frac{1}{2} \int_{\mathbf{T}^3} \frac{v_1^2(s) ds}{w_2(p, s) - z}, \\ D_i(p; z) &= 1 - a_{ii}(p; z), \quad \Delta_i(p; z) = 1 - b_i(p; z), \quad i = 1, 2, 3, \\ \Delta_4(p; z) &= \begin{vmatrix} D_0(p; z) & c_1(p; z) & c_2(p; z) & c_3(p; z) \\ c_1(p; z) & D_1(p; z) & a_{12}(p; z) & a_{13}(p; z) \\ c_2(p; z) & a_{21}(p; z) & D_2(p; z) & a_{23}(p; z) \\ c_3(p; z) & a_{31}(p; z) & a_{32}(p; z) & D_3(p; z) \end{vmatrix}. \end{aligned}$$

Substituting the expression (5.5) for f_1 into the first equation of system (3.5) and the equalities (5.4), we get that the equality

$$\det[E + A_0(p; z)] = \prod_{i=1}^4 \Delta_i(p; z).$$

Therefore, by the equality (3.8) we obtain that

$$\sigma_{\text{disc}}(h(p)) = \{z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h(p)) : \prod_{i=1}^4 \Delta_i(p; z) = 0\}, \quad p \in \mathbf{T}^3.$$

By Lemma 3.4 and Theorem 2.3 we have that

$$\sigma_{\text{ess}}(H) = \bigcup_{p \in \mathbf{T}^3} \sigma_{\text{disc}}(h(p)) \cup [m; M].$$

Note that if the function $w_2(\cdot, \cdot)$ has form (5.2), then we have that $[m; M] = [0; 12]$.

Acknowledgments. The authors are grateful to Prof. Dr. H. Spohn for stimulating discussions on the results of the paper and to the referee for useful comments. This paper

was completed during the visit of the second author at the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy. He would like to thank to ICTP for the kind hospitality and support, and the Commission on Development and Exchanges of the International Mathematical Union for the travel grant.

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Received 12/02/2010; Revised 30/05/2010