SOME CLASS OF REAL SEQUENCES HAVING INDEFINITE HANKEL FORMS

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ABSTRACT. In this paper we generalize the results given in [14] about real sequences which are not necessarily positive (i.e, they are not sequences of power moments) but can be mapped, by a difference operator, into a power moment sequence. We prove by elementary methods that the integro-polynomial representation of such sequences remains after dropping the condition on its growth imposed in the mentioned article. Some additional results on the uniqueness of the representation are included.

1. INTRODUCTION

The general power moment problem is stated as follows: Given a real sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$, it is required to find a non-decreasing function $\sigma(t)$ $(t \in I)$ such that

$$(1) \qquad \gamma_n = \int_I t^n d\sigma(t) \quad (n = 0, 1, 2, \ldots).$$

In the case that such a function exists, its uniqueness is also to be studied. If there is an *essentially unique* solution (which means that the difference of any two solutions is constant at the points where it is continuous) $\sigma(t)$ to (1) then the moment problem is called *determinate*; otherwise (two or more solutions) the moment problem is called *indeterminate*. For details on these terms we refer to [2], page 3.

There are essentially 3 types of the (classical) power moment problem: the *Hamburger moment problem*, in which $I = \mathbb{R}$; the *Stieltjes moment problem*, corresponding to $I = [0, \infty)$ and the *Hausdorff moment problem*, in the case of a finite (bounded) interval I (interval which can be taken, without loss of generality, as $[0, 1]$). The results in this paper concerns mainly the Hamburger moment problem.

It is well known now that in order for such a function $\sigma(t)$ to exist (for the Hamburger moment problem) it is necessary and sufficient that all Hankel forms

$$(2) \qquad \sum_{i,k=0}^{m} x_i x_k \gamma_{i+k}$$

be non-negative. If these forms are positive (non-negative) then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called positive (non-negative). The term positive (non-negative) relative to the axis is also used.

Thus, solvable moment problems (i.e., Hamburger moment sequences) correspond to non-negative sequences and vice versa. Let us note that the (strict) positivity of all the Hankel forms (2) guarantees the existence of an infinite number of points of increase for the solutions of the moment problem and that the integral in (1) doesn't degenerate into a finite sum (see [12], theorem 1.2).

In [14], sequences which can be mapped to a Hausdorff moment sequence by a difference operator are studied. Such sequences are called there *definitizable* (for exact

definitions see Section 2). In the mentioned article it is stated that definitizable sequences with an additional condition on its growth have an integro-polynomial representation (instead of (1)) and it is sketched there how to obtain the elements involved in this representation. It is worth to note that the resulting moment problem (which can be called *definitizable power moment problem*) arose naturally, as well as the *indefinite power moment problem* (we refer to [3] for origin and details of this problem. See also [11], [5] and [6]) from the spectral theory of operators in spaces with an indefinite metric (see [14] for the *definitizable* case and, for instance, [3] or [7] or [8] and [9] for the *indefinite* case). They differ in that the *indefinite moment problem* is limited to the case where the underlying space is a Pontryagin space (denoted by π_κ) whereas the *definitizable moment problem* suits Krein spaces as well as Pontryagin spaces.

In the present paper, we consider sequences that can be mapped, by a difference operator, to a Hamburger moment sequence and with no restrictions on its growth. In Section 2 we prove (by elementary methods instead of the operator approach given in [14]) that such sequences have the same representation (except for the interval of integration). In Section 3 we prove some results on uniqueness and Section 4 contains some results concerning the particular case of (real) sequences being mapped (by a difference operator) to a Hausdorff moment sequence. An analog treatment but for the trigonometric moment problem was given in [10].

2. Solubility

In this section we will give a precise definition of the type of sequences we will be dealing with in this paper and prove a representation (which generalizes the representation of the Hamburger moment sequences) for such sequences.

Definition 1. Let $\{c_k\}_{k=0}^\infty$ be a real sequence. If there exist real numbers $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that the sequence $\{\tilde{c}_k\}_{k=0}^\infty$ defined by

$$\tilde{c}_k = \sum_{i=0}^n \gamma_i c_{i+k} \quad (k = 0, 1, \ldots)$$

is a Hamburger moment sequence then $\{c_k\}_{k=0}^\infty$ will be called *Hamburger-definitizable* and the polynomial $\mathfrak{Q}(z) = \sum_{i=0}^n \gamma_i z^i$ will be called the *definitizing polynomial* of the sequence $\{c_k\}_{k=0}^\infty$. We can assume (and we will) that $\gamma_n \neq 0$.

To avoid confusions, the sequences called *definitizable* in [14] will be called here *Hausdorff-definitizable*. We can define analogously for the other moment problems.

We will also make the agreement that whenever the underlying moment problem is not explicitly specified, we will be confined to the Hamburger case. In other words, we will take *definitizable* as *Hamburger-definitizable*.

The next theorem settles the existence of solutions to our problem (a generalization of the Hamburger moment problem).

Theorem 1. *Let $\{c_k\}_{k=0}^\infty$ be a* definitizable *sequence with* definitizing *polynomial $\mathfrak{Q}(z) = \sum_{i=0}^n \gamma_i z^i$ ($\gamma_n \neq 0$). Then there exist a polynomial $Q(z)$ of degree m, $m \leq n$, and a piecewise monotone function $\rho(t)$ defined on $\mathbb{R}$ except on the real roots of Q such that*

$$(3) \qquad c_k = \int_{-\infty}^\infty (t^k - P_k(t))d\rho(t) + \sum_{i=0}^{m-1} a_i^{(k)} c_i \quad (k = 0, 1, 2, \ldots),$$

where $P_k(z) = \sum_{i=0}^{m-1} a_i^{(k)} z^i$ is, for $k = 0, 1, \ldots$, the interpolating polynomial of z^k at the zero multiset of Q. The integral in (3) is an improper (Stieltjes) one with singular points the real roots of Q.

Proof. Let $Q(z) = \sum_{i=0}^{n} \gamma_i z^i$ (notice we are taking $m = n$ and $Q = \mathfrak{Q}$). Then $P_k(z)$, the interpolating polynomial of the function z^k $(k = 0, 1, \ldots)$ at the zero multiset of Q, satisfies

$$z^k = Q(z) R_k(z) + P_k(z) \quad (k = 0, 1, \ldots),$$

being $R_k(z)$ some polynomial (for each $k = 0, 1, \ldots$).

As the degree of Q is n then we have

$$P_k(z) = z^k, \quad k = 0, 1, \ldots, n - 1;$$

(4)
$$P_n(z) = \sum_{i=0}^{n-1} -\frac{\gamma_i}{\gamma_n} z^i.$$

Now, from

(5)
$$z^n = \frac{1}{\gamma_n} Q(z) + P_n(z)$$

we have

$$z^{n+j} = \frac{1}{\gamma_n} z^j Q(z) - \sum_{i=0}^{n-1} \frac{\gamma_i}{\gamma_n} z^{i+j} = \frac{1}{\gamma_n} z^j Q(z) + \sum_{k=j}^{n+j-1} \frac{-\gamma_{k-j}}{\gamma_n} z^k$$

$$= \frac{1}{\gamma_n} z^j Q(z) + \sum_{k=j}^{n+j-1} \frac{-\gamma_{k-j}}{\gamma_n} \left(Q(z) R_k(z) + P_k(z) \right)$$

$$= Q(z) \left[\frac{z^j}{\gamma_n} + \sum_{k=j}^{n+j-1} \frac{-\gamma_{k-j}}{\gamma_n} R_k(z) \right] + \sum_{k=j}^{n+j-1} \frac{-\gamma_{k-j}}{\gamma_n} P_k(z).$$

Thus

(6)
$$P_{n+j}(z) = \sum_{k=j}^{n+j-1} \frac{-\gamma_{k-j}}{\gamma_n} P_k(z) \quad (j = 0, 1, \ldots).$$

We now prove our assertion by (generalized) induction.

First, note that because of (4) the representation (3) holds for $c_0, \ldots, c_{n-1}$. Since $\{\tilde{c}_k\}_{k=0}^{\infty}$ is positive definite we have

$$\tilde{c}_k = \int_{-\infty}^{\infty} t^k d\sigma(t) \quad (k = 0, 1, \ldots).$$

In the other hand, $\tilde{c}_k = \sum_{i=0}^{n} \gamma_i c_{i+k}$, so in particular for $k = 0$ we have

$$\int_{-\infty}^{\infty} d\sigma(t) = \tilde{c}_0 = \sum_{i=0}^{n} \gamma_i c_i,$$

from which follows

$$c_n = \frac{1}{\gamma_n} \left[\int_{-\infty}^{\infty} d\sigma(t) - \sum_{i=0}^{n-1} \gamma_i c_i \right] = \int_{-\infty}^{\infty} (t^n - P_n(t)) \, d\rho(t) + \sum_{i=0}^{n-1} \frac{-\gamma_i}{\gamma_n} c_i,$$

where $d\rho(t) = \dfrac{d\sigma(t)}{\gamma_n (t^n - P_n(t))}$.

Thus (3) also holds for $c_m = c_n$.

Suppose now that (3) holds for $k = 0, 1, \ldots, n, \ldots, n + j - 1$, with some fixed $j \in \mathbb{N}$. Then, since $\{\tilde{c}_k\}_{k=0}^{\infty}$ is a Hamburger moment sequence, we have

$$\int_{-\infty}^{\infty} t^j d\sigma(t) = \tilde{c}_j = \sum_{i=0}^{n} \gamma_i c_{i+j} = \sum_{k=j}^{n+j} \gamma_{k-j} c_k.$$

Also, by the definition of $d\rho(t)$ and (5) we have

$$\int_{-\infty}^{\infty} t^j \, d\sigma(t) = \gamma_n \int_{-\infty}^{\infty} t^j \left(t^n - P_n(t)\right) d\rho(t) = \gamma_n \int_{-\infty}^{\infty} \frac{t^j}{\gamma_n} Q(t) \, d\rho(t).$$

Thus, $\int_{-\infty}^{\infty} t^j Q(t) \, d\rho(t) = \sum_{k=j}^{n+j} \gamma_{k-j} c_k$, from which follows

$$c_{n+j} = \frac{1}{\gamma_n} \left[\int_{-\infty}^{\infty} t^j \sum_{i=0}^{n} \gamma_i t^i \, d\rho(t) - \sum_{k=j}^{n+j-1} \gamma_{k-j} c_k \right]$$

$$= \int_{-\infty}^{\infty} t^{n+j} \, d\rho(t) + \sum_{i=0}^{n-1} \frac{\gamma_i}{\gamma_n} \int_{-\infty}^{\infty} t^{i+j} \, d\rho(t) - \sum_{k=j}^{n+j-1} \frac{\gamma_{k-j}}{\gamma_n} c_k.$$

Thus, by the inductive hypothesis we have

$$c_{n+j} = \int_{-\infty}^{\infty} t^{n+j} \, d\rho(t) + \sum_{i=0}^{n-1} \frac{\gamma_i}{\gamma_n} \int_{-\infty}^{\infty} t^{i+j} \, d\rho(t)$$

$$+ \sum_{k=j}^{n+j-1} -\frac{\gamma_{k-j}}{\gamma_n} \left[\int_{-\infty}^{\infty} \left(t^k - P_k(t)\right) d\rho(t) + \sum_{i=0}^{n-1} a_i^{(k)} c_i \right]$$

$$= \int_{-\infty}^{\infty} \left[t^{n+j} + \sum_{i=0}^{n-1} \frac{\gamma_i}{\gamma_n} t^{i+j} + \sum_{k=j}^{n+j-1} -\frac{\gamma_{k-j}}{\gamma_n} \left(t^k - P_k(t)\right) \right] d\rho(t)$$

$$+ \sum_{k=j}^{n+j-1} -\frac{\gamma_{k-j}}{\gamma_n} \sum_{i=0}^{n-1} a_i^{(k)} c_i,$$

which with the help of (6) can be written as

$$c_{n+j} = \int_{-\infty}^{\infty} \left(t^{n+j} + \sum_{k=j}^{n+j-1} \frac{\gamma_{k-j}}{\gamma_n} P_k(t) \right) d\rho(t) + \sum_{i=0}^{n-1} a_i^{(n+j)} c_i$$

$$= \int_{-\infty}^{\infty} \left(t^{n+j} - P_{n+j}(t) \right) d\rho(t) + \sum_{i=0}^{n-1} a_i^{(n+j)} c_i,$$

i.e., representation (3) holds also for c_{n+j}. □

Remark 1. We have proved our theorem taking $m = n$ and $Q(z) = \sum_{i=0}^{n} \gamma_i z^i$. Sometimes it is possible to obtain the same representation but with another polynomial Q of degree $m < n$ (see, for instance, theorem 4 below).

Remark 2. From $d\rho(t) = \dfrac{d\sigma(t)}{\gamma_n(t^n - P_n(t))} = \dfrac{d\sigma(t)}{Q(t)}$ we conclude that contrary to the situation presented in the classical moment problem, the function $\rho(t)$ is, in general, unbounded. In fact, it can have an infinite jump at any of the zeros of Q. Actually, the role of polynomial Q is to extinguish the unboundedness of ρ in order to make convergent the integrals in (3). The function $\rho(t)$ is readily seen to be a piecewise monotone function.

Remark 3. The possibility that the degree of Q be zero is not excluded. In this case the function $\rho(t)$ is bounded and the interpolating polynomial is absent from the representation. We shall assume in this case that $Q(z) \equiv 1$.

3. Uniqueness

In this section we present some results on the uniqueness of the elements involved in (3).

Having proved that we can define $\rho(t)$ by means of $d\rho(t) = \dfrac{d\sigma(t)}{\gamma_n(t^n - P_n(t))} = \dfrac{d\sigma(t)}{Q(t)}$, we can affirm that if the moment problem for the sequence $\{\tilde{c}_k\}_{k=0}^{\infty}$ is indeterminate then the function $\rho(t)$ is not unique. And we have to take into account also that the polynomial $Q(t)$ is not unique, as the next theorems show.

Let's start proving that we can add real roots to Q without affecting representation (3).

Theorem 2. *Let $\{c_k\}_{k=0}^{\infty}$ be a real sequence admitting the representation (3) and let $R(z) = z - z_0$, with $z_0 \in \mathbb{R}$ fixed. If we denote $\tilde{Q}(z) = Q(z)R(z)$ and $\tilde{m} = \deg(\tilde{Q}(z)) = m + 1$ then we have*

$$(7) \qquad c_k = \int_{-\infty}^{\infty} \left(t^k - \tilde{P}_k(t) \right) d\rho(t) + \sum_{i=0}^{\tilde{m}-1} \tilde{a}_i^{(k)} c_i \quad (k = 0, 1, \ldots),$$

where $\tilde{P}_k(z) = \sum_{i=0}^{\tilde{m}-1} \tilde{a}_i^{(k)} z^i$ is the interpolating polynomial of z^k at the zero multiset of $\tilde{Q}$.

Proof. It is clear that the first $\tilde{m}$ elements of the sequence, $c_0, c_1, \ldots, c_{\tilde{m}-1}$, have the representation (7).

To prove the representation for $c_{\tilde{m}} = c_{m+1}$ we start noting that from the equation $z^m = Q(z)R_m(z) + P_m(z)$ we have $z^{m+1} = \tilde{Q}(z)R_m(z) + zP_m(z) + z_0(z^m - P_m(z))$ and, as a consequence,

$$(8) \qquad \tilde{P}_{m+1}(z) = zP_m(z) + z_0(z^m - P_m(z)).$$

Now, noting that

$$\left(z + a_{m-1}^{(m)} \right) P_m(z) = \left(z + a_{m-1}^{(m)} \right) \left(a_{m-1}^{(m)} z^{m-1} + a_{m-2}^{(m)} z^{m-2} + \cdots + a_0^{(m)} \right)$$

$$= a_{m-1}^{(m)} z^m + \cdots + \left(a_0^{(m)} + a_{m-1}^{(m)} a_1^{(m)} \right) z + a_{m-1}^{(m)} a_0^{(m)},$$

we conclude that $\left(z + a_{m-1}^{(m)} \right) P_m(z) - a_{m-1}^{(m)} z^m$ is a polynomial of degree less or equal to $m - 1$ which at the zero multiset of $Q(z)$ coincides with z^{m+1}, in other words

$$(9) \qquad P_{m+1}(z) = \left[z + a_{m-1}^{(m)} \right] P_m(z) - a_{m-1}^{(m)} z^m = zP_m(z) - a_{m-1}^{(m)} \left[z^m - P_m(z) \right].$$

Combining (8) and (9) we get

$$(10) \qquad \tilde{P}_{m+1}(z) = P_{m+1}(z) + \left(z_0 + a_{m-1}^{(m)} \right) (z^m - P_m(z)).$$

In this way we have

$$\int_{-\infty}^{\infty} \left(t^{m+1} - \tilde{P}_{m+1}(t) \right) d\rho(t)$$

$$= \int_{-\infty}^{\infty} \left(t^{m+1} - P_{m+1}(t) \right) d\rho(t) - \int_{-\infty}^{\infty} \left(z_0 + a_{m-1}^{(m)} \right) (t^m - P_m(t)) d\rho(t)$$

$$= c_{m+1} - \sum_{i=0}^{m-1} a_i^{(m+1)} c_i - \left(z_0 + a_{m-1}^{(m)} \right) \left(c_m - \sum_{i=0}^{m-1} a_i^{(m)} c_i \right),$$

from which follows by (10) the representation (7) for c_{m+1}.

The rest of the proof is straightforward. $\qquad \square$

We can also add pairs of conjugate non-real roots to Q, without affecting representation (3), as the next theorem shows. The addition of non-real roots has to be done by conjugate pairs in order to ensure that the coefficients of Q remain real.

Theorem 3. *Let $\{c_k\}_{k=0}^{\infty}$ be a real sequence admitting representation (3) and let $R(z) = (z - z_0)(z - \overline{z_0})$, with $z_0 \in \mathbb{C} \setminus \mathbb{R}$ fixed. Then, denoting $\tilde{Q}(z) = Q(z)R(z)$ (and $\tilde{m} = \deg(\tilde{Q}(z)) = m + 2$), the sequence $\{c_k\}_{k=0}^{\infty}$ admits the representation (7).*

Proof. The proof, again, goes by induction. We really need to prove only that the representation (7) holds for c_{m+2}.

We start from $z^m = Q(z)R_m(z) + P_m(z)$. Denoting $R(z) = (z - z_0)(z - \bar{z}_0) = z^2 + \alpha z + \beta$, we have $z^m(z^2 + \alpha z + \beta) = Q(z)R_m(z)R(z) + (z^2 + \alpha z + \beta)P_m(z)$, from which follows

$$z^{m+2} = \tilde{Q}(z)R_m(z) + z^2 P_m(z) + \alpha z\left(P_m(z) - z^m\right) + \beta\left(P_m(z) - z^m\right),$$

(8') i.e., $\tilde{P}_{m+2}(z) = z^2 P_m(z) + (\alpha z + \beta)\left(P_m(z) - z^m\right).$

Now, with a and b arbitrary consider the expression

$$
\begin{aligned}
(z^2 + az + b)P_m(z) &= (z^2 + az + b)\left(a_{m-1}^{(m)} z^{m-1} + \cdots + a_1^{(m)} z + a_0^{(m)}\right) \\
&= a_{m-1}^{(m)} z^{m+1} + (a a_{m-1}^{(m)} + a_{m-2}^{(m)}) z^m \\
&\quad + \left(b a_{m-1}^{(m)} + a a_{m-2}^{(m)} + a_{m-3}^{(m)}\right) z^{m-1} + \cdots
\end{aligned}
$$

It follows that $(z^2 + az + b)P_m(z) - a_{m-1}^{(m)} z^{m+1} - \left(a a_{m-1}^{(m)} + a_{m-2}^{(m)}\right) z^m$ is a polynomial of degree less or equal to $m-1$ which at the zero multiset of Q coincides with $(z^2 + az + b)z^m - a_{m-1}^{(m)} z^{m+1} - \left[a a_{m-1}^{(m)} + a_{m-2}^{(m)}\right] z^m = z^{m+2} + \left[a - a_{m-1}^{(m)}\right] z^{m+1} + \left[b - a a_{m-1}^{(m)} - a_{m-2}^{(m)}\right] z^m$, which is just z^{m+2} if we take $a = a_{m-1}^{(m)}$ and $b = a a_{m-1}^{(m)} + a_{m-2}^{(m)} = \left(a_{m-1}^{(m)}\right)^2 + a_{m-2}^{(m)}$. Thus

(9') $$P_{m+2}(z) = \left[z^2 + az + b\right] P_m(z) - az^{m+1} - bz^m,$$

where $a = a_{m-1}^{(m)}$ and $b = a a_{m-1}^{(m)} + a_{m-2}^{(m)} = \left(a_{m-1}^{(m)}\right)^2 + a_{m-2}^{(m)}.$

Now, by (8'), (9') and (9) we have

(11)
$$
\begin{aligned}
\tilde{P}_{m+2}(z) &= P_{m+2}(z) + (\alpha - a)\left[P_{m+1}(z) + az^m - aP_m(z)\right] - bP_m(z) \\
&\quad + az^{m+1} + bz^m - \alpha z^{m+1} + \beta\left[P_m(z) - z^m\right] \\
&= \tilde{P}_{m+2}(z) = P_{m+2}(z) + (a - \alpha)\left[z^{m+1} - P_{m+1}(z)\right] \\
&\quad + (a(\alpha - a) + b - \beta)\left[z^m - P_m(z)\right].
\end{aligned}
$$

In this way,

$$
\int_{-\infty}^{\infty} \left(t^{M+2} - \tilde{P}_{M+2}(t)\right) d\rho(t)
$$

$$
= c_{M+2} - \sum_{i=0}^{M-1} a_i^{(M+2)} c_i - (a - \alpha)\left[c_{M+1} - \sum_{i=0}^{M-1} a_i^{(M+1)} c_i\right]
$$

$$
+ (a(\alpha - a) - b + \beta)\left[c_M - \sum_{i=0}^{M-1} a_i^{(M)} c_i\right],
$$

whence follows (thanks to (11)) the representation (7) for c_{M+2}. □

The next two theorems states (as can be guessed from the above theorems) that it is also possible, in some cases, to remove some interpolating points without affecting the representation of $\{c_k\}_{k=0}^{\infty}$. Even non-real roots of the polynomial Q can be removed (sometimes) but this has to be done by conjugate pairs. We note that the remotion of roots of Q has to be done with care as it can compromise the convergence of the integrals in the representation (3) of the sequence $\{c_k\}_{k=0}^{\infty}$.

Theorem 4. *Let $\{c_k\}_{k=0}^{\infty}$ be a real sequence admitting the representation (3). Suppose that there exists some polynomial $\check{Q}(z) = \sum_{i=0}^{\check{m}} \check{\gamma}_i z^i$ such that $Q(z) = \check{Q}(z)(z - z_0)$ (with*

$z_0 \in \mathbb{R}$), the integral $\int_{-\infty}^{\infty} \check{Q}(t)\, d\rho(t)$ is convergent and

$$(12) \qquad c_{\check{m}} = \frac{1}{\check{\gamma}_{\check{m}}} \int_{-\infty}^{\infty} \check{Q}(t)\, d\rho(t) + \sum_{i=0}^{\check{m}-1} \check{a}_i^{(\check{m})} c_i,$$

where $\check{P}_{\check{m}}(z) = \sum_{i=0}^{\check{m}-1} \check{a}_i^{(\check{m})} z^i$ is the interpolating polynomial of $z^{\check{m}}$ at the zero multiset of $\check{Q}(z)$. Then $\{c_k\}_{k=0}^{\infty}$ admits the representation

$$(13) \qquad c_k = \int_{-\infty}^{\infty} \left(t^k - \check{P}_k(t)\right) d\rho(t) + \sum_{i=0}^{\check{m}-1} \check{a}_i^{(k)} c_i \quad (k = 0, 1, \ldots),$$

where $\check{P}_k(z) = \sum_{i=0}^{\check{m}-1} \check{a}_i^{(k)} z^i$ is the interpolating polynomial of z^k at the zero multiset of $\check{Q}$.

Proof. It suffices to show that representation (13) holds for $c_{\check{m}} = c_{m-1}$. But since

$$\check{P}_{m-1}(z) = \sum_{i=0}^{m-2} \frac{-\check{\gamma}_i}{\check{\gamma}_{m-1}} z^i = -\frac{1}{\check{\gamma}_{m-1}} \left[\check{Q}(z) - \check{\gamma}_{m-1} z^{m-1} \right],$$

then representation (13) for c_{m-1} is just (12). □

Clearly, in order to remove a pair of conjugate non-real roots of Q without affecting the representation we just need that the representation (13) holds for c_{m-2}. Thus we have

Theorem 5. *With the same notation used in theorem 4, if $\{c_k\}_{k=0}^{\infty}$ is a real sequence admitting representation (3) and $\check{Q}(z) = \frac{Q(z)}{(z-z_0)(z-\overline{z}_0)}$ is a polynomial satisfying (12) (the convergence of the integral is required) then $\{c_k\}_{k=0}^{\infty}$ admits the representation (13).*

The question on the uniqueness of the function ρ (the density of the representation) is subtler, as we can conclude from the following example.

Example 1. Let $\{d_k\}_{k=0}^{\infty}$ be a Hamburger moment sequence such that the corresponding moment problem is indeterminate. Let σ be an N-extremal solution to the problem (see [1] or [13]). Then σ corresponds to the spectral measure of some selfadjoint extension of the symmetric operator associated to the moment problem and the set $\{1, x, x^2, \ldots\}$ is dense in L_{σ}^2. As the deficiency index of the operator is $(1,1)$ (details can be consulted in [1], [12] or [13]), the set $\{(x+i), (x+i)x, (x+i)x^2, \ldots\}$ is not dense in L_{σ}^2.

Let $\{c_k\}_{k=0}^{\infty}$ be the sequence $c_k = \int_{-\infty}^{\infty} x^k \frac{1}{x^2+1} d\sigma(x) = \int_{-\infty}^{\infty} t^k d\nu(t)$, where $d\nu(t) = \frac{d\sigma(t)}{t^2+1}$. Then we have

(1) $\{c_k\}_{k=0}^{\infty}$ is a Hamburger moment sequence.

(2) The set $\{1, x, x^2, \ldots\}$ is dense in L_{ν}^2. Indeed, denoting $\psi_{[a,b]}$ (with $a < b$) the characteristic function

$$\psi_{[a,b]}(t) = \begin{cases} 1 & t \in [a,b] \\ 0 & t \notin [a,b], \end{cases}$$

there exists a sequence of polynomials $\{p_n(t)\}_{n=0}^{\infty}$ converging in L_{σ}^2 to $\psi_{[a,b]}(t)$, from which follows that

$$\int_{-\infty}^{\infty} |p_n(t) - \psi_{[a,b]}(t)|^2 d\nu(t) \leq \int_{-\infty}^{\infty} |p_n(t) - \psi_{[a,b]}(t)|^2 d\sigma(t) \overset{n \to \infty}{\to} 0.$$

(3) The set $\{(x+i), (x+i)x, \ldots\}$ is also dense in L_{ν}^2 since

$$\int_{-\infty}^{\infty} |(t+i)q_n(t) - \psi_{[a,b]}(t)|^2 d\nu(t) = \int_{-\infty}^{\infty} \left| q_n(t) - \frac{1}{t+i}\psi_{[a,b]}(t) \right|^2 d\sigma(t) \overset{n \to \infty}{\to} 0,$$

where $\{q_n(t)\}_{n=0}^{\infty}$ is a polynomial sequence converging to $\frac{1}{t+i}\psi_{[a,b]}(t) \in \mathrm{L}_{\sigma}^2$.

Now, these conditions implies that ν is a solution to a moment problem (for the sequence $\{c_k\}_{k=0}^\infty$) whose associated operator has dense domain and deficiency index $(0,0)$. Hence, the moment problem for the sequence $\{c_k\}_{k=0}^\infty$ is determinate.

Noticing that

$$c_k + c_{k+2} = \int_{-\infty}^{\infty} \frac{t^k + t^{k+2}}{t^2 + 1} d\sigma(t) = d_k,$$

we conclude that any real sequence that can be mapped, by a difference operator, to the sequence $\{c_k\}_{k=0}^\infty$ (whose associated moment problem is determinate) can also be mapped to the sequence $\{d_k\}_{k=0}^\infty$ (whose associated moment problem is indeterminate).

The uniqueness of the function ρ (and the question on the *constructability of solutions*) will be treated in a next paper.

Theorem 1 shows that definitizable sequences $\{c_k\}_{k=0}^\infty$ have representation (3). We proceed now to show that the sequences having that representation are precisely the definitizable ones.

Theorem 6. *Let $\{c_k\}_{k=0}^\infty$ be a real sequence having representation (3), where $P_k(z) = \sum_{i=0}^{m-1} a_i^{(k)} z^i$ is (for each $k = 0, 1, \ldots$) the interpolating polynomial of the function z^k at zero multiset of some polynomial Q of degree m and $\rho(t)$ is a piecewise monotone function defined on $\mathbb{R}$ except the real roots of Q. Then $\{c_k\}_{k=0}^\infty$ is definitizable.*

Proof. As ρ is piecewise monotone there exists real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_\mu$ such that the function ρ is monotone in each of the intervals $(-\infty, \alpha_1)$, $(\alpha_1, \alpha_2), \ldots$, $(\alpha_\mu, +\infty)$. We take now any polynomial Q_ρ such that the product $Q(t)Q_\rho(t)$ is non-negative in those intervals in which ρ is non-decreasing and negative in the other case (we impose no conditions at the boundary of each interval). Let $\tilde{Q}(t)$ denote the polynomial $\tilde{Q}(t) = Q(t)Q_\rho(t)$.

By theorem 2, representation (7) holds and we have

$$c_{\tilde{m}} = \int_{-\infty}^{\infty} \left(t^{\tilde{m}} - \tilde{P}_{\tilde{m}}(t)\right) d\rho(t) + \sum_{i=0}^{\tilde{m}-1} \tilde{a}_i^{(\tilde{m})} c_i = \frac{1}{\tilde{\gamma}_{\tilde{m}}} \int_{-\infty}^{\infty} \tilde{Q}(t) \, d\rho(t) + \sum_{i=0}^{\tilde{m}-1} \tilde{a}_i^{(\tilde{m})} c_i,$$

where we have used the notation of theorem 2 (the symbol $\tilde{}$ refer to polynomial $\tilde{Q}$).

One important consequence of representation (7) is that the integral $\int_{-\infty}^{\infty} t^k \tilde{Q}(t) \, d\rho(t)$ converges for each $k = 0, 1, \ldots$ Thus the sequence $\{\tilde{c}_k\}_{k=0}^\infty$, defined by

$$\tilde{c}_k = \int_{-\infty}^{\infty} t^k \tilde{Q}(t) \, d\rho(t) \quad (k = 0, 1, \ldots)$$

is a Hamburger moment sequence satisfying

$$\tilde{c}_k = \sum_{i=0}^{\tilde{m}} \tilde{\gamma}_i c_{i+k} \quad \forall \, k = 0, 1, 2, \ldots,$$

from which follows that $\{c_k\}_{k=0}^\infty$ is definitizable $\qquad\qquad\square$

4. The Hausdorff case

Hausdorff-definitizable sequences were introduced in [14]. In order to obtain an integro-polynomial representation of such sequences, it was imposed there an additional condition on the growth of the sequence. We start this section by proving that the restriction on the growth of the sequence is, actually, a consequence of its definitizability.

Theorem 7. *Let $\{c_k\}_{k=0}^\infty$ be a Hausdorff-definitizable sequence with definitizing polynomial $\mathfrak{Q}(z) = \sum_{i=i}^{n} \gamma_i z^i$, $\tilde{c}_k = \sum_{i=0}^{n} \gamma_i c_{i+k}$ and $\tilde{c}_k = \int_a^b t^k d\sigma(t)$ with $[a, b] \subset [-1, 1]$. Then*

$$|c_{n+k}| \leq \Lambda \cdot \omega^k \quad \forall \, k = 0, 1, \ldots,$$

where $\Lambda = \frac{\tilde{c}_0 + \hat{\gamma} \sum_{i=0}^{n-1} |c_i|}{|\gamma_n|}$ *and* $\omega = 1 + \frac{\hat{\gamma}}{|\gamma_n|}$, *being* $\hat{\gamma} = \max_{i=0,1,\dots,n-1} |\gamma_i|$.

Proof. First note that since $[a,b] \subset [-1,1]$ then we have

$$(14) \qquad |\tilde{c}_k| \leq \int_a^b |t^k| d\sigma(t) \leq \int_a^b d\sigma(t) = \tilde{c}_0 \quad \forall\, k = 0,1,\dots$$

The proof goes by (generalized) induction.

From $\tilde{c}_0 = \sum_{i=0}^n \gamma_i c_{i+0}$ we get $c_n = \frac{1}{\gamma_n}\big[\tilde{c}_0 - \sum_{i=0}^{n-1} \gamma_i c_i\big]$. Hence,

$$|c_n| \leq \frac{1}{|\gamma_n|}\Big[\tilde{c}_0 + \hat{\gamma} \sum_{i=0}^{n-1} |c_i|\Big] = \Lambda,$$

which proves that the thesis holds for $k = 0$.

Suppose now the thesis holds for $k = 0,1,\dots,l$. To prove that the thesis holds for $k = l+1$ we start noting that from $d_{l+1} = \sum_{i=0}^n \gamma_i c_{i+l+1}$ we get

$$c_{n+l+1} = \frac{1}{\gamma_n}\left[d_{l+1} - \sum_{i=0}^{n-1} \gamma_i c_{i+l+1}\right].$$

Hence

$$|c_{n+l+1}| \leq \frac{1}{|\gamma_n|}\Big[|\tilde{c}_{l+1}| + \hat{\gamma} \sum_{i=0}^{n-1} |c_{i+l+1}|\Big] \leq \frac{1}{|\gamma_n|}\Big[\tilde{c}_0 + \hat{\gamma} \sum_{j=l+1}^{n+l} |c_j|\Big]$$

$$\leq \frac{1}{|\gamma_n|}\Big[\tilde{c}_0 + \hat{\gamma} \sum_{j=0}^{n+l} |c_j|\Big] = \frac{1}{|\gamma_n|}\Big[\tilde{c}_0 + \hat{\gamma} \sum_{j=0}^{n-1} |c_j|\Big] + \frac{\hat{\gamma}}{|\gamma_n|} \sum_{i=0}^{l} |c_{n+i}|.$$

Thus, by the inductive hypothesis we have

$$|c_{n+l+1}| \leq \Lambda + \frac{\hat{\gamma}}{|\gamma_n|} \sum_{i=0}^{l} \Lambda \omega^i = \Lambda \omega^{l+1},$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is well known that any Hausdorff moment problem is *determinate* (unless it has no solution). Thus it is natural to ask: Is it possible to have a real sequence $\{c_k\}_{k=0}^\infty$ which can be mapped simultaneously (by difference operators) to a Hausdorff moment sequence (hence determinate) and to an indeterminate Hamburger (or Stieltjes) moment sequence? This question, which is answered in the negative in the next theorem, acquire special interest from a theorem of Boas (see [4] or [15], chapter 3, section 14 or [12], sections 11–13), according to which any real sequence is a Stieltjes moment sequence if we relax the condition on the monotonicity of the distribution function σ to require only it to be of bounded variation. This result is valid also for the Hamburger moment problem but not in the Hausdorff case (the proofs can be found in the references just given). In our case, the functions ρ are not non-decreasing but they can change its monotonicity only a finite number of times and this fact prevents the trivial possibility we would get from applying Boas's result.

Theorem 8. *Let* $\{c_k\}_{k=0}^\infty$ *be a definitizable sequence with definitizing polynomial* $\mathfrak{Q}(z) = \sum_{i=0}^n \gamma_i z^i$. *If* $d\rho(t)$ *has not compact support then there can not exist constants* Λ *and* ω *such that* $|c_k| \leq \Lambda \omega^k \quad \forall\, k = n, n+1,\dots$

Proof. Suppose there exist Λ and ω (real) constants such that $|c_k| \leq \Lambda \omega^k \quad \forall\, k = n, n+1, \dots$ Then, from $\tilde{c}_k = \sum_{i=0}^n \gamma_i c_{i+k} \quad (k = 0,1,\dots)$ we get

$$|\tilde{c}_k| \leq \sum_{i=0}^n |\gamma_i|\, |c_{i+k}| \leq \sum_{i=0}^n \hat{\gamma} \Lambda \omega^{i+k} = \hat{\gamma} \Lambda \omega^k \frac{1 - \omega^{n+1}}{1 - \omega},$$

where $\hat{\gamma} = \max_{i=0,1,\ldots,n} |\gamma_i|$ (as before). Now, denoting $\alpha = \hat{\gamma}\Lambda\dfrac{1 - \omega^{n+1}}{1 - \omega}$ we have

$$|\tilde{c}_k| \leq \alpha\omega^k \quad \forall\, k = 0, 1, \ldots \tag{15}$$

In the other hand, since $|\tilde{c}_k| = \left| \int_{-\infty}^{\infty} t^k d\sigma(t) \right|$ we have

$$|\tilde{c}_{2k}| = \int_{-\infty}^{-\lambda} t^{2k} d\sigma(t) + \int_{\lambda}^{\infty} t^{2k} d\sigma(t) + \int_{-\lambda}^{\lambda} t^{2k} d\sigma(t)$$
$$\geq \lambda^{2k}\mathrm{Var}_{(-\infty,-\lambda)}\sigma + \lambda^{2k}\mathrm{Var}_{(\lambda,\infty)}\sigma,$$

being λ any non-negative (real) number. But according to (15) this is not possible, unless $d\sigma(t)$ is supported in $[-\omega, \omega]$. $\qquad\square$

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