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ON FINE STRUCTURE OF SINGULARLY CONTINUOUS PROBABILITY MEASURES AND RANDOM VARIABLES WITH INDEPENDENT $\widetilde{Q}$-SYMBOLS

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Abstract. We introduce a new fine classification of singularly continuous probability measures on R^1 on the basis of spectral properties of such measures (topological and metric properties of the spectrum of the measure as well as local behavior of the measure on subsets of the spectrum). The theorem on the structural representation of any one-dimensional singularly continuous probability measure in the form of a convex combination of three singularly continuous probability measures of pure spectral type is proved.

We introduce into consideration and study a $\widetilde{Q}$-representation of real numbers and a family of probability measures with independent $\widetilde{Q}$-symbols. Topological, metric and fractal properties of the above mentioned probability distributions are studied in details. We also show how the methods of $\widetilde{P}-\widetilde{Q}$-measures can be effectively applied to study properties of generalized infinite Bernoulli convolutions.

1. Introduction

As is well known there exist only three types of pure probability distributions: discrete, absolutely continuous and singularly continuous (w.r.t. the Lebesgue measure). During a long period mathematicians had a rather low interest in singular probability distributions, which was mainly caused by the following two reasons: the absence of effective analytic tools and the widely spread point of view that such distributions do not have any applications, in particular in physics, and are interesting only for theoretical reasons. The interest in singularly continuous probability distributions increased however in 1990's due their deep connections with the theory of fractals. On the other hand, recent investigations show that singularity is generic for many classes of random variables, and absolutely continuous and discrete distributions arise only in exceptional cases (see, e.g. [12, 19]). Possible applications in the spectral theory of self-adjoint operators [18] is an additional reason in the intensive investigation of singularly continuous measures. It was proved that Schrödinger type operators with singular continuous spectra are generic for some classes of potentials [6]. Moreover, by using the fractal analysis of the corresponding spectral singularly continuous measures, it is possible to analyze the dynamical properties of the corresponding quantum systems [10].

This paper is devoted to the study and classification of one-dimensional singular measures. Unfortunately, usually singular probability distributions are associated only with Cantor-like distributions. We show that even in the R^1 case the family of singularly continuous probability measures is rather rich and diverse. In this paper we introduce into consideration three pure spectral types of singularly continuous probability measures

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and prove that any singularly continuous measure on the real line can be represented as a convex combination of probability measures of the above spectral types.

To give a simple way to construct classes of singularly continuous probability measures we introduce into consideration and study the $\widetilde{Q}$-representation of real numbers (which is a convenient tool for the construction of a wide class of fractals) and a family of probability measures with independent $\widetilde{Q}$-symbols. Topological, metric and fractal properties of the above mentioned probability distributions are studied in details. This family contains all possible pure spectral types of singular continuous measures, and (as a very particular case) the class of all self-similar measures on $[0,1]$ satisfying the open set condition (see Section 3 for details).

We also show how the methods of $\widetilde{P}-\widetilde{Q}$-measures can be effectively applied to study properties of generalized infinite Bernoulli convolutions (see, e.g., [2, 11, 14] for the survey on Bernoulli convolutions, related applications and problems).

An additional reason for the investigation of the distribution of the random variables with independent $\widetilde{Q}$-symbols is to extend the famous Jessen-Wintner theorem (see, e.g., [8]) to the case of sums of random variables which are not independent.

The paper is organized as follows. In Section 2 we study $\widetilde{Q}$-representation of real numbers and properties of related fractal sets. In Section 3 we study the structure and properties of probability measures with independent $\widetilde{Q}$-symbols and show how the obtained results can be applied to study properties of generalized infinite Bernoulli convolutions. Section 4 is devoted to a classification of singularly continuous measures and fine structure of such measures.

2. $\widetilde{Q}$-representation of real numbers and related fractals

Let us consider an $\mathbf{N}_k\times\mathbf{N}-$matrix $\widetilde{Q}=\|q_{ik}\|$, $i\in\mathbf{N}_k$, $k\in\mathbf{N}$, where $\mathbf{N}_k=\{0,1,\dots,N_k\}$, with $0<N_k\le\infty$. We suppose that $\sum_{i\in\mathbf{N}_k}q_{ik}=1$, ; $q_{ik}>0$, $\forall i\in\mathbf{N}_k$, $k\in\mathbf{N}$; and

$$\prod_{k=1}^{\infty}\max_{i\in\mathbf{N}_k}\{q_{ik}\}=0. \tag{1}$$

Given a $\widetilde{Q}-$matrix we consecutively perform decompositions of the segment $[0,1]$ as follows.

Step 1. We decompose $[0,1]$ (from the left to the right) into the union of closed intervals $\Delta^{\widetilde{Q}}_{i_1}$, $i_1\in\mathbf{N}_1$ (without common interior points) of the length $\left|\Delta^{\widetilde{Q}}_{i_1}\right|=q_{i_11}$,

$$[0,1]=\bigcup_{i_1\in\mathbf{N}_1}\Delta^{\widetilde{Q}}_{i_1}.$$

Each interval $\Delta^{\widetilde{Q}}_{i_1}$ is called a *1-rank interval*.

Step $k\ge 2$. We decompose (from the left to the right) each closed $(k-1)$-rank interval $\Delta^{\widetilde{Q}}_{i_1i_2\dots i_{k-1}}$ into the union of closed $k-$*rank intervals* $\Delta^{\widetilde{Q}}_{i_1i_2\dots i_k}$,

$$\Delta^{\widetilde{Q}}_{i_1i_2\dots i_{k-1}}=\bigcup_{i_k\in\mathbf{N}_k}\Delta^{\widetilde{Q}}_{i_1i_2\dots i_k},$$

where their lengths

$$\left|\Delta^{\widetilde{Q}}_{i_1i_2\dots i_k}\right|=q_{i_11}\cdot q_{i_22}\cdots q_{i_kk}=\prod_{s=1}^{k}q_{i_ss} \tag{2}$$

are related as follows

$$\left|\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_{k-1} 0}\right| : \left|\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_{k-1} 1}\right| : \cdots : \left|\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_{k-1} i_k}\right| : \cdots = q_{0k} : q_{1k} : \cdots : q_{i_k k} : \cdots$$

For any sequence of indices $\{i_k\}$, $i_k \in \mathbf{N}_k$, there corresponds the sequence of embedded closed intervals

$$\Delta^{\widetilde{Q}}_{i_1} \supset \Delta^{\widetilde{Q}}_{i_1 i_2} \supset \cdots \supset \Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_k} \supset \cdots$$

such that $|\Delta^{\widetilde{Q}}_{i_1 \ldots i_k}| \to 0$, $k \to \infty$, due to (1) and (2). Therefore, there exists a unique point $x \in [0,1]$ belonging to all intervals $\Delta^{\widetilde{Q}}_{i_1}$, $\Delta^{\widetilde{Q}}_{i_1 i_2}$, $\ldots$, $\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_k}, \ldots$ Conversely, for any point $x \in [0,1)$ there exists a sequence of embedded intervals $\Delta^{\widetilde{Q}}_{i_1} \supset \Delta^{\widetilde{Q}}_{i_1 i_2} \supset \cdots \supset \Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_k} \supset \cdots$ containing x, i.e.,

$$x = \bigcap_{k=1}^{\infty} \Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_k} = \bigcap_{k=1}^{\infty} \Delta^{\widetilde{Q}}_{i_1(x) i_2(x) \ldots i_k(x)} =: \Delta^{\widetilde{Q}}_{i_1(x) i_2(x) \ldots i_k(x) \ldots} \tag{3}$$

Notation (3) is called the $\widetilde{Q}-$*representation* of the point $x \in [0,1]$.

Remark 1. The correspondence $[0,1] \in x \Leftrightarrow \{(i_1(x), i_2(x), \ldots, i_k(x), \ldots)\}$ in (3) is one-to-one, i.e., the $\widetilde{Q}-$representation is unique for every point $x \in [0,1]$, provided that the $\widetilde{Q}$-matrix contains an infinite number of columns with an infinite number of elements. However in the case, where $N_k < \infty$, $\forall k > k_0$ for some k_0, there exists a countable set of points $x \in [0,1]$ having two different $\widetilde{Q}-$representations. Precisely, this is the set of all end-points of intervals $\Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_k}$ with $k > k_0$.

Remark 2. If $q_{ik} = q_i$, $k \in \mathbf{N}$, then the $\widetilde{Q}-$representation coincides with the $Q-$representation (see [15]); moreover, if $q_{ik} = \frac{1}{s}$ for some natural number $s > 1$, then the $\widetilde{Q}-$representation coincides with the classical $s-$adic expansion.

The $\widetilde{Q}$-representation allows to construct in a convenient way a wide class of fractals on R^1 and other mathematical objects with fractal properties. Firstly we consider compact fractals from R^1. Let $\mathbf{V} := \{\mathbf{V}_k\}_{k=1}^{\infty}$, $\mathbf{V}_k \subseteq \mathbf{N}_k$, and let us consider the set

$$\Gamma_{\widetilde{Q}(\mathbf{V})} \equiv \Gamma := \left\{ x \in [0,1] : \; x = \Delta^{\widetilde{Q}}_{i_1 i_2 \ldots i_k \ldots}, \; i_k \in \mathbf{V}_k \right\}, \tag{4}$$

i.e., Γ consists of points, which can be $\widetilde{Q}-$represented by using only symbols i_k from the set $\mathbf{V}_k$ on each k-th position of their $\widetilde{Q}-$representation.

If $\mathbf{V}_k \neq \mathbf{N}_k$ at least for one $k < k_0$, and $\mathbf{V}_k = \mathbf{N}_k$ for all $k \geq k_0$ with some fixed $k_0 > 1$, then Γ is a union of closed intervals. In this case one can get Γ removing from $[0,1]$ all open intervals $\dot{\Delta}^{\widetilde{Q}}_{i_1 \ldots i_k}$, $k < k_0$ with $i_k \notin \mathbf{V}_k$ (where a point over Δ means that an interval is open). If the condition $\mathbf{V}_k \neq \mathbf{N}_k$ holds for infinitely many values of k, then obviously Γ is a nowhere dense set.

We shall study the metric properties of the sets $\Gamma_{\widetilde{Q}(\mathbf{V})}$. Let $S_k(\mathbf{V})$ denote the sum of all elements q_{ik} such that $i_k \in \mathbf{V}_k$, i.e., $S_k(\mathbf{V}) := \sum_{i \in \mathbf{V}_k} q_{ik}$. We note that $0 < S_k(\mathbf{V}) \leq 1$.

Lemma 1. *The Lebesgue measure* $\lambda(\Gamma)$ *of the set* Γ *is equal to*

$$\lambda(\Gamma) = \prod_{k=1}^{\infty} S_k(\mathbf{V}). \tag{5}$$

Proof. Let $\Gamma_n := \bigcup_{i_k \in \mathbf{V}_k} \Delta_{i_1 \dots i_n}$. It is easy to see that $\Gamma_n \subseteq \Gamma_{n-1}$ and $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$. From the definition of the sets Γ_n and from (2), it follows that $\lambda(\Gamma_n) = \prod_{k=1}^{n} S_k(\mathbf{V})$, and, therefore, $\lambda(\Gamma) = \lim_{n \longrightarrow \infty} \lambda(\Gamma_n) = \prod_{k=1}^{\infty} S_k(\mathbf{V})$. □

Corollary. *Let $W_k(\mathbf{V}) = 1 - S_k(\mathbf{V}) \geq 0$. The set Γ is of zero Lebesgue measure if and only if*

$$\sum_{k=1}^{\infty} W_k(\mathbf{V}) = \infty \ . \tag{6}$$

The above mentioned procedure allows to construct nowhere dense compact fractal sets E with a desirable Hausdorff-Besicovitch dimension (including the anomalously fractal case ($\alpha_0(E) = 0$) and the superfractal case ($\alpha_0(E) = 1$)) in a very easy formal way.

Theorem 1. *Let $\mathbf{N}_k = N_{s-1}^0 := \{0, 1, \dots, s-1\}$ $k \in N$, let $\mathbf{V_0} = \{v_1, v_2, \dots, v_m\} \subset N_{s-1}^0$ and let the matrix $\widetilde{Q}$ have the following asymptotic property:*

$$\lim_{k \to \infty} q_{ik} = q_i, \quad i \in N_{s-1}^0.$$

Then

1) the Hausdorff-Besicovitch dimension of the set $\Gamma_{\widetilde{Q}(\mathbf{V_0})}$ coincides with the root of the following equation:

$$\sum_{i \in \mathbf{V_0}} q_i^x = 1; \tag{7}$$

2) if

$$M[\widetilde{Q}, (\nu_0, \dots, \nu_{s-1})] = \left\{ x : \Delta^{\widetilde{Q}}_{\alpha_1(x) \dots \alpha_k(x) \dots}, \ \lim_{k \to \infty} \frac{N_i(x,k)}{k} = \nu_i, \ i \in N_{s-1}^0 \right\},$$

where $N_i(x,k)$ is the number of symbols "i" among the first k symbols of the $\widetilde{Q}$-representation of x, then

$$\alpha_0(M[\widetilde{Q}, (\nu_0, \dots, \nu_{s-1})]) = \frac{\sum_{i=0}^{s-1} \nu_i \ln \nu_i}{\sum_{i=0}^{s-1} \nu_i \ln q_i}. \tag{8}$$

Proof. Firstly we consider the particular case where the matrix $\widetilde{Q}$ has exactly s rows and all its columns are the same: $q_{ik} = q_i$. In such a simple case the $\widetilde{Q}$-representation reduces to the Q-representation studied in [12]. One can prove (see, e.g., [15]), that to calculate the Hausdorff-Besicovitch dimension of any subset $E \subset [0,1]$ it is sufficient to consider a class of cylinder sets of different ranks generated by Q-partitions of the unit interval. The Billingsley theorem (see, e.g., [4], p. 141) admits a generalization to the class of Q-cylinders, and, from this theorem it follows that in the case of usual Q-representation, the Hausdorff-Besicovitch dimension of the set $M[Q, (\nu_0, \dots, \nu_{s-1})]$ is equal to the right-side expression in (8).

In the Q-case the set $\Gamma_{Q(\mathbf{V_0})}$ is a self-similar set satisfying the open set condition. Therefore, the Hausdorff-Besicovitch dimension of this set is the root of equation (7).

Now let us consider a general case of theorem 1. To this end we introduce into consideration the following transformation f of $[0,1]$:

$$f(x) = f(\Delta^{Q}_{\alpha_1(x) \dots \alpha_k(x) \dots}) = \Delta^{\widetilde{Q}}_{\alpha_1(x) \dots \alpha_k(x) \dots}.$$

This transformation belongs to the DP-class (see, e.g., [1, 17]), i.e., f preserves the Hausdorff-Besicovitch dimension of any subset of $[0,1]$. Since $f(\Gamma_{Q(\mathbf{V_0})}) = \Gamma_{\widetilde{Q}(\mathbf{V_0})}$ and $f(M[Q,(\nu_0,\ldots,\nu_{s-1})]) = M[\widetilde{Q},(\nu_0,\ldots,\nu_{s-1})]$, we get the desired formulas under general assumptions of theorem 1. $\square$

Example 1. If $\mathbf{N}_k = \{0,1,2\}, \mathbf{V}_k = \{0,2\}, q_{1k} \to 0$, but $\sum_{k=1}^{\infty} q_{1k} = \infty$ with $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, then Γ is a nowhere dense set of zero Lebesgue measure. From Theorem 1 it follows that the Hausdorff dimension of this set is equal 1. In the terminology of [12] a set of this kind is called a superfractal set.

Example 2. If $\mathbf{N}_k = \{0,1,2\}, \mathbf{V}_k = \{0,2\}$, $q_{1k} \to 1$ (but $\prod_{k=1}^{\infty} q_{1k} = 0$), and $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, then Γ is a nowhere dense set of zero Lebesgue measure and of zero Hausdorff dimension, i.e., Γ is an anomalously fractal set (see [12]).

3. Random variables with independent $\widetilde{Q}$−symbols

Let $\{\xi_k\}$ be a sequence of independent random variables with the following distributions:

$$P(\xi_k = i) := p_{ik} \geq 0 \quad \text{with} \quad \sum_{i\in\mathbf{N}_k} p_{ik} = 1, \quad k \in \mathbf{N}.$$

By using ξ_k and the $\widetilde{Q}$-representation we construct a random variable ξ as follows:

$$\xi := \Delta^{\widetilde{Q}}_{\xi_1\xi_2\ldots\xi_k\ldots} \, . \tag{9}$$

The distribution of ξ is completely determined by two matrices: $\widetilde{Q}$ and $\widetilde{P} = ||p_{ik}||$, where some elements of the matrix $\widetilde{P}$ possibly are equal to zero. Of course, all sets $\mathbf{N}_k$ are the same as those in the $\widetilde{Q}$-matrix. Let μ_ξ be the measure corresponding the distribution of the random variable ξ with independent $\widetilde{Q}$-symbols.

If $q_{ik} = q_i$ and $p_{ik} = p_i$ $\forall j \in N$, $i \in N^0_{s-1}$ (i.e., ξ is a random variable with independent identically distributed Q-digits), then the measure μ_ξ is the self-similar measure associated with the list $(S_0,\ldots,S_{s-1},p_0,\ldots,p_{s-1})$, where S_i is the similarity with the ratio q_i ($\sum_{i=0}^{s-1} q_i = 1$), and the list $(S_0,\ldots,S_{s-1})$ satisfies the open set condition. More precisely, μ_ξ is the unique Borel probability measure on $[0,1]$ such that

$$\mu_\xi = \sum_{i=0}^{s-1} p_i \cdot \mu_\xi \circ S_i^{-1}$$

(see, e.g., [7] for details). In the so-called "Q^*- case" we construct the measure μ_ξ in a similar way but with the possibility of changing of the ratios and probabilities from the list $(S_0,\ldots,S_{s-1},p_0,\ldots,p_{s-1})$ at each stage of the construction. In our general "$\widetilde{Q}-$ case" we may additionally choose the number of contracting similarities (including a countable number) at each stage of the construction.

The random variable ξ can be represented as a sum of an a.s. convergent series of discretely distributed random variables *which are not independent.* Nevertheless the distribution of ξ is of pure type.

Theorem 2. *The measure μ_ξ is of pure type, i.e., it is either purely absolutely continuous, resp., purely point, resp., purely singular continuous. Precisely,*

1) μ_ξ is purely absolutely continuous if and only if

$$\rho := \prod_{k=1}^{\infty} \Big\{ \sum_{i\in\mathbf{N}_k} \sqrt{p_{ik}\cdot q_{ik}} \Big\} > 0; \tag{10}$$

2) μ_ξ is purely point if and only if

$$P_{max} := \prod_{k=1}^{\infty} \max_{i \in \mathbf{N}_k} \{p_{ik}\} > 0; \tag{11}$$

3) μ_ξ is purely singularly continuous in all other cases, i.e., if and only if

$$\rho = 0 = P_{\max}. \tag{12}$$

Proof. Let $\Omega_k = \mathbf{N}_k$, $\mathcal{A}_k = 2^{\Omega_k}$. We define measures μ_k and ν_k in the following way:

$$\mu_k(i) = p_{ik}; \quad \nu_k(i) = q_{ik}, \quad i \in \Omega_k.$$

Let

$$(\Omega, \mathcal{A}, \mu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, \mu_k), \quad (\Omega, \mathcal{A}, \nu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, \nu_k)$$

be the infinite products of probability spaces, and let us consider the measurable mapping $f : \Omega \to [0; 1]$ defined as follows:

$$\forall \omega = (\omega_1, \omega_2, \ldots, \omega_k, \ldots) \in \Omega, \quad f(\omega) = x = \Delta_{i_1(x) i_2(x) \ldots i_k(x) \ldots}$$

with $\omega_k = i_k(x)$, $k \in N$.

We define the measures μ^* and ν^* as the image measure of μ resp. ν under f:

$$\mu^*(B) := \mu(f^{-1}(B)), \quad \nu^*(B) = \nu(f^{-1}(B)), \quad B \in \mathcal{B}.$$

It is easy to see that ν^* coincides with Lebesgue measure λ on $[0, 1]$, and $\mu^* \equiv \mu_\xi$. In general, the mapping f is not bijective, but there exists a countable set Ω_0 such that $\nu(\Omega_0) = 0, \mu(\Omega_0) = 0$ and the mapping $f : \Omega \setminus \Omega_0 \to [0, 1]$ is bijective.

Therefore, the measure μ_ξ is absolutely continuous (singular) with respect to the Lebesgue measure if and only if the measure μ is absolutely continuous (singular) with respect to the measure ν. Since, $q_{ik} > 0$, we conclude that $\mu_k \ll \nu_k$, $\forall k \in N$. By using Kakutani's theorem [9], we have

$$\mu_\xi \ll \lambda \;\Leftrightarrow\; \prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k > 0 \;\Leftrightarrow\; \prod_{k=1}^{\infty} \Big(\sum_{i \in \mathbb{N}_k} \sqrt{p_{ik} q_{ik}} \Big) > 0, \tag{13}$$

$$\mu_\xi \perp \lambda \;\Leftrightarrow\; \prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k = 0 \;\Leftrightarrow\; \prod_{k=1}^{\infty} \Big(\sum_{i \in \mathbb{N}_k} \sqrt{p_{ik} q_{ik}} \Big) = 0. \tag{14}$$

Of course, a singularly distributed random variable ξ can also be distributed discretely. For any point $x \in [0, 1]$ the set $f^{-1}(x)$ consists of at most two points from Ω. Therefore, the measure μ_ξ is an atomic measure if and only if the measure μ is atomic.

If $\prod\limits_{k=1}^{\infty} \max\limits_i p_{ik} = 0$, then

$$\mu(\omega) = \prod_{k=1}^{\infty} p_{\omega_k k} \le \prod_{k=1}^{\infty} \max_i p_{ik} = 0 \quad \text{for any} \quad \omega \in \Omega,$$

and μ is continuous.

If $\prod\limits_{k=1}^{\infty} \max\limits_i p_{ik} > 0$, then we consider the subset $A_+ = \{\omega : \mu(\omega) > 0\}$. The set A_+ contains the point $\omega^* = (\omega_1^*, \omega_2^*, \ldots, \omega_k^*, \ldots)$ such that $p_{\omega_k^* k} = \max\limits_i p_{ik}$. It is easy to see that for all $\omega \in A_+$ the condition $p_{\omega_k k} \ne \max\limits_i p_{ik}$ holds only for a finite amount of values

k. This means that A_+ is a countable set and the event "$\omega \in A_+$" does not depend on any finite coordinates of ω. Therefore, by using Kolmogorov's "0 and 1" theorem, we conclude that $\mu(A_+) = 0$ or $\mu(A_+) = 1$. Since $\mu(A_+) \geq \mu(\omega^*) > 0$, we have $\mu(A_+) = 1$, which proves the discreteness of the measure μ. □

Remark 3. If there exists a positive number q^+ such that $q_{ik} \geq q^+, \forall k \in \mathbf{N}$, $\forall i \in \mathbf{N}_k$, then condition (13) is equivalent to the convergence of the following series:

$$\sum_{k=1}^{\infty} \Big\{ \sum_{i \in \mathbf{N}_k} (1 - \frac{p_{ik}}{q_{ik}})^2 \Big\} < \infty. \tag{15}$$

If $\lim_{\overline{k \to \infty}} q_{ik} = 0$, then, generally speaking, conditions (13) and (15) are not equivalent. For example, let us consider the matrices $\widetilde{Q}$ and $\widetilde{P}$ as follows: $\mathbf{N}_k = \{0, 1, 2\}, q_{1k} = \frac{1}{2^k}, \quad q_{0k} = q_{2k} = \frac{1 - q_{1k}}{2}, \quad p_{1k} = 0, \qquad p_{0k} = p_{2k} = \frac{1}{2}$. In this case condition (13) holds, but (15) does not hold.

Let us show how the obtained results can be applied to the study of generalized infinite Bernoulli convolutions, i.e., probability distributions of the random variables of the following form:

$$\eta = \sum_k \eta_k a_k, \tag{16}$$

where $\{\eta_k\}$ is a sequence of independent random variables taking values 0 and 1 with probabilities p'_{0k} and p'_{1k} respectively, and $\sum_{k=1}^{\infty} a_k$ is a convergent positive series.

For the simplicity let us firstly consider the case where $a_k > r_k := a_{k+1} + a_{k+2} + \cdots$, $\forall k \in N$.

In such a case the probability measure μ_η is a measure with independent $\widetilde{Q}$-symbols, and the matrices $\widetilde{Q}$ and $\widetilde{P}$ are of the following form:

$$q_{0k} = \frac{r_k}{r_{k-1}}, \quad q_{1k} = \frac{a_k - r_k}{r_{k-1}}, \quad q_{2k} = \frac{r_k}{r_{k-1}},$$

$$p_{0k} = p'_{0k}, \quad p_{1k} = 0, \quad p_{2k} = p'_{1k}.$$

Applying the previous theorem, we obtain the following conclusions.

Proposition 1. If $a_k > r_k, \forall k \in N$, then

the random variable η is purely discretely distributed if and only if

$$\prod_{k=1}^{\infty} \max_i p'_{ik} > 0;$$

the random variable η is purely absolutely continuously distributed if and only if

$$\prod_{k=1}^{\infty} \sum_i \sqrt{p'_{ik} \frac{r_k}{r_{k-1}}} = \lim_{k \to \infty} \sqrt{r_k} \prod_{j=1}^{k} (\sqrt{p'_{0j}} + \sqrt{p'_{1j}}) > 0;$$

the random variable η is purely singularly continuously distributed in all other cases.

A relative simplicity of the latter class of Bernoulli convolutions ($a_k > r_k, \forall k \in N$) can be explained by the following observations: two cylindrical sets of rank k (i.e., sets of the form $[c_1 a_1 + \cdots + c_k a_k,\ c_1 a_1 + \cdots + c_k a_k + r_k], c_i \in \{0, 1\}$) either coincide or they have no common interior points. Sometimes such Bernoulli convolutions are said to be Bernoulli convolutions without "large intersections".

By using the $\widetilde{P}-\widetilde{Q}$ approach it is also possible to analyze properties of Bernoulli convolutions with "large intersections". As an example, let us consider the case where

$$\begin{cases} a_{3k-2}=a_{3k-1}+a_{3k}, \\ r_{3k-1}<a_{3k-1}, \\ r_{3k}<a_{3k},\ k\in N. \end{cases} \tag{17}$$

In such a case the probability measure μ_η is also a measure with independent $\widetilde{Q}$-symbols. The matrix $\widetilde{Q}$ is of the following form:

$$q_{0k}=q_{2k}=q_{4k}=q_{6k}=q_{8k}=q_{10,k}=q_{12,k}=\frac{r_{3k}}{r_{3k-3}},$$

$$q_{1k}=q_{5k}=q_{7k}=q_{11,k}=\frac{a_{3k}-r_{3k}}{r_{3k-3}},$$

$$q_{3k}=q_{9k}=\frac{a_{3k-1}-r_{3k-1}}{r_{3k-3}};$$

and the matrix $\widetilde{P}$ has the following structure:

$$p_{1k}=p_{3k}=p_{5k}=p_{7k}=p_{9k}=p_{11,k}=p_{13,k}=0,$$

$$p_{0k}=p^{'}_{0,3k-2}p^{'}_{0,3k-1}p^{'}_{0,3k},\quad p_{2k}=p^{'}_{0,3k-2}p^{'}_{0,3k-1}p^{'}_{1,3k},\quad p_{4k}=p^{'}_{0,3k-2}p^{'}_{1,3k-1}p^{'}_{0,3k},$$

$$p_{6k}=p^{'}_{0,3k-2}p^{'}_{1,3k-1}p^{'}_{1,3k}+p^{'}_{1,3k-2}p^{'}_{0,3k-1}p^{'}_{0,3k},$$

$$p_{8k}=p^{'}_{1,3k-2}p^{'}_{0,3k-1}p^{'}_{1,3k},\quad p_{10,k}=p^{'}_{1,3k-2}p^{'}_{1,3k-1}p^{'}_{0,3k},\quad p_{12,k}=p^{'}_{1,3k-2}p^{'}_{1,3k-1}p^{'}_{1,3k}.$$

Applying the previous theorem, we get necessary and sufficient conditions for the discreteness, absolute continuity and singular continuity. Taking into account that for the above matrices $\widetilde{Q}$ and $\widetilde{P}$ the infinite product $\prod\limits_{k=1}^{\infty}(\sum_i\sqrt{p_{ik}q_{ik}})$ always diverges to zero, we obtain the following conclusion.

Proposition 2. If condition (17) holds, then the random variable η is either purely discretely distributed (if $\prod\limits_{k=1}^{\infty}\max_i p^{'}_{ik}>0$) or it is purely singularly continuously distributed (in all other cases).

4. On fine structure of singularly continuous probability measures

Let us remind that the set $S_\mu=\{x:\ \mu(x-\varepsilon,x+\varepsilon)>0,\forall\varepsilon>0\}$ is said to be the spectrum (topological support) of a measure μ. It is the minimal closed support of μ.

Definition 1. A singularly continuous probability measure μ on R^1 is said to be of the pure GC-type (generalized Cantor type), if there exists a nowhere dense subset E such that

$$\begin{cases} E\subset S_\mu, \\ \mu(E)=1, \\ \forall x\in E\ \exists\varepsilon(x)>0:\ [x-\varepsilon(x),x+\varepsilon(x)]\cap S_\mu \ \text{ is a subset of zero Lebesgue measure.} \end{cases}$$

Example 1.

a) Let $\xi=\sum_{k=1}^{\infty}\frac{\xi_k}{3^k}$, where ξ_k are independent identically distributed random variables taking values 0 and 2 with probabilities p and q respectively, $p+q=1,p\in(0,1)$. For any choice of $p\in(0,1)$ the probability measure μ_ξ is singularly continuous measure of GC-type. This measure can be represented as a measure with independent $\widetilde{Q}$-symbols (in this case $q_{0k}=q_{1k}=q_{2k}=\frac{1}{3};p_{0k}=p,p_{1k}=0,p_{2k}=q$). Its spectrum coincides with the "classical" Cantor set C_0 and the spectrum itself can be taken instead of the set

E, which has been mentioned in the definition. If $p = \frac{1}{2}$ we get the "classical" Cantor measure *on the unit interval*.

b) Let $I = [0,1]$, $\{(a_i, b_i)\}$ be a sequence of intervals without common limit points such that $(a_i, b_i) \subset I$, $\sum_{i=1}^{\infty}(b_i - a_i) = a_0 < 1$, and the set $P = I \setminus \bigcup_i (a_i, b_i)$ is perfect nowhere dense of positive Lebesgue measure (for instance one can choose $\widetilde{Q}$ with $q_{0k} = q_{2k} = \frac{1}{2} - \frac{1}{2^{k+1}}, q_{1k} = \frac{1}{2^k}; V_k = \{0, 2\}$ and put $P = \Gamma_{\widetilde{Q}(\mathbf{V})}$). Let $d_i := b_i - a_i$, and let us construct the measure ν in the following way:

$$\nu = \sum_{i=1}^{\infty} \frac{\nu_i}{2^i},$$

where the measure ν_i coincides with the "classical" probability Cantor measure on the closed interval $[a_i + \frac{1}{4}d_i, a_i + \frac{3}{4}d_i]$ (S_{ν_i} is geometrically similar to the Cantor set with $k = \frac{1}{2}d_i, \inf S_{\nu_i} = a_i + \frac{1}{4}d_i, \sup S_{\nu_i} = a_i + \frac{3}{4}d_i$). The measure ν is a probability one by the construction, and its spectrum consists of the union of the spectra S_{ν_i} and points which belongs to the closure of the above union, i.e.,

$$S_\nu = \left(\bigcup_i S_{\nu_i}\right) \bigcup P.$$

The measure ν is of pure GC-type (the set $\bigcup_i S_{\nu_i}$ can be taken instead of the set E, which has been mentioned in the definition). In such a case the spectrum of the measure ν is of positive Lebesgue measure ($\lambda(S_\nu) = 1 - a_0 > 0$).

Remark 4. The spectrum of singularly continuous probability measure of GC-type can be of zero as well as of positive Lebesgue measure.

Definition 2. A singularly continuous probability measure μ is said to be of the pure GP-type, if there exists a nowhere dense set E such that

$$\begin{cases} E \subset S_\mu, \\ \mu(E) = 1, \\ \forall x \in E \ \forall \varepsilon > 0: \ [x - \varepsilon, x + \varepsilon] \cap S_\mu \ \text{ is a set of positive Lebesgue measure.} \end{cases}$$

Example 2.

a) Let $\psi = \sum_{k=1}^{\infty} \psi_k a_k$, where $a_k = \frac{9}{10}\left(\frac{1}{2^k} + \frac{1}{10^k}\right)$, and ψ_k are i.i.d. random variables taking the values 0 and 1 with probabilities p and q, $p + q = 1, p \neq q, p \in (0,1)$. The measure μ_ψ can also be thought as a probability measure with independent $\widetilde{Q}-$symbols. In such a case $q_{0k} = q_{2k} = \frac{9 \cdot 5^k + 1}{18 \cdot 5^k + 10}, q_{1k} = \frac{8}{18 \cdot 5^k + 10}; p_{0k} = p, p_{1k} = 0, p_{2k} = q$.

For any choice of $p \in (0, \frac{1}{2}) \bigcup (\frac{1}{2}, 1)$ the probability measure μ_ψ is singularly continuous. Its spectrum is a nowhere dense set of positive Lebesgue measure. Moreover, the intersection of the spectrum with any ε-vicinity of any point from the spectrum is also a nowhere dense set of positive Lebesgue measure. Therefore, μ_ψ is singularly continuous of the pure GP-type (the spectrum itself can be taken instead of the set E, which has been mentioned in the definition). This measure is called the "classical" measure of the GP-type *on the unit interval*.

b) Let $I = [0,1]$, and let $\{(f_i, g_i)\}$ be a sequence of intervals without common limit points such that $(f_i, g_i) \subset I$, and $P_1 = I \setminus \bigcup_i (f_i, g_i)$ is a nowhere dense perfect subset (it can be of zero as well as of positive Lebesgue measure). Let $h_i := g_i - f_i$, and let us construct the measure μ in the following way:

$$\mu = \sum_{i=1}^{\infty} \frac{\mu_i}{2^i},$$

where the measure μ_i coincides with the "classical" probability measure of the pure GP-type on the closed interval $[f_i + \frac{1}{4}h_i, f_i + \frac{3}{4}h_i]$ (the spectrum S_{μ_i} is geometrically similar to the spectrum of the above constructed measure μ_ψ with the coefficient of similarity $k = \frac{1}{2}h_i$, $\inf S_{\mu_i} = f_i + \frac{1}{4}h_i$, $\sup S_{\mu_i} = f_i + \frac{3}{4}h_i$). It is clear that μ is a probability measure which are singularly continuous, and its spectrum consists of the union of the spectra S_{μ_i} and points, which are limit points of this union, i.e.,

$$S_\mu = \Big(\bigcup_i S_{\mu_i}\Big) \bigcup P_1.$$

The measure μ is of the pure GP-type (the set $\bigcup_i S_{\nu_i}$ can be taken instead of the set E, which has been mentioned in the definition). In this case we have $P_1 \subset S_\mu$ and $\mu(P_1) = 0$ independently of the Lebesgue measure of the set P_1.

Definition 3. A singularly continuous probability measure μ is said to be of the pure GS-type if there exists a sequence of closed intervals $\{[a_i, b_i]\}$ such that

$$\begin{cases} [a_i, b_i] \subset S_\mu, \\ \mu\left(\bigcup_i [a_i, b_i]\right) = 1. \end{cases}$$

Example 3.

a) Let $\eta = \sum_{k=1}^{\infty} \frac{\eta_k}{2^k}$, where η_k are i.i.d. random variables taking values 0 and 1 with probabilities p and q, $p + q = 1, p \neq q, p \in (0, 1)$. The measure μ_η can be thought as a probability measure with independent $\widetilde{Q}-$symbols. In such a case $q_{0k} = q_{2k} = \frac{1}{2}; p_{0k} = p, p_{1k} = q$. From theorem 2 it follows that for any choice of $p \in (0, \frac{1}{2}) \bigcup (\frac{1}{2}, 1)$ the probability measure μ_η is singularly continuous. Its spectrum coincides with the whole unit interval $[0, 1]$. Therefore, μ_η is of the pure GS-type (the unit interval itself can be chosen instead of the set E, which has been mentioned in the definition). This measure is called the "classical" measure of the GS-type *on the unit interval*.

b) Let $\{(a_i, b_i)\}$ be a sequence of subintervals, which are adjacent to the Cantor set C_0. Let $d_i := b_i - a_i$, and let us construct the measure m in the following way:

$$m = \sum_{i=1}^{\infty} \frac{m_i}{2^i},$$

where the measure m_i coincides with the "classical" probability measure of the pure GS-type on $[a_i + \frac{1}{4}d_i, a_i + \frac{3}{4}d_i]$ (in such a case the spectrum S_{m_i} coincides with the closed interval $[a_i + \frac{1}{4}d_i, a_i + \frac{3}{4}d_i]$). The measure m is a probability one by the construction, and its spectrum consists of the union of the spectra S_{μ_i} and points, which are limit points of this union, i.e.,

$$S_m = \Big(\bigcup_i S_{m_i}\Big) \bigcup C_0.$$

The measure m is of the pure GS-type (the closed intervals S_{m_i} can be chosen instead of the intervals, which have been mentioned in the definition). In this case we have $C_0 \subset S_m, C_0 \bigcap (\bigcup_i S_{m_i}) = \emptyset, \lambda(S_m) = \lambda(\bigcup_i S_{m_i})$, and $m(C_0) = 0$.

c) Let the set P_2 coincides with the spectrum of the measure μ_ψ mentioned in the example 2a), and let $\{(a_i, b_i)\}$ be the sequence of subintervals, which are adjacent to the set P_2. Let $d_i := b_i - a_i$ let us construct the measure m^* in the following way:

$$m^* = \sum_{i=1}^{\infty} \frac{m_i^*}{2^i},$$

where the measure m_i^* coincides with the "classical" probability measure of the pure GS-type on $[a_i + \frac{1}{4}d_i, a_i + \frac{3}{4}d_i]$ (in such a case the spectrum S_{m_i} also coincides with the whole closed interval $[a_i + \frac{1}{4}d_i, a_i + \frac{3}{4}d_i]$). The probability measure m^* is singularly continuous, and its spectrum is of the following form:

$$S_{m^*} = \Big(\bigcup_i S_{m_i^*}\Big) \bigcup P_2.$$

The measure m^* is of the pure GS-type (as before, the closed intervals S_{m_i} can be chosen instead of the intervals, which have been mentioned in the definition). At the same time we have $\lambda(S_{m^*}) > \lambda(\bigcup_i S_{m_i^*})$, and $m^*(P_2) = 0$.

There exist, of course, singularly continuous measures on R^1, which do not belong to any of the above mentioned types. Nevertheless, the following theorem establishes the spectral structure of any one-dimensional singularly continuous probability measure.

Theorem 3. *Any singularly continuous probability measure μ on R^1 can be represented in the following form*

$$\mu = \alpha_1 \mu^{GS} + \alpha_2 \mu^{GC} + \alpha_3 \mu^{GP}, \tag{18}$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$; μ^{GS}, μ^{GC} and μ^{GP} are singularly continuous probability measures of the pure $GS-$, $GC-$ resp. GP-type.

Proof. The proof of the theorem can be split naturally into the proofs of the following two lemmas.

Lemma 2. *Any singularly continuous probability measure μ on R^1 can be represented in the following form:*

$$\mu = \beta_1 \mu^{GS} + \beta_2 \mu^{T^*}, \tag{19}$$

where $\beta_1 \geq 0$, $\beta_2 \geq 0, \beta_1 + \beta_2 = 1$, μ^{GS} is a singularly continuous probability measure of the pure GS-type, and μ^{T^*} is a singularly continuous probability measure with a nowhere dense spectrum.

Proof. 1. If μ is of GS-type, then $\beta_1 = 1$, $\mu^{GS} = \mu$, $\beta_2 = 0$ and the "classical" Cantor measure can be chosen instead of the measure μ^{T^*}.

2. If S_μ is a nowhere dense set, then $\beta_1 = 0$, $\beta_2 = 1$, $\mu^{T^*} = \mu$ and one can choose any measure of the pure GS-type instead of the measure μ^{GS}.

3. Now let μ be not of the pure GS-type and let its spectrum S_μ be not nowhere dense. Then S_μ, being a closed set, contains at least one closed interval. A closed interval $[a, b] \subset S_\mu$ is said to be "full" if there is no any closed interval $[c, d]$ with $[a, b] \subset [c, d] \subset S_\mu$ (i.e., for any $\varepsilon > 0$ intervals $(a - \varepsilon, a)$ and $(b, b + \varepsilon)$ contain points, which do not belong to the spectrum S_μ).

Let $\{[a_i, b_i]\}$ be a family of all "full" closed intervals from S_μ, $[a_i, b_i] \bigcap [a_j, b_j] = \emptyset, i \neq j$, and let $S = \bigcup_i [a_i, b_i]$. From the latter assumption it follows that:

$$\mu(S) = \mu\Big(\bigcup_i [a_i, b_i]\Big) = \mu\Big(\bigcup_i (a_i, b_i)\Big) \in (0, 1).$$

Let us denote $\beta_1 = \mu(S)$ and

$$\mu^{GS}(E) := \frac{1}{\mu(S)} \cdot \mu(E \cap S), \ \forall E \in \mathcal{B}(R^1).$$

The measure μ^{GS} is a probability one and its property of being singularly continuous follows from the singular continuity of μ. The measure μ^{GS} is of the pure GS-type by definition, because the set $S = \bigcup_i [a_i, b_i]$ is a subset of the topological support of the measure μ^{GS} and $\mu^{GS}(S) = 1$.

Let $T := S_\mu \setminus S$. Since the set S_μ is perfect, and S contains all interior points from the spectrum S_μ, it is clear that T is a nowhere dense subset.

Set $\beta_2 = \mu(T) = 1 - \mu(S) \in (0, 1)$ and

$$\mu^{T^*}(E) := \frac{1}{\mu(T)} \cdot \mu(E \cap T), \quad \forall E \in \mathcal{B}(R^1).$$

The measure μ^{T^*} is a probability one and its singular continuity follows from the singular continuity of μ. It is clear that $\mu^{T^*}(T) = 1$. Therefore, the set $S_{\mu^{T^*}}$, being a subset of the closure of a nowhere dense set T, is nowhere dense. ☐

Lemma 3. *Let μ^{T^*} be any singularly continuous probability measure on R^1 with a nowhere dense support. Then the measure μ^{T^*} can be represented in the following form:*

$$\mu^{T^*} = \gamma_1 \mu^{GC} + \gamma_2 \mu^{GP}, \tag{20}$$

where $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $\gamma_1 + \gamma_2 = 1$, μ^{GC} are singularly continuous probability measure of the pure GC-type, and μ^{GP} is a singularly continuous probability measure of the pure GP-type.

Proof. Let $S_{\mu^{T^*}}$ be the spectrum of the measure μ^{T^*}. Every point of the spectrum $S_\mu^{T^*}$ belongs to one of the following sets:

$$T_C = \{x : \ x \in S_{\mu^{T^*}} \text{ and } (\exists \varepsilon(x) > 0 : \ \lambda(S_{\mu^{T^*}} \cap (x - \varepsilon(x), x + \varepsilon(x))) = 0)\},$$

$$T_P = \{x : \ x \in S_{\mu^{T^*}} \text{ and } (\forall \varepsilon > 0 : \ \lambda(S_{\mu^{T^*}} \cap (x - \varepsilon, x + \varepsilon)) > 0)\}.$$

It is obvious that $T_C \cap T_P = \emptyset$ and $T_C \cup T_P = S_{\mu^{T^*}}$.

Let us show that T_C is a Borel subset of zero Lebesgue measure. For any point $x \in T_C$ we define

$$\varepsilon_1(x) = \sup\{\varepsilon : \ \lambda(S_{\mu^{T^*}} \cap (x - \varepsilon, x]) = 0\},$$

$$\varepsilon_2(x) = \sup\{\varepsilon : \ \lambda(S_{\mu^{T^*}} \cap [x, x + \varepsilon)) = 0\}.$$

Let $A_x = (x - \varepsilon_1(x), x + \varepsilon_2(x)) \cap S_{\mu^{T^*}}$. From the construction of the set A_x it follows that it is a nonempty nowhere dense subset of zero Lebesgue measure for any $x \in T_C$.

Let us consider the set $C = \bigcup_{x \in T_C} A_x$. If $x \in T_C$ and $y \in T_C$, then either $A_x \equiv A_y$ (if there are no points from the set T_P between x and y) or $A_x \cap A_y = \emptyset$ (if there exists a point from the set T_P between x and y). So, the latter union contains at most countable number of *different* subsets A_x, $x \in T_C$. Since all subsets A_x are Borel ones (as an intersection of two Borel subsets), we conclude that C is also a Borel subset. Moreover, $\lambda(C) = 0$, because C is a union of an at most countable number of zero-sets.

If $x \in T_C$, then $x \in A_x$. So $T_C \subset C$.

If $x \in C$, then $x \in A_y$ for some $y \in T_C$. So, $x \in (y - \varepsilon_1(y), y + \varepsilon_2(y))$ and $x \in S_{\mu^{T^*}}$.

Therefore, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset (y - \varepsilon_1(y), y + \varepsilon_2(y))$ and, hence, $\lambda((x - \varepsilon, x + \varepsilon) \cap S_{\mu^{T^*}}) = 0$. Therefore, $x \in T_C$, and $C \subset T_C$. So, $C = T_C$.

Since T_C is a Borel subset with $\lambda(T_C) = 0$, we conclude that $T_P = S_{\mu^{T^*}} \setminus T_C$ is also a Borel subset. Let us remark that the set T_P is closed. To show this, let us assume that $\{x_n\}$ is a sequence of points from T_P, which converges to some point x_0. $x_0 \in S_{\mu^{T^*}}$ since the set $S_{\mu^{T^*}}$ is closed as the spectrum. Therefore, the point x_0 belongs either to T_P or to T_C. Suppose that $x_0 \in T_C$. Then there exists $\varepsilon(x_0) > 0$ such that $\lambda((x_0 - \varepsilon(x_0), x_0 + \varepsilon(x_0)) \bigcap S_{\mu^{T^*}}) = 0$. On the other hand, there exists $N_0 \in N$ such that $x_n \in (x_0 - \varepsilon(x_0), x_0 + \varepsilon(x_0))$ for all $n > N_0$. Let us choose $n > N_0$ and $\varepsilon_1 > 0$ such that $(x_n - \varepsilon_1, x_n + \varepsilon_1) \subset (x_0 - \varepsilon(x_0), x_0 + \varepsilon(x_0))$. Since $x_n \in T_P$, we have $\lambda((x_0 - \varepsilon(x_0), x_0 + \varepsilon(x_0)) \bigcap S_{\mu^{T^*}}) \geq \lambda((x_n - \varepsilon_1, x_n + \varepsilon_1) \bigcap S_{\mu^{T^*}}) > 0$. This contradiction shows that $x_0 \in T_P$ and proves that the set T_P is closed.

If $\mu^{T^*}(T_C) = 1$, then the measure μ^{T^*} is of the pure GC-type (by the definition). In such a case we can set $\gamma_1 = 1$, $\gamma_2 = 0$; $\mu^{GC} = \mu^{T^*}$ (any singularly continuous measure of the pure GP-type can be chosen instead of the measure μ^{GP}.).

If $\mu^{T^*}(T_P) = 1$, then the measure μ^{T^*} is of the pure GP-type (by the definition). In such a case we set $\gamma_1 = 0$, $\gamma_2 = 1$; $\mu^{GP} = \mu^{T^*}$ (the "classical" Cantor measure on $[0, 1]$ can be chosen instead of the measure μ^{GC}).

If $0<\mu^{T^*}(T_C)<1$, then we define the measure

$$\mu^{GP}(E)=\frac{1}{\mu^{T^*}(T_P)}\cdot\mu^{T^*}(E\cap T_P),$$

$$\mu^{GC}(E)=\frac{1}{\mu^{T^*}(T_C)}\cdot\mu^{T^*}(E\cap T_C),\ \forall E\in\mathcal{B}(R^1).$$

It is clear that μ^{GP} and μ^{GC} are probability measures. Their singular continuity follows directly from the singular continuity of the measure μ^{T^*}.

Let $T_P^*=T_P\bigcap S_{\mu^{GP}}$. It is clear that $\mu^{GP}(T_P^*)=1$. The set T_P is closed, and it is a subset of $S_{\mu^{GP}}$. On the other hand the spectrum $S_{\mu^{GP}}$ is the **minimal** closed support of μ^{GP}. Therefore, $T_P^*=S_{\mu^{GP}}$ (the set T_P itself , generally speaking, can contain $S_{\mu^{GP}}$ as a proper subset). So, μ^{GP} is a measure of GP-type (one can choose the set T_P^* instead of the set E, which was mentioned in the definition of a measure of the pure GP-type).

Let $T_C^*=T_C\bigcap S_{\mu^{GC}}$. It is also clear that $\mu^{GC}(T_C^*)=1$, and $T_C^*\subset S_{\mu^{GC}}$. Therefore, μ^{GC} is a measure of the pure GC-type (one can choose the set T_C^* instead of the set E, which was mentioned in the definition of a measure of the pure GC-type).

Let $\gamma_1:=\mu^{T^*}(T_C)$ and $\gamma_2:=\mu^{T^*}(T_P)$. Then $\forall E\in\mathcal{B}(R^1)$:

$$\begin{aligned}\mu^{T^*}(E)&=\mu^{T^*}(E\cap(T_C\cup T_P))\\&=\gamma_1\cdot\frac{1}{\mu^{T^*}(T_C)}\cdot\mu(E\cap T_C)+\gamma_2\cdot\frac{1}{\mu^{T^*}(T_P)}\cdot\mu(E\cap T_P)\\&=\gamma_1\mu^{GC}(E)+\gamma_2\mu^{GP}(E),\end{aligned}$$

which proves the Lemma. □

The theorem is a direct corollary of two latter Lemmas. □

Example 4. Let the measure φ coincides with the singularly continuous measure, which was considered in Example 2a) and let $P_2=I\setminus\bigcup_i(a_i,b_i)$ be its spectrum ($[a_i,b_i]\bigcap[a_j,b_j]=\emptyset,\ i\neq j$). Let $d_i:=b_i-a_i$, and let us construct the measure ν_i and μ_i in the following way. The measure ν_i coincides with the "classical" measure of the pure GC-type on the closed interval $S_{\nu_i}=[a_i+\frac{1}{7}d_i,a_i+\frac{2}{7}d_i]$ (see, e.g., Example 1a)), and the measure μ_i coincides with the "classical" measure of the pure GS-type on the closed interval $S_{\mu_i}=[a_i+\frac{5}{7}d_i,a_i+\frac{6}{7}d_i]$ (see, e.g., Example 3a)).

Let us define $\nu=\sum_{i=1}^{\infty}\frac{\nu_i}{2^i}$, $\mu=\sum_{i=1}^{\infty}\frac{\mu_i}{2^i}$.

Then the measure

$$\mu^*=\frac{1}{3}(\varphi+\nu+\mu)$$

is a singularly continuous measure, which is a mixture of measures from the above defined pure spectral classes. The spectrum of μ^* coincides with the union of spectra of the measures ν and μ, because the spectrum of the measure φ coincides with the intersection of the sets S_μ and S_ν.

Remark 5. The latter theorem can be obviously generalized to the family of finite measures.

Remark 6. The latter theorem can be generalized to the multidimensional case, which will be treated in a forthcoming paper.

Theorem 4. *A singularly continuously distributed random variable ξ with independent $\widetilde{Q}-$symbols is of the pure spectral type.*

1) It is of the pure GS-type if and only if the matrix $\widetilde{P}$ contains only a finite number of columns containing zero elements.

2) It is of the pure GC-type if and only if the matrix $\widetilde{P}$ contains infinitely many columns having some elements $p_{ik} = 0$, and

$$\sum_{k=1}^{\infty} \Big(\sum_{i: p_{ik}=0} q_{ik} \Big) = \infty \ . \tag{21}$$

3) It is of the pure GP-type if and only if the matrix $\widetilde{P}$ contains infinitely many columns having zero elements, and

$$\sum_{k=1}^{\infty} \Big(\sum_{i: p_{ik}=0} q_{ik} \Big) < \infty. \tag{22}$$

Proof. Let us consider the set $\Gamma \equiv \Gamma_{\widetilde{Q}(\mathbf{V})}$ (see, e.g., Section 2) with $\mathbf{V} = \{\mathbf{V}_k\}_{k=1}^{\infty}$ defined by the $\widetilde{P}-$matrix as follows: $\mathbf{V}_k = \{i \in \mathbf{N}_k : \ p_{ik} \neq 0\}$. It is easy to see that the spectrum of the measure μ_ξ coincides with the closure of set Γ (in such a case the difference $(\Gamma)^{cl} \setminus \Gamma$ is at most countable). Therefore, to examine the metric and topological structure of the set S_ξ we may apply the results of Section 2. So, if the matrix $\widetilde{P}$ contains only finite number of zero elements, then $\mathbf{V}_k = \mathbf{N}_k, \ k > k_0$ for some $k_0 > 0$. In such a case, Γ is a union of an at most countable number of closed intervals and at most countable set of points, which are limit ones for these intervals. Therefore, the measure μ_ξ is of the pure GS-type.

In the opposite case the matrix $\widetilde{P}$ contains an infinite number of columns containing zero elements, and, therefore, Γ is a nowhere dense set (see Sec. 2). The Lebesgue measure of the set Γ by Lemma 1 is equal to

$$\lambda(\Gamma) = \prod_{k=1}^{\infty} S_k(\mathbf{V}) = \prod_{k=1}^{\infty} \Big(\sum_{i \in \mathbf{V}_k} q_{ik} \Big) = \prod_{k=1}^{\infty} \Big(1 - \sum_{i: p_{ik}=0} q_{ik} \Big).$$

Then, by the Corollary after Lemma 1, either $\lambda(\Gamma) = 0$, provided that condition (21) fulfilled, or $\lambda(\Gamma) > 0$, if condition (22) holds. Thus the measure μ_ξ either is of the pure GC-type, or it is of the pure GP-type.

Since the conditions 1), 2) and 3) of this theorem are mutually exclusive and one of them always holds, we conclude that the distribution of the random variable ξ with independent $\widetilde{Q}$-symbols is always of the pure spectral type. □

Remark 7. By using the latter theorem and theorem 2 one can easily construct singularly continuous probability measures of any pure spectral type.

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