Methods of Functional Analysis and Topology
Vol. 17 (2011), no. 2, pp. 168–173

ON C^*-ALGEBRA GENERATED BY FOCK REPRESENTATION OF WICK ALGEBRA WITH BRAIDED COEFFICIENTS

D. PROSKURIN

Abstract. We consider C^*-algebras $\mathcal{W}(T)$ generated by operators of Fock representations of Wick $*$-algebras with a braided coefficient operator T. It is shown that for any braided T with $||T|| < 1$ one has the inclusion $\mathcal{W}(0) \subset \mathcal{W}(T)$. Conditions for existence of an isomorphism $\mathcal{W}(T) \simeq \mathcal{W}(0)$ are discussed.

1. Introduction

In this note we study the Fock representation of the Wick $*$-algebra $W(T)$ with a braided coefficient operator. Recall, see [4], that

$$W(T) = \mathbb{C}\big\langle a_i,\ a_i^*,\ i = \overline{1,d} \mid a_i a_j^* = \delta_{ij} 1 + \sum_{k,l=1}^{d} T_{ij}^{kl} a_l^* a_k,\ T_{ij}^{kl} = \overline{T_{ji}^{lk}}\ \big\rangle. \tag{1}$$

Let $\mathcal{H} = \langle e_1, \ldots, e_d \rangle$ and put

$$T\colon \mathcal{H}\otimes\mathcal{H} \mapsto \mathcal{H}\otimes\mathcal{H}, \quad Te_k\otimes e_l = \sum_{i,j=1}^{d} T_{ik}^{lj} e_i \otimes e_j, \quad T = T^*,$$
$$T_i\colon \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}, \quad T_i = \underbrace{\mathbf{1}_{\mathcal{H}}\otimes\cdots\otimes\mathbf{1}_{\mathcal{H}}}_{i-1} \otimes T \otimes \underbrace{\mathbf{1}_{\mathcal{H}}\otimes\cdots\otimes\mathbf{1}_{\mathcal{H}}}_{n-i-1}, \quad i = 1,\ldots,n-1.$$

An operator T is called a **coefficient operator** for $W(T)$. We say that T satisfies the **braid** relation if on $\mathcal{H}^{\otimes 3}$

$$T_1T_2T_1 = T_2T_1T_2. \tag{2}$$

The Fock representation of W(T), see [1, 4], is defined on the full tensor space over $\mathcal{H}$,

$$\mathcal{T}(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots, \quad ||\Omega|| = 1$$

, by the following rules, see

$$\pi_F(a_i^*) e_{i_1}\otimes\cdots\otimes e_{i_n} = e_i \otimes e_{i_1}\otimes\cdots\otimes e_{i_n},$$
$$\pi_F(a_i) e_{i_1}\otimes\cdots\otimes e_{i_n} = \mu(e_i) M_n e_{i_1}\otimes\cdots\otimes e_{i_n}, \quad n \geq 1,$$
$$\pi_F(a_i)\Omega = 0, \quad i = 1,\ldots,d,$$

where

$$\mu(e_i) e_{i_1}\otimes\cdots\otimes e_{i_n} = \delta_{ii_1} e_{i_2}\otimes\cdots\otimes e_{i_n}, \quad \mu(e_i)\Omega = 0, \quad i = 1,\ldots,d$$

, and

$$M_n\colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}, \quad M_n = \mathbf{1}_{\mathcal{H}^{\otimes n}} + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}, \quad n \geq 2,$$

$M_0 = 1$, $M_1 = \mathbf{1}_{\mathcal{H}}$.

2000 *Mathematics Subject Classification.* Primary 46L65; Secondary 81T05.

Key words and phrases. Fock representation, Cuntz-Topelitz algebra, Braid relation.
The author was supported by DFG grant SCHM1009/4-1.

Theorem 1. ([4]). *There exists a unique Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_T$ on $\mathcal{T}(\mathcal{H})$ such that*

$$\langle \pi_F(a_i^*)X, Y\rangle_T = \langle X, \pi_F(a_i)Y\rangle_T, \quad X, Y \in \mathcal{T}(\mathcal{H}), \quad i = 1, \dots, d.$$

More precisely,

$$\langle X, Y\rangle_T = \langle X, P_n Y\rangle,\ X, Y \in \mathcal{H}^{\otimes n}, \quad \langle X, Y\rangle_T = 0,\ X \in \mathcal{H}^{\otimes n}, \quad Y \in \mathcal{H}^{\otimes m}, \quad n \neq m,$$

where

$$\begin{aligned} &P_n \colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}, \quad P_n^* = P_n, \\ &P_0 = 1, \quad P_1 = \mathbf{1}_{\mathcal{H}}, \\ &P_n = (\mathbf{1}_{\mathcal{H}} \otimes P_{n-1})M_n, \quad n \geq 2. \end{aligned}$$

The following statement is a combination of the main results of [1] and [3].

Theorem 2. *Let T be braided and $\|T\| < 1$, then $P_n > 0$, $n \geq 2$, hence $\langle\cdot,\cdot\rangle_T$ is positive definite and the Fock representation π_F of $W(T)$ can be extended to a *-representation on the Hilbert space. Moreover, the Fock representation is a faithful bounded *-representation of the *-algebra $W(T)$ and*

$$\|\pi_F(a_i)\|^2 \leq \frac{1}{1 - \|T\|}, \quad i = 1, \dots, d.$$

Below we denote by $\mathcal{T}(\mathcal{H})_0$ the closure of $\mathcal{T}(\mathcal{H})$ with respect to the standard scalar product, and by $\mathcal{T}(\mathcal{H})_T$ the closure with respect to the Fock scalar product $\langle\cdot,\cdot\rangle_T$. Also $\mathcal{H}^{\otimes n}$ will denote the n-th tensor component of $\mathcal{T}(\mathcal{H})_0$, and $\mathcal{H}_T^{\otimes n}$ the corresponding component of $\mathcal{T}(\mathcal{H})_T$.

In paper [2], the authors proved that for any $q \in (0,1)$ the C^*-algebra $\mathcal{E}_q^{(d)}$ generated by the Fock representation of q-CCR with d generators contains the C^*-algebra $\mathcal{E}_0^{(d)}$ that is isomorphic to the Cuntz-Topelitz C^*-algebra $\mathcal{O}_d^{(0)}$. It was also shown in [2] that for q satisfying the condition

$$q^2 < \prod_{k=1}^{\infty} \frac{1 - q^k}{1 + q^k} = \sum_{k=-\infty}^{+\infty} (-1)^k q^{k^2}, \tag{3}$$

one has an isomorphism $\mathcal{E}_q^{(d)} \simeq \mathcal{E}_0^{(d)}$.

In this note we show that results of [2] can easily be extended to the C^*-algebras $\mathcal{W}(T)$ generated by operators of Fock representations of $W(T)$ with braided T and $||T|| = q$, $q < 1$.

2. C^*-algebra generated by Fock representation of $W(T)$

Let $W(T)$ be the Wick $*$-algebra with braided T and $||T|| = q < 1$. Consider the C^*-algebra $\mathcal{W}(T)$ generated by the operators $\pi_F(a_i)$, $\pi_F(a_i^*)$, $i = 1, \dots, d$, of the Fock representation of $W(T)$.

Following [2] construct the unitary operator $U \colon \mathcal{T}(\mathcal{H})_T \to \mathcal{T}(\mathcal{H})_0$.

Namely, consider the operators

$$U_n \colon \mathcal{H}_T^{\otimes n} \to \mathcal{H}^{\otimes n}$$

defined by

$$U_n = (\mathbf{1}_{\mathcal{H}} \otimes U_{n-1}) M_n^{\frac{1}{2}}, \quad n \geq 2, \quad U_0\Omega = \Omega, \quad U_1 = \mathbf{1}_{\mathcal{H}}. \tag{4}$$

It is easy to verify that on $\mathcal{H}_T^{\otimes n}$, $n \geq 2$, one has

$$M_n = \sum_{i=1}^{d} \pi_F(a_i^*)\pi_F(a_i) > 0,$$

so $M_n^{\frac{1}{2}}$ is well defined.

Proposition 1. *Put* $U = \bigoplus_{n=0}^{\infty} U_n$. *Then*

$$U : \mathcal{T}(\mathcal{H})_T \to \mathcal{T}(\mathcal{H})_0$$

is a unitary operator.

Proof. As in [2] one has to verify that $[U_n]^*[U_n] = [P_n]$, where $[U_n]$, $[P_n]$ denote matrices of the corresponding operators with respect to the standard basis of $\mathcal{H}^{\otimes n}$. Note also that $[P_n]$ is just the Gramm matrix for the standard basis of $\mathcal{H}_T^{\otimes n}$.

Indeed, using induction on d and the definition of P_n one can get

$$\begin{aligned}
[U_n]^*[U_n] &= [M_n^{\frac{1}{2}}]^*[\mathbf{1}_{\mathcal{H}} \otimes U_{n-1}]^*[\mathbf{1}_{\mathcal{H}} \otimes U_{n-1}][M_n^{\frac{1}{2}}] \\
&= [P_n][M_n^{\frac{1}{2}}][P_n]^{-1}(\mathbf{1}_d \otimes [P_{n-1}])[M_n^{\frac{1}{2}}] \\
&= [P_n][M_n^{\frac{1}{2}}][M_n]^{-1}[M_n^{\frac{1}{2}}] = [P_n].
\end{aligned}$$

□

Set $R = UM^{\frac{1}{2}}U^* : \mathcal{T}(\mathcal{H})_0 \to \mathcal{T}(\mathcal{H})_0$.

Theorem 3. *1.* $U\pi_F(a_i)U^* = V_i R$, *where*

$$V_i^* e_{i_1} \otimes \cdots \otimes e_{i_n} = e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_n}, \quad n \geq 1, \quad V_i^*\Omega = e_i, \quad i = 1, \ldots, d.$$

2. R *is a unique linear operator on* $\mathcal{T}(\mathcal{H})_0$ *leaving* $\mathcal{H}^{\otimes n}$ *invariant,* $R_{\mathbb{C}} = 0$, *and satisfying the relation*

$$R^2 = P + \sum_{i,j,k,l=1}^{d} T_{ij}^{kl} V_i^* R V_l^* V_k R V_j, \tag{5}$$

where $P = \sum_{j=1}^{d} V_i^* V_i$.

Proof. The proof is essentially the same as the similar result in [2].

1.

$$\begin{aligned}
(U\pi_F(a_i)U^*R^{-1})_{|\mathcal{H}^{\otimes n}} &= U_{n-1}\mu(e_i)M_nU_n^*U_nM_n^{-\frac{1}{2}}U_n^* = \mu(e_i)(\mathbf{1}_{\mathcal{H}} \otimes U_{n-1})M_n^{\frac{1}{2}}U_n^* \\
&= \mu(e_i)U_nU_n^* = \mu(e_i) = V_i, \quad i = 1, \ldots, d.
\end{aligned}$$

2. The second statement follows from induction arguments. □

Evidently, the operators V_i, $i = 1, \ldots, d$, determine the Fock representation of $W(0)$, i.e., $\mathcal{W}(0) = C^*(V_i, V_i^*,\ i = 1, \ldots, d)$.

Let us show that $\mathcal{W}(0) \subset U\mathcal{W}(T)U^*$ for any braided T with $||T|| = q$, $q < 1$. To prove the inclusion above it is sufficient to show that $V_i \in U\mathcal{W}(T)U^*$ for any $i = 1, \ldots, d$. Indeed, $R = UM^{\frac{1}{2}}U^*$ and

$$M = \sum_{i=1}^{d} \pi_F(a_i^*)\pi_F(a_i)$$

imply $R \in U\mathcal{W}(T)U^*$.

Since $R\Omega = 0$ and $V_i\Omega = 0$, to show that $V_i \in U\mathcal{W}(T)U^*$, it is sufficient to prove that R_n, the component of R corresponding to $\mathcal{H}^{\otimes n}$, is invertible and $||R_n^{-1}|| \leq C$ for some fixed $C > 0$ and any $n \in \mathbb{N}$. First we need some auxiliary results.

Lemma 1. *Let* T *be braided. Then for any* $n \geq 2$ *and* $1 \leq k \leq n-1$, *we have*

$$(T_1T_2\cdots T_n)T_1T_2\cdots T_k = T_1T_2\cdots T_{k+1}(T_1T_2\cdots T_n).$$

Proof. Indeed from relations $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ and $T_iT_j = T_jT_i$, $|i-j| \geq 2$, it follows that

$$(T_1T_2\cdots T_n)T_i = T_{i+1}(T_1T_2\cdots T_n), \quad 1 \leq i \leq n-1,$$

proving the statement of the lemma. □

Lemma 2. *Let T satisfy the braid relation. Then, for any $n \in \mathbb{N}$,*

$$M_{n+1}T_1T_2\cdots T_n = T_1T_2\cdots T_n + T_1^2T_2\cdots T_n(M_n \otimes \mathbf{1}_{\mathcal{H}}). \tag{6}$$

Proof. Using the previous lemma we get

$$\begin{aligned} &M_{n+1}T_1T_2\cdots T_n \\ &\quad = T_1T_2\cdots T_n + T_1(\mathbf{1}_{\mathcal{H}^{\otimes n+1}} + T_2 + T_2T_3 + \cdots + T_2T_3\cdots T_n)T_1T_2\cdots T_n \\ &\quad = T_1T_2\cdots T_n + T_1^2T_2\cdots T_n(\mathbf{1}_{\mathcal{H}^{\otimes n+1}} + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}) \\ &\quad = T_1T_2\cdots T_n + T_1^2T_2\cdots T_n(M_n \otimes \mathbf{1}_{\mathcal{H}}). \end{aligned}$$

□

As an immediate corollary we obtain the following result.

Lemma 3. *For any $n \geq 2$,*

$$M_{n+1}(\mathbf{1}_{\mathcal{H}^{\otimes n+1}} - T_1T_2\cdots T_n) = (\mathbf{1}_{\mathcal{H}^{\otimes n+1}} - T_1^2T_2\cdots T_n)(M_n \otimes \mathbf{1}_{\mathcal{H}}). \tag{7}$$

Proof.

$$\begin{aligned} &M_{n+1} - M_{n+1}T_1T_2\cdots T_n \\ &\quad = M_n \otimes \mathbf{1}_{\mathcal{H}} + T_1T_2\cdots T_n - T_1T_2\cdots T_n - T_1^2T_2\cdots T_n(M_n \otimes \mathbf{1}_{\mathcal{H}}) \\ &\quad = (\mathbf{1}_{\mathcal{H}^{\otimes n+1}} - T_1^2T_2\cdots T_n)(M_n \otimes \mathbf{1}_{\mathcal{H}}). \end{aligned}$$

□

Now we can show that the main estimate in [2] is true in our case.

Proposition 2. *Let T be braided, $\|T\| = q < 1$, and $R_n = R_{|\mathcal{H}^{\otimes n}}$. Then*

$$\frac{1}{1-q}\prod_{k=1}^{\infty}\frac{1-q^k}{1+q^k} \leq R_n^2 \leq \frac{1}{1-q}, \quad n \geq 1.$$

Proof. The idea of proof is the same as in [2].

Since $R^2 = UMU^*$,

$$M = \bigoplus_{n\geq 0} M_n, \quad M_0 = 0, \quad M_1 = \mathbf{1}_{\mathcal{H}},$$

it is sufficient to prove the required inequalities for M. Note that $\|T\| = \|T_i\| = q$. Further, since M_n is self-adjoint on $\mathcal{H}_T^{\otimes n}$, its norm equals to spectral radius and

$$\|M_n\|_T = r(M_n) \leq \|M_n\|$$

(here by $||\cdot||_T$ we denote the operator norm corresponding to the norm on $\mathcal{H}_T^{\otimes n}$). Analogously, $\|M_n^{-1}\|_T \leq \|M_n^{-1}\|$. Then, as in [2], we get

$$M_n \leq \|M_n\|_T \leq \|\mathbf{1}_{\mathcal{H}^{\otimes n}} + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}\| \leq \frac{1}{1-q}. \tag{8}$$

From (7) we have

$$M_n^{-1} = (\mathbf{1}_{\mathcal{H}^{\otimes n}} - T_1T_2\cdots T_{n-1})(M_{n-1}^{-1} \otimes \mathbf{1}_{\mathcal{H}})(\mathbf{1}_{\mathcal{H}^{\otimes n}} - T_1^2T_2\cdots T_{n-1})^{-1}. \tag{9}$$

This implies that

$$\|M_n^{-1}\| \leq \frac{1+q^{n-1}}{1-q^n}\|M_{n-1}^{-1}\|. \tag{10}$$

Hence,

$$\|M_n^{-1}\|_T \le \|M_n^{-1}\| \le (1-q)\prod_{i=1}^{n}\frac{1+q^i}{1-q^i} \tag{11}$$

and

$$M_n^{-1} \le (1-q)\prod_{i=1}^{\infty}\frac{1+q^i}{1-q^i}.$$

□

Corollary 1. *Let T be braided and $||T|| = q < 1$. Then, for any $j = 1,\dots,d$, $V_j \in C^*(U\pi_F(a_i)U^*,\ i = 1,\dots,d)$, i.e., $\mathcal{W}(0) \subset U\mathcal{W}(T)U^*$.*

Proof. The same as in [2]. □

Finally let us make a few remarks about the isomorphism $\mathcal{W}(0) \simeq \mathcal{W}(T)$ if $||T|| = q$ satisfies (3). The crucial step here is to show that under this assumption one has $R \in \mathcal{W}(0)$.

As in [2] denote by R_n the restriction of R to $\mathcal{H}^{\otimes n}$ and put $X_n \colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ to be

$$X_n = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus R_n \otimes \mathbf{1}_{\mathcal{H}} \oplus R_n \otimes \mathbf{1}_{\mathcal{H}} \otimes \mathbf{1}_{\mathcal{H}} \oplus \cdots \tag{12}$$

Note that Theorem 3 gives

$$R_{n+1}^2 = \mathbf{1}_{\mathcal{H}^{\otimes n+1}} + \sum_{i,j,k,l=1}^{d} T_{ij}^{kl} V_i^* R_n V_l^* V_k R_n V_j, \quad n \in \mathbb{N}. \tag{13}$$

Since

$$V_i^* R_n = (\mathbf{1}_{\mathcal{H}} \otimes R_n)V_i^*, \quad R_n V_i = V_i(\mathbf{1}_{\mathcal{H}} \otimes R_n), \quad i = 1,\dots,d,$$

one can present (13) in the following form:

$$R_{n+1}^2 = \mathbf{1}_{\mathcal{H}^{\otimes n+1}} + \sum_{i,j,k,l=1}^{d} T_{ij}^{kl}(\mathbf{1}_{\mathcal{H}} \otimes R_n) V_i^* V_l^* V_k V_j (\mathbf{1}_{\mathcal{H}} \otimes R_n), \quad n \in \mathbb{N}. \tag{14}$$

Since

$$\sum_{i,j,k,l=1}^{d} T_{ij}^{kl} V_i^* V_l^* V_k V_j|_{\mathcal{H}^{\otimes 2}} = T,$$

we have

$$\sum_{i,j,k,l=1}^{d} T_{ij}^{kl} V_i^* V_l^* V_k V_j|_{\mathcal{H}^{\otimes n}} = T_1.$$

Then

$$R_{n+1}^2 = \mathbf{1}_{\mathcal{H}^{\otimes n+1}} + (\mathbf{1}_{\mathcal{H}} \otimes R_n) T_1 (\mathbf{1}_{\mathcal{H}} \otimes R_n), \quad n \ge 2. \tag{15}$$

Evidently, $\{X_n,\ n \ge 1\}$ converges to R weakly and $X_n \in \mathcal{W}(0)$ for any $n \in \mathbb{N}$. So, the idea is to examine when $\{X_n,\ n \ge 1\}$ is norm convergent. Then evident modifications of the technique given in [2], using (15), allow to show that the following statement is true.

Proposition 3. *Let $\alpha_n(q)$ denote the smallest eigenvalue of R_n^2. Then*

$$\|X_{n+2} - X_{n+1}\| \le \frac{q}{\sqrt{(1-q)\min(\alpha_{n+1}(q), \alpha_{n+2}(q))}}\|X_{n+1} - X_n\|, \quad n \ge 1.$$

Further, recall some considerations from [2]. The condition

$$\liminf_{n\to\infty} \alpha_n(q) > \frac{q^2}{1-q}, \quad n \geq 1, \tag{16}$$

is sufficient for the sequence $\{X_n,\ n \geq 1\}$ to be norm convergent. Indeed, with the condition above one has that

$$\sum_{n\to\infty} \|X_{n+2} - X_{n+1}\| < \infty.$$

Further, since $\alpha_n(q) \geq \frac{1}{1-q}\prod_{k=1}^{\infty}\frac{1-q^k}{1+q^k}$, $n \geq 2$, see Proposition 2, one has that $\frac{1}{1-q}\prod_{k=1}^{\infty}\frac{1-q^k}{1+q^k} > \frac{q^2}{1-q}$. So, under the following assumption, $\alpha_n(q)$ satisfies (16):

$$q^2 < \prod_{k=1}^{\infty}\frac{1-q^k}{1+q^k} = \sum_{k=-\infty}^{+\infty}(-1)^k q^{k^2}. \tag{17}$$

I.e. the main result of [2] is true for $\mathcal{W}(T)$ with braided T, $||T|| = q$.

Theorem 4. *Let $\mathcal{W}(T)$ be the C^*-algebra generated by the operators of Fock representation of the Wick $*$-algebra $W(T)$ with coefficient operator T. Then if T satisfies the braid relation and $\|T\| = q$ with q satisfying (17), then $\mathcal{W}(T) \simeq \mathcal{W}(0)$.*

Acknowledgments. The work on the paper was initiated during my visit to Chalmers University of Technology. The warm hospitality and stimulating atmosphere are gratefully appreciated. I am grateful also to Prof. Lyudmila Turowska for stimulating discussions on the subject.

References

1. M. Bożejko and R. Speicher, *Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces*, Mat. Ann. **300** (1994), 97–120.
2. K. Dykema and A. Nica, *On the Fock representation of q-commutation relations*, J. Reine Angew. Math. **440** (1993), 201–212.
3. P. E. T. Jørgensen, D. P. Proskurin and Yu. S. Samoĭlenko, *The kernel of Fock representation of Wick algebras with braided operator of coefficients*, Pacific J. Math. **198** (2001), 109–122.
4. P. E. T. Jørgensen, L. M. Schmitt, and R. F. Werner, *Positive representations of general commutation relations allowing Wick ordering*, J. Funct. Anal. **134** (1995), 33–99.
5. P. E. T. Jørgensen, L. M. Schmitt, and R. F. Werner, *q-canonical commutation relations and stability of the Cuntz algebra*, Pacific J. Math. **163** (1994), no. 1, 131–151.

Kyiv Taras Shevchenko University, Cybernetics Department, 64 Volodymyrska, Kyiv, 01033, Ukraine

E-mail address: prosk@univ.kiev.ua

Received 20/09/2010