Methods of Functional Analysis and Topology
Vol. 17 (2011), no. 2, pp. 112–117

COMPLEMENT ON ORDER WEAKLY COMPACT OPERATORS

BELMESNAOUI AQZZOUZ AND JAWAD H'MISHANE

Abstract. We generalize a result on the duality property for order weakly compact operators and use it to establish some characterizations of positive operators.

1. Introduction and notation

An operator T from a Banach lattice E into a Banach space G is called order weakly compact if for each $x \in E^+$, $T([0, x])$ is a relatively weakly compact subset of G where $E^+ = \{x \in E : 0 \leq x\}$. Note that the class of order weakly compact operators does not satisfy the duality property. And in [4], we studied this property, by giving sufficient and necessary conditions under which the order weak compactness of an operator implies the order weak compactness of its adjoint and conversely. But our last result Theorem 2.8 of [4] is proved under the strong conditions that E and F are Dedekind σ-complete Banach lattices. The aim of this note is to give a generalization of our Theorem 2.8 in [4], by proving that if E and F are two Banach lattices such that F is Dedekind σ-complete, then each order bounded operator T from E into F is order weakly compact whenever its adjoint T' from F' into E' is order weakly compact if and only if the norm of E or F is order continuous. Also, we will give an example which shows that the condition "F is Dedekind σ-complete" is essential. After that we will use our generalization to give a short proof of Theorem 2.2 of [6], which gave necessary and sufficient conditions for which a semi-compact operator is order weakly compact. Also, we will exploit our generalization to prove necessary and sufficient conditions under which an order bounded weak Dunford-Pettis operator is order weakly compact. Finally, we will establish other results which use order weakly compact operators.

Recall that a vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

The term operator $T : E \longrightarrow F$ between two Banach lattices means a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. For unexplained terminology on Banach lattice theory and positive operators, we refer to the excellent book of Aliprantis and Burkinshaw [3].

2. Main results

To give our major result, we need to recall the following Lemma which is just Lemma 3.4 of [5].

Lemma 2.1. *Let E be a Banach lattice and let (x_n) be a disjoint sequence of E. If (f_n) is a sequence of E', then there exists a disjoint sequence (g_n) of E' with $|g_n| \leq |f_n|$ such that $g_n(x_n) = f_n(x_n)$ for all n and $g_n(x_m) = 0$ for $n \neq m$.*

2000 *Mathematics Subject Classification.* 46A40, 46B40, 46B42.
Key words and phrases. Order weakly compact operator, order continuous norm.

Moreover, if (f_n) is a positive sequence of E' then we may take (g_n) in $(E')^+$.

Our major result is a generalization of Theorem 2.8 of [4].

Theorem 2.2. *Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following conditions are equivalent.*

1. Each order bounded operator T from E into F is order weakly compact.

2. Each order bounded operator T from E into F is order weakly compact whenever its adjoint T' from F' into E' is order weakly compact.

3. One of the following assertions is valid:

i) the norm of E is order continuous,

ii) the norm of F is order continuous.

Proof. 1) $\Longrightarrow$ 2) Obvious.

2) $\Longrightarrow$ 3) Assume that the norms of E and F are not order continuous.

As the norm of E is not order continuous, it follows from Theorem 10.1 of [1] and Lemma 2.1, the existence of an order bounded disjoint sequence (x_n) in E^+ with $\|x_n\| = 1$ for all n and there exists a disjoint sequence of positive elements (g_n) of E' with $\|g_n\| \leq 1$ for each n, such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

We consider the operator S defined by the following:

$$S : E \longrightarrow l^\infty, \ x \longmapsto S(x) = (g_n(x))_{n=1}^\infty .$$

It is clear that S is positive.

On the other hand, since the norm of F is not order continuous, then it follows from Theorem 2.4.2 of [8] that there exists some $y \in F^+$ and an order bounded disjoint sequence (y_n) in F such that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for all n.

Now, as F is Dedekind σ-complete, it follows from the proof of Theorem 117.3 of [9] that the operator

$$\varphi : l^\infty \longrightarrow F, \ (\lambda_1, \lambda_2, \ldots) \longmapsto \varphi\left((\lambda_1, \lambda_2, \ldots)\right) = \sum_{i=1}^{\infty} \lambda_i y_i$$

defines a positive operator from l^∞ into F where the convergence is in the sense of the order.

We consider the operator product $T = \varphi \circ S : E \longrightarrow F$ defined by

$$T(x) = \sum_{i=1}^{\infty} g_i(x) y_i \quad \text{for each} \quad x \in E$$

is well defined and positive. Its adjoint $T' = S' \circ \varphi'$ is order weakly compact but the operator T is not order weakly compact. In fact, since $T(x_n) = \sum_{i=1}^{\infty} g_i(x_n) y_i = y_n$ for all n and (x_n) is a positive order bounded disjoint sequence in E^+ such that $\|x_n\| = 1$ for all n and since $\|T(x_n)\| = \|y_n\| = 1$ for all n, it follows from Theorem 1.2 of [2] that T is not order weakly compact. This completes the proof of 2) $\Longrightarrow$ 3).

3) $\Longrightarrow$ 1). It is just a consequence of Theorem 2.4 of [4]. □

Remark 1. The condition "F is Dedekind σ-complete" is essential in Theorem 2.2. In fact, each operator from l^∞ into c (which is not Dedekind σ-complete) is weakly compact, and hence is order weakly compact. Also, its adjoint is order weakly compact but the norms of the Banach lattices l^∞ and c are not order continuous.

Remark 2. The condition "order bounded operator" is essential for the proof of the implication 3) $\Longrightarrow$ 1) of Theorem 2.2. In fact, it follows from Lemma 2.4 of [4] the existence of an operator T from c into c_0 which is not order bounded and hence not order weakly compact. However, the norm of the Banach lattice c_0 is order continuous.

For our following result, we need to introduce the class of semi-compact operators. Recall that an operator T from a Banach space E into a Banach lattice F is said to be semi-compact if for each $\varepsilon > 0$, there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where B_H is the closed unit ball of $H = E$ or F.

Note that there exists an operator which is semi-compact but not order weakly compact. In fact, the identity operator of l^∞ is semi-compact but not order weakly compact. And conversely, there exists an operator which is order weakly compact but not semi-compact. In fact, the identity operator of the Banach lattice $L^1[0,1]$ is order weakly compact but not semi-compact.

Now, we use our previous generalization to give another proof, which is short compared to that of Theorem 2.2 of [6], of necessary and sufficient conditions under which a semi-compact operator is order weakly compact.

Theorem 2.3. *Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following conditions are equivalent.*

1. Each order bounded semi-compact operator T from E into F is order weakly compact.

2. One of the following assertions is valid:

i) The norm of E is order continuous.

ii) The norm of F is order continuous.

Proof. 1) $\Longrightarrow$ 2) Assume that the norms of E and F are not order continuous.

As the norm of E is not order continuous, it follows from Theorem 10.1 of [1] and Lemma 2.1 the existence of an order bounded disjoint sequence (x_n) in E^+ with $\|x_n\| = 1$ for all n and there exists a disjoint sequence of positive elements (g_n) of E' with $\|g_n\| \leq 1$ for each n, such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

We consider the operator S defined as follows:

$$S : E \longrightarrow l^\infty, \ x \longmapsto S(x) = (g_n(x))_{n=1}^\infty .$$

It is clear that S is positive and is not order weakly compact.

Now, as the norm of F is not order continuous and F is Dedekind σ-complete, it follows from Corollary 2.4.3 of [8] that F contains a positively complemented closed sublattice which is order and topologically isomorphic to l^∞. Let $P : F \longrightarrow l^\infty$ be the positive projection and $i : l^\infty \longrightarrow F$ the canonical injection.

We consider the operator product $T = i \circ S$. Since $T = i \circ \mathrm{Id}_{l^\infty} \circ S$ and Id_{l^∞} are semi-compact, the operator T is semi-compact but it is not order weakly compact. If not, the operator $S = P \circ T$ would be order weakly compact, which is a contradiction.

2) $\Longrightarrow$ 1) Let $T : E \longrightarrow F$ be an order bounded semi-compact It follows from Proposition 3.6.18 of [8] that the adjoint $T' : F' \longrightarrow E'$ is order weakly compact, and hence Theorem 2.2 implies that the operator $T : E \longrightarrow F$ is order weakly compact. □

Let us recall that a Banach space X has the Dunford-Pettis property if $\lim_n x'_n(x_n) = 0$ whenever (x_n) converges weakly to zero in X and (x'_n) converges weakly to zero in X'.

It follows from Theorem 5.82 of [3] that a Banach space X has the Dunford-Pettis property if and only if every weakly compact operator from X to an arbitrary Banach space is Dunford-Pettis.

On the other hand, an operator T from a Banach space E into another F is said to be weak Dunford-Pettis if the operator product $S \circ T$ is Dunford-Pettis for each weakly compact operator S from F into G, for some Banach space G.

Alternatively, T is weak Dunford-Pettis if $(y'_n(T(x_n)))$ converges to 0 whenever (x_n) converges weakly to 0 in E and (y'_n) converges weakly to 0 in F.

Note that the class of weak Dunford-Pettis operators is a two sided ideal. Also, without any difficulties, we can establish that a Banach space X has the Dunford-Pettis property if and only if each operator from X into X is weak Dunford-Pettis.

On the other hand, there exists a weak Dunford-Pettis operator which is not order weakly compact. In fact, the identity operator of l^∞ is weak Dunford-Pettis but not order weakly compact. And conversely, there exists an order weakly compact operator which is not weak Dunford-Pettis. In fact, the identity operator of the Banach lattice l^2 is order weakly compact but not weak Dunford-Pettis.

Also, we utilize our previous generalization and the same proof as Theorem 2.3, to establish necessary and sufficient conditions under which an order bounded weak Dunford-Pettis operator is order weakly compact.

Theorem 2.4. *Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following conditions are equivalent.*

1. Every order bounded weak Dunford-Pettis operator T from E into F is order weakly compact.

2. One of the following assertions is valid:

i) the norm of E is order continuous,

ii) the norm of F is order continuous.

Proof. 1) $\Longrightarrow$ 2) Assume that the norms of E and F are not order continuous.

As the norm of E is not order continuous, it follows from Theorem 10.1 of [1] and Lemma 2.4 of [5] the existence of an order bounded disjoint sequence (x_n) in E^+ with $\|x_n\| = 1$ for all n and there exists a disjoint sequence of positive elements (g_n) of E' with $\|g_n\| \leq 1$ for each n, such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

We consider the operator S defined by

$$S : E \longrightarrow l^\infty,\ x \longmapsto S(x) = (g_n(x))_{n=1}^\infty .$$

It is clear that S is positive and is not order weakly compact.

Now, as the norm of F is not order continuous and F is Dedekind σ-complete, it follows from Corollary 2.4.3 of [8] that F contains a positively complemented closed sublattice which is order and topologically isomorphic to l^∞. Let $P : F \longrightarrow l^\infty$ the positive projection and $i : l^\infty \longrightarrow F$ the canonical injection. We consider the operator product $T = i \circ S$. Since $T = i \circ \mathrm{Id}_{l^\infty} \circ S$ and Id_{l^∞} is weak Dunford-Pettis (because l^∞ has the Dunford-Pettis property), it follows that the operator T is weak Dunford-Pettis. But the operator T is not order weakly compact. Otherwise, the operator $S = P \circ T$ is order weakly compact, which is a contradiction.

2) $\Longrightarrow$ 1) Follows immediately from Theorem 2.4 of [4]. □

Remark 3. The same example in the above Remark shows that the condition "F is Dedekind σ-complete" is essential in Theorem 2.4.

Recall from [8] that an operator T from a Banach lattice E into a Banach space X is said to be M–weakly-compact if $\lim \|Tx_n\| = 0$ holds for every norm bounded disjoint sequence (x_n) of E.

Theorem 2.5. *Let E and F be two Banach lattices such that E is Dedekind σ-complete and let T be an operator from E into F. Then the following conditions are equivalent.*

1) The operator T carries order bounded subsets of E into relatively weakly compact subsets of F.

2) For each semi-compact operator S from a Banach lattice G into E, the composed operator $T \circ S : G \longrightarrow F$ is weakly compact.

3) For each positive operator S from l^∞ into E, the composed operator $T \circ S : l^\infty \longrightarrow F$ is M-weakly compact.

4) One of the following assertions is valid:
i) the operator T is order weakly compact,
ii) the norm of E is order continuous.

Proof. 1) $\Longrightarrow$ 2) Let S be a semi-compact operator from a Banach lattice G into E, then $S(B_G)$ is an almost order bounded subset of E i.e. for each $\varepsilon > 0$ there exists some $u \in E^+$ such that $S(B_G) \subset [-u, u] + \varepsilon B_E$ where B_H is the closed unit ball of $H = E, G$. It follows from our hypothesis and Theorem 3.44 of [3] that $T \circ S(B_G)$ is relatively weakly compact. So, the operator $T \circ S$ is weakly compact.

2) $\Longrightarrow$ 3) Since l^∞ is an AM-space with unit, its closed unit ball B_{l^∞} is an order interval and hence B_{l^∞} is almost order bounded. So, the identity operator of l^∞ is semi-compact, and then each positive operator S from l^∞ into E is semi-compact and hence the operator $T \circ S : l^\infty \longrightarrow F$ is weakly compact. As $(l^\infty)'$ has the positive Schur property (i.e. $\lim_n \|x_n\| = 0$ for every positive weakly null sequence (x_n) of $(l^\infty)'$), it follows from Theorem 3.3 of [7] that the operator $T \circ S$ is M-weakly compact.

3) $\Longrightarrow$ 4) Suppose that the norm of E is not order continuous, and let (x_n) be a disjoint order bounded sequence of E. Consider the positive operator S from l^∞ into E defined by $S\left((\lambda_n)_{n=1}^\infty\right) = \sum_{n=1}^{\infty} \lambda_n x_n$.It follows from hour hypothesis that the operator $T \circ S$ is M-weakly compact.

Now, consider the unit vector basis (e_n) of c_0, and note that c_0 is a closed sublattice of l^∞. We have $T \circ S(e_n) = T(x_n)$. As (e_n) is a norm bounded disjoint sequence in l^∞, then $(T(x_n))$ is norm convergent to 0. Hence the operator T is order weakly compact.

4) $\Longrightarrow$ 1) Since the norm of E is order continuous, the operator $T : E \longrightarrow F$ is order weakly compact and hence T carries order bounded subsets of E into relatively weakly compact subsets of F. □

Remark 4. In Theorem 2.5, the assumption that the Banach lattice E is Dedekind σ-complete, is essential. In fact, if c is the Banach lattice of all convergent sequences, every positive operator from l^∞ into c is M-weakly compact, but the identity operator of c is not order weakly compact and the norm of c is not order continuous.

Finally, we give a result which generalizes Theorem 5.48 of [3].

Theorem 2.6. *Let E be a Banach space, and let F and G be Banach lattices. Consider operators $T : E \longrightarrow F$ and $S : F \longrightarrow G$. If T is dominated by a weakly compact operator and S is order weakly compact, then $S \circ T$ is weakly compact.*

Proof. Since S is order weakly compact, it follows from Theorem 5.58 of [3] that S factor through a Banach lattice X with an order continuous norm, i.e. there exists two operators $Q : F \longrightarrow X$ and $R : X \longrightarrow G$ such that $S = R \circ Q$ where Q is a lattice homomorphism. Since the positive operator $Q \circ T : E \longrightarrow X$ is dominated by a weakly compact operator, then Theorem 5.31 of [3] implies that the positive operator $Q \circ T$ is weakly compact. Hence, $S \circ T = R \circ Q \circ T$ is weakly compact. □

As a consequence of Theorem 2.6, we obtain the best Theorem 5.32 of [3].

Corollary 2.7. (Theorem 5.32 of [3]). *If a positive operator S defined on a Banach lattice E is dominated by a weakly compact operator, then its second power operator S^2 is weakly compact.*

References

1. C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces*, Academic Press, New York—London, 1978.
2. C. D. Aliprantis and O. Burkinshaw, *Dunford-Pettis operators on Banach lattices*, Trans. Amer. Math. Soc. **274** (1982), no. 1, 227–238.

3. C. D. Aliprantis and O. Burkinshaw, *Positive Operators* (Reprint of the 1985 Academic Press original ed.), Springer, Dordrecht, 2006.
4. B. Aqzzouz and J. Hmichane, *The duality problem for the class of order weakly compact operators*, Glasgow Math. J. **51** (2009), no. 1, 101–108.
5. B. Aqzzouz, A. Elbour, and J. Hmichane, *The duality problem for the class of b-weakly compact operators*, Positivity **13** (2009), no. 4, 683–692.
6. B. Aqzzouz and A. Elbour, *Characterization of the order weak compactness of semi-compact operators*, J. Math. Anal. Appl. **355** (2009), no. 2, 541–547.
7. Z. L. Chen and A. W. Wickstead, *L-weakly and M-weakly compact operators*, Indag. Math. (N.S.) **10** (1999), no. 3, 321–336.
8. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, 1991.
9. A. C. Zaanen, *Riesz Spaces* II, North Holland Publishing Company, Amsterdam, 1983.

Université Mohammed V-Souissi, Faculté des Sciences Economiques, Juridiques et Sociales, Département d'Economie, B.P. 5295, SalaAljadida, Morocco
E-mail address: baqzzouz@hotmail.com

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra, Morocco

Received 16/09/2010; Revised 12/01/2011