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UNCONDITIONAL BASES OF DE BRANGES SPACES AND INTERPOLATION PROBLEMS CORRESPONDING TO THEM

G. M. GUBREEV AND M. G. VOLKOVA

Abstract. In this paper the unconditional bases of de Branges spaces are constructed from the values of reproducing kernels. Appropriate problems of interpolation by entire functions are also considered. The paper is a continuation of papers [2, 3].

1. Let us recall a number of facts from the theory of de Branges spaces [1], which are used in the course of the work.

In the sequel, $\mathbb{C}_+$ ($\mathbb{C}_-$) is an upper (lower) half-plane. A function f meromorphic in $\mathbb{C}_+$ is called a function of bounded type if $f(z) = f_1(z)/f_2(z)$, $f_1, f_2 \in H_+^\infty$. The mean type of a function of bounded type is defined by the formula

$$h(f) = \limsup_{y\to+\infty} y^{-1} \log |f(iy)|.$$

For an arbitrary entire function F we use the following notation $F^*(z) = \overline{F(\bar{z})}$. Let E be an entire function that satisfies the condition

$$|E^*(z)| < |E(z)|, \quad z \in \mathbb{C}_+. \tag{1}$$

Let us denote by $\mathcal{H}(E)$ a linear space of entire functions F such that:
1) $F(z)/E(z)$, $F^*(z)/E(z)$ are functions of bounded type and nonpositive mean type in $\mathbb{C}_+$;
2) $\int_{\mathbb{R}} |F(x)/E(x)|^2 dx < \infty$.

The space $\mathcal{H}(E)$ is a Hilbert space with respect to the inner product

$$(F, G) := \int_{\mathbb{R}} F(x)G^*(x)/|E(x)|^2 dx.$$

Assumed that

$$a(z) = \frac{1}{2}(E^*(z) + E(z)), \quad b(z) = \frac{1}{2i}(E^*(z) - E(z)),$$

we get the equality $E(z) = a(z) - ib(z)$, where a, b are entire real functions ($a = a^*$, $b = b^*$). The reproducing kernel of the space $\mathcal{H}(E)$ is expressed as

$$k(z, \lambda) := \pi^{-1} \left(b(z)\overline{a(\lambda)} - a(z)\overline{b(\lambda)} \right) / (z - \bar{\lambda}),$$

i.e., if $F \in \mathcal{H}(E)$, then

$$(F(x), k(x, \lambda)) = F(\lambda), \quad \lambda \in \mathbb{C}.$$

Then, an entire function S is called associated to de Branges space, if
a) $S(z)/E(z)$, $S^*(z)/E(z)$ are functions of bounded type and have nonpositive mean type

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in $\mathbb{C}_+$;
b) the following integral converges:

$$\int_{\mathbb{R}} |S(x)/E(x)|^2(1+x^2)^{-1}dx.$$

Let $\mathcal{H}(E)$ be an arbitrary de Branges space, $k(x,\lambda)$ its reproducing kernel, Λ an infinite sequence of complex numbers with the unique limit point ∞. The problem of description of the unconditional bases in the spaces $\mathcal{H}(E)$ of the form

$$\{k(z,\lambda_k)\,:\,\lambda_k\in\Lambda\} \tag{2}$$

was considered in [2, 3].

Unconditional bases from values of reproducing kernels are of interest in the theory of de Branges spaces [1]. In addition to this, criteria for completeness and basis property of the families (2) have an application to the theory of nonself-adjoint operators with discrete spectrum [3, 4]. For example, such an approach is efficient while studying boundary problems generated by canonic second-order differential systems of equations on a finite interval.

Then, let us assume that $\Lambda\cap\mathbb{R}=\varnothing$, i.e., $\Lambda=\Lambda_+\cup\Lambda_-$, where $\Lambda_\pm=\Lambda\cap\mathbb{C}_\pm$. Recall that [5] a sequence $\{\mu_k\}_1^\infty$, $\mu_k\in\mathbb{C}_+$ satisfies the Carleson condition if the following holds:

$$\inf_k\prod_{j\neq k}\left|\frac{\mu_k-\mu_j}{\mu_k-\bar\mu_j}\right|>0.$$

Let us consider a sequence $\bar\Lambda_-:=\{\bar\lambda_k,\,\lambda_k\in\Lambda_-\}$ in the upper half-plane. The following result takes place.

Theorem. ([2, 3]). *Let the sequence $\Lambda=\Lambda_+\cup\Lambda_-$ and*

$$\sup_{\lambda_k\in\Lambda_+}\left|\frac{E^*(\lambda_k)}{E(\lambda_k)}\right|<1,\qquad \sup_{\mu_k\in\bar\Lambda_-}\left|\frac{E^*(\mu_k)}{E(\mu_k)}\right|<1. \tag{3}$$

For family (2) to be an unconditional basis in the space $\mathcal{H}(E)$ it is necessary and sufficient that Λ coincide with the set of simple roots of a function S, associated to the space, satisfying the following conditions:
1) $h(S/E)=h(S^*/E)=0$;
2) the weight $w^2(x):=|S(x)/E(x)|^2$ satisfies the condition (A_2) on $\mathbb{R}$, i.e.,

$$\sup_\Delta\{M(w^2)M(w^{-2})\}<\infty,\qquad M(w^{\pm2}):=|\Delta|^{-1}\int_\Delta w^{\pm2}(x)dx,$$

where Δ is an arbitrary interval in $\mathbb{R}$, $|\Delta|$ its length;
3) the sequences Λ_+, $\bar\Lambda_-$ satisfy the Carleson condition.
Further, if only the conditions 1)–2) are fulfilled, then the family (2) is complete in the space $\mathcal{H}(E)$.

Let us remark that we have from (1) the factorization

$$\frac{E^*(z)}{E(z)}=ce^{i\alpha z}B(z),\quad \alpha>0,\quad |c|=1,\quad z\in\mathbb{C}_+,$$

where B is the Blaschke product in $\mathbb{C}_+$. Therefore the condition (3) puts constrains on the behaviour of Λ near the real line. For example, if $\alpha>0$, then the condition $\inf\limits_{\lambda_k\in\Lambda}|\operatorname{Im}\lambda_k|>0$ ensures that the inequalities (3) are true.

There exist unconditional bases of de Branges spaces in the form (2) with real sequences Λ [1]. It is clear that the formulated theorem is inapplicable to such bases. The approach proposed in this paper removes this disadvantage in some way. The results obtained here do not give a complete description of bases (2), though they have a significant usage in applications.

2. If $F \in \mathcal{H}(E)$, then F/E belongs to a Hardy class H^2_+ [1] and therefore

$$\int_{\mathbb{R}+i\varepsilon} \left|\frac{F(z)}{E(z)}\right|^2 dz \le \|F\|^2_{\mathcal{H}(E)}, \quad \varepsilon > 0,$$

where $\mathbb{R} + i\varepsilon := \{x + i\varepsilon,\, x \in \mathbb{R}\}$. The mentioned approach is based on the notion of an isotropic space $\mathcal{H}(E)$.

Definition. The de Branges space is called isotropic, if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\|F\|^2_{\mathcal{H}(E)} \le \delta(\varepsilon) \int_{\mathbb{R}+i\varepsilon} \left|\frac{F(z)}{E(z)}\right| dz.$$

If a sequence of complex numbers, Λ, do not intersect some straight line $\mathbb{R}+i\varepsilon$, $\varepsilon > 0$, then we introduce the following notations:

$$\Lambda^\varepsilon_+ := \{\lambda_k - i\varepsilon,\, \operatorname{Im}\lambda_k > \varepsilon,\, \lambda_k \in \Lambda\}, \quad \Lambda^\varepsilon_- := \{\lambda_k - i\varepsilon,\, \operatorname{Im}\lambda_k < \varepsilon,\, \lambda_k \in \Lambda\}.$$

Now let S be associated with the de Branges space $\mathcal{H}(E)$. Let us denote by $\Lambda = \{\lambda_k\}$ the set of roots of the function S, moreover, multiplicity of root λ_k is equal to m_k. Let us assume that the normalization condition $E(0) = S(0)$ is fulfilled. Therefore the function

$$u(z) := z^{-1}(E(z) - S(z))$$

belongs to the space $\mathcal{H}(E)$. The following theorem takes place.

Theorem 1. *Let a function S be associated to the isotropic space $\mathcal{H}(E)$, and the set of its roots, Λ, do not intersect some straight line $\mathbb{R} + i\varepsilon$, $\varepsilon > 0$.*

The family of functions

$$\mathcal{S}^j_k(z) := \frac{j!S(z)}{(z-\lambda_k)^{j+1}}, \quad 0 \le j < m_k, \quad \lambda_k \in \Lambda, \tag{4}$$

forms an unconditional basis in the closure of its linear span in $\mathcal{H}(E)$ if the following conditions are fulfilled:
1) $h(S/E) = h(S^*/E) = 0$;
2) the weight $w^2_\varepsilon(x) := |S(x + i\varepsilon)/E(x + i\varepsilon)|^2$ satisfies the condition (A_2) on $\mathbb{R}$;
3) the multiplicities m_k satisfy the condition $\sup_k m_k < \infty$;
4) the sequences Λ^ε_+, $\overline{\Lambda^\varepsilon_-}$ satisfy the Carleson condition.
Conditions 1)–4) are necessary and sufficient for the family (4) to form an unconditional base $\mathcal{H}(E)$, if the function $u(z + i\varepsilon)/S(z + i\varepsilon)$ is a function of bounded type in half-plane $\mathbb{C}_-$.

Let us point out the main steps of the proof of Theorem 1. First of all it follows from the condition 2) that S has no roots on the straight line $\mathbb{R} + i\varepsilon$. Due to isotropy of the space $\mathcal{H}(E)$, the operator

$$(V_\varepsilon F)(x) := F(x + i\varepsilon)/S(x + i\varepsilon), \quad F \in \mathcal{H}(E),$$

maps $\mathcal{H}(E)$ isomorphically on some subspace of the space L_2 on $\mathbb{R}$ with a measure $w^2_\varepsilon(x)dx$, and functions of the family (4) transform into rational functions

$$r^j_k(x) := \frac{j!}{(x-\mu_k)^{j+1}}, \quad 0 \le j < m_k, \quad \mu_k := \lambda_k - i\varepsilon, \quad \lambda_k \in \Lambda. \tag{5}$$

It is known [6] that the conditions 2)–4) guarantee the unconditional basis property of the family (5) in closure of its linear span. Since V_ε is an isomorphism, the family (4) is also a basis in the closure of its linear span in the space $\mathcal{H}(E)$.

Secondly, applying the arguments used in the proof of Theorem 2.1 in [3], we can conclude that completeness of the family (4) in $\mathcal{H}(E)$ is equivalent to the inclusion

$$u \in \operatorname*{closspan}_{\mathcal{H}(E)} \{\mathcal{S}_k^j \,:\, 0 \le j < m_k\,,\, \lambda_k \in \Lambda\}. \tag{6}$$

And finally it is proved that inclusion (6) is equivalent to the fact that the function $u(z+i\varepsilon)/S(z+i\varepsilon)$ is a function of bounded type in $\mathbb{C}_-$. Let us notice that the description of a closed linear span of the family (5) in the space $L_2(\mathbb{R})$ with the measure $w_\varepsilon^2(x)dx$ [6] is substantionally used while proving this fact.

Now let us consider the unconditional bases problem in form (2) for the space $\mathcal{H}(E)$. Let us assume that the associated function S in Theorem 1 has only simple roots, i.e., $m_k = 1$ for all $\lambda_k \in \Lambda$. Then the family of functions

$$\mathcal{S}_k(z) := \frac{S(z)}{(z-\lambda_k)S'(\lambda_k)}, \quad \lambda_k \in \Lambda,$$

and also the family (2) are mutually biorthogonal. As a result they form unconditional bases only at the same time. Thus, we have the following.

Theorem 2. *Let S be associated with an isotropic de Branges space $\mathcal{H}(E)$, Λ the set of its roots that are supposed to be simple. If for some $\varepsilon > 0$, the following conditions are fulfilled:*
1) $h(S/E) = h(S^*/E) = 0$,
2) the weight $|S(x+i\varepsilon)/E(x+i\varepsilon)|^2$ satisfies the condition (A_2) on $\mathbb{R}$,
3) the sequences Λ_+^ε, $\overline{\Lambda_-^\varepsilon}$ satisfy the Carleson condition,
4) $u(z+i\varepsilon)/S(z+i\varepsilon)$ is the function of bounded type in the half-plane $\mathbb{C}_-$,
then the family (2) forms an unconditional basis of the space $\mathcal{H}(E)$.

Thus, in case of the isotropic space $\mathcal{H}(E)$, the formulated theorem gives a basis property criterion for functions of the form (2) without the rather restrictive assumption $\Lambda \cap \mathbb{R} = \varnothing$.

3. In order to apply Theorems 1, 2, it is necessary to have simple criteria of isotropy of the de Branges space $\mathcal{H}(E)$. In this section we will formulate two such results that will allow us to know if isotropy of $\mathcal{H}(E)$ holds, using the properties of the function E.

Let us point out that due to (1) the roots of the function

$$S_\xi(z) := E^*(z) - \xi E(z), \quad |\xi| > 1 \tag{7}$$

are located in the half-plane $\mathbb{C}_-$. Let us consider a complex number ξ_+ which is calculated by formula

$$(\xi_+)^{-1} = \lim_{y\to+\infty} \frac{E^*(iy)}{E(iy)},$$

under the condition that this limit exists. If it is equal to 0, let us consider that $\xi_+ = \infty$ and assume the following in the formula (7):

$$S_\infty(z) = E(z).$$

Theorem 3. *Let the sequence of roots, Λ, of the function $S_\xi(z) = E^*(z) - \xi E(z)$ satisfy, with some $\xi \ne \xi_+ (|\xi| > 1)$, the following conditions:*
1) the multiplicities of the roots m_k of the function S_ξ satisfy the condition $\sup_k m_k < \infty$,
2) the sequence $\{\bar\lambda_k \,:\, \lambda_k \in \Lambda\}$ satisfies the Carleson condition,

3) $\sup\limits_{\lambda_k \in \Lambda} \operatorname{Im} \lambda_k < 0$,
then the de Branges space $\mathcal{H}(E)$ *is isotropic.*

Let us point out that an analog of Theorem 3, in which functions in the form (7) are present at $|\xi| < 1$, is true.

A disadvantage of the formulated theorem is that it is necessary to have an information about the arrangement of the roots of functions S_ξ. The next result is free from this deficiency.

Theorem 4. *Let* E *be an entire function of exponential type and the weight* $|E(x)|^2$ *satisfy the condition* (A_2) *on* $\mathbb{R}$. *Then the corresponding space* $\mathcal{H}(E)$ *is isotropic.*

If E satisfies the conditions of Theorem 4, then all elements of the space $\mathcal{H}(E)$ also have the exponential type that does not exceed $h(E)$ and belongs to the Cartright class [7]. Indeed, since $E^2(x)(1+x^2)^{-1}$ is summable on $\mathbb{R}$, E belongs to the Cartright class and so [8] it is a function of bounded type in $\mathbb{C}_+$. If $F \in \mathcal{H}(E)$, then F/E, F^*/E are functions of bounded type in $\mathbb{C}_+$. Consequently, the functions F, F^* also belong to this class. Under the Krein theorem [8], the function F has finite exponential type and belongs to the Cartright class.

4. In this section we formulate an interpolation corollary from unconditional bases from the values of reproducing kernels theorem (see p. 1). Let us point out that every theorem concerning bases of the form (2) generates a solution of an appropriate interpolation problem.

Let again, as in item 1, $\Lambda \cap \mathbb{R} = \varnothing$, $\Lambda = \Lambda_+ \cup \Lambda_-$ and let us suppose that parts of $\Lambda_\pm$ are numbered using subsets of the sets $\mathbb{Z}_+ := \{k \in \mathbb{Z}\,,\ k > 0\}$, $\mathbb{Z}_- := \{k \in \mathbb{Z}\,,\ k < 0\}$. Let us remark that the cases where Λ_+ or Λ_- are empty sets are not eliminated. Though, Λ is numbered using the set $\mathbb{Z}_0 \subseteq \mathbb{Z}_+ \cup \mathbb{Z}_-$, which is naturally ordered.

Let us denote the weight space of all complex sequences $c := \{c_j,\ j \in \mathbb{Z}_0\}$ by $l_2(b_j)$ with the norm

$$\|c\|^2 = \sum_{j \in \mathbb{Z}_0} |c_j|^2 b_j, \quad b_j > 0.$$

Now let $\mathcal{H}(E)$ be an arbitrary de Branges space. Using the given sequence Λ let us construct an interpolation operator J_Λ that associates to every function $F \in \mathcal{H}(E)$ the sequence

$$J_\Lambda F := \{F(\lambda_j)\ :\ j \in \mathbb{Z}_0\}. \tag{8}$$

The problem of solvability of interpolation problems of the type

$$F(\lambda_j) = c_j, \quad j \in \mathbb{Z}_0, \quad F \in \mathcal{H}(E), \tag{9}$$

we are interested in, will be formulated in the following way.

What kind of sequence should Λ and weight sequence $\{b_j\}$ be in order to make the operator J_Λ map the space $\mathcal{H}(E)$ on the weight space $l_2\{b_j\}$ bijectively and bicontinuously?

Let the sequence Λ be such that inequalities (3) hold. Let us call it interpolational if it coincides with the set of simple roots of some function S associated to the space $\mathcal{H}(E)$ satisfying the following conditions:
1) $h(S/E) = h(S^*/E) = 0$;
2) the weight $|S(x)/E(x)|^2$ satisfies the condition (A_2);
3) Λ_+ and $\bar{\Lambda}_-$ satisfy the Carleson condition.
In the sequel bijective and bicontinuous linear operators are called isomorphisms.

Theorem 5. *1) If an operator J_Λ, which is determined by formula (8), is an isomorphism of $\mathcal{H}(E)$ on the space $l_2(b_j)$, then the sequence $\{b_j\}$ satisfies the two-sided estimates*

$$(10)\qquad b_j \asymp (\operatorname{Im}\lambda_j)|E(\lambda_j)|^{-2}, \quad j \in \mathbb{Z}_+; \quad b_j \asymp |\operatorname{Im}\lambda_j||E(\bar{\lambda}_j)|^{-2}, \quad j \in \mathbb{Z}_-.$$

2) Let the sequence $\{b_j\}$ satisfy the estimates (10). Then the operator J_Λ is an isomorphism of $\mathcal{H}(E)$ on $l_2(b_j)$ if and only if the sequence Λ is interpolational.

3) If the conditions of the previous statement are fulfilled, then a solution of the interpolational problem (9) for every sequence $\{c_{ij}\}$ from the space $l_2(b_j)$ is given by the series

$$(11)\qquad F(z) = \sum_{\lambda_j \in \Lambda} c_j \frac{S(z)}{S'(\lambda_i)(z-\lambda_j)},$$

where the associated function S is a part of the definition of the interpolational sequence Λ.

Let us give a sketch of the proof of Theorem 5. Denote by e_k the sequence which for $k \in \mathbb{Z}_0$ is equal to 1 and 0 for all other $j \in \mathbb{Z}_0$. If the operator J_Λ is an isomorphism, then the family of preimages $F_k(z) := J_\Lambda^{-1} e_k$, $k \in \mathbb{Z}_0$, forms an unconditional basis $\mathcal{H}(E)$. Therefore, the biorthogonal family $\{k(z, \lambda_j) : \lambda_j \in \mathbb{Z}_0\}$ also forms an unconditional basis, and thus the sequence Λ is interpolational. From the definition of the function F_k, it follows that $\|F_k\|^2 \asymp \|e_k\|^2 = b_k$ and, since $\|F_k\|^2 \|k(x, \lambda_k)\|^2 = 1$, we come to the two-sided estimates $b_k \asymp \|k(x, \lambda_j)\|^{-2}$, $j \in \mathbb{Z}_0$. Since inequalities (3) take place, estimates [3] are true,

$$\|k(x, \lambda_j)\|^2 \asymp (\operatorname{Im}\lambda_j)^{-1}|E(\lambda_j)|^2, \quad \lambda_j \in \Lambda_+,$$
$$\|k(x, \lambda_j)\|^2 \asymp |\operatorname{Im}\lambda_j|^{-1}|E(\bar{\lambda}_j)|^2, \quad \lambda_j \in \Lambda_-.$$

Conditions (10) follow from these estimates.

Let us also remark that series (11) converges in the norm of the space $\mathcal{H}(E)$ and uniformly on compacts of the complex plane.

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Poltava National Technical University, 24 Pervomaiskii prospect, Poltava, 36011, Ukraine
E-mail address: gubreev@yandex.ru

South Ukrainian Pedagogical University, 26 Staroportofrankivs'ka, Odesa, 65091, Ukraine
E-mail address: varv@rambler.ru

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