Methods of Functional Analysis and Topology
Vol. 17 (2011), no. 2, pp. 118–125

BOUNDARY PROBLEMS AND INITIAL-BOUNDARY VALUE PROBLEMS FOR ONE CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH LÉVY LAPLACIAN

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Abstract. We develop a method to construct a solution to a boundary problem and an initial-boundary value problem in a fundamental domain of a Hilbert space for a class of nonlinear parabolic equations not containing explicitly the unknown function,

$$\frac{\partial U(t,x)}{\partial t} = f(t, \Delta_L U(t,x)),$$

where Δ_L is the infinite dimensional Lévy Laplacian.

1. Introduction

In the paper by M. N. Feller [1] (see also [2]) we have constructed a solution of the Cauchy problem for a nonlinear parabolic equations with the Lévy Laplacian Δ_L,

$$\frac{\partial U(t,x)}{\partial t} = f(t, \Delta_L U(t,x)), \quad U(0,x) = U_0(x),$$

where $f(t,\zeta)$ is a function on R^2.

The present paper is devoted to solution of the boundary value problem for a nonlinear parabolic equations with the Lévy Laplacian,

$$\frac{\partial U(t,x)}{\partial t} = f(t, \Delta_L U(t,x)) \quad \text{in} \quad \Omega, \quad U(t,x) = G(t,x) \quad \text{on} \quad \Gamma,$$

and the initial-boundary value problem for a nonlinear parabolic equations with the Lévy Laplacian,

$$\frac{\partial U(t,x)}{\partial t} = f(t, \Delta_L U(t,x)) \quad \text{in} \quad \Omega,$$
$$U(0,x) = U_0(x), \quad U(t,x) = G(t,x) \quad \text{on} \quad \Gamma,$$

in a fundamental domain $\Omega \cup \Gamma$ of a Hilbert space H.

2. Preliminaries

Let H be a separable real Hilbert space, $F(x)$ be a scalar function defined on H, $x \in H$.

An infinite-dimensional Laplacian was introduced by P. Lévy [3]. For a function $F(x)$ twice strongly differentiable at the point x_0 the Lévy Laplacian in this point is defined (when it exists) by the formula

$$\Delta_L F(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (F''(x_0) f_k, f_k)_H, \tag{1}$$

where $F''(x)$ is the Hessian of the function $F(x)$, $\{f_k\}_1^\infty$ is an orthonormal basis in H.

2000 *Mathematics Subject Classification.* Primary 35R15, 45G05.

Key words and phrases. Lévy Laplacian, nonlinear parabolic equations, boundary problems, initial-boundary value problems.

In general, the Lévy Laplacian depends on the choice of the basis. However for specific classes of functions $\mathcal{F}$, the Lévy Laplacian often either does not depend on the choice of a basis in H (for example, if $\mathcal{F}$ is the Shilov class of functions [4]). Otherwise it can be independent of a basis from some set $\mathcal{B}$ of bases in H (for example, if $H = L_2(0,1)$, $\mathcal{F}$ is the class of functions whose second differential has the normal form, $\mathcal{B}$ is the set of uniformly dense bases in $L_2(0,1)$ [2], [3]).

In the sequel we will need a property of the Lévy Laplacian studied in [3] (see as well [2]). Set

$$F(x) = f(U_1(x), \dots, U_m(x)),$$

where $f(u_1, \dots, u_m)$ is a twice continuously differentiable function with m arguments defined on the domain $\{U_1(x), \dots, U_m(x)\} \subset R^m$, where $(U_1(x), \dots, U_m(x))$ is a vector of values of the functions $U_1(x), \dots, U_m(x)$. Assume that $U_j(x)$ are uniformly continuous in a bounded domain $\Omega \subset H$ and twice strongly differentiable functions and $\Delta_L U_j(x)$ exist $(j = 1, 2, \dots, m)$. Then $\Delta_L F(x)$ exists and

$$\Delta_L F(x) = \sum_{j=1}^{m} \frac{\partial f}{\partial u_j}\Big|_{u_j = U_j(x)} \Delta_L U_j(x). \tag{2}$$

Let Ω be a bounded domain in H (that is a bounded open set in H), while $\overline{\Omega} = \Omega \bigcup \Gamma$ – be a closed domain in H with the boundary Γ.

Define a domain Ω in H with a surface Γ as follows:

$$\Omega = \{x \in H : 0 \le Q(x) < R^2\}, \quad \Gamma = \{x \in H : Q(x) = R^2\},$$

where $Q(x)$ is a twice continuously differentiable function such that $\Delta_L Q(x) = \gamma$ and γ is a strictly positive constant. A domain of this type is called a fundamental domain.

Let us give some examples of fundamental domains.

1) A ball $\overline{\Omega} = \{x \in H : \|x\|_H^2 \le R^2\}$.

2) An ellipsoid $\overline{\Omega} = \{x \in H : (Bx, x)_H \le R^2\}$, where $B = \gamma E + S(x)$, E is an identity operator, $S(x)$ is a compact operator in H.

Introduce a function $T(x) = \frac{R^2 - Q(x)}{\gamma}$. The function $T(x)$ possesses the following properties:

$$0 < T(x) \le \frac{R^2}{\gamma}, \quad \Delta_L T(x) = -1 \quad \text{if} \quad x \in \Omega,$$
$$T(x) = 0 \quad \text{if} \quad x \in \Gamma.$$

3. The equation $\frac{\partial U(t,x)}{\partial t} = f(\Delta_L U(t,x))$

First we consider the nonlinear equation

$$\frac{\partial U(t,x)}{\partial t} = f(\Delta_L U(t,x)),$$

where $f(\zeta)$ is a given function of one argument.

3.1. **Boundary problem.** Consider the boundary value problem

$$\frac{\partial U(t,x)}{\partial t} = f(\Delta_L U(t,x)) \quad (x \in \Omega), \tag{3}$$

$$U(t,x) = G(t,x) \quad (x \in \Gamma), \tag{4}$$

where $f(\zeta)$ is a given function of one variable, $G(t,x)$ is a given function.

Theorem 1. *We assume the following.*

The function $f(\zeta)$ is a continuous function, twice differentiable in the domain $\{\Delta_L U(t,x)\} \subset R^1$. The equation $f(\zeta) = z$ has a solution $\zeta = \varphi(z)$.

The domain $\bar{\Omega}$ is fundamental.

In a some functional class $\mathcal{F}$ *there exists a solution* $V(\tau,x)$ *of the boundary value problem for the heat equation*

$$\frac{\partial V(\tau,x)}{\partial\tau}=\Delta_L V(\tau,x)\quad (x\in\Omega),\quad V(\tau,x)|_\Gamma=G(\tau,x). \tag{5}$$

The equation

$$f'\Big(\varphi\Big(\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=X+T(x)}\Big)\Big)[t-X]-T(x)=0 \tag{6}$$

has a solution $X=\chi(t,x)$ *such that* $\chi(t,x)|_\Gamma=t$.

Then the solution of the boundary problem (3), (4) has the form

$$U(t,x)=f(\psi(\chi(t,x)))[t-\chi(t,x)]-\psi(\chi(t,x))T(x)+V(\chi(t,x)+T(x),x), \tag{7}$$

where

$$\psi(\chi(t,x))=\varphi\Big(\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=X+T(x)}\Big) \tag{8}$$

($\psi(z)$ *is a function on* R^1 *).*

Proof. From (7) we deduce

$$\begin{aligned}\frac{\partial U(t,x)}{\partial t}&=f(\psi(\chi(t,x)))-f(\psi(\chi(t,x)))\frac{\partial\chi(t,x)}{\partial t}\\&+f'(\psi(\chi(t,x)))\psi'_z(\chi(t,x))\frac{\partial\chi(t,x)}{\partial t}[t-\chi(t,x)]\\&-\psi'_z((\chi(t,x)))\frac{\partial\chi(t,x)}{\partial t}T(x)+\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=\chi(t,x)+T(x)}\frac{\partial\chi(t,x)}{\partial t}\\&=f(\Psi(t,x))+\{f'(\psi(\chi(t,x)))[t-\chi(t,x)]-T(x)\}\psi'_z(\chi(t,x))\frac{\partial\chi(t,x)}{\partial t}\\&-\Big[f(\psi(\chi(t,x)))-\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=\chi(t,x)+T(x)}\Big]\frac{\partial\chi(t,x)}{\partial t}.\end{aligned}$$

Since $f=\varphi^{-1}$ we deduce from (8) that

$$f(\psi(\chi(t,x)))=\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=\chi(t,x)+T(x)}$$

and finally we get

$$\frac{\partial U(t,x)}{\partial t}=f(\psi(\chi(t,x))) \tag{9}$$

as far as $\chi(t,x)$ satisfies (6).

Due to (2) we deduce from (7) that

$$\begin{aligned}\Delta_L U(t,x)&=-f(\psi(\chi(t,x)))\Delta_L\chi(t,x)+f'(\psi(\chi(t,x)))\psi'_z(\chi(t,x))\Delta_L\chi(t,x)[t-\chi(t,x)]\\&-\psi(\chi(t,x))\Delta_L T(x)-\psi'_z(\chi(t,x))\Delta_L\chi(t,x)T(x)+\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=\chi(t,x)+T(x)}\Delta_L\chi(t,x)\\&+\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=\chi(t,x)+T(x)}\Delta_L T(x)+\Delta_L V(\tau,x)|_{\tau=\chi(t,x)+T(x)}.\end{aligned}$$

Recall that $\Delta_L T(x)=-1$, hence

$$\begin{aligned}\Delta_L U(t,x)&=\psi(\chi(t,x))+\{f'(\psi(\chi(t,x)))[t-\chi(t,x)]-T(x)\}\psi'_z(\chi(t,x))\Delta_L\chi(t,x)\\&-\Big[f(\psi(\chi(t,x)))-\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=\chi(t,x)+T(x)}\Big]\Delta_L\chi(t,x)\\&-\Big[\frac{\partial V(\tau,x)}{\partial\tau}-\Delta_L V(\tau,x)\Big]\Big|_{\tau=\chi(t,x)+T(x)}.\end{aligned}$$

As far as due to (8) we have $f(\psi(\chi(t,x))) = \frac{\partial V(\tau,x)}{\partial \tau}|_{\tau=\chi(t,x)+T(x)}$, $\chi(t,x)$ satisfies (6) and due to (5) we have $\frac{\partial V(\tau,x)}{\partial \tau} = \Delta_L V(\tau,x)$, then we deduce

$$\Delta_L U(t,x) = \psi(\chi(t,x)). \tag{10}$$

Substituting (9) and (10) into (3) we obtain the identity.

At the surface Γ we have $T(x) = 0$ and $\chi(t,x) = t$. Choosing $T(x) = 0$ and $\chi(t,x) = t$ in (7) and taking into consideration that $V(t,x)|_\Gamma = G(t,x)$ we have

$$U(t,x)|_\Gamma = V(t,x)|_\Gamma = G(t,x).$$

□

3.2. **Initial-boundary value problem.** Consider the initial-boundary value problem

$$\frac{\partial U(t,x)}{\partial t} = f(\Delta_L U(t,x)) \quad (x \in \Omega,\ t \in (0,\mathcal{T}]), \tag{11}$$

$$U(0,x) = 0, \tag{12}$$

$$U(t,x) = G(t,x) \quad (x \in \Gamma), \tag{13}$$

where $f(\zeta)$ is a function on R^1, $G(t,x)$ is a given function.

We denote by $\mathfrak{M}\Phi$ a mean value of the function $\Phi(y)$ over the sphere $\|y\|_H^2 = 1$.

Theorem 2. *We assume the following.*

The function $f(\zeta)$ is a continuous twice differentiable function defined in $\{\Delta_L U(t,x)\} \subset R^1$. There exists a solution $\zeta = \varphi(z)$ of the equation $f(\zeta) = z$ such that $\varphi(0) = 0$.

The domain $\bar{\Omega}$ is fundamental.

In a some functional class $\mathcal{F}$ there exists a solution $V(\tau,x)$ of the initial-boundary value problem for the heat equation

$$\frac{\partial V(\tau,x)}{\partial \tau} = \Delta_L V(\tau,x) \quad (x \in \Omega,\ \tau \in (0,\mathcal{T}]), \quad V(0,x) = 0, \quad V(\tau,x)|_\Gamma = G(\tau,x).$$

The function $G(t,x)$ is uniformly continuous in $\bar{\Omega}$ for each $t \in [0,\mathcal{T}]$, possesses a meal value $\mathfrak{M}G(t, x + \sqrt{2T(x)}y)$ $\forall t \in [0,\mathcal{T}]$ and besides $G(t,x) = 0$, $G'_t(t,x) = 0$ if $t \le r$ $(r > 0)$.

The equation

$$f'\Big(\varphi\Big(\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=X+T(x)}\Big)\Big)[t - X] - T(x) = 0 \tag{14}$$

has a solution $X = \chi(t,x)$ such that $\chi(t,x)|_\Gamma = t$ and $\chi(0,x) < r$.

Then the solution of the initial-boundary value problem (11)–(13) has the form

$$U(t,x) = f(\psi(\chi(t,x)))[t - \chi(t,x)] - \psi(\chi(t,x))T(x) + V(\chi(t,x) + T(x), x), \tag{15}$$

where $\psi(\chi(t,x)) = \varphi(\frac{\partial V(\tau,x)}{\partial \tau}|_{\tau=\chi(t,x)+T(x)})$.

Proof. We prove that (15) satisfies the equation (11) in Ω and $U(t,x) = G(t,x)$ on the surface Γ in the same way as it has been done in the proof of theorem 1.

Let us show that $U(0,x) = 0$.

To this end we check that if $G(\tau,x) = 0$ for $\tau \le 0$ then under the theorem conditions the solution $V(\tau,x)$ of the problem

$$\frac{\partial V(\tau,x)}{\partial \tau} = \Delta_L V(\tau,x) \quad \text{in} \quad \Omega, \quad V(0,x) = 0, \quad V(\tau,x)\Big|_\Gamma = G(\tau,x)$$

with $G(\tau,x) = 0$ for $\tau \le 0$ can be written in the form

$$V(t,x) = \mathfrak{M}G(t - T(x), x + \sqrt{2T(x)}y). \tag{16}$$

In fact at one hand
$$\frac{\partial V(t,x)}{\partial t} = \frac{\partial \mathfrak{M}G(t-T(x), x+\sqrt{2T(x)}y)}{\partial t}.$$
At the other hand due to (2) we have
$$\begin{aligned}\Delta_L V(t,x) &= -\frac{\partial \mathfrak{M}G(t-T(x), x+\sqrt{2T(x)}y)}{\partial t}\Delta_L T \\ &+ \Delta_L \mathfrak{M}G(\tau, x+\sqrt{2T(x)}y)|_{\tau=t-T(x)} \\ &= \frac{\partial \mathfrak{M}G(t-T(x), x+\sqrt{2T(x)}y)}{\partial t} + \Delta_L \mathfrak{M}G(\tau, x+\sqrt{2T(x)}y)|_{\tau=t-T(x)}\end{aligned}$$
since $\Delta_L T = -1$.

It is known that if a uniformly continuous in Ω function $F(x)$ possesses the mean value $\mathfrak{M}F(x+\sqrt{2T(x)}y)$ then the function $\mathfrak{M}F(x+\sqrt{2T(x)}y)$ is a harmonic function in Ω (see [5]), that is
$$\Delta_L \mathfrak{M}F(x+\sqrt{2T(x)}y) = 0.$$
Hence,
$$\Delta_L V(t,x) = \frac{\partial \mathfrak{M}G(t-T(x), x+\sqrt{2T(x)}y)}{\partial t}.$$
Substituting the above expressions for $\frac{\partial V(t,x)}{\partial t}$ and $\Delta_L V(t,x)$ into the heat equation $\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x)$ we obtain the identity
$$\frac{\partial \mathfrak{M}G(t-T(x), x+\sqrt{2T(x)}y)}{\partial t} = \frac{\partial \mathfrak{M}G(t-T(x), x+\sqrt{2T(x)}y)}{\partial t}.$$
Putting $t=0$ in (16) we get $V(0,x) = \mathfrak{M}G(-T(x), x+\sqrt{2T(x)}y) = 0$, since by theorem condition, $G(\tau,x)=0$ if $\tau \le 0$.

At the surface Γ $T(x)=0$ and it yields from (16) that $V(t,x)\Big|_\Gamma = \mathfrak{M}G(t,x) = G(t,x)$.

By (16), we get
$$V(\chi(t,x)+T(x), x) = \mathfrak{M}G(\chi(t,x), x+\sqrt{2T(x)}\,y)$$
and
$$V(\chi(0,x)+T(x), x) = \mathfrak{M}G(\chi(0,x), x+\sqrt{2T(x)}\,y) = 0$$
since by the theorem conditions $\chi(0,x) < r$, and $G(\tau,x)=0$ if $\tau \le r$.

By theorem conditions and (16) we can check that
$$\psi(\chi(t,x)) = \varphi\Big(\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}\Big) = \varphi\Big(\mathfrak{M}G'_\tau(\chi(t,x), x+\sqrt{2T(x)}y)\Big)$$
and
$$\begin{aligned}f(\psi(\chi(t,x))) &= f\Big(\varphi\Big(\frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)}\Big)\Big) \\ &= \frac{\partial V(\tau,x)}{\partial \tau}\Big|_{\tau=\chi(t,x)+T(x)} \\ &= G'_\tau(\chi(t,x), x+\sqrt{2T(x)}y).\end{aligned}$$
That is why
$$\psi(\chi(0,x)) = \varphi\Big(\mathfrak{M}G'_\tau(\chi(0,x), x+\sqrt{2T(x)}y)\Big) = \varphi(0) = 0,$$
$$f(\psi(\chi(0,x))) = G'_\tau(\chi(0,x), x+\sqrt{2T(x)}y) = 0.$$
Here we have used the fact that by the theorem conditions $\chi(0,x) < r$, $G'_\tau(\tau,x) = 0$ for $\tau \le r$ and $\varphi(0) = 0$.

Choosing $t = 0$ in (15) we obtain

$$U(0, x) = -f(\psi(\chi(0, x))\chi(0, x)) - \psi(\chi(0, x))T(x) + V(\chi(0, x) + T(x), x) = 0.$$

□

4. The equation $\frac{\partial U(t,x)}{\partial t} = f(t, \Delta_L U(t, x))$

Next we consider the nonlinear equation

$$\frac{\partial U(t, x)}{\partial t} = f(t, \Delta_L U(t, x)),$$

where $f(t, \zeta)$ is a given function of two variables.

Consider the boundary value problem

$$\frac{\partial U(t, x)}{\partial t} = f(t, \Delta_L U(t, x)) \quad (x \in \Omega), \tag{17}$$

$$U(t, x) = G(t, x) \quad (x \in \Gamma), \tag{18}$$

where $f(t, \zeta)$ is a given function of two variables and $G(t, x)$ is given function.

Theorem 3. *We assume the following.*

The function $f(t, \zeta)$ is a continuous function, differentiable in t and twice differentiable in ζ. The equation $f(t, \zeta) = z$ can be solved with respect to ζ, $\zeta = \varphi(t, z)$.

The domain $\overline{\Omega}$ is fundamental.

In a some functional class $\mathcal{F}$ there exists a solution $V(\tau, x)$ of the boundary problem for the heat equation

$$\frac{\partial V(\tau, x)}{\partial \tau} = \Delta_L V(\tau, x) \quad in \quad \Omega, \quad V(\tau, x)|_\Gamma = G(\tau, x).$$

The equation

$$\int_X^t f'_\zeta\Big(s, \varphi\Big(t, \frac{\partial V(\tau, x)}{\partial \tau}\Big|_{\tau = X + T(x)}\Big)\Big) ds - T(x) = 0$$

has a solution $X = \chi(t, x)$, such that $\chi(t, x)|_\Gamma = t$.

Then the solution of the boundary problem (17), (18) has the form

$$U(t, x) = \int_{\chi(t,x)}^t f(s, \psi(\chi(t, x)))\, ds - \psi(\chi(t, x))T(x) + V(\chi(t, x) + T(x), x), \tag{19}$$

where

$$\psi(\chi(t, x)) = \varphi\Big(\chi(t, x),\ \frac{\partial V(\tau, x)}{\partial \tau}\Big|_{\tau = \chi(t,x) + T(x)}\Big).$$

Consider the initial-boundary value problem

$$\frac{\partial U(t, x)}{\partial t} = f(t, \Delta_L U(t, x)) \quad (x \in \Omega,\ t \in (0, \mathcal{T}]) \tag{20}$$

$$U(0, x) = 0, \tag{21}$$

$$U(t, x) = G(t, x) \quad (x \in \Gamma), \tag{22}$$

where $f(t, \zeta)$ is a given function of two variables and $G(t, x)$ is a given function.

Theorem 4. *We assume the following.*

The function $f(t, \zeta)$ is a continuous function differentiable in t and twice differentiable in ζ. The equation $f(t, \zeta) = z$ can by solved with respect to ζ, $\zeta = \varphi(t, z)$, and $f(t, 0) = 0$.

The domain $\overline{\Omega}$ is fundamental.

In a some functional class $\mathcal{F}$ there exists a solution $V(\tau,x)$ of the initial-boundary problem for the heat equation

$$\frac{\partial V(\tau,x)}{\partial\tau}=\Delta_L V(\tau,x)\quad (x\in\Omega,\ \tau\in(0,\mathcal{T}]),\quad V(0,x)=0,\quad V(\tau,x)|_\Gamma=G(\tau,x).$$

The function $G(t,x)$ is a uniformly continuous function in $\overline{\Omega}$ for each $t\in[0,\mathcal{T}]$, having a mean value $\mathfrak{M}G(t,x+\sqrt{2T}y)\quad \forall t\in[0,\mathcal{T}]$ and $G(t,x)=0$, $G'_t(t,x)=0$ for $t\le r\quad (r>0)$;

The equation

$$\int_X^t f'_\zeta\Big(s,\varphi\Big(t,\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=X+T(x)}\Big)\Big)ds-T(x)=0$$

has a solution $X=\chi(t,x)$, such that $\chi(t,x)|_\Gamma=t$ and $\chi(0,x)<r$.

Then the solution of the initial-boundary value problem (20)–(22) has the form

$$U(t,x)=\int_{\chi(t,x)}^t f(s,\psi(\chi(t,x)))\,ds-\psi(\chi(t,x))T(x)+V(\chi(t,x)+T(x),x),\tag{23}$$

where

$$\psi(\chi(t,x))=\varphi\Big(\chi(t,x),\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=\chi(t,x)+T(x)}\Big).$$

Proofs of Theorems 3, 4 are similar to the proofs of Theorems 1, 2.

5. Example

Let us construct a solution of the initial-boundary value problem in a ball of the Hilbert space H $\overline{\Omega}=\{x\in H:\ \|x\|_H^2\le R^2\}$

$$\frac{\partial U(t,x)}{\partial t}=\sqrt{\Delta_L U(t,x)}\quad\text{in}\quad\Omega,\tag{24}$$

$$U(0,x)=0,\tag{25}$$

$$U(t,x)\Big|_{\|x\|_H^2=R^2}=g\Big(t-\frac12\|x\|_H^2\Big),\tag{26}$$

where $g(\lambda)=\frac12\lambda^2$ for $\lambda\ge0$, $g(\lambda)=0$ for $\lambda\le0$.

Equation (24) corresponds to the case $f(\zeta)=\sqrt{\zeta}$ and hence, $\varphi(z)=z^2$.

For the ball $\|x\|_H^2\le R^2$ the function $T(x)$ has the form $T(x)=\frac{R^2-\|x\|_H^2}{2}$.

The solution of the initial-boundary value problem for the heat equation

$$\frac{\partial V(\tau,x)}{\partial\tau}=\Delta_L V(\tau,x)\quad\text{in}\ \Omega,\quad V(0,x)=0,\quad V(\tau,x)|_{\|x\|_H^2=R^2}=g\Big(\tau-\frac12\|x\|_H^2\Big)$$

is given by

$$V(\tau,x)=g\Big(\tau+\frac12\|x\|_H^2-R^2\Big).$$

Hence

$$\psi(X)=\varphi\Big(\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=X+T(x)}\Big)=\Big(\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=X+T(x)}\Big)^2$$
$$=\Big(g'\Big(X-\frac{R^2}{2}\Big)\Big)^2=\Big(X-\frac{R^2}{2}\Big)^2.$$

But $f'(\zeta)=\frac{1}{2\sqrt{\zeta}}$ that yields

$$f'\Big(\varphi\Big(\frac{\partial V(\tau,x)}{\partial\tau}\Big|_{\tau=X+T(x)}\Big)\Big)=\frac{1}{2(X-\frac{R^2}{2})}$$

and as the result (14) is reduced to

$$\frac{1}{2\left(X-\frac{R^2}{2}\right)}[t-X]-T(x)=0.$$

Its solution is given by

$$X=\chi(t,x)=\frac{t+T(x)R^2}{1+2T(x)},$$

where $\chi(t,x)|_{\|x\|_H^2=R^2}=t$.

As far as

$$f(\chi(t,x))=\Big(\chi(t,x)-\frac{R^2}{2}\Big),$$
$$\psi(\chi(t,x))=\Big(\chi(t,x)-\frac{R^2}{2}\Big)^2,$$

$$V(\tau,x)|_{\tau=\chi(t,x)+T(x)}=\frac{1}{2}\Big(\chi(t,x)-\frac{R^2}{2}\Big)^2,$$

we get according to (15) that the solution $U(t,x)$ of the problem (24)–(26) has the form

$$U(t,x)=\frac{g\left(t-\frac{R^2}{2}\right)}{1+R^2-\|x\|_H^2}.$$

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Received 19/06/2009