Methods of Functional Analysis and Topology
Vol. 12 (2006), no. 1, pp. 84–96

ON EQUIANGULAR CONFIGURATIONS OF SUBSPACES OF A HILBERT SPACE

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Abstract. In this paper, we find τ, $0 < \tau < 1$, such that there exists an equiangular (Γ, τ)-configuration of one-dimensional subspaces, and describe (Γ, τ)-configurations that correspond to unicyclic graphs and to some graphs that have cyclomatic number satisfying $\nu(\Gamma) \geq 2$.

0. Introduction

There are numerous publications, see [1] and references therein, studying systems $S = (H; H_1, H_2, \ldots, H_n)$ of subspaces H_i, $i = 1, \ldots, n$, of a complex separable Hilbert space H that may be finite dimensional or have countable dimension.

Denote by P_i an orthogonal projection of H onto the corresponding subspace H_i, $i = 1, \ldots, n$.

A system of subspaces is called *irreducible* if any operator $C \in \mathcal{B}(H)$ that commutes with all orthogonal projections, $CP_i = P_iC$, $i = 1, \ldots, n$, is a scalar operator, $C = \lambda I$, $\lambda \in \mathbb{C}$.

Two systems $S = (H; H_1, \ldots, H_n)$ and $S' = (H'; H'_1, \ldots, H'_n)$ are called *unitary equivalent* if there is a unitary operator $U \in \mathcal{B}(H, H')$ such that $U(H_i) = H'_i$ for all $i = 1, \ldots, n$ or, equivalently, if $UP_i = P_iU$, $i = 1, \ldots, n$.

To give a description of irreducible systems of n subspaces of a Hilbert space up to unitary equivalence for $n \geq 3$ is an unmanageable task, see [2, 3, 4].

In this paper, we consider equiangular (Γ, τ)-configurations of subspaces corresponding to a connected simple (without loops or multiple edges) undirected graph $\Gamma = (V_\Gamma, E_\Gamma)$, where V_Γ denotes the set of vertices of the graph, E_Γ is the set of its edges, and $\tau \in \mathbb{R}$, $0 < \tau < 1$. An equiangular (Γ, τ)-configuration is a system $S = (H; H_1, \ldots, H_n)$ of subspaces, $n = |V_\Gamma|$, such that the orthogonal projections corresponding to each pair of subspaces H_i, H_j satisfy the relation

$$\begin{cases} P_iP_jP_i = \tau^2 P_i, \ P_jP_iP_j = \tau^2 P_j, & \text{if there is an edge } \gamma_{ij} \in E_\Gamma, \\ P_iP_j = P_jP_i = 0, & \text{if } \gamma_{ij} \notin E_\Gamma. \end{cases}$$

Here we define an angle between each pair of subspaces H_i and H_j to be $\theta = \arccos \tau$, $0 < \theta < \pi/2$, if $\gamma_{ij} \in E_\Gamma$, and consider H_i and H_j to be orthogonal otherwise, that is, if $\gamma_{ij} \notin E_\Gamma$. Let us remark that if $\Gamma = K_n$ is a complete graph, such (Γ, τ)-equiangular one-dimensional configurations of subspaces in a Euclidean space were studied in [5].

If Γ is a tree or a cycle, all (Γ, τ)-irreducible configurations are described in, e.g., [6]. An irreducible (Γ, τ)-configuration corresponding to a tree or a unicycle graph is a configuration of one-dimensional subspaces [1].

2000 *Mathematics Subject Classification.* 47A62, 16G20.

Key words and phrases. System of subspaces, equiangular configurations, unicyclic graphs, **S**-index of graph.

In this paper, by finding τ such that there exists a (Γ,τ)-configuration of one-dimensional subspaces, we describe irreducible equiangular (Γ,τ)-configurations, of one-dimensional subspaces, corresponding to the graphs that are cactuses (Theorem 4), and give a complete description, up to unitary equivalence, of all irreducible (Γ,τ)-configurations corresponding to an arbitrary unicyclic graph (Theorem2).

1. Preliminaries

Consider an equiangular configuration of n one-dimensional subspaces, which corresponds to a simple undirected graph $\Gamma = (V_\Gamma, E_\Gamma)$, where V_Γ denotes the set of vertices of Γ and E_Γ is the set of edges of the graph Γ. Let Φ be a mapping defined on edges of the graph giving a grading of the graph, $\Phi(\gamma_{kj}) = e^{i\phi_{kj}}$, $k<j$, $\gamma_{kj}\in E_\Gamma$. Such a pair (Γ,Φ) will be called an **S**-*signed graph*.

An *adjacency matrix* $A_{(\Gamma,\Phi)} = (a_{kj})_{k,j=1}^n$ of an **S**-signed graph (Γ,Φ) is defined to be

$$a_{kj} = \begin{cases} e^{i\phi_{kj}}, & \gamma_{kj}\in E_\Gamma,\ k<j,\\ e^{-i\phi_{kj}}, & \gamma_{kj}\in E_\Gamma,\ k>j,\\ 0, & \gamma_{kj}\notin E_\Gamma. \end{cases}$$

Spectrum, $\sigma(\Gamma)$, and *index*, $\operatorname{ind}(\Gamma,\Phi)$, of an **S**-signed graph refers to the spectrum and the largest eigenvalue of the matrix $A_{(\Gamma,\Phi)}$, respectively.

Introduce an *sw*-equivalence, a switching equivalence, on the set of all **S**-signatures of a graph by defining two **S**-sign graphs (Γ,Φ_1) and (Γ,Φ_2) to be equivalent if there exists a function $\psi\colon V_\Gamma\to S^1$ such that

$$\Phi_2(\gamma_{kj}) = e^{i\psi_k}\Phi_1(\gamma_{kj})e^{-i\psi_j}.$$

Lemma 1. *Let $|\Gamma| = n$, and $\nu(\Gamma)$ be the cyclomatic number of the graph Γ. Then any* **S**-*signature of the graph Γ is sw-equivalent to some* **S**-*signature with the function Φ taking the value 1 on $n-\nu(\Gamma)$ edges, so that the corresponding ϕ_{kj} satisfy $\phi_{kj}=0$, and the remaining $\nu(\Gamma)$ edges can be indexed so that Φ takes the values $e^{i\phi_j}$, $j=1,\dots,\nu(\Gamma)$. Thus any* **S**-*signature can be parametrized with $\nu(\Gamma)$ parameters ϕ_j.*

This means that if Γ is a tree, then all **S**-signatures are sw-equivalent to the **S**-signature $\Phi(\gamma_{kj}) = 1$ for all $\gamma_{kj}\in E_\Gamma$.

If Γ is a unicyclic graph, see Section 3, then there are only the following two *sw*-nonequivalent **S**-signatures: the **S**-signature $\Phi(\gamma_{kj}) = 1$ for all $\gamma_{kj}\in E_\Gamma$, and the **S**-signature $\Phi(\gamma_{kj}) = 1$ for all $\gamma_{kj}\in E_\Gamma$ but one edge γ in the cycle, and for this edge, $\Phi(\gamma) = e^{i\phi}$, $\phi\in[0,2\pi)$.

If Γ is a cactus with k cycles, see Section 3, then by indexing cycles of the graph in a certain order, one can parametrize the set of all *sw*-nonequivalent **S**-signatures with k parameters, $(\phi_1,\dots,\phi_k)$, such that $\Phi(\gamma)=1$ on all edges $\gamma\in E_\Gamma$ except for edges in the set formed by picking one edge γ_j in each cycle C_j, where $\Phi(\gamma_j) = e^{i\phi_j}$, $j=1,\dots,k$. Here $\operatorname{ind}(\Gamma,\Phi)$ does not depend on the set $\{\gamma_j\colon \gamma_j\in C_j\}$.

Definition. The quantity

$$\operatorname{ind}_{\mathbf{S}}\Gamma = \inf_{\phi\in\Omega_\Phi}\operatorname{ind}(\Gamma,\Phi)$$

is called an **S**-*index* of the graph, where Ω_Φ is the set of all values of the parameters $(\phi_1,\dots,\phi_k)$, $\phi_j\in[0,2\pi)$, $j=1,\dots,k$.

If the graph is connected, then all subspaces of the simple system S corresponding to the graph have the same dimension, see [1].

The main problem is to give a description of all irreducible unitary nonequivalent equiangular configurations S corresponding to the graph Γ with a fixed τ, $0<\tau<1$.

In what follows, we will only consider equiangular configurations of one-dimensional subspaces, $\dim H_i = 1$, $i = 1, \dots, n$.

The following theorem describes τ, $0 < \tau < 1$, for which there exist (Γ, τ)-configurations of one-dimensional subspaces.

Theorem 1. ([7]). *Let Γ be an arbitrary fixed graph. There exist (Γ, τ)-configurations of one-dimensional subspaces if and only if $\tau \le \frac{1}{\operatorname{ind}_{\mathbf{S}} \Gamma}$.*

Proposition 1. ([6, 9]). *If Γ is a tree, then the following assertions hold:*

- *if there exists an irreducible configuration S corresponding to a signed graph (Γ, ϕ), then all the subspaces H_i, $i = 1, \dots, n$, are one-dimensional;*
- *an irreducible configuration S exists only if $\tau \le \frac{1}{\operatorname{ind} \Gamma}$ and, for each τ, such a configuration is unique. Moreover, $\dim H = n$ if $\tau < \frac{1}{\operatorname{ind} \Gamma}$, and $\dim H = n - 1$ if $\tau = \frac{1}{\operatorname{ind} \Gamma}$.*

2. Equiangular configurations of subspaces corresponding to unicyclic graphs

A *unicyclic graph of girth g* is a graph $\Gamma = (C_g; T_1, T_2, \dots, T_g)$ obtained from a cycle C_g of length g by identifying the i-th vertex of the cycle with the root vertex of some tree T_i.

In the case where Γ is a unicyclic graph there can be infinitely many irreducible configurations for some values of τ, but all such configurations are n-tuples of one-dimensional subspaces [6] and are parametrized with $\phi \in \Phi_\tau \subseteq [0, 2\pi)$. Introduce a matrix $B_{\Gamma,\tau,\phi}$ for the unicyclic graph as follows: $B_{\Gamma,\tau,\phi} = I - \tau A_{\Gamma,\phi}$, where $A_{\Gamma,\phi} = (a_{ij})_{i,j=1}^n$ is the $\mathbf{S}$-signed adjacency matrix of the graph,

$$a_{ij} = \begin{cases} 0, & \text{if } \gamma_{ij} \notin E_\Gamma, \\ 1, & \text{if } \gamma_{ij} \in E_\Gamma,\ (i,j) \notin \{(1,g),(g,1)\}, \\ e^{i\phi}, & \text{if } (i,j) = (1,g), \\ e^{-i\phi}, & \text{if } (i,j) = (g,1). \end{cases}$$

Then the formula for the $\mathbf{S}$-index of the unicyclic graph becomes

$$\operatorname{ind}_{\mathbf{S}} \Gamma = \inf_{\phi \in [0,2\pi)} \operatorname{ind}(\Gamma, \phi).$$

In the sequel, we will need the following facts from the theory of graphs.

Proposition 2. ([13]).

(1) *If Γ is a connected graph and x is an arbitrary vertex of the graph, then* $\operatorname{ind}(\Gamma - x) < \operatorname{ind} \Gamma$.

(2) *Let Γ be a connected graph and H a proper spanning subgraph of Γ, which is a subgraph that differs from the entire graph, constructed over a subset V_H of vertices of the graph $\Gamma - V_\Gamma$, and containing all edges from E_Γ the endpoints of which belong to V_H (the subgraph H is not necessarily connected). Then for all $\lambda \ge \operatorname{ind} \Gamma$,*

$$P_H(\lambda) > P_\Gamma(\lambda),$$

where $P_\Gamma(\lambda)$ is the characteristic polynomial of the graph Γ, and $P_H(\lambda)$ is the characteristic polynomial of the graph H.

Lemma 2. *The Schwenk formula for the characteristic polynomial of a unicyclic $\mathbf{S}$-signed graph Γ has the form*

$$P_{\Gamma,\phi}(\lambda) = \lambda P_{\Gamma - v_1}(\lambda) - \sum_{u \sim v_1,\ u \in V_\Gamma} P_{\Gamma - v_1 - u}(\lambda) - 2\cos\phi \prod_{i=1}^{g} P_{T_i - v_i}(\lambda),$$

where v_1 is a vertex of the cycle of the graph and $\{u\colon u \sim v_1\}$ denotes the set of vertices of the graph Γ neighboring the vertex v_1, and $\Gamma - v$ is the subgraph of the graph Γ obtained by removing the vertex v.

Proof. Let Γ be a cycle C_g of length g. Then, directly evaluating the determinant of the corresponding adjacency matrix of the $\mathbf{S}$-signed graph (Γ, ϕ) we get

$$P_{(C_g,\phi)}(\lambda) = \lambda P_{g-1}(\lambda) - 2P_{g-2}(\lambda) - 2\cos\phi,$$

where P_n is the characteristic polynomial of the Dynkin graph A_n, a chain with n vertices.

Let now Γ be a cycle C_g with a root vertex of a tree T attached to one of the vertices, denoted by v_1, and let the valency of the root vertex be 1. Denote the vertex of the tree T neighboring to the root vertex by v. Then the graph Γ contains a bridge between the vertices v_1 and v, and we use the decomposition formula for the characteristic polynomial of the graph with respect to the bridge $\gamma_{v_1 v}$,

$$P_{(\Gamma,\phi)}(\lambda) = P_{(C_g,\phi)}(\lambda)P_T(\lambda) - P_{g-1}(\lambda)P_{T-v}(\lambda).$$

Substituting the expression for the characteristic polynomial of the cycle we get

$$P_{(\Gamma,\phi)}(\lambda) = \lambda P_{g-1}(\lambda)P_T(\lambda) - \big(2P_{g-2}(\lambda)P_T(\lambda) + P_{g-1}(\lambda)P_{T-v}(\lambda)\big) - 2\cos\phi P_T(\lambda),$$

which coincides with the required formula for the graph Γ,

$$P_{\Gamma,\phi}(\lambda) = \lambda P_{\Gamma-v_1}(\lambda) - \sum_{u\sim v_1,\ u\in V_\Gamma} P_{\Gamma-w-u}(\lambda) - 2\cos\phi P_{T-v}(\lambda).$$

Now, using the same argument we can extend this formula to the case where there is a tree with the root having an arbitrary valency attached to a vertex of the cycle. Then, extend it to an arbitrary unicyclic graph. □

Proposition 3. *Let Γ be a unicyclic $\mathbf{S}$-signed graph. Then*

$$\operatorname{ind}_{\mathbf{S}}\Gamma = \operatorname{ind}(\Gamma, \pi).$$

Proof. To simplify the notations, denote

$$f(\lambda) = \lambda P_{\Gamma-v_1}(\lambda) - \sum_{u\sim v_1,\ u\in V_\Gamma} P_{\Gamma-v_1-u}(\lambda)$$

and

$$h(\lambda) = \prod_{i=1}^{g} P_{T_i - v_i}(\lambda).$$

Then $P_{\Gamma,\phi}(\lambda) = f(\lambda) - 2\cos\phi \cdot h(\lambda)$. Consider this polynomial on the segment $[\operatorname{ind}(\Gamma,\pi); \operatorname{ind}(\Gamma,0)]$.

Since $h(\lambda)$ is a characteristic polynomial of the induced subgraph of the graph Γ, by Proposition 2 (2), $h\big(\operatorname{ind}(\Gamma,\pi)\big) > 0$ and $h\big(\operatorname{ind}(\Gamma,0)\big) > 0$. We have

$$\begin{aligned} P_{\Gamma,\phi}\big(\operatorname{ind}(\Gamma,\pi)\big) &= f\big(\operatorname{ind}(\Gamma,\pi)\big) - 2\cos\phi\, h\big(\operatorname{ind}(\Gamma,\pi)\big) \\ &= -2(1+\cos\phi)h\big(\operatorname{ind}(\Gamma,\pi)\big) \le 0, \end{aligned}$$

$$\begin{aligned} P_{\Gamma,\phi}\big(\operatorname{ind}(\Gamma,0)\big) &= f\big(\operatorname{ind}(\Gamma,0)\big) - 2\cos\phi\, h\big(\operatorname{ind}(\Gamma,0)\big) \\ &= 2(1-\cos\phi)h\big(\operatorname{ind}(\Gamma,0)\big) \ge 0. \end{aligned}$$

Then, by the Weierstrass theorem, the polynomial $P_{\Gamma,\phi}(\lambda)$ has a root on the segment $\big[\operatorname{ind}(\Gamma,\pi); \operatorname{ind}(\Gamma,0)\big]$. This means that the $\mathbf{S}$-signed graph (Γ,ϕ) has the least index for $\phi = \pi$. □

Proposition 4. ([1]). *For a unicyclic graph Γ there exists an irreducible simple n-tuple of subspaces corresponding to a pair (Γ, ϕ) if and only if the set Φ_τ of parameters for which the matrix $B_{\Gamma,\tau,\phi}$ is nonnegative definite is not empty. In such a case, for every $\phi \in \Phi_\tau$ there exists a unique, up to unitary equivalence, nonzero irreducible simple n-tuple of subspaces, $S_{\tau,\phi}$, and all of them are unitary nonequivalent.*

All subspaces of the system $S_{\tau,\phi}$ are one-dimensional, and $\dim H = n$ if the matrix $B_{\Gamma,\tau,\phi}$ is positive definite, and $\dim H = n-1$ or $\dim H = n-2$ otherwise.

Theorem 2. *Let Γ be a unicyclic graph with n vertices.*

(1) *If $\tau < \frac{1}{\operatorname{ind}\Gamma}$, then for the pair (Γ,τ) there exists a corresponding irreducible simple system $S_{\tau,\phi}$ of subspaces for any $\phi \in [0, 2\pi)$, and $\dim H = n$.*
(2) *If $\tau = \frac{1}{\operatorname{ind}\Gamma}$, then there exists an infinite family of irreducible simple configurations $S_{\tau,\phi}$, parametrized with $\phi \in [0, 2\pi)$, and $\dim H = n$ for all $\phi \neq 0$, and $\dim H = n-1$ for $\phi = 0$.*
(3) *If $\frac{1}{\operatorname{ind}\Gamma} < \tau < \frac{1}{\operatorname{ind}_{\mathbf{S}}\Gamma}$, then there exists an infinite family of irreducible simple configurations $S_{\tau,\phi}$ parametrized with ϕ,*

$$\phi \in \Big[\arccos\Big(\frac{1}{2}\frac{f(\tau^{-1})}{h(\tau^{-1}}\Big), 2\pi - \arccos\Big(\frac{1}{2}\frac{f(\tau^{-1}}{h(\tau^{-1})}\Big)\Big],$$

where

$$f(\lambda) = \lambda P_{\Gamma - v_1} - \sum_{u\sim v_1,\ u\in V_\Gamma} P_{\Gamma - v_1 - u}(\lambda),$$

$$h(\lambda) = \prod_{i=1}^{g} P_{T_i - v_i}(\lambda).$$

For

$$\phi \in \Big(\arccos\Big(\frac{1}{2}\frac{f(\tau^{-1})}{h(\tau^{-1}}\Big), 2\pi - \arccos\Big(\frac{1}{2}\frac{f(\tau^{-1}}{h(\tau^{-1})}\Big)\Big),$$

$\dim H = n$, *and* $\dim H = n-1$ *for*

$$\phi = \arccos\Big(\frac{1}{2}\frac{f(\tau^{-1})}{h(\tau^{-1}}\Big) \quad \text{or} \quad \phi = 2\pi - \arccos\Big(\frac{1}{2}\frac{f(\tau^{-1})}{h(\tau^{-1})}\Big).$$

(4) *If $\tau = \frac{1}{\operatorname{ind}_{\mathbf{S}}\Gamma}$, then there is a unique configuration S corresponding to (Γ,τ) for $\phi = \pi$, and the dimension of the space is $n-2$ if the graph is a cycle, and is $n-1$ otherwise.*
(5) *If $\tau > \frac{1}{\operatorname{ind}_{\mathbf{S}}\Gamma}$, then no corresponding configuration exists.*

Proof. Let $\Gamma = (C_g; T_1, \dots, T_g)$ be a unicyclic graph with n vertices.

First, by using the Schwenk formula,

$$P_{\Gamma,\phi}(\lambda) = \lambda P_{\Gamma - v_1}(\lambda) - \sum_{u\sim v_1,\, u\in V_\Gamma} P_{\Gamma - v_1 - u}(\lambda) - 2\cos\phi \prod_{i=1}^{g} P_{T_i - v_i}(\lambda),$$

we get for the characteristic polynomial of the signed adjacency **S**-matrix of a unicyclic graph that, as the value of ϕ increases in the segment $[0,\pi]$, the corresponding value of the index monotonically decrease, and $\operatorname{ind}(\Gamma,\phi) = \operatorname{ind}(\Gamma, 2\pi - \phi)$. Then, for every value of τ, $\tau \le \frac{1}{\operatorname{ind}_{\mathbf{S}}\Gamma}$, the matrix $B_{\Gamma,\tau,\phi}$ is nonnegative definite for all values of the parameter lying in the interval

$$\phi \in \Big[\arccos\Big(\frac{1}{2}\frac{f(\tau^{-1})}{h(\tau^{-1})}\Big), 2\pi - \arccos\Big(\frac{1}{2}\frac{f(\tau^{-1})}{h(\tau^{-1})}\Big)\Big].$$

Now, let the graph Γ be unicyclic and not just a cycle but having at least one nontrivial tree T_1. Then the matrix $B_{U,\tau,\phi}$ for this graph is

$$B_{\Gamma,\tau,\phi} = \begin{pmatrix} 1 & * & * \\ * & B_{T_1-v_1,\tau} & 0 \\ * & 0 & B_{\Gamma\setminus T_1,\tau} \end{pmatrix}.$$

Removing the first row and the first column we obtain a block diagonal matrix that does not depend on ϕ and having blocks corresponding to $B_{T_1-v_1}$ and $B_{\Gamma\setminus T_1}$,

$$\operatorname{rank} B_{\Gamma,\tau,\phi} \geq \operatorname{rank} B_{T_1-v_1,\tau} + \operatorname{rank} B_{\Gamma\setminus T_1,\tau} \geq n-2.$$

Suppose that $\operatorname{rank} B_{\Gamma,\tau,\phi} = n-2$, which is possible if $\tau = \frac{1}{\operatorname{ind}_S \Gamma}$. Then $\operatorname{ind}(\Gamma,\pi)$ must be a root of the characteristic polynomial $P_{\Gamma\setminus T_1}(\lambda)$, that is, $P_{\Gamma\setminus T_1}(\operatorname{ind}(\Gamma,\pi)) = 0$. We have

$$\begin{aligned} P_{\Gamma,\phi}(\lambda) &= \lambda P_{T_1-v_1}(\lambda) P_{\Gamma\setminus T_1}(\lambda) - \left(\sum_{u\sim v_1\, u\in V_{T_1}} P_{T_1-v_1-u}(\lambda) \right) P_{\Gamma\setminus T_1}(\lambda) \\ &\quad - \left(\sum_{w\sim v_1,\, w\in V_{\Gamma\setminus T_1}} P_{\Gamma\setminus T_1 - w}(\lambda) \right) P_{T_1-v_1}(\lambda) - 2\cos\phi \prod_{i=1}^{g} P_{T_i-v_i}(\lambda). \end{aligned}$$

For $\lambda = \operatorname{ind}(\Gamma,\pi)$, we have

$$\begin{aligned} 0 = P_{\Gamma,\pi}(\operatorname{ind}(\Gamma,\pi)) = \Bigg(& 2\prod_{i=2}^{g} P_{T_i-v_i}(\operatorname{ind}(\Gamma,\pi)) \\ & - \sum_{w\sim v_1,\, w\in V_{\Gamma\setminus T_1}} P_{\Gamma\setminus T_1-w}(\operatorname{ind}(\Gamma,\pi)) \Bigg) P_{T_1-v_1}(\operatorname{ind}(\Gamma,\pi)). \end{aligned}$$

Since removing a vertex from a connected graph we have $\operatorname{ind}(\Gamma - v) < \operatorname{ind}\Gamma$, it follows from Proposition 2 (2) that $P_{T_1-v_1}(\operatorname{ind}(\Gamma,\pi)) > 0$. The expression in the round brackets must then be equal to zero,

$$2\prod_{i=2}^{g} P_{T_i-v_i}(\operatorname{ind}(\Gamma,\pi)) - \sum_{w\sim v_1,\, w\in V_{\Gamma\setminus T_1}} P_{\Gamma\setminus T_1-\omega}(\operatorname{ind}(\Gamma,\pi)) = 0.$$

On the other hand,

$$\begin{aligned} & 2\prod_{i=2}^{g} P_{T_i-v_i}(\operatorname{ind}(\Gamma,\pi)) - \sum_{w\sim v_1,\, w\in V_{\Gamma}\setminus T_1} P_{\Gamma\setminus T_1-w}(\operatorname{ind}(\Gamma,\pi) \\ &= 2\prod_{i=2}^{g} P_{T_i-v_i}(\operatorname{ind}(\Gamma,\pi)) - P_{T_2-v_2}(\operatorname{ind}(\Gamma,\pi)) P_{\Gamma\setminus T_1\cup T_2}(\operatorname{ind}(\Gamma,\pi)) \\ &- P_{T_g-v_g}(\operatorname{ind}(\Gamma,\pi)) P_{\Gamma\setminus T_1\cup T_g}(\operatorname{ind}(\Gamma,\pi)) \\ &= P_{T_2-v_2}(\operatorname{ind}(\Gamma,\pi)) \big[P_{T_3-v_3}(\operatorname{ind}(\Gamma,\pi)) \cdot \ldots \cdot P_{T_g-v_g}(\operatorname{ind}(\Gamma,\pi)) \\ &- P_{\Gamma\setminus T_1\cup T_2}(\operatorname{ind}(\Gamma,\pi)) \big] \\ &+ P_{T_g-v_g}(\operatorname{ind}(\Gamma,\pi)) \big[P_{t_2-v_2}(\operatorname{ind}(\Gamma,\pi)) \cdot \ldots \cdot P_{T_{g-1}-v_{g-1}}(\operatorname{ind}(\Gamma,\pi)) \\ &- P_{\Gamma\setminus T_1\cup T_g}(\operatorname{ind}(\Gamma,\pi)) \big] > 0, \end{aligned}$$

since

$$P_{T_2-v_2}(\lambda) \cdot \ldots \cdot P_{t_{g-1}-v_{g-1}}(\lambda) > P_{\Gamma\setminus T_1\cup T_g}(\lambda)$$

and

$$P_{t_3-v_3}(\lambda) \cdot \ldots \cdot P_{t_g-v_g}(\lambda) > P_{\Gamma\setminus T_1\cup T_2}(\lambda)$$

for all values of λ satisfying

$$\lambda > \max\Big\{\prod_{i=2}^{g-1} \operatorname{ind}(T_i - v_i), \prod_{i=3}^{g} \operatorname{ind}(T_i - v_i)\Big\}.$$

Hence, we get $P_{\Gamma\setminus T_1}(\operatorname{ind}(\Gamma,\pi)) \neq 0$ and $\dim H = n-1$. $\square$

Remark. In case when the graph is a cycle, the dimension at the endpoint becomes $n-2$. Indeed,

$$P_{C_n,\phi}(\lambda) = \lambda P_{n-1}(\lambda) - 2P_{n-2}(\lambda) - 2\cos\phi,$$

and $\operatorname{ind}_{\mathbf{S}} C_n = \operatorname{ind} A_{n-1}$, $P_{n-2}(\operatorname{ind} A_{n-1}) = 1$, since, for $\lambda < 2$, the characteristic polynomial $P_{n-2}(\lambda)$ for the Dynkin graph A_{n-2} has the form

$$P_{n-2}(\lambda) = \frac{\sin\big((n-1)\arccos\frac{\lambda}{2}\big)}{\sqrt{1-\big(\frac{\lambda}{2}\big)^2}},$$

and, for $\lambda = \operatorname{ind} A_{n-1} = 2\cos\frac{\pi}{n}$,

$$P_{n-2}(\lambda) = \frac{\sin\big((n-1)\frac{\pi}{n}\big)}{\sin\frac{\pi}{n}} = 1.$$

Example 1. Let $\Gamma = C_n$.

- If $\tau < \frac{1}{2}$, then for any pair (Γ,τ) there exists a corresponding irreducible simple system $S_{\tau,\phi}$ of subspaces for any $\phi \in [0,2\pi)$, and $\dim H = n$.
- If $\tau = \frac{1}{2}$, then there exists an infinite family of irreducible simple configurations $S_{\tau,\phi}$, parametrized with $\phi \in [0,2\pi)$, whereas $\dim H = n$ for all $\phi \neq 0$, and $\dim H = n-1$ for $\phi = 0$.
- If $\frac{1}{2} < \tau < \frac{1}{2\cos\frac{\pi}{n}}$, then there exists an infinite family $S_{\tau,\phi}$ of irreducible simple configurations parametrized with $\phi \in [n\alpha; 2\pi - n\alpha]$, with $\dim H = n$ for $\phi \in (n\alpha; 2\pi - n\alpha)$, $\dim H = n-1$ for $\phi = n\alpha$ and $\phi = 2\pi - n\alpha$, where α is a root of the equation $\tau = \frac{1}{2\cos\alpha}$.
- If $\tau = \frac{1}{2\cos\frac{\pi}{n}}$, then there is a unique configuration S corresponding to (Γ,τ) for $\phi = \pi$, and the dimension of the space is $n-2$.
- If $\tau > \frac{1}{2\cos\frac{\pi}{n}}$, then no corresponding configurations exist.

Example 2. Let $\Gamma = (C_4; m_1, 0, 0, 0)$ be the graph consisting of a square with a tree having the root attached to one of the corners, where the tree is a star with m_1 rays and the root is located in the vertex having the maximal valency.

- If $\tau < \sqrt{\frac{2}{4+m_1+\sqrt{m_1^2+16}}}$, then for any $\phi \in [0,2\pi)$ there exists an irreducible simple system $S_{\tau,\phi}$ of subspaces, corresponding to the pair (Γ,τ), and $\dim H = m_1+4$.
- If $\tau = \sqrt{\frac{2}{4+m_1+\sqrt{m_1^2+16}}}$, then there is an infinite family of irreducible simple configurations $S_{\phi,\tau}$ parametrized with $\phi \in [0,2\pi)$, and $\dim H = m_1+4$ for all $\phi \neq 0$, and $\dim H = m_1+3$ for $\phi = 0$.
- If $\sqrt{\frac{2}{4+m_1+\sqrt{m_1^2+16}}} < \tau < \sqrt{\frac{1}{m_1+2}}$, then there is an infinite family $S_{\phi,\tau}$ of irreducible simple configurations parametrized with
$$\phi \in \Big[\arccos\Big(\tfrac{2m_1\tau^4-(m_1+4)\tau^2+1}{2\tau^4}\Big); 2\pi - \arccos\Big(\tfrac{2m_1\tau^4-(m_1+4)\tau^2+1}{2\tau^4}\Big)\Big].$$
We have $\dim H = m_1+4$ if
$$\phi \in \Big(\arccos\Big(\tfrac{2m_1\tau^4-(m_1+4)\tau^2+1}{2\tau^4}\Big); 2\pi - \arccos\Big(\tfrac{2m_1\tau^4-(m_1+4)\tau^2+1}{2\tau^4}\Big)\Big),$$
and $\dim H = m_1+3$ if
$$\phi = \arccos\Big(\tfrac{2m_1\tau^4-(m_1+4)\tau^2+1}{2\tau^4}\Big)$$

or

$$\phi = 2\pi - \arccos\Big(\tfrac{2m_1\tau^4-(m_1+4)\tau^2+1}{2\tau^4}\Big).$$

- If $\tau = \sqrt{\frac{1}{m_1+2}}$, then there is a unique configuration S corresponding to (Γ, τ) for $\phi = \pi$, and the dimension of the space equals $m_1 + 3$.
- If $\tau > \sqrt{\frac{1}{m_1+2}}$, then no corresponding configurations exist.

3. Equiangular configurations, of one-dimensional subspaces, connected with cactuses

A graph in which every two cycles have no more than 1 common vertex will be called a cactus.

It follows from Lemma 1 that all irreducible equiangular $(\Gamma\,\tau)$-configurations, of one-dimensional subspaces, connected with a cactus having k cycles can be parametrized with k parameters by picking one edge γ_j in every cycle C_j in an arbitrary way and setting $\Phi(\gamma_j) = e^{i\phi_j}$ on these edges, and $\Phi(\gamma) = 1$ for other edges. Such a parametrization ϕ will be denoted by $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_n)$.

Lemma 3. *Let Γ be a cactus and $w \in E_\Gamma$. Then the characteristic polynomial for Γ satisfies the following modification of the Schwenk formula:*

$$P_{\Gamma,\phi}(\lambda) = \lambda P_{\Gamma-w,\phi'}(\lambda) - \sum_{u\sim w,\, u\in V_\Gamma} P_{\Gamma-u-w,\phi''}(\lambda) - 2\sum_{C_j\in\mathcal{C}(w)} P_{\Gamma-C_j,\phi'''}(\lambda)\cos\phi_j,$$

where ϕ', ϕ'', ϕ''' are restrictions of ϕ to the corresponding subgraphs of the graph Γ, $\mathcal{C}(w)$ is the set of all cycles of the graph containing the vertex w.

Proof. The proof is obtained by induction similarly to the proof of the Schwenk formula for a unicyclic graph. □

Lemma 4. *Let $\lambda > \min\{\operatorname{ind}(\Gamma,\vec{\phi}), \operatorname{ind}(\Gamma,\vec{\chi})\}$, $\vec{\phi}, \vec{\chi} \in \mathbb{R}^k$. Then, if $\cos(\phi_j) > \cos(\chi_j)$ for $j = 1, \dots, k$, then*

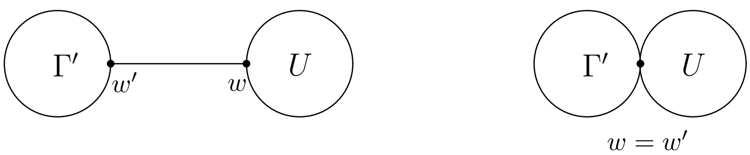
$$\begin{cases} P_{\Gamma,\vec{\phi}}(\lambda) > P_{\Gamma,\vec{\chi}}(\lambda), & \text{if} \quad \operatorname{ind}(\Gamma,\vec{\phi}) < \operatorname{ind}(\Gamma,\vec{\chi}),\\ P_{\Gamma,\vec{\phi}}(\lambda) < P_{\Gamma,\vec{\chi}}(\lambda), & \text{if} \quad \operatorname{ind}(\Gamma,\vec{\phi}) > \operatorname{ind}(\Gamma,\vec{\chi}).\end{cases}$$

This lemma can be easily proved by induction.

Theorem 3. *Let Γ be a cactus. Then $\operatorname{ind}_{\mathbf{S}}\Gamma = \operatorname{ind}\big(\Gamma, (\pi, \pi, \dots, \pi)\big)$.*

Proof. The proof will be carried out by induction on the number of cycles in the graph.

If the graph is unicyclic, the claim is clear. Let it also hold for a cactus with $k-1$ cycles. Consider a cactus with k cycles and having one of the following forms:



where Γ' is a cactus with $(k-1)$ cycles connected to a unicyclic graph U with a bridge (a) or a common vertex (b), with w being a vertex of the cycle C_1 of the graph U. Then the characteristic polynomial for the graph Γ has the following form:

$$P_{\Gamma,\vec{\phi}}(\lambda) = P_{\Gamma',\vec{\phi}'}(\lambda)P_{U-w}(\lambda) - \Big(\sum_{u\sim w,\, u\in V_U} P_{U-w}(\lambda) + 2P_{U-C_1}(\lambda)\cos\phi_1\Big)P_{\Gamma'-w',\vec{\phi}''}(\lambda).$$

Let for some set $(\tilde{\phi}_1, \ldots, \tilde{\phi}_k)$, distinct from $(\pi, \ldots, \pi)$, the index of the $\mathbf{S}$-signed graph $(\Gamma, \vec{\tilde{\phi}})$ be the smallest. Then $P_{\Gamma,(\pi,\ldots,\pi)}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})) < 0$. Consider

$$P_{\Gamma,\vec{\tilde{\phi}}}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}}) - P_{\Gamma,(\pi,\ldots,\pi)}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})).$$

We get

$$\begin{aligned}
&P_{\Gamma,\vec{\tilde{\phi}}}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})) - P_{\Gamma,(\pi,\ldots,\pi)}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})) \\
&\quad = \Big(P_{\Gamma', \vec{\tilde{\phi}}}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})) - P_{\Gamma',(\pi,\ldots,\pi)}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}}))\Big) P_{U-w}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})) \\
&\quad + 2(1-\cos\phi_1) P_{U-C_1}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}}))\Big(P_{\Gamma'-w', \vec{\tilde{\phi}}''}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})) - P_{\Gamma'-w',(\pi,\ldots,\pi)}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}}))\Big).
\end{aligned}$$

Since the graphs Γ' and $\Gamma' - w$ are cactuses with numbers of cycles less than k, by the inductive assumption we have that $\operatorname{ind}(\Gamma, \vec{\tilde{\phi}}) > \operatorname{ind}(\Gamma, (\pi, \ldots, \pi))$. Using Lemma 4 and setting $\vec{\phi} = \vec{\tilde{\phi}}$ and $\vec{\chi} = (\pi, \ldots, \pi)$ we get

$$P_{\Gamma,(\pi,\ldots,\pi)}(\operatorname{ind}(\Gamma, \vec{\tilde{\phi}})) > 0,$$

which is a contradiction. □

Theorem 4. *Let K be a cactus with k cycles, $K \neq C_n$. Suppose that for some τ_0,*

$$\frac{1}{\operatorname{ind} K} \leq \tau_0 \leq \frac{1}{\operatorname{ind}_{\mathbf{S}} K},$$

and

$$\vec{\phi} \in \Sigma_{\tau_0} = \{\vec{\phi} = (\phi_1, \ldots, \phi_k)\colon\ \operatorname{ind}(K, \vec{\phi}) \leq \tau_0^{-1}\} \neq \emptyset$$

there exists a corresponding irreducible one-dimensional configuration $(K, \tau_0, \vec{\phi})$.

Then

$$\dim H = \begin{cases} n, & if \operatorname{ind}(K, \vec{\phi}) < \frac{1}{\tau_0}, \\ n-1, & if \operatorname{ind}(K, \vec{\phi}) = \frac{1}{\tau_0}. \end{cases}$$

Proof. **1.** Let K be a bundle of cycles, i.e., a set of cycles having a common point, such that the cycle C_j contains m_j points, $j = 1, \ldots, k$. Then

$$B_{K,\tau,\phi} = \left(\begin{array}{c|c|c|c}
1 & -\tau_0 e^{i\phi_1} 0 \ldots 0 - \tau_0 & \ldots & -\tau_0 e^{i\phi_g} 0 \ldots 0 - \tau_0 \\
\hline
\begin{matrix} -\tau_0 e^{-i\phi_1} \\ 0 \\ \vdots \\ 0 \\ -\tau_0 \end{matrix} & B_{A_{m_1 - 1}} & \ldots & 0 \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
\begin{matrix} -\tau_0 e^{-i\phi_g} \\ 0 \\ \vdots \\ 0 \\ -\tau_0 \end{matrix} & 0 & \ldots & B_{A_{m_g - 1}}
\end{array}\right).$$

So,

$$\operatorname{rank} B_{K,\vec{\phi},\tau_0} \geq \sum_{j=1}^{k} \operatorname{rank} B_{A_{m_j - 1}}.$$

The indices of the chain satisfy $\operatorname{ind}(A_m) \nearrow 2$ for $m \to \infty$, that is, $\operatorname{ind}(A_{m_j-1}) < 2$ for all $j = 1, \dots, g$.

Let us show that $\operatorname{ind}(K, \vec{\phi}) > 2$. By using the Schwenk formula for the characteristic polynomial for the bundle, we get

$$\begin{aligned} P_K(\lambda) = \lambda \prod_{j=1}^{k} P_{m_j-1}(\lambda) - 2\sum_{i=1}^{k}\Big(\prod_{j=1,\, j\neq i}^{k} P_{m_j-1}(\lambda)\Big) P_{m_i-2}(\lambda) \\ - 2\sum_{i=1}^{k}\Big(\prod_{j=1,\, j\neq i}^{k} P_{m_j-1}(\lambda)\Big)\cos\phi_i, \end{aligned}$$

where $P_{m_j-1}(\lambda)$ is the characteristic polynomial for the adjacency matrix of the graph A_{m_j-1}.

If the graph Γ is a chain with n vertices, then $P_n(2) = n + 1$, see [8]. Then

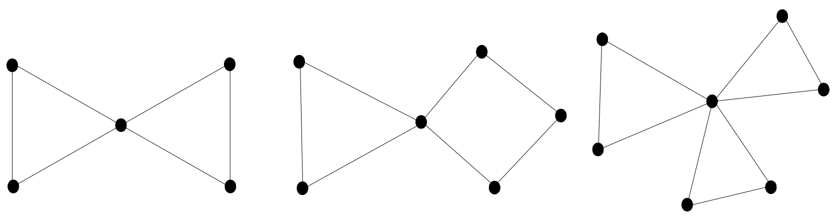
$$\begin{aligned} P_K(2) &= 2m_1 \dots m_k - 2\sum_{i=1}^{k} m_1 \dots m_{i-1} m_{i+1} \dots m_k (m_i - 1) \\ &\quad - 2\sum_{i=1}^{k} m_1 \dots m_{i-1} m_{i+1} \dots m_g \cos\phi_i \\ &= 2(1-k) m_1 \dots m_k + (1 - \cos\phi_1) m_2 \dots m_g + \dots + (1 - \cos\phi_g) m_1 \dots m_{g-1}. \end{aligned}$$

This expression takes its maximal value at $\phi_1 = \phi_2 = \dots = \phi_g = \pi$,

$$\begin{aligned} (1-k) m_1 \dots m_k + 2m_2 \dots m_g + \dots + 2m_1 \dots m_{g-1} \\ = m_1 \dots m_k + (2 - m_1) m_2 \dots m_g + \dots + (2 - m_g) m_1 \dots m_{g-1}. \end{aligned}$$

For sufficiently large values of m_j, $m_j \geq 3$, we have $P_K(2) < 0$. This is true for all bundles except for the following ones:

- $C_{3,3}$, $P_{C_{3,3}}(2) = 3 \cdot 3 - 3 - 3 = 3 > 0$;
- $C_{3,4}$, $P_{C_{3,4}} = 3 \cdot 4 - 2 \cdot 3 - 4 = 2 > 0$;
- $C_{3,3,3}$, $P_{C_{3,3,3}}(2) = 3 \cdot 3 \cdot 3 - 3 \cdot 3 - 3 \cdot 3 - 3 \cdot 3 = 0$.

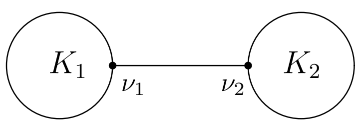


$C_{3,3}$ $C_{3,4}$ $C_{3,3,3}$

This means that $\operatorname{ind}(K, \vec{\phi}) > 2$ for all $\vec{\phi}$, and the claim is proved in this case.

Consider the exceptions. For $C_{3,3}$ and $C_{3,3,3}$, we have $\operatorname{ind} K > 1$, $\operatorname{ind} A_2 = 1$, and, for $C_{3,4}$, $\operatorname{ind}(K, \vec{\phi}) \geq 1.813606503\dots$, and $\operatorname{ind} A_3 = 2\cos\frac{\pi}{4} = \sqrt{2} < \operatorname{ind}(K, \vec{\phi})$.

2. K is a "Christmas tree", that is, a graph which is a cactus such that any two bundles are connected with one edge. Such a graph contains at least one edge that is a bridge,

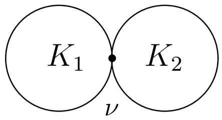


Then

$$P_K(\lambda) = P_{K_1}(\lambda)P_{K_2}(\lambda) - P_{K_1-v_1}(\lambda)P_{K_2-v_2}(\lambda).$$

(1) Let K_1, K_2 be bundles. If $P_{K_1}(\lambda_K) = 0$, then $P_{K_2-v_2}(\mathrm{ind}(K,\vec{\phi})) = 0$ and, hence, $P_{K_2}(\mathrm{ind}(K,\vec{\phi})) = 0$, which is not true for a bundle.
(2) Let K_1 , K_2 be "Christmas trees" such that all brunches have "leaves" (no hanging edges with endpoints having valency 1 are allowed). Similarly to the above, we use induction.
(3) Let K be a "Christmas tree" with hanging edges. The most simple case of such a graph is a unicyclic graph. In this case, we use Theorem 2 and continue by induction.

3. Let K be a "thread with beads"–type graph, which is a cactus consisting only of cycles and not having bridge edges,



In this case,

$$\begin{aligned} P_{K,\vec{\phi}}(\mathrm{ind}(K,\vec{\phi})) &= \lambda P_{K_1-v,\vec{\phi}'_1}(\mathrm{ind}(K,\vec{\phi}))P_{K_2-v,\vec{\phi}'_2}(\mathrm{ind}(K,\vec{\phi})) \\ &- \Big(\sum_{u\sim v,\, u\in V_{K_1}} P_{K_1-v-u,\vec{\phi}''_{1,u}}(\mathrm{ind}(K,\vec{\phi})) \\ &+ 2\sum_{C_j\in\mathcal{C}_1} P_{K_1\setminus C_j,\vec{\phi_{1,j}}'''}(\mathrm{ind}(K,\vec{\phi}))\cos\phi_j\Big) P_{K_2-v,\vec{\phi}'_2}(\mathrm{ind}(K,\vec{\phi})) \\ &- \Big(\sum_{w\sim v,\, w\in V_{K_2}} P_{K_2-v-w,\vec{\phi}''_{2,w}}(\mathrm{ind}(K,\vec{\phi})) \\ &+ 2\sum_{C_i\in\mathcal{C}_2} P_{K_2\setminus C_i,\vec{\phi}'''_{2,i}}(\mathrm{ind}(K,\vec{\phi}))\cos\phi_i\Big) P_{K_1-v,\vec{\phi}'_1}(\mathrm{ind}(K,\vec{\phi})) = 0. \end{aligned}$$

Let, for example, $P_{K_1-v}(\mathrm{ind}(K,\vec{\phi})) = 0$. Then $P_{K_2-v}(\mathrm{ind}(K,\vec{\phi})) = 0$ or

$$\sum_{u\sim v,\, u\in V_{K_1}} P_{K_1-v-u,\vec{\phi}''_{1,u}}(\mathrm{ind}(K,\vec{\phi})) + 2\sum_{C_j\in\mathcal{C}_1} P_{K_1\setminus C_j,\vec{\phi}'''_{1,j}}(\mathrm{ind}(K,\vec{\phi}))\cos\phi_j = 0.$$

(1) Let $P_{K_2-v}(\mathrm{ind}(K,\vec{\phi})) \neq 0$. We remind that $\mathcal{C}_1$ is the set of cycles belonging to the graph Γ_1 that contain the vertex v. In each of the cycles belonging to the set $\mathcal{C}_1$ there exist two vertices $u_j^{(1)}$ and $u_j^{(2)}$ adjacent to v. Without any loss of generality one might assume that $P_{K_1-v-u_j^{(2)},\vec{\phi}''_{1,u_j^{(1)}}}(\mathrm{ind}(K,\vec{\phi})) \geq P_{K_1-v-u_j^{(2)},\vec{\phi}''_{1,u_j^{(2)}}}(\mathrm{ind}(K,\vec{\phi}))$. Define the set a set $\tilde{V}_{K_1} = \{u_j^{(1)} | u_j^{(1)} \in C_j, C_j \in \mathcal{C}_1\}$. Then $|\tilde{V}_{K_1}| = |\mathcal{C}_1|$ and since the induction hypothesis is valid for K_1,

$$\mathrm{ind}(K_1 - v - u_j, \vec{\phi}''_{1,u_1}) > \mathrm{ind}(K \setminus C_j, \vec{\phi}'''_{1,j}).$$

Then

$$\begin{aligned}
&\sum_{u\sim v,\, u\in V_{K_1}} P_{K_1-v-u,\vec{\phi}''_{1,u}}(\operatorname{ind}(K,\vec{\phi}))\\
&\quad + 2\sum_{C_j\in\mathcal{C}_1} P_{K_1\setminus C_j,\vec{\phi}'''_{1,j}}(\operatorname{ind}(K,\vec{\phi}))\cos\phi_j\\
&\quad \geq 2\Bigg(\sum_{u\sim v,\, u\in \tilde{V}_{K_1}} P_{K_1-v-u,\vec{\phi}''_{1,u}}(\operatorname{ind}(K,\vec{\phi}))\\
&\quad + \sum_{C_j\in\mathcal{C}_1} P_{K_1\setminus C_j,\vec{\phi}'''_{1,j}}(\operatorname{ind}(K,\vec{\phi}))\cos\phi_j\Bigg) > 0,
\end{aligned}$$

which is a contradiction.

(2) Let now $P_{K_2-v}(\operatorname{ind}(K,\vec{\phi})) = 0$. Let us first show that, by adding a cycle to a "thread with beads", the index of the **S**-signed graph strictly increases. The proof of this will be carried out by induction starting with the simplest case where K_1 is a bundle of $k-1$ cycles such that there is a vertex v of the bundle, distinct from the common vertex, that is identified with a vertex of the k-th cycle C_k. The characteristic polynomial for such a cactus will be

$$P_{K,\vec{\phi}}(\lambda) = P_{K_1,\vec{\phi}_1}(\lambda)P_{m_k-1}(\lambda) - 2(P_{m_k-2}(\lambda)+\cos\phi_k)P_{K_1-v,\vec{\phi}'_1}(\lambda).$$

For $\lambda = \operatorname{ind}(K_1,\vec{\phi}_1)$, we get

$$P_{K,\vec{\phi}}(\operatorname{ind}(K_1,\vec{\phi}_1)) = -2(P_{m_k-2}(\operatorname{ind}(K_1,\vec{\phi}_1))+\cos\phi_k)P_{K_1-v,\vec{\phi}'_1}(\operatorname{ind}(K_1,\vec{\phi}_1)) \leq 0,$$

that is, $P_{K,\vec{\phi}}(\operatorname{ind}(K_1,\vec{\phi}_1)) = 0$ if and only if $P_{K_1-v,\vec{\phi}'_1)}(\operatorname{ind}(K_1,\vec{\phi}_1)) = 0$, but since K_1 was assumed to be a bundle, we get that $P_K(\operatorname{ind}(K_1,\vec{\phi}_1)) < 0$ and $\operatorname{ind}(K,\vec{\phi}) > \operatorname{ind}(K_1,\vec{\phi}_1)$.

Continuing now by induction we prove the needed claim that, by adding a cycle to a "thread with beads", the **S**-index of the signed graph strictly increases.

The proof in the general case is finished by induction. □

Theorem 5. *Let Γ be a bundle of k cycles C_j of lengths m_j.*

- *If $\tau < \frac{1}{\operatorname{ind}\Gamma}$, then for the pair (Γ,ϕ) there exists a simple system $S_{\tau,\phi}$ of one-dimensional subspaces for any $\phi\in[0,2\pi)$, and* $\dim H = n$.
- *If $\tau = \frac{1}{\operatorname{ind}\Gamma}$, then there exists an infinite family $S_{\tau,\phi}$ parametrized with $\vec{\phi}\in[0,2\pi)\times\cdots\times[0,2\pi)$ of irreducible simple configurations, and* $\dim H = n$ *for all $\vec{\phi}\neq\vec{0}$, and* $\dim H = n-1$ *for $\vec{\phi} = (0,0,\dots,0)$.*
- *If $\frac{1}{\operatorname{ind}\Gamma} < \tau < \frac{1}{\operatorname{ind}_{\mathbf{S}}\Gamma}$, then there exist infinite families $S_{\tau,(\phi_1,\dots,\phi_k)}$ of irreducible simple configurations, where*

$$\begin{aligned}
(\phi_1,\dots,\phi_k)\in\Phi_\tau = \Big\{(\phi_1,\dots,\phi_k)\;\Big|\;& \tau^{-1}\prod_{j=1}^{k}P_{m_j-1}(\tau^{-1})\\
&-2\sum_{i=1}^{k}\Big(\prod_{j=1,\,j\neq i}P_{m_j-1}(\tau^{-1})\Big)P_{m_i-2}(\tau^{-1})\\
&\qquad > 2\sum_{i=1}^{k}\Big(\prod_{j=1,\,j\neq i}P_{m_j-1}(\tau^{-1})\Big)\cos\phi_i\Big\}.
\end{aligned}$$

- *If $\tau = \frac{1}{\mathrm{ind_S}\,\Gamma}$, then there exists a unique configuration S corresponding to Γ such that $\phi_i = \pi$ for all $i = 1, \dots, k$, and dimension of the space equals $n-2$ if the graph is a cycle, and equals $n-1$ in other cases.*
- *If $\tau > \frac{1}{\mathrm{ind_S}\,\Gamma}$, then there are no corresponding configurations.*

Proof. The proof is similar to the proof of Theorem 2. Note that the set Φ_τ is nonempty for all τ, $\tau \leq \frac{1}{\mathrm{ind_S}\,\Gamma}$. In particular, if Γ is a bundle of cycles of equal lengths, then the set Φ_τ is defined by

$$\Phi_\tau = \Big\{(\phi_1, \dots, \phi_k) \;\Big|\; \sum_{i=1}^{k} \cos\phi_i < \frac{1}{2\tau} P_{m-1}\big(\frac{1}{\tau}\big) - kP_{m-2}\big(\frac{1}{\tau}\big)\Big\}.$$

□

References

1. A. V. Strelets, Yu. S. Samoilenko, *On simple n-tuples of a Hilbert space*, Ukrain. Mat. Zh. **61** (2009), no. 1, 48–60.
2. Yu. S. Samoilenko, S. A. Kruglyak, *On complexity of description of representations of $*$-algebras generated by idempotents*, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1655–1664.
3. Yu. S. Samoilenko, S. A. Kruglyak, *On unitary equivalence of collections of self-adjoint operators*, Funktsional. Anal. i Prilozhen. **14** (1998), no. 1, 60–62.
4. V. L. Ostrovskyi, Yu. S. Samoilenko, *Introduction to the Theory of Representations of Finitely Presented $*$-Algebras,* I. *Representations by Bounded Operators*, Rev. Math. Math. Phys., **11**, Harwood Academic Publishers, Amsterdam, 1999.
5. J. J. Seidel, P. W. H. Lemmens, *Equiangular lines*, J. Algebra **24** (1973), 494–512.
6. N. D. Popova, M. A. Vlasenko, *On configurations of subspaces of a Hilbert space with fixed angles between the subspaces*, Ukrain. Mat. Zh. **56** (2004), no. 5, 606–615.
7. Yu. S. Samoilenko, R. V. Grushevoy, *Systems of one-dimensional subspaces of a Hilbert space*, Methods Funct. Anal. Topology **16** (2010), no. 2, 131–139.
8. Z. Radosavljevic' and M. Rasajski, *Multicyclic treelike reflexive graphs*, Discrete Math. **296** (2005), no. 1, 43–57.
9. N. D. Popova, Yu. S. Samoilenko, *On the existence of configurations of subspaces in a Hilbert space with fixed angles*, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA) **55** (2006), 1–5.
10. A. J. Schwenk, *Computing the characteristic polynomial of a graph*, Lecture Notes in Mathematics, **406**, Springer, Berlin, 1974, pp. 153–172.
11. N. D. Popova, *On finite-dimensional representations of one algebra of Temperley-Lieb type*, Methods Funct. Anal. Topology **7** (2001), no. 3, 80-92.
12. P. J. Cameron, J. J. Seidel, and S. V. Tsaranov, *Signed graphs, root lattices, and Coxeter groups*, J. Algebra **164** (1994), 173–209.
13. S. Simic, *On the largest eigenvalue of unicyclic graphs*, Publ. Inst. Mat. (Beograd) **42** (1987), 13-19.

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Received 05/03/2010