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POLARIZATION FORMULA FOR (p,q)-POLYNOMIALS ON A COMPLEX NORMED SPACE

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Abstract. The aim of this paper to give some analogues of polarization formulas and the polarization inequality for (p,q)-polynomials between complex normed spaces. Obtained results are useful for investigation of real-differentiable mappings on complex spaces.

Introduction

Let X and Y be complex linear spaces and $X^n = X \times \cdots \times X$ be the Cartesian power of X. An *n-homogeneous polynomial* P_n from X to Y may be defined as a restriction to the diagonal of an n-linear map $B_n \colon X^n \to Y$. That is,

$$P_n(x) = B_n(x, \ldots, x).$$

It is well known that there is a unique *symmetric* n-linear map B_n which generates P_n. The map B_n can be recovered from P_n using the following *polarization formula*:

$$B_n(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_n P_n\Big(\sum_{j=1}^{n} \varepsilon_j x_j \Big).$$

The polarization formula had been known since 1931 [3] and it was rediscovered later in various forms by many authors (e.g. [6, 7]).

The polarization formula is an important tool in the theory of polynomials on normed spaces. Using it we can get the *polarization inequality.* If X and Y are normed spaces, then there is a constant $c(n,X)$ such that for all n-homogeneous polynomials P_n

$$\|P_n\| \le \|B_n\| \le c(n,X)\|P_n\|$$

(see e.g. [5] for details). The minimal constant satisfying the inequality in general can be estimated as $1 \le c(n,X) \le \frac{n^n}{n!}$. More precise estimations depends on geometrical properties of X (see [10]). For example if $X = \ell_1$, then $c(n,X)$ can not be less than $\frac{n^n}{n!}$ but if $X = \ell_2$, then we can take $c(n,X) = 1$.

The purpose of the paper to consider the case of so-called (p,q) linear maps (which are linear with respect to the first p components and anti-linear with respect to the last q components) and corresponding (p,q)-polynomials. We prove various versions of polarization formulas and a polarization inequality for this case. The polarization formulas for (p,q)-polynomials should be applicable for real-differentiable mappings on complex Banach spaces because the Taylor formula expansion of a real-differentiable mapping is a linear span of (p,q)-polynomials.

A detail information about polynomials can be found in [5, 9]. (p,q)-polynomials have been considered in [8, 9, 11].

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1. Basic Concepts

We denote by S_n the group of all permutations on $\{1, \ldots, n\}$.

Definition 1.1. *The mapping $B_{p,q}(x_1, \ldots, x_p; x_{p+1}, \ldots, x_{p+q})$, $B_{p,q} : X^{p+q} \to Y$ is called a (p,q)-linear symmetric mapping if it has the following properties:*
$1°$. $\forall\, i \in \{1, \ldots, p+q\} \quad \forall\, x_i', x_i'' \in X$

$$B_{p,q}(x_1, \ldots, x_{i-1}, x_i' + x_i'', x_{i+1}, \ldots, x_{p+q}) \\ = B_{p,q}(x_1, \ldots, x_{i-1}, x_i', x_{i+1} \ldots, x_{p+q}) + B_{p,q}(x_1, \ldots, x_{i-1}, x_i'', x_{i+1} \ldots, x_{p+q}).$$

$2°$. $\forall\, i \in \{1, \ldots, p\} \quad \forall\, \lambda \in \mathbb{C}$

$$B_{p,q}(x_1, \ldots, \lambda x_i, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}) = \lambda B_{p,q}(x_1, \ldots, x_i, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}).$$

$3°$. $\forall\, i \in \{p+1, \ldots, p+q\} \quad \forall\, \lambda \in \mathbb{C}$

$$B_{p,q}(x_1, \ldots, x_p; x_{p+1}, \ldots, \lambda x_i, \ldots, x_{p+q}) = \overline{\lambda} B_{p,q}(x_1, \ldots, x_p; x_{p+1}, \ldots, x_i, \ldots, x_{p+q}).$$

$4°$. $\forall \sigma \in S_p$

$$B_{p,q}(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}; x_{p+1}, \ldots, x_{p+q}) = B_{p,q}(x_1, x_2, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}).$$

$5°$. $\forall \sigma \in S_q$

$$B_{p,q}(x_1, \ldots, x_p; x_{p+\sigma(1)}, x_{p+\sigma(2)}, \ldots, x_{p+\sigma(q)}) = B_{p,q}(x_1, \ldots, x_p; x_{p+1}, x_{p+2}, \ldots, x_{p+q}).$$

In other words, $B_{p,q}$ is linear and symmetric with respect to $x_1, \ldots, x_p$, and anti-linear and symmetric with respect to $x_{p+1}, \ldots, x_{p+q}$.

Definition 1.2. *We define a (p,q)-polynomial as the restriction of a (p,q)-linear symmetric mapping $B_{p,q}$ onto the diagonal*

$$P_{p,q}(x) = B_{p,q}(\underbrace{x, \ldots, x}_{p}; \underbrace{x, \ldots, x}_{q}).$$

Note that any $(p,0)$-polynomial is just a p-homogeneous polynomial and any $(p,0)$-linear symmetric mapping is just a p-linear symmetric mapping.

The aim of this work is to find a method of recovering a (p,q)-linear symmetric mapping from its restriction onto the diagonal. To do it we use the techniques of (classical) Rademacher functions (see [10]) and generalized Rademacher functions (see [1] and [2]). Note that using this approach in [4] were proved analogues of the polarization formula for nonhomogeneous polynomials and analytic mappings.

Definition 1.3. *The i-th Rademacher function $r_i(t)$ is defined on $[0,1]$ by $r_i(t) = \operatorname{sign} \sin 2^i \pi t$, $i \in \mathbb{N}$.*

Rademacher functions have the following properties:
$1°$. $(r_i(t))^{2n} = 1$;
$2°$. $(r_i(t))^{2n+1} = r_i(t)$;
$3°$. Let $m_1, \ldots, m_n \in \mathbb{Z}$. Then

$$\int_0^1 (r_1(t))^{m_1} (r_2(t))^{m_2} \ldots (r_n(t))^{m_n}\, dt = \begin{cases} 1, & \text{if all } m_1, \ldots, m_n \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.4. (see [2]) *For a given integer $n \geq 2$ let $\alpha_j = e^{2\pi i \frac{j-1}{n}}$ be the complex nth roots of the unity, $j = 1, \ldots, n$ and $I_j = \left(\frac{j-1}{n}, \frac{j}{n}\right)$. The generalized Rademacher function $S_1^{[n]} : [0,1] \to \mathbb{C}$ is defined by setting $S_1^{[n]}(t) = \alpha_j$ for $t \in I_j$ where $1 \leq j \leq n$. We set $S_1^{[n]}(t) = 1$ for all endpoints t.*

The function $S_1^{[n]}(t)$ has the following properties:
1°. Since $\forall t \in [0,1]: \ |S_1^{[n]}(t)| = 1$,

$$\overline{S_1^{[n]}(t)} = \left(S_1^{[n]}(t)\right)^{-1};$$

2°.

$$\int_0^1 \left(S_1^{[n]}(t)\right)^m dt = \begin{cases} 1, & \text{if } m = 0 \mod n\ , \\ 0, & \text{otherwise.} \end{cases}$$

We denote

$$\begin{aligned} &B_{p,q}((x_1)^{n_1}, \dots, (x_s)^{n_s}; (x_{s+1})^{n_{s+1}}, \dots, (x_k)^{n_k}) \\ &\quad = B_{p,q}(\underbrace{x_1, \dots, x_1}_{n_1}, \dots, \underbrace{x_s, \dots, x_s}_{n_s}; \underbrace{x_{s+1}, \dots, x_{s+1}}_{n_{s+1}}, \dots, \underbrace{x_k, \dots, x_k}_{n_k}). \end{aligned}$$

The case $n_j = 0$ for some j means that x_j does not belong to the list of arguments at the position.

Remark 1.1. Let $c_1, \dots, c_{p+q} \in \mathbb{C}$. It is easy to see that

$$\begin{aligned} &P_{p,q}(c_1x_1 + \dots + c_{p+q}x_{p+q}) = B_{p,q}\left((c_1x_1 + \dots + c_{p+q}x_{p+q})^{p+q}\right) \\ &= \sum_{\substack{k_1 \ge 0, \dots, k_{p+q} \ge 0 \\ k_1 + \dots + k_{p+q} = p}} \frac{p!}{k_1! \dots k_{p+q}!} \sum_{\substack{l_1 \ge 0, \dots, l_{p+q} \ge 0 \\ l_1 + \dots + l_{p+q} = q}} \frac{q!}{l_1! \dots l_{p+q}!} \\ &\times B_{p,q}\left((c_1x_1)^{k_1}, \dots, (c_{p+q}x_{p+q})^{k_{p+q}}; (c_1x_1)^{l_1}, \dots, (c_{p+q}x_{p+q})^{l_{p+q}}\right) \\ &= \sum_{\substack{k_1 \ge 0, \dots, k_{p+q} \ge 0 \\ k_1 + \dots + k_{p+q} = p}} c_1^{k_1} c_2^{k_2} \dots c_{p+q}^{k_{p+q}} \frac{p!}{k_1! \dots k_{p+q}!} \sum_{\substack{l_1 \ge 0, \dots, l_{p+q} \ge 0 \\ l_1 + \dots + l_{p+q} = q}} \overline{c_1^{l_1} c_2^{l_2} \dots c_{p+q}^{l_{p+q}}} \cdot \frac{q!}{l_1! \dots l_{p+q}!} \\ &\times B_{p,q}\left((x_1)^{k_1}, \dots, (x_{p+q})^{k_{p+q}}; (x_1)^{l_1}, \dots, (x_{p+q})^{l_{p+q}}\right). \end{aligned}$$

2. Polarization formula for (p,q)-linear mappings

Theorem 2.1. *Let $B_{p,q}(x_1, \dots, x_{p+q})$ be a (p,q)-linear symmetric mapping and $P_{p,q}(x)$ be the corresponding (p,q)-polynomial. Then*

$$\begin{aligned} &B_{p,q}(x_1, \dots, x_{p+q}) \\ &= \frac{1}{p!q!} \int_0^1 \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p} r_1(\theta) r_2(\theta) \dots r_{p+q}(\theta) \\ &\times P_{p,q}\Big(S_1^{[2q+1]}(t)\left(r_1(\theta)x_1 + \dots + r_p(\theta)x_p\right) \\ &+ \left(r_{p+1}(\theta)x_{p+1} + \dots + r_{p+q}(\theta)x_{p+q}\right)\Big)\, dt\, d\theta. \end{aligned} \tag{2.1}$$

Proof. Let us denote the right-hand side of (2.1) by A. From remark 1.1 we have

$$\begin{aligned} A = &\sum_{\substack{k_1 \ge 0, \dots, k_{p+q} \ge 0 \\ k_1 + \dots + k_{p+q} = p}} \frac{1}{k_1! \dots k_{p+q}!} \sum_{\substack{l_1 \ge 0, \dots, l_{p+q} \ge 0 \\ l_1 + \dots + l_{p+q} = q}} \frac{1}{l_1! \dots l_{p+q}!} \\ &\times \int_0^1 \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p} r_1(\theta) r_2(\theta) \dots r_{p+q}(\theta) \\ &\times \left(S_1^{[2q+1]}(t)\right)^{k_1 + \dots + k_p} \left(\overline{S_1^{[2q+1]}(t)}\right)^{l_1 + \dots + l_p} \end{aligned}$$

$$\times (r_1(\theta))^{k_1+l_1}(r_2(\theta))^{k_2+l_2}\dots(r_{p+q}(\theta))^{k_{p+q}+l_{p+q}}$$
$$\times B_{p,q}\left((x_1)^{k_1},\dots,(x_{p+q})^{k_{p+q}};(x_1)^{l_1},\dots,(x_{p+q})^{l_{p+q}}\right)\,dt\,d\theta$$
$$= \sum_{\substack{k_1\geq 0,\dots,k_{p+q}\geq 0\\ k_1+\dots+k_{p+q}=p}} \frac{1}{k_1!\dots k_{p+q}!} \sum_{\substack{l_1\geq 0,\dots,l_{p+q}\geq 0\\ l_1+\dots+l_{p+q}=q}} \frac{1}{l_1!\dots l_{p+q}!}$$
$$\times B_{p,q}\left((x_1)^{k_1},\dots,(x_{p+q})^{k_{p+q}};(x_1)^{l_1},\dots,(x_{p+q})^{l_{p+q}}\right)$$
$$\times \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p+k_1+\dots+k_p-l_1-\dots-l_p}\,dt$$
$$\times \int_0^1 (r_1(\theta))^{1+k_1+l_1}(r_2(\theta))^{1+k_2+l_2}\dots(r_{p+q}(\theta))^{1+k_{p+q}+l_{p+q}}\,d\theta.$$

Let us consider the integral

$$\int_0^1 (r_1(\theta))^{1+k_1+l_1}(r_2(\theta))^{1+k_2+l_2}\dots(r_{p+q}(\theta))^{1+k_{p+q}+l_{p+q}}\,d\theta.$$

If there exists a natural number i such that $k_i+l_i=0$, then there is a multiplier $(r_i(\theta))^1$ in the above expression and therefore the property 3° of Rademacher functions implies that the integral is equal to zero.

So for non-zero elements of the sum $k_i+l_i\geq 1,\quad i=1,\dots,p+q$. Thus $\sum\limits_{i=1}^{p+q}(k_i+l_i)\geq p+q$ and the equality holds only if $k_i+l_i=1$, $i=1,\dots,p+q$. But we know that $k_1+\dots+k_{p+q}=p$ and $l_1+\dots+l_{p+q}=q$, that is why $\sum\limits_{i=1}^{p+q}(k_i+l_i)=p+q$. Hence, for non-zero elements of the sum $k_i+l_i=1$, $i=1,\dots,p+q$ and

$$\int_0^1 (r_1(\theta))^{1+k_1+l_1}(r_2(\theta))^{1+k_2+l_2}\dots(r_{p+q}(\theta))^{1+k_{p+q}+l_{p+q}}\,d\theta$$
$$= \int_0^1 (r_1(\theta))^2(r_2(\theta))^2\dots(r_{p+q}(\theta))^2\,d\theta = 1.$$

We can simplify the representation of A taking into account that for non-zero elements $l_i=1-k_i$, $i=1,\dots,p+q$.

$$A = \sum_{\substack{0\leq k_1\leq 1,\dots,0\leq k_{p+q}\leq 1\\ k_1+\dots+k_{p+q}=p}} \frac{1}{k_1!\dots k_{p+q}!}\times\frac{1}{(1-k_1)!\dots(1-k_{p+q})!}$$
$$\times B_{p,q}\left((x_1)^{k_1},\dots,(x_{p+q})^{k_{p+q}};(x_1)^{1-k_1},\dots,(x_{p+q})^{1-k_{p+q}}\right) \tag{2.2}$$
$$\times \int_0^1 \left(S_1^{[2q+1]}(t)\right)^{2q+1-p+k_1+\dots+k_p-(1-k_1)-\dots-(1-k_p)}\,dt.$$

Let us find values of k_i such that the integral is not equal to zero.

$$\begin{aligned} &2q+1-p+k_1+\dots+k_p-(1-k_1)-\dots-(1-k_p)\\ &\quad=2q+1-p+k_1+\dots+k_p-p+k_1+\dots+k_p\\ &\quad=2q+1-2(p-k_1-\dots-k_p)=2q+1-2(k_{p+1}+\dots+k_{p+q}).\end{aligned}$$

By the property 3° of generalized Rademacher functions the integral in (2.2) is not equal to zero if and only if $2q+1$ is a divisor of $2q+1-2(k_{p+1}+\dots+k_{p+q})$. But since $0\leq k_i\leq 1$, it will be only if $2q+1-2(k_{p+1}+\cdots+k_{p+q}) = 2q+1$, that is, $k_{p+1}=\cdots=k_{p+q}=0$. Hence $p = k_1+\cdots+k_p+k_{p+1}+\cdots+k_{p+q} = k_1+\cdots+k_p$, so $k_1=\cdots=k_p=1$.

Finally, we can rewrite A as

$$A = B_{p,q}\Big((x_1)^1,\ldots,(x_p)^1,(x_{p+1})^0,\ldots,(x_{p+q})^0;(x_1)^0,\ldots,(x_p)^0,(x_{p+1})^1,\ldots,(x_{p+q})^1\Big)$$

$$\times \int_0^1 \Big(S_1^{[2q+1]}(t)\Big)^{2q+1}\,dt = B_{p,q}(x_1,\ldots,x_p;x_{p+1},\ldots,x_{p+q}).$$

□

3. Polarization inequality

We can get an another form of the polarization formula from formula (2.1).

Theorem 3.1.

$$\begin{aligned} &B_{p,q}(x_1,\ldots,x_{p+q})\\ &= \frac{1}{2^{p+q}p!q!}\sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm1}\varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_{p+q}\sum_{i=1}^{2q+1}\frac{1}{2q+1}\alpha_i^{2q+1-p}\\ &\quad\times P_{p,q}\Big(\alpha_i\varepsilon_1x_1+\alpha_i\varepsilon_2x_2+\cdots+\alpha_i\varepsilon_px_p+\varepsilon_{p+1}x_{p+1}+\cdots+\varepsilon_{p+q}x_{p+q}\Big). \end{aligned} \tag{3.1}$$

Proof.

$$\begin{aligned} B_{p,q}(x_1,\ldots,x_{p+q}) &= \frac{1}{p!q!}\int_0^1\int_0^1\Big(S_1^{[2q+1]}(t)\Big)^{2q+1-p} r_1(\theta)r_2(\theta)\ldots r_{p+q}(\theta)\\ &\quad\times P_{p,q}\Big(S_1^{[2q+1]}(t)\,(r_1(\theta)x_1+\cdots+r_p(\theta)x_p)\\ &\quad\quad + (r_{p+1}(\theta)x_{p+1}+\cdots+r_{p+q}(\theta)x_{p+q})\Big)\,dt\,d\theta \end{aligned}$$

$$\begin{aligned} &= \frac{1}{p!q!}\int_0^1 r_1(\theta)r_2(\theta)r_3(\theta)\ldots r_{p+q}(\theta)\int_0^1\Big(S_1^{[2q+1]}(t)\Big)^{2q+1-p}\\ &\quad\times P_{p,q}\Big(S_1^{[2q+1]}(t)r_1(\theta)x_1+S_1^{[2q+1]}(t)r_2(\theta)x_2+\cdots\\ &\quad\quad + S_1^{[2q+1]}(t)r_p(\theta)x_p + r_{p+1}(\theta)x_{p+1}+\cdots+r_{p+q}(\theta)x_{p+q}\Big)\,dt\,d\theta \end{aligned}$$

$$\begin{aligned} &= \frac{1}{p!q!}\sum_{\varepsilon_1=\pm1}\int_0^{1/2}\varepsilon_1 r_2(\theta)r_3(\theta)\ldots r_{p+q}(\theta)\int_0^1\Big(S_1^{[2q+1]}(t)\Big)^{2q+1-p}\\ &\quad\times P_{p,q}\Big(S_1^{[2q+1]}(t)\varepsilon_1x_1+S_1^{[2q+1]}(t)r_2(\theta)x_2+\cdots\\ &\quad\quad + S_1^{[2q+1]}(t)r_p(\theta)x_p + r_{p+1}(\theta)x_{p+1}+\cdots+r_{p+q}(\theta)x_{p+q}\Big)\,dt\,d\theta \end{aligned}$$

$$\begin{aligned} &= \frac{1}{p!q!}\sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm1}\int_0^{1/2^{p+q}}\varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_{p+q}\int_0^1\Big(S_1^{[2q+1]}(t)\Big)^{2q+1-p}\\ &\quad\times P_{p,q}\Big(S_1^{[2q+1]}(t)\varepsilon_1x_1+S_1^{[2q+1]}(t)\varepsilon_2x_2+\cdots\\ &\quad\quad + S_1^{[2q+1]}(t)\varepsilon_px_p + \varepsilon_{p+1}x_{p+1}+\cdots+\varepsilon_{p+q}x_{p+q}\Big)\,dt\,d\theta \end{aligned}$$

$$= \frac{1}{2^{p+q}p!q!}\sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm1}\varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_{p+q}\int_0^1\Big(S_1^{[2q+1]}(t)\Big)^{2q+1-p}$$

$$\times P_{p,q}\Big(S_1^{[2q+1]}(t)\varepsilon_1 x_1 + S_1^{[2q+1]}(t)\varepsilon_2 x_2 + \cdots + S_1^{[2q+1]}(t)\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}\Big)\,dt$$

$$= \frac{1}{2^{p+q}p!q!}\sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1}\varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_{p+q}\sum_{i=1}^{2q+1}\frac{1}{2q+1}\alpha_i^{2q+1-p} \times P_{p,q}\Big(\alpha_i\varepsilon_1 x_1 + \alpha_i\varepsilon_2 x_2 + \cdots + \alpha_i\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}\Big).$$

□

Let X and Y be normed spaces. Let us define norms of (p,q)-polynomials and (p,q)-linear symmetric mappings respectively by

$$\|P_{p,q}\|_1 = \sup\{\|P_{p,q}(x)\|_Y : \ x \in \mathcal{B}\}$$

and

$$\|B_{p,q}\|_2 = \sup\{\|B_{p,q}(x_1,\ldots,x_{p+q})\|_Y : \ x_1,\ldots,x_{p+q} \in \mathcal{B}\},$$

where $\mathcal{B}$ is the closed unit ball of X.

From formula (3.1) we have the following estimations:

$$\|B_{p,q}\|_2 = \sup_{\|x_1\|_X\le 1,\ldots,\|x_{p+q}\|_X\le 1}\Big\|\frac{1}{2^{p+q}p!q!}\sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1}\varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_{p+q}\sum_{i=1}^{2q+1}\frac{1}{2q+1}\alpha_i^{2q+1-p} \times P_{p,q}\Big(\alpha_i\varepsilon_1 x_1 + \alpha_i\varepsilon_2 x_2 + \cdots + \alpha_i\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}\Big)\Big\|_Y$$

$$\le \frac{1}{2^{p+q}p!q!}\sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1}\sum_{i=1}^{2q+1}\frac{1}{2q+1}\times\sup_{\|x_1\|_X\le 1,\ldots,\|x_{p+q}\|_X\le 1}\Big\|(p+q)^{p+q} \times P_{p,q}\Big(\frac{\alpha_i\varepsilon_1 x_1 + \alpha_i\varepsilon_2 x_2 + \cdots + \alpha_i\varepsilon_p x_p + \varepsilon_{p+1}x_{p+1} + \cdots + \varepsilon_{p+q}x_{p+q}}{p+q}\Big)\Big\|_Y$$

$$\le \frac{1}{2^{p+q}p!q!}\cdot 2^{p+q}\cdot\frac{2q+1}{2q+1}\cdot(p+q)^{p+q}\|P_{p,q}\|_1 = \frac{(p+q)^{p+q}}{p!q!}\|P_{p,q}\|_1.$$

Hence we have the polarization inequality:

Theorem 3.2. *Let $B_{p,q}(x_1,\ldots,x_{p+q})$ be a (p,q)-linear symmetric mapping and $P_{p,q}(x)$ be the corresponding (p,q)-polynomial between normed spaces X and Y. Then*

$$\|P_{p,q}\|_1 \le \|B_{p,q}\|_2 \le \frac{(p+q)^{p+q}}{p!q!}\|P_{p,q}\|_1.$$

Note that the first part of this inequality is trivial. The following example shows that the second part of the inequality is sharp.

Example 3.1. Let $X = \ell_1$.

For given $p > 0$ and $q > 0$ let $B : X^{p+q} \to \mathbb{C}$ be the following (p,q)-linear map $B(x^1,\ldots,x^p;x^{p+1},\ldots,x^{p+q}) = x_1^1 x_2^2\ldots x_p^p\,\overline{x_{p+1}^{p+1}}\ldots\overline{x_{p+q}^{p+q}}$, where $x^j \in \ell_1$, $j = 1,\ldots,p+q$.

The (p,q)-symmetrization of B,

$$B_s(x^1,\ldots,x^p;x^{p+1},\ldots,x^{p+q}) = \frac{1}{p!q!}\sum_{\sigma_1\in S_p}x_1^{\sigma_1(1)}\ldots x_p^{\sigma_1(p)}\sum_{\sigma_2\in S_q}\overline{x_{p+1}^{p+\sigma_2(1)}\ldots x_{p+q}^{p+\sigma_2(q)}}$$

is a (p,q)-linear symmetric mapping.

$$
\begin{aligned}
&|B_s(x^1,\dots,x^p;x^{p+1},\dots,x^{p+q})| \\
&\qquad \le \frac{1}{p!q!}\sum_{\sigma_1\in S_p}|x_1^{\sigma_1(1)}|\dots|x_p^{\sigma_1(p)}|\sum_{\sigma_2\in S_q}|x_{p+1}^{p+\sigma_2(1)}|\dots|x_{p+q}^{p+\sigma_2(q)}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le \frac{1}{p!q!}\|x^1\|_{\ell_1}\dots\|x^{p+q}\|_{\ell_1}.
\end{aligned}
$$

Hence $\|B_s\|_2 \le \frac{1}{p!q!}$. On the other hand,

$$B_s(e^1,\dots,e^{p+q}) = \frac{1}{p!q!},$$

where

$$e^i = (\underbrace{0,\dots,0}_{i-1},1,0,\dots).$$

Therefore, $\|B_s\|_2 = \frac{1}{p!q!}$.

Let $\widehat{B}_s(x) = B_s(x,\dots,x) = x_1\dots x_p\overline{x_{p+1}}\dots\overline{x_{p+q}}$, where $x=(x_1,\dots,x_{p+q},\dots)$. Then $\widehat{B}_s$ is a (p,q)-polynomial. Since the geometric mean of positive numbers is always less than or equal to the arithmetic mean, we have

$$|\widehat{B}_s(x)| = |x_1|\dots|\overline{x_{p+q}}| \le \frac{1}{(p+q)^{p+q}}(|x_1|+\dots+|x_{p+q}|)^{p+q}.$$

Thus $\|\widehat{B}_s\|_1 \le \frac{1}{(p+q)^{p+q}}$. If we take $x = \Big(\underbrace{\frac{1}{p+q},\dots,\frac{1}{p+q}}_{p+q},0,\dots\Big)$, we obtain $|\widehat{B}_s(x)| = \frac{1}{(p+q)^{p+q}}$.

This shows that $\|\widehat{B}_s\|_1 = \frac{1}{(p+q)^{p+q}}$, and hence

$$\|B_s\|_2 = \frac{(p+q)^{p+q}}{p!q!}\|\widehat{B}_s\|_1.$$

The following example shows that in the case of Hilbert space the (p,q)-polarization constant is greater than 1.

Example 3.2. Let $X=\ell_2$, $p>1$, $q>1$, $B_{p,q}: X^{p+q}\to\mathbb{C}$,

$$B_{p,q}(x^1,\dots,x^p;x^{p+1},\dots,x^{p+q}) = x_1^1x_1^2\dots x_1^p\overline{x_2^{p+1},\dots,x_2^{p+q}}.$$

Obviously, $B_{p,q}$ is a (p,q)-linear symmetric mapping. From the following estimations

$$
\begin{aligned}
|B_{p,q}(x^1,\dots,x^{p+q})| &= |x_1^1||x_1^2|\dots|x_1^p||\overline{x_2^{p+1}}|\dots|\overline{x_2^{p+q}}| \\
&= \sqrt{|x_1^1|^2}\sqrt{|x_1^2|^2}\dots\sqrt{|x_1^p|^2}\sqrt{|x_2^{p+1}|^2}\dots\sqrt{|x_2^{p+q}|^2} \\
&\qquad\qquad \le \|x^1\|_{\ell_2}\|x^2\|_{\ell_2}\dots\|x^{p+q}\|_{\ell_2}
\end{aligned}
$$

we have that $\|B_{p,q}\|_2 \le 1$. On the other hand $B_{p,q}(e^1,\dots,e^1;e^2,\dots,e^2) = 1$ and so $\|B_{p,q}\|_2 = 1$. The map $\widehat{B}_{p,q}(x) = (x_1)^p(\overline{x_2})^q$ is a (p,q)-polynomial and

$$|\widehat{B}_{p,q}(x)| = |x_1|^p|\overline{x_2}|^q = |x_1|^p|x_2|^q.$$

Hence

$$\|\widehat{B}_{p,q}\|_1 = \sup_{\|x\|_{\ell_2}\le 1}|\widehat{B}_{p,q}(x)| = \sup_{|x_1|^2+|x_2|^2\le 1}|x_1|^p|x_2|^q = \max_{0\le t\le 1} t^p\left(\sqrt{1-t^2}\right)^q.$$

It is easy to check that the function

$$f(t) = t^p\left(\sqrt{1-t^2}\right)^q$$

has a maximal value at the point

$$t = \sqrt{\frac{p}{p+q}} \in [0,1]$$

and

$$f\left(\sqrt{\frac{p}{p+q}}\right) = \left(\frac{p}{p+q}\right)^{\frac{p}{2}} \left(\sqrt{1-\frac{p}{p+q}}\right)^{q} = \left(\frac{p}{p+q}\right)^{\frac{p}{2}} \left(\frac{q}{p+q}\right)^{\frac{q}{2}}.$$

Thus

$$\|\widehat{B}_{p,q}\|_1 = \frac{p^{p/2}q^{q/2}}{(p+q)^{\frac{p+q}{2}}}$$

and

$$\|B_{p,q}\|_2 = \frac{(p+q)^{\frac{p+q}{2}}}{p^{p/2}q^{q/2}} \|\widehat{B}_{p,q}\|_1.$$

Let $\mathcal{P}_{p,q}(X;Y)$ be the normed space of all continuous (p,q)-polynomials from X to Y with the norm $\|\cdot\|_1$ and $\mathcal{B}_{p,q}(X^n;Y)$ the normed space of all continuous symmetric (p,q)-linear maps from X^n to Y, $p+q=n$, with the norm $\|\cdot\|_2$.

Theorem 3.2 implies the following corollary:

Corollary 3.1. *The space $\mathcal{P}_{p,q}(X;Y)$ is isomorphic to $\mathcal{B}_{p,q}(X^n;Y)$.*

Proof. The polarization formula gives us the required linear isomorphism from $\mathcal{P}_{p,q}(X;Y)$ onto $\mathcal{B}_{p,q}(X^n;Y)$ and the polarization inequality implies its continuity. □

Remark 3.1. In [11] the authors proved some another forms of the polarization formula using different approach

$$\begin{aligned} &B_{p,q}(x_1,\ldots,x_p;x_{p+1},\ldots,x_{p+q}) \\ &= \frac{1}{2^m p\,!\,q\,!} \sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=0}^{1} (-1)^{p+q-(\varepsilon_1+\cdots+\varepsilon_{p+q})} \sum_{\mu_1,\ldots,\mu_m=0}^{1} (r_1^{\mu_1} r_2^{\mu_2}\ldots r_m^{\mu_m})^q \\ &\times P_{p,q}\big((x'+\varepsilon_1 x_1+\cdots+\varepsilon_p x_p) \\ &+ (r_1^{\mu_1} r_2^{\mu_2}\ldots r_m^{\mu_m})(x''+\varepsilon_{p+1}x_{p+1}+\cdots+\varepsilon_{p+q}x_{p+q})\big), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &B_{p,q}(x_1,\ldots,x_p;x_{p+1},\ldots,x_{p+q}) \\ &= \frac{1}{p!q!2^{p+q}} \sum_{\varepsilon_1,\ldots,\varepsilon_{p+q}=\pm 1} \varepsilon_1\varepsilon_2\ldots\varepsilon_{p+q} \sum_{\mu_1,\ldots,\mu_m=0}^{1} \frac{1}{2^m} (r_1^{\mu_1} r_2^{\mu_2}\ldots r_m^{\mu_m})^q \\ &\times P_{p,q}\big((x'+\varepsilon_1 x_1+\cdots+\varepsilon_p x_p) \\ &+ (r_1^{\mu_1} r_2^{\mu_2}\ldots r_m^{\mu_m})(x''+\varepsilon_{p+1}x_{p+1}+\cdots+\varepsilon_{p+q}x_{p+q})\big), \end{aligned} \tag{3.3}$$

where

$$r_k = \cos\frac{\pi}{2^{k-1}} + i\sin\frac{\pi}{2^{k-1}},$$

$$m = \big[\log_2(p+q)\big] + 1$$

and x', x'' are arbitrary elements of X.

Note that from formula (3.3) we can also get the polarization inequality.

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