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FACTOR REPRESENTATIONS OF INFINITE SEMI-DIRECT PRODUCTS

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Abstract. In this article, we propose a new method to study unitary representations of inductive limits of locally compact groups. For the group of infinite upper triangular matrices, we construct a family of type III factorial representations. These results are complements to previous results of A. V. Kosyak, and Albeverio and Kosyak [1, 5].

1. Introduction and notations

In recent works [1, 5] a class of unitary representations of the group of upper triangular real matrices $B_0^{\mathbf{N}}$ of arbitrarily large size have been defined and studied. The space of representations is an inductive limit of Hilbert spaces, which can be described as an infinite tensor product along a vector Ω. It is shown that, under reasonable conditions, the von Neumann algebra of $B_0^{\mathbf{N}}$ is a type III factor. In this paper, we propose a new method to study these representations. Although our results have some overlap with Kosyak's conditions, our approach is completely different, and we hope, will be useful to study infinite semi direct products of a large class of groups. Our conditions are different, and are, in some sense, complements to conditions from [5]. The paper is organized as follows. In Section 2, we study inductive limit representations, and establish some properties that will be used in the sequel. In Section 3, we give a short argument to prove that our representations are factorial. In Section 4, we compute the spectrum of the von Neumann algebra $\mathcal{M}$, generated by the representations introduced in Section 2. Using a result of G. K. Pedersen, the problem boils down to constructing operators X in $\mathcal{M}$, and Y, in $\mathcal{M}'$, fulfilling suitable approximate commutation relations. The real Heisenberg group, $H_3(\mathbf{R})$, embeds into $B_0^{\mathbf{N}}$ in a natural way. We construct such operators, associated to every copy of $H_3(\mathbf{R})$ contained in $B_0^{\mathbf{N}}$. This gives us operators associated to finite dimensional subgroups of $B_0^{\mathbf{N}}$. We finally show that, for the representations introduced in Section 2, we can extend our construction to the whole inductive limit, and fulfill Pedersen's conditions. The algebra $\mathcal{M}$ is a factor of type III_1.

Let us now fix notations: We shall denote by $\{e_{i,j}\}$ the usual matrix units, and by $\mathbf{1}$ the identity matrix.

For any $1 \leq i < j$, we let $G_{i,j}$ be the subgroup $\mathbf{1} + \mathbf{R}e_{i,j}$.

For any strictly positive integer n, we shall denote by G_n the subgroup generated by $\{G_{i,n}, 1 \leq i < n\}$.

We define inductively the groups $(G^n)_{n>0}$ as follows: $G^1 = \{e\}$ is the trivial group. For any $n \geq 1$, we set $G^{n+1} = G^n \ltimes G_{n+1}$. The group G^n acts on G^{n+1} through matrix multiplication.

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In the rest of the paper, we suppose that a unit vector $\omega \in L^2(\mathbf{R})$ is chosen, and a set $\{b_{i,j}, 1 \leq i < j\}$ of strictly positive integers is given. We define $\Omega_{i,j}(x) = b_{i,j}^{1/4}\omega(x\sqrt{b_{i,j}})$. We define the functions:
$\Omega_n = \otimes_{i=1}^{n-1}\Omega_{i,n}$, $n > 1$, $\Omega^n = \otimes_{k=2}^{n}\Omega_k$, $n > 1$,
$\mathcal{H}_n = L^2(G_n)$, $n \geq 2$ (with the Haar measure),
$\mathcal{H}_{i,j} = L^2(\mathbf{R}_{i,j})$, $1 \leq i < j$,
$\mathcal{H}^n = \otimes_{j=2}^{n}\mathcal{H}_j \simeq L^2(G^n)$,
$\Omega = \otimes_{n=2}^{\infty}\Omega_n$.

We shall also use Fourier transform on the spaces $\mathcal{H}_{i,j}$.The Fourier transform of a function $\xi \in \mathcal{H}_{i,j}$ will be denoted by $\mathcal{F}_{i,j}(\xi) \in \widehat{\mathcal{H}}_{i,j}$, or, simply, by $\widehat{\xi}$. In these notations, $\mathcal{F}_{i_1,j_1} \otimes \mathcal{F}_{i_2,j_2} \otimes \cdots \otimes \mathcal{F}_{i_n,j_n}$ is an isometry of $\mathcal{H}$ onto $\widehat{\mathcal{H}}_{i_1,j_1} \otimes \widehat{\mathcal{H}}_{i_2,j_2} \otimes \cdots \otimes \widehat{\mathcal{H}}_{i_n,j_n} \otimes \bigotimes_{(a,b)\notin\{(i_1,j_1),\ldots,(i_n,j_n)\}} \mathcal{H}_{a,b}$. If no confusion can arise, we shall keep the same symbols to denote operators, representations, etc... on $\mathcal{H}$, and on the partial Fourier transformed space. Explicitly, the operator $T \in \mathcal{L}(\mathcal{H})$, and the operator $\mathcal{F}_{i,j}T\mathcal{F}_{i,j}^*$ on the space $\widehat{\mathcal{H}}$ will be denoted by the same symbol T. The same conventions apply to representations of groups, algebras, etc...

For each integer $n > 1$, the isometry $\xi \to \xi \otimes \Omega_{n+1}$, from $\mathcal{H}^n$ to $\mathcal{H}^{n+1}$ will be denoted by v_n. Given $m > n$, the composition $v_{m-1}v_{m-2}\ldots v_n : \mathcal{H}^n \hookrightarrow \mathcal{H}^m$ is denoted by v_n^m.

The Hilbert space inductive limit of the system $(\mathcal{H}^n, v_n)_{n\in\mathbf{N}}$ is the infinite tensor product of the spaces $\mathcal{H}_n$ along the vector Ω. It will be denoted by $\mathcal{H} = \bigotimes_{n=2}^{\infty}(\mathcal{H}_n, \Omega_n)$.

2. Operators on inductive limit spaces

In this section, we recall some known facts about infinite tensor products that will be needed in the sequel.

We shall denote by $\mathcal{H}$ the infinite tensor product of the Hilbert spaces $(\mathcal{H}_n)_{n>1}$, along the vector $\Omega = \bigotimes_{n=2}^{\infty}\Omega_n$. The space $\mathcal{H}$ is the inductive limit of an increasing sequence of subspaces $(\mathcal{H}^n)_{n\in\mathbf{N}}$, as in section 1. Any operator $T \in \mathcal{L}(\mathcal{H}^n)$ has a canonical extension (amplification) $T \otimes \mathrm{Id}$ to $\mathcal{H}$.

2.1. **Infinite tensor product of representations.** We now suppose that a unitary, strongly continuous representation, $u^{(n)} : \mathbf{R} \to \mathcal{L}(\mathcal{H}^n)$ is given for each integer $n > 1$. For any $s \in \mathbf{R}$, the unitary $u^{(n)}(s)$ will be denoted by $u_s^{(n)}$.

Lemma 2.1. *Let $\mathcal{H} = \bigotimes_{n=2}^{\infty}(\mathcal{H}_n, \Omega_n)$. Assume that the scalar products $\{(u_s^{(n)}\Omega_n|\Omega_n)$, $n > 1$, $s \in \mathbf{R}\}$ are reals. Suppose that, for each $n > 1$, there exists a positive constant C_n, such that $1 - (u_s^{(n)}\Omega_n|\Omega_n) \leq C_n|s|$, for any real s. If $\sum_{n=2}^{\infty} C_n < \infty$, then, for any real s, the sequence $(\bigotimes_{k=2}^{n} u_s^{(k)} \otimes \mathrm{Id})_{n\in\mathbf{N}}$ converges strongly in $\mathcal{L}(\mathcal{H})$ to a unitary u_s. Moreover, the map $s \to u_s$ is a strongly continuous unitary representation of $\mathbf{R}$ in $\mathcal{L}(\mathcal{H})$.*

Proof. Let $\xi = \xi_0 \otimes (\bigotimes_{k=n+1}^{\infty}\Omega_k)$, with $\xi_0 \in \mathcal{H}^n$. Let $q > p > n$, and $s \in \mathbf{R}$. We have: $\|(\bigotimes_{k=2}^{p} u_s^{(k)} \otimes \mathrm{Id} - \bigotimes_{k=2}^{q} u_s^{(k)} \otimes \mathrm{Id})\xi\|^2 = \|\xi_0\|^2\|\bigotimes_{k=p+1}^{q}\Omega_k - \bigotimes_{k=p+1}^{q} u_s^{(k)}\Omega_k\|^2 = 2\|\xi_0\|^2(1 - \prod_{k=p+1}^{q}(u_s^{(k)}\Omega_k|\Omega_k)) = 2\|\xi_0\|^2(1 - \prod_{k=p+1}^{q}(1 - (1 - (u_s^{(k)}\Omega_k|\Omega_k)))$. Moreover, for every sequence of positive reals $(a_k)_{k\in\mathbf{N}}$, such that $\sum_{k=1}^{\infty} a_k < 1/2$, we have: $\prod_{k=1}^{\infty}(1 - a_k) \geq 1 - 2\sum_{k=1}^{\infty} a_k$. With $a_k = 1 - (u_s^{(k)}\Omega_k|\Omega_k)$, we obtain: $\|(\bigotimes_{k=2}^{p} u_s^{(k)} \otimes \mathrm{Id} - \bigotimes_{k=2}^{q} u_s^{(k)} \otimes \mathrm{Id})\xi\|^2 \leq 4\|\xi_0\|^2\sum_{k=p+1}^{q}(1 - (\Omega_k|u_s^{(k)}\Omega_k)) \leq 4|s|\|\xi_0\|^2\sum_{k=p+1}^{q} C_k$, for all sufficiently large p. It follows that the sequence $(\bigotimes_{k=2}^{n} u_s^{(k)} \otimes \mathrm{Id})_{n\in\mathbf{N}}$ converges strongly to a unitary u_s. Let us now show the strong continuity of the representation. It suffices to evaluate $\|u_s(\xi) - \xi\|$, for sufficiently small $|s|$. Let $\epsilon > 0$ be given. It follows from the above that there exists $p \in \mathbf{N}$, such that $\|u_s(\xi) - \bigotimes_{k=2}^{p} u_s^{(k)}\xi \otimes \bigotimes_{l=p+1}^{\infty}\Omega_l\| < \epsilon$, for every s such that $|s| < 1$. Since the representations $u^{(k)}$ are supposed to be strongly continuous, the result follows. □

2.2. **The left and right regular representations.**

2.2.1. *Existence of the left representation.* In this section, we shall give a sufficient condition for the existence of analogs of the left regular representation of the group $G = \cup_{1\leq i<j} G_{i,j}$, with $G_{i,j}$ as in section 1. (Note that the condition we shall be using $(B1)$ is more restrictive than the criteria established in [4].) The space $\mathcal{H}$ of these regular representations is an inductive limit of the spaces $\mathcal{H}^n \simeq L^2(G^n) = \bigotimes_{1\leq i<j\leq n} \mathcal{H}_{i,j}$ along the vectors $\Omega_{i,j}$, as described in section 1. The right regular representation is always well defined. For every triple of integers $i < j < k$, the group $G_{i,j} \simeq \mathbf{1}+\mathbf{R}e_{i,j}$ acts from the left on $\mathcal{H}_{i,k} \otimes \mathcal{H}_{j,k} \simeq L^2(\mathbf{R}^2)$ through the unitaries $(u_{i,j}(s)\xi)(x, y) = \xi(x-sy, y)$, $s, x, y \in \mathbf{R}$. The existence of a left action of $G_{i,j}$ on the whole space $\mathcal{H}$ amounts to the existence of an infinite tensor product of such unitaries. We shall study the convergence of these unitaries, acting on the infinite tensor product $\bigotimes_{k\geq j}(\mathcal{H}_{i,k} \otimes \mathcal{H}_{j,k})$, along the vectors $\Omega_{i,k} \otimes \Omega_{j,k}$. Our vectors $\Omega_{i,j}$ are the composition of a fixed function $\omega \in L^2(\mathbf{R})$ with a linear transformation of the real line. With a view to using the conditions of 2.1, we shall make the following assumptions on ω.

Assumptions on ω:

(1) ω is a unit vector in $L^2(\mathbf{R})$.
(2) ω is a real valued, positive function.
(3) There exists a constant $C_\omega > 0$, such that: for any $s \in \mathbf{R}$,
$\|\omega(x + sy)\omega(y) - \omega(x)\omega(y)\|^2 = 2|1 - (\omega(x + sy)\omega(y)|\omega(x)\omega(y))| \leq C_\omega|s|$.

(These requirements are fulfilled by the standard Gaussian function $\omega(x) = \pi^{-1/4}e^{-x^2/2}$ considered by Kosyak.)

Definition 2.2. Suppose we are given, for each pair of integers (i, j), with $0 < i < j$, a strictly positive real number $b_{i,j}$. Let $\omega \in L^2(\mathbf{R})$ be a unit vector, fulfiling assumptions 1 and 2 above.

(1) Define, for each pair (i, j), the function $\Omega_{i,j}(x) = b_{i,j}^{1/4}\omega(x\sqrt{b_{i,j}})$, $x \in \mathbf{R}$.
We define $\mathcal{H}$ as the infinite tensor product $\bigotimes_{0<i<j} \mathcal{H}_{i,j}$, along the vector $\Omega = \bigotimes_{0<i<j} \Omega_{i,j}$, with $\mathcal{H}_{i,j} = L^2(\mathbf{R})$, for every pair (i, j).
(2) For each integer $n \geq 2$, set $\Omega_n = \bigotimes_{0<i<n} \Omega_{i,n}$, and $\Omega^n = \bigotimes_{1<l\leq n} \Omega_l$. The spaces $\mathcal{H}_n = \bigotimes_{1\leq i<n} \mathcal{H}_{i,n}$, with $\mathcal{H}_{i,n} \simeq L^2(\mathbf{R})$, and $\mathcal{H}^n = \bigotimes_{2\leq l\leq n} \mathcal{H}_l$ are defined accordingly.
(3) We shall say that condition $B1$ is fulfilled, if and only if, for any pair of integers (i, j), with $0 < i < j$, we have : $\sum_{\{k/\, j<k\}} \sqrt{\frac{b_{i,k}}{b_{j,k}}} < \infty$. We shall say that condition $B2$ is fulfilled, if and only if, $\sum_{\{i,j,k/\, 0<i<j<k\}} \sqrt{\frac{b_{i,k}}{b_{i,j}b_{j,k}}} < \infty$.

(Note that condition $B2$ is stronger than condition $B1$.)

Definition 2.3. For every integer $n > 1$, we shall denote by $\lambda^{(n)}$ the left regular representation of G^n on $\mathcal{H}^n \simeq L^2(G^n)$, and by $\lambda^{(n)} \otimes \mathbf{1}$, the amplification of $\lambda^{(n)}$ to the space $\mathcal{H}$. (Explicitly, $\lambda^{(n)} \otimes \mathbf{1}(g) = \lambda^{(n)}(g) \otimes \mathrm{Id}$, for every $g \in G^n$.)

Lemma 2.4. *If condition $B1$ is fulfilled (see definition 2.2) , then, for every $g \in \bigcup_{l>1} G^l$, the sequence $(\lambda^{(n)}\otimes\mathbf{1}(g))_n$ converges strongly in $\mathcal{L}(\mathcal{H})$. We shall denote by $\lambda(g)$ the strong limit of this sequence.*

Proof. It suffices to show the strong convergence of the sequence $(\lambda^{(n)} \otimes \mathbf{1}(\mathbf{1}+se_{i,j}))_{n\geq j}$, for every $0 < i < j$, and every $s \in \mathbf{R}$. We have $(\lambda^{(n)} \otimes \mathbf{1}(\mathbf{1} + se_{i,j})) = \lambda^{(j)}(\mathbf{1} +$

$se_{i,j})\otimes\bigotimes_{j<k\leq n} u_s^{(k)}\otimes \mathrm{Id}$, with $u_s^{(k)} \in \mathcal{L}(\mathcal{H}_k)$. It follows from our assumptions on ω, that $1-(u_s^{(k)}\Omega_k|\Omega_k) < |s| C_\omega \sqrt{\frac{b_{i,k}}{b_{j,k}}}$. We conclude using lemma 2.1. □

Definition 2.5. Let G be endowed with the inductive limit topology. With ω and $\mathcal{H}$ as above, we shall denote by λ the weakly continuous unitary representation of G in $\mathcal{L}(\mathcal{H})$, described in lemma 2.4. We shall call λ the left regular representation of G on $\mathcal{H}$, relative to Ω. For every integer $n>1$, let $\rho^{(n)}$ be the right regular representation of G^n on $\mathcal{H}^n$. We define the right regular representation, ρ, of G on $\mathcal{H}$, by $\rho(g)(\xi\otimes\eta)=\rho^{(n)}(g)\xi\otimes\eta$, for every $g\in G^n$, $\xi\in\mathcal{H}^n$, and $\eta\in\bigotimes_{k>n}\mathcal{H}_k$. (See also [5].) We set $\mathcal{M}=\lambda(G)''$, and $\mathcal{N}=\rho(G)''$.

When no confusion can arise, we shall abbreviate notations, and call λ simply the left regular representation of G.

2.2.2. *Cuts of the left regular representation.*

Note 2.6. We use the notations of [6]. Given a locally compact group Γ, and a Hilbert space H, we shall denote by $L^2(\Gamma,H)$ the Hilbert space of square integrable functions on Γ, with values in H. This space is isometric to the completed tensor product of Hilbert spaces $L^2(\Gamma)\otimes H$.

In this section, we give an equivalent construction of the left regular representation of the group G. This is based on the absorption property of the regular representation of locally compact groups. Although this is standard material, we recall the proof for the convenience of the reader. Given a locally compact group Γ, and a unitary representation, π, of Γ on a separable Hilbert space H, there exists a unitary intertwining $\lambda\otimes\pi$, and $\lambda\otimes\mathbf{1}$. (Details can be found in [3].) Here, λ denotes the regular representation of Γ, and $\mathbf{1}$ denotes the trivial representation. Explicitly, define $\mathcal{V}$ on $L^2(\Gamma,H)$, such that $(\mathcal{V}(\xi\otimes\eta))(g)=\xi(g)\pi(g)\eta$, for every $\xi\in L^2(\Gamma)$, $\eta\in H$,$g\in\Gamma$. It is easily checked that $\mathcal{V}$, defined that way, extends to a unitary on $L^2(\Gamma,H)$, given by $\mathcal{V}\zeta(g)=\pi(g)\zeta(g)$, $\zeta\in L^2(\Gamma,H)$, $g\in\Gamma$. Let $\zeta\in L^2(\Gamma,H)$, $\gamma\in\Gamma$, and $g\in\Gamma$ be given. We compute : $(\mathcal{V}^*\lambda\otimes\pi(\gamma)\mathcal{V}\zeta)(g)=\pi(g^{-1})[(\lambda\otimes\pi(\gamma)\mathcal{V}\zeta)(g)]=\pi(g^{-1})\pi(\gamma)[\mathcal{V}\zeta(\gamma^{-1}g)]=\pi(g^{-1}\gamma)[\pi(\gamma^{-1}g)\zeta(\gamma^{-1}g)]=[\lambda\otimes\mathbf{1}(\gamma)\zeta](g)$.

Since our group G is the inductive limit of the locally compact groups G^n, we get, for each n, a unitary intertwining the regular representation of G^n, and the restriction to G^n of the regular representation of G. The study of this sequence of unitaries will lead to an alternate definition of the left regular representation of G. The following decomposition pf $\mathcal{V}$ will be used :

Definition 2.7. (1) Let $k\geq n>1$ be integers. For every $h\in G^n$, we define the $k-rest$ of $(\lambda(h))$ as the unitary $(\lambda^{(k)}\otimes\mathbf{1}(h))^{-1}\lambda(h)$. We shall denote by $\lambda_{(k)}$ the $k-rest$ of $(\lambda(h))$, so that $\lambda(h)=\lambda^{(k)}(h)\otimes\lambda_{(k)}(h)$, on $\mathcal{H}\simeq L^2(G^k)\otimes\bigotimes_{p>k}\mathcal{H}_p$.

(2) Let $n\geq i>1$ be a pair of integers. We decompose $\mathcal{H}$ as $\mathcal{H}=L^2(G^n,\bigotimes_{p>n}\mathcal{H}_p)$. Each g, element of G^n can be written uniquely as $g=g_ng_{n-1}\ldots g_2$, with $g_j\in G_j$, for $2\leq j\leq n$. (Recall that in our notations, $G^1=G_1$ is the trivial group.) We define the unitary $\mathcal{V}_{i,n}$, on $\mathcal{H}$, by

$$\mathcal{V}_{i,n}:\begin{cases}\mathcal{V}_{i,n}: L^2(G^n,\bigotimes_{p>n}\mathcal{H}_p)\to L^2(G^n,\bigotimes_{p>n}\mathcal{H}_p),\\ \\ (\mathcal{V}_{i,n}\xi)(g_ng_{n-1}\ldots g_2)=[\lambda_{(n)}(g_i))](\xi(g_ng_{n-1}\ldots g_2)).\end{cases}$$

Lemma 2.8. *Set, for every integer $n>0$, $\mathcal{V}^{(n)}=\mathcal{V}_{n,n}\mathcal{V}_{n-1,n}\ldots\mathcal{V}_{2,n}$. We have ${\mathcal{V}^{(n)}}^*\lambda(\gamma)\mathcal{V}^{(n)}=\lambda^{(n)}(\gamma)\otimes\mathbf{1}$, for every $\gamma\in G^n$.*

Proof. This follows from the above discussion, with $\mathcal{V} = \mathcal{V}^{(n)}, \Gamma = G^n, H = \bigotimes_{p>n} \mathcal{H}_p$, $\pi = \lambda_{(n)}$. $\square$

Definition 2.9. For every $T \in \mathcal{M}$, and $n \in \mathbf{N}$, we shall denote by $n\!-\!cut(T)$ the operator $\mathcal{V}^{n*} T \mathcal{V}^n$.

Lemma 2.10. *If condition $B2$ is fulfilled (see definition 2.2), the sequence $(\mathcal{V}^{(n)})_{n\in\mathbf{N}}$ converges strongly to the identity operator on $\mathcal{H}$, as n goes to infinity.*

Proof. Let $\xi \in \mathcal{H}$. We may assume that $\xi = \eta \otimes \bigotimes_{p>n} \Omega_p$, with η a unit vector in $\mathcal{H}^n$. It suffices to evaluate $\sum_{j=2}^{p} \|\mathcal{V}_{j,p}(\xi) - \xi\|$, for any integer $p > n$. Write $\sum_{j=2}^{p} \|\mathcal{V}_{j,p}(\xi) - \xi\| = \sum_{j=2}^{n} \|\mathcal{V}_{j,p}(\xi) - \xi\| + \sum_{j=n+1}^{p} \|\mathcal{V}_{j,p}(\xi) - \xi\|$. For $j \le n$, we have: $\|\mathcal{V}_{j,p}(\xi) - \xi\|^2 \le \int_{g_n \dots g_j \dots g_2 \in G^n} |\eta(g_n \dots g_j \dots g_2)|^2 \, \|[\lambda_{(p+1)}(g_j)]\Omega - \Omega\|^2 d(g_n \dots g_j \dots g_2)$.

Since $\|[p+1\!-\!rest(\lambda(g_j))]\Omega - \Omega\|$ tends to 0 when p goes to infinity, the first term of the sum tends to 0, when p goes to infinity. Assume now that $j > n$.

We have

$$\|\mathcal{V}_{j,p}(\xi) - \xi\|^2 \le \int_{g_j \in G_j} |\Omega_j(g_j)|^2 \, \|[\lambda_{(p+1)}(g_j)]\Omega - \Omega\|^2 dg_j \le C_\omega \sum_{1\le i<j} \sum_{k>p} \sqrt{\frac{b_{i,k}}{b_{i,j} b_{j,k}}}.$$

Thus, the second term of the sum is less or equal to $\sum_{j=n+1}^{p} (\sum_{1\le i<j,\, k>p} \sqrt{\frac{b_{i,k}}{b_{i,j} b_{j,k}}})$. It follows from condition $B2$ that this last quantity tends to 0, when p goes to infinity. $\square$

2.2.3. *Cut of the right regular representation, and the Heisenberg group H_3.* We shall denote by H_3 the real Heisenberg group of dimension 3. For each integer $k \ge 3$, the subgroup of G generated by $\{\mathbf{1} + e_{1,k-1},\ \mathbf{1} + e_{1,k},\ \mathbf{1} + e_{k-1,k}\}$ is a copy of H_3, which we shall denote by $H_3(k)$. There is a Hilbert space $\mathcal{H}'(k)$, and a decomposition $\mathcal{H} \simeq L^2(H_3(k), \mathcal{H}'(k))$. The Hilbert space $\mathcal{H}'(k)$ is endowed with a unitary representation $\gamma \to u_\gamma$, of $H_3(k)$, such that, for every $\xi \in L^2(H_3(k), \mathcal{H}'(k))$, and $h \in H_3(k)$, we have: $(\rho(\gamma)\xi)(h) = u_\gamma(\xi(h\gamma))$. Hence, there exists a canonical unitary $\mathcal{W}^{(k)}$, on $\mathcal{H}$, such that $\mathcal{W}^k \xi(\gamma) = u_\gamma^* \xi(\gamma)$, and : $(\mathcal{W}^{(k)^*} \rho(\gamma) \mathcal{W}^{(k)} \xi)(h) = \xi(h\gamma)$, for every $\xi \in L^2(H_3(k), \mathcal{H}'(k))$, and γ, $h \in H_3(k)$.

Lemma 2.11. *If condition $B2$ is fulfilled, the sequence $(\mathcal{W}^{(k)})_{k\in\mathbf{N}}$ converges strongly to the identity, when k goes to infinity.*

Proof. We remark that $\mathcal{W}^{(k)}$ is of the form $\mathbf{1} \otimes w_k \otimes \mathbf{1}$, in the decomposition $\mathcal{H} = (\mathcal{H}^{k-2}) \otimes (\mathcal{H}_{k-1} \otimes \mathcal{H}_k) \otimes \bigotimes_{p>k} \mathcal{H}_p$, with w_k a unitary on $\mathcal{H}_{k-1} \otimes \mathcal{H}_k$. The proof is similar to the proof of lemma 2.10. $\square$

Definition 2.12. For every $k \ge 3$, and every $T \in \rho(H_3(k))'' \subset \mathcal{N}$, we set $cut(T) = \mathcal{W}^{(k)^*} T \mathcal{W}^{(k)}$.

3. Factorial representations

In this section, we shall prove that the representations introduced in section 2 are factorial. We first study the case of the finite dimensional groups G^{2n}, generated by the unit matrix $\{\mathbf{1} + e_{i,j}, 1 \le i < j \le 2n\}$. Let G_1^{2n} be the subgroup generated by $\{\mathbf{1}+e_{i,j}, 1 \le i < j \le n\}$, G_2^{2n} be the subgroup generated by $\{\mathbf{1}+e_{i,j}, n+1 \le i < j \le 2n\}$, and A^{2n} be the subgroup generated by $\{\mathbf{1} + e_{i,j}, 1 \le i \le n \text{ and } n+1 \le j \le 2n\}$. We have a decomposition $G^{2n} \simeq (G_1^{2n} \times G_2^{2n}) \ltimes A^{2n}$. The group G_1^{2n} (resp. G_2^{2n}) acts by matrix multiplication from the left (resp from the right) on A^{2n}. We denote by $\mathcal{L}^{2n}$ the von Neumann algebra generated by the left regular representation of G^{2n}, and by $\mathcal{A}^{2n}$ the subalgebra of $\mathcal{L}^{2n}$ generated by A^{2n}. We have the following:

Lemma 3.1. *The action of $G_1^{2n} \times G_2^{2n}$ on $\mathcal{A}^{2n}$ is free, in the sense that the stabilizer of almost every point $\hat{a}$, in the spectrum $\hat{A}^{2n}$ of $\mathcal{A}^{2n}$, is trivial. (See* [7]*, section 2, for details.)*

Proof. Under Fourier transform, $\mathcal{A}^{2n}$ is isomorphic to $L^\infty(\widehat{\mathbf{R}}^{n^2})$. We shall denote by $\hat{\mathcal{A}}^{2n}$ this algebra, and by $\hat{A}^{2n}$ the space $\mathbf{R}^{n^2}$. The groups G_1^{2n} and G_2^{2n} are both isomorphic to the matrix group G^n. Under this isomorphism, G_1^{2n} and G_2^{2n} act on $\hat{A}^{2n}$ by left and right multiplication with transposed matrices. Explicitly, let $({}^tM)$ denote the transposed matrix of a matrix M. Let $\hat{a} \in \hat{A}^{2n}$. A matrix $M \in G^n \simeq G_1^{2n}$ sends $\hat{a}$ to $({}^tM)\hat{a}$. A matrix $N \in G^n \simeq G_2^{2n}$ sends $\hat{a}$ to $\hat{a}({}^tN)$. It follows that the study of the stabilizers of a point $\hat{a} \in \hat{A}^{2n}$ amounts to finding two matrices M, and N, in G^n, such that $({}^tM)\hat{a} = \hat{a}({}^tN^{-1})$. Replacing $\hat{a}$ by ${}^t\hat{a}$, the problem boils down to the following : Given an $n \times n$ matrix $\hat{a}$, find two matrices in G^n, that we shall still denote by M and N, such that $M\hat{a} = \hat{a}N$. We shall proceed by induction. Assume that the equality $M'\hat{b} = \hat{b}N'$, with M', $N' \in G^{n-1}$ implies $M' = N' = \mathbf{1}$, for almost every $(n-1) \times (n-1)$ matrix $\hat{b}$ (this assumption is trivial for $n-1 = 1$). Suppose we are given an $n \times n$ matrix $\hat{a}$. Let $\hat{b}$ be the submatrix of $\hat{a}$, obtained by removing the first line and the last column of $\hat{a}$. The set of equations $M\hat{a} = \hat{a}N$, with M, $N \in G^n$ contains the equation $M'\hat{b} = \hat{b}N'$, where M', and N' are submatrices of M and N, respectively. It follows from our induction hypothesis, that M is of the form $M = \mathbf{1} + \sum_{2 \le j \le n} M_{1,j} e_{1,j}$, and $N = \sum_{1 \le i \le n-1} \mathbf{1} + N_{i,n} e_{i,n}$. Moreover, the vector $(M_{1,j})_{2 \le j \le n}$ has to be orthogonal to every column of the submatrix $\hat{b}$. This shows that, for almost every $\hat{a}$, we have $M = \mathbf{1}$. A similar argument shows that $N = 1$. □

We shall also need the following lemma:

Lemma 3.2. *Let $\mathcal{H} = \otimes_{n=2}^\infty \mathcal{H}_n$ be an infinite tensor product of Hilbert spaces, along a vector $\Omega = \otimes_{n=2}^\infty \Omega_n$. Then $\cap_{n>1}(\mathbf{1}_{\mathcal{H}^n} \otimes \mathcal{L}(\bigotimes_{p>n} \mathcal{H}_p)) = \mathbf{C}.\mathbf{1}$.*

Proof. Denote by $\mathcal{K}(\mathcal{H})$ (resp. $\mathcal{K}(\mathcal{H}^n)$) the C^*-algebra of compact operators on $\mathcal{H}$ (resp. on $\mathcal{H}^n$). For any integer $n > 1$, let $\Pi^n \in \mathcal{L}(\otimes_{p=n+1}^\infty \mathcal{H}_p)$ be the (rank one) projection on $\mathbf{C}.(\otimes_{p=n+1}^\infty \Omega_p)$. Then $\mathcal{K}(\mathcal{H})$ is the norm closure of $\cup_{n \ge 2}(\mathcal{K}(\mathcal{H}^n) \otimes \Pi^n)$. It follows that $\mathcal{L}(\mathcal{H})$ is the strong closure of $\cup_{n \ge 2}(\mathcal{L}(\mathcal{H}^n) \otimes \Pi^n)$. Now, one checks easily that this is equal to the strong closure of $\cup_{n \ge 2}(\mathcal{L}(\mathcal{H}^n) \otimes \mathbf{1})$. The proof is completed by computing the commutant of $\cap_{n>1}(\mathbf{1}_{\mathcal{H}^n} \otimes \mathcal{L}(\bigotimes_{p>n} \mathcal{H}_p))$. □

Corollary 3.3. *Under the assumptions of lemma 2.4, $\mathcal{M}$ is a factor.*

Proof. Let B be a central element of $\mathcal{M}$. Let $n > 0$ be an integer. Let $\mathcal{V}^{(2n)}$ be the unitary defined in 2.8. Then ${\mathcal{V}^{(2n)}}^* B \mathcal{V}^{(2n)}$ is in the commutant of ${\mathcal{V}^{(2n)}}^* \lambda(G^{2n}) \mathcal{V}^{(2n)} = \lambda^{(2n)}(G^{2n}) \otimes \mathbf{1}$. (In the last equality, we use the decomposition $\mathcal{H} \simeq L^2(G^{2n}) \otimes (\bigotimes_{p>2n} \mathcal{H}_p)$.) Moreover, ${\mathcal{V}^{(2n)}}^* B \mathcal{V}^{(2n)}$ commutes with ${\mathcal{V}^{(2n)}}^* \rho(G^{2n})'' \mathcal{V}^{(2n)} = \rho(G^{2n})''$. Denote by $\mathcal{Z}_n$ the center of $\rho(G^{2n})''$. It follows from lemma 3.1, and from lemma 2.4 of [7], that $\mathcal{Z}_n \subset \rho(A^{2n})$". Hence, ${\mathcal{V}^{(2n)}}^* B \mathcal{V}^{(2n)} \in \mathbf{1}_{\mathcal{H}^n} \otimes \mathcal{L}(\bigotimes_{p>n} \mathcal{H}_p)$. Since the unitaries $(\mathcal{V}^{2n})_{n \ge 1}$ converge to the identity, B is in $\cap_{n>0}(\mathbf{1}_{\mathcal{H}^n} \otimes \mathcal{L}(\bigotimes_{p>n} \mathcal{H}_p)) = \mathbf{C}.\mathbf{1}$. □

4. Computing $\Gamma(\mathcal{M})$

In this section, we prove that $\mathcal{M}$ is of type III_1. We shall use the following characterization of the Connes spectrum:

Theorem 4.1. ([6], Theorem 8.15.9, p. 389). *Let $\mathcal{M}$ be a σ-finite von Neumann algebra on a Hilbert space $\mathcal{H}$. Then $\Gamma(\mathcal{M})$ consists of those real numbers s such that, for each* *unit vector $\xi \in \mathcal{H}$, and every $\alpha > 0$, there are elements $x \in \mathcal{M}$, and $y \in \mathcal{M}'$ with $\|x\xi\| > 1$, and*

$$\|x\xi - y\xi\| < \alpha, \quad \|x^*\xi - e^s y^*\xi\| < \alpha.$$

4.1. **The algebra A_μ.** Let H_3 be the real Heisenberg group of dimension 3. Since H_3 can be decomposed as a semi-direct product $\mathbf{R} \ltimes \mathbf{R}^2$ of amenable groups, H_3 is amenable, hence, the full and reduced group C^*-algebras of H_3 coincide. For each integer $k \geq 3$, the subgroup of G generated by $\{\mathbf{1} + e_{1,k-1},\, \mathbf{1} + e_{1,k},\, \mathbf{1} + e_{k-1,k}\}$ is a copy of H_3, which we shall denote by $H_3(k)$. For the sake of simplicity, we shall denote in the same way a subset of H_3, and its identical image in $H_3(k)$. In particular, $H_3(3)$ is identical to G^3. Under this identification, the space $L^2(H_3)$ is isometric to $L^2(\mathbf{R}_{1,2}) \otimes L^2(\mathbf{R}_{1,3}) \otimes L^2(\mathbf{R}_{2,3})$. Desintegrating $C^*(H_3)$ amounts to compose the regular representation λ of H_3, with the Fourier transformation $\mathcal{F}$, on $\mathbf{R}_{1,2}$ and $\mathbf{R}_{1,3}$. In the definition below, the variable μ parametrizes the center of $C^*(H_3)$, and is dual to $t_{1,3}$. The variable ν is dual to $t_{1,2}$, and the variable $t_{2,3}$ is simply denoted by t. The C^*-algebra $\mathcal{F}\lambda(C^*(H_3))\mathcal{F}^*$ is a continuous field of C^*-algebras. We write $C^*(H_3) = (A_\mu)_{\mu \in \mathbf{R}}$ the desintegration of $C^*(H_3)$ over its center. We shall determine, for each $\epsilon > 0$, $s \in \mathbf{R}$, and $\eta = (\xi_\mu)_\mu$, a real number $\mu_0 > 0$, depending on ϵ, s, and η such that there are operators fulfilling the conditions of theorem 4.1, provided that the support of η is contained in $[-\mu_0, \mu_0]$. Then, we shall see that these operators "almost extend" to operators on $\mathcal{H}$, belonging to $\mathcal{M}$, and $\mathcal{N}$ respectively. This will allow to compute $\Gamma(\mathcal{M})$. We start by describing the algebras A_μ.

Definition 4.2. Let μ be a real number. The C^*- algebra $A_\mu$is the C^*- crossed product $C_0(\widehat{\mathbf{R}}) \rtimes \mathbf{R}$, where $\widehat{\mathbf{R}}$ acts on $C_0(\mathbf{R})$ through the automorphisms $(\alpha_s^\mu(f))(t) = f(t - \mu s)$, for any reals t, s.

Definition 4.3. Let μ be a real number. Define, for every $f \in C_0(\mathbf{R})$, $s, t \in \mathbf{R}$, $\nu \in \widehat{\mathbf{R}}$, and $\xi \in L^2(\widehat{\mathbf{R}}) \otimes L^2(\mathbf{R})$:

$$\pi_L^\mu : \begin{cases} \pi_L^\mu(f)\xi(\nu, t) = f(\nu + \mu t)\xi(\nu, t), \\ \\ \pi_L^\mu(s)\xi(\nu, t) = \xi(\nu, t - s), \end{cases}$$

and

$$\pi_R^\mu : \begin{cases} \pi_R^\mu(f)\xi(\nu, t) = f(\nu)\xi(\nu, t), \\ \\ \pi_R^\mu(s)\xi(\nu, t) = \xi(\nu + \mu s, t - s). \end{cases}$$

The maps π_L^μ (resp. π_R^μ) define commuting representation (resp. anti-representation) of A_μ on $L^2(\widehat{\mathbf{R}}) \otimes L^2(\mathbf{R})$. We shall denote by $\mathcal{K}_\mu$ the Hilbert space $L^2(\widehat{\mathbf{R}}) \otimes L^2(\mathbf{R})$, endowed with the left and right representations π_L^μ, π_R^μ, of A_μ. The C^*-algebra $C^*(H_3)$ is (non spatially) isomorphic to sections of the continuous field $(\pi_L^\mu(A_\mu))_{\mu \in \mathbf{R}}$. The maps π_L^μ, π_R^μ extend to normal representations of the von Neumann crossed product $L^\infty(\widehat{\mathbf{R}}) \rtimes \mathbf{R}$. We denote by $\mathcal{L}^\mu$ (resp. $\mathcal{R}^\mu$) the von Neumann algebra of $\pi_L^\mu(L^\infty(\widehat{\mathbf{R}}) \rtimes \mathbf{R})$ (resp. of $\pi_R^\mu(L^\infty(\widehat{\mathbf{R}}) \rtimes \mathbf{R})$).

4.2. **The operators X and Y.** We shall now prove the following proposition:

Proposition 4.4. *Let $\mu \in \mathbf{R}$, $\xi = \xi_1 \otimes \xi_2 \in \mathcal{K}_\mu$ be an elementary tensor, and $\epsilon > 0$, $\lambda > 0$ be given. There exists $C > 0$, and $\mu_0 > 0$, such that, if $0 < |\mu| < \mu_0$, there are elements $X_\mu \in \mathcal{L}^\mu$, and $Y_\mu \in \mathcal{R}^\mu$, with*

$$(C1): \quad \|\sqrt{\lambda}X_\mu\xi - Y_\mu^*\xi\| < \epsilon\|X_\mu\xi\|,$$

and

$$(C2): \quad \|X_\mu^*\xi - \sqrt{\lambda}Y_\mu\xi\| < \epsilon\|X_\mu\xi\|.$$

Moreover, $\|X_\mu\|$, and $\|Y_\mu\|$ are less or equal to C, and $\|X_\mu \xi\| \geq \|\xi\|/2$.

We may assume that $\|\xi\| = 1$. The above operators X_μ and Y_μ fulfill the conditions of theorem 4.1, with $\lambda = e^s$, $x = 2X_\mu/\|X_\mu\xi\|$, $y = 2Y_\mu^*/\sqrt{\lambda}\|X_\mu\xi\|$, and $\alpha = 2\epsilon(1 + 1/\sqrt{\lambda})$.

4.2.1. *Definition of X and Y for ξ, an elementary tensor, nowhere vanishing function.* We first assume that ξ is an elementary tensor, $\xi(\nu, t) = \xi_1(\nu)\xi_2(t)$, with ξ_1 a (nowhere vanishing) continuous, real-valued function, tending to 0 at infinity. Choose two reals, ν_1, ans ν_2, such that $\xi_1(\nu_1) = \sqrt{\lambda}\xi_1(\nu_2) \neq 0$. Choose a sufficiently small real ρ_0, such that

$$(A1): \quad \forall \nu \in \mathbf{R},\ |\nu - \nu_1| < \rho_0 \Rightarrow |\xi_1(\nu) - \xi_1(\nu_1)| < \epsilon|\xi_1(\nu)|/4(1+\epsilon)$$

and

$$(A2): \quad \forall \nu \in \mathbf{R},\ |\nu - \nu_1| < \rho_0 \Rightarrow |\xi_1(\nu + \nu_2 - \nu_1) - \xi_1(\nu_2)| < \epsilon|\xi_1(\nu)|/4(1+\epsilon).$$

Definition 4.5. Let y be the characteristic function of the interval $[\nu_1 - \rho_0, \nu_1 + \rho_0]$. Set $\kappa = \mu^{-1}(\nu_1 - \nu_2)$. We define

$$X_\mu = \pi_L^\mu(-\kappa)\pi_L^\mu(y), \quad Y_\mu = \pi_R^\mu(y)\pi_R^\mu(\kappa).$$

We have

Proposition 4.6. *Given an elementary tensor $\xi \in \mathcal{K}_\mu$, and a real $\epsilon > 0$ as above, the operators X_μ and Y_μ, defined in 4.5, fulfill conditions $C1$ and $C2$, provided that $|\mu|$ is sufficiently small.*

Proof. (1) On computes

$$\begin{cases} X_\mu\xi(\nu, t) = y(\nu + \mu(t + \kappa))\xi(\nu, t + \kappa), \\ Y_\mu^*\xi(\nu, t) = y(\nu)\xi(\nu - \mu\kappa, t + \kappa) \end{cases}$$

and

$$\begin{cases} X_\mu^*\xi(\nu, t) = y(\nu + \mu t)\xi(\nu, t - \kappa), \\ Y_\mu\xi(\nu, t) = y(\nu + \mu\kappa)\xi(\nu + \mu\kappa, t - \kappa). \end{cases}$$

(2) Consider the approximate vector $(X_\mu\xi)_{\text{app}}(\nu, t) = y(\nu)\xi(\nu, t + \kappa)$. Then $\|(X_\mu\xi)_{\text{app}}\|^2 = \|\xi_2\|^2 \int |y(\nu)\xi_1(\nu)|^2 d\nu$ is strictly positive, and does not depend on μ. Set $(X_\mu^*\xi)_{\text{app}}(\nu, t) = y(\nu + \mu\kappa)\xi(\nu, t - \kappa)$. Observe that $\|(X_\mu^*\xi)_{\text{app}} - \sqrt{\lambda}Y_\mu\xi\| = \|Y_\mu^*\xi - \sqrt{\lambda}(X_\mu\xi)_{\text{app}}\|$, since the first quantity is obtained from the second through a translation of $\mu\kappa$ in the first variable, and of 2κ in the second variable. It will be sufficient to check the following condition C_{app}, and evaluate $\|X_\mu\xi - (X_\mu\xi)_{\text{app}}\|$, $\|X_\mu^*\xi - (X_\mu^*\xi)_{\text{app}}\|$:

$$(C_{\text{app}}) \qquad \|\sqrt{\lambda}(X_\mu\xi)_{\text{app}} - Y_\mu^*\xi\| < \epsilon\|(X_\mu\xi)_{\text{app}}\|/2(1+\epsilon).$$

(3) It follows from assumptions $A1$, and $A2$, that, for every reals ν, and t, with $y(\nu)\xi_2(t) \neq 0$, the following hold:

$$(A'1): \quad |y(\nu)(\xi_1(\nu) - \xi_1(\nu_1))\xi_2(t)| < \epsilon|y(\nu)\xi_1(\nu)\xi_2(t)|/4(1+\epsilon),$$

and

$$(A'2): \quad |y(\nu)(\xi_1(\nu + \nu_2 - \nu_1) - \xi_1(\nu_2))\xi_2(t)| < \epsilon|y(\nu)\xi_1(\nu)\xi_2(t)|/4(1+\epsilon).$$

Assumptions $A'1$ and $A'2$ show that C_{app} is fulfilled (recall that $\xi_1(\nu_1) = \sqrt{\lambda}\xi_2(\nu_2)$.)

(4) We have: $\|X_\mu\xi - (X_\mu\xi)_{\rm app}\| = \|(y(\nu+\mu t) - y(\nu))\xi(\nu,t)\|$. Since $\|(X_\mu\xi)_{\rm app}\|$ is strictly positive, and independent of μ, there is a real $\mu_1 > 0$, such that $\|X_\mu\xi - (X_\mu\xi)_{\rm app}\| < \epsilon\|X_\mu\xi_{\rm app}\|/2(1+\epsilon)$, for every $|\mu| < \mu_1$. Using the change of variables $t \to t+\kappa$, and $\nu \to \nu - \mu\kappa$, we obtain: $\|X^*_\mu\xi - (X^*_\mu\xi)_{\rm app}\| = \|(y(\nu+\mu t) - y(\nu))\xi(\nu-\nu_2-\nu_1,t)\|$. Hence, there is a real $\mu_2 > 0$, such that $\|X^*_\mu\xi - (X^*_\mu\xi)_{\rm app}\| < \epsilon\|(X_\mu\xi)_{\rm app}\|/2(1+\epsilon)$, for every $|\mu| < \mu_2$. We let μ_0 be the minimum of $\{\mu_1,\, \mu_2\}$.

(5) Let μ be a real, with $|\mu| < \mu_0$. Since $\|X_\mu\xi - (X_\mu\xi)_{\rm app}\| < \epsilon\|X_\mu\xi_{\rm app}\|/2(1+\epsilon)$, it follows from the triangle inequality that $\|X_\mu\xi_{\rm app}\| < (1+\epsilon)\|X_\mu\xi\|$. The results in 2 and 4 show that $C1$, and $C2$ are fulfilled.

This finishes the proof. □

Remarks 4.7. i) Condition $C_{\rm app}$ is fulfilled for every μ. The choice of μ_0 insures that the approximate vector $(X_\mu\xi)_{\rm app}$ is sufficiently close to the actual vector $X_\mu\xi$.
ii) The hypothesis that ξ_1 is real-valued is not essential. Dropping this assumption, we may still find two points ν_1, and ν_2, and a real $\theta \in [0, 2\pi[$, with $\xi_1(\nu_1) = e^{i\theta}\sqrt{\lambda}\xi_2(\nu_2)$. It suffices to replace X_μ with $e^{-i\theta}X_\mu$.

4.2.2. *Removing the "nowhere vanishing" assumption.* We want to extend the result of 4.2.1 to arbitrary elementary tensors ξ in the unit ball of $\mathcal{H}$. We shall construct, for every ξ in some dense subset S of the unit ball of $\mathcal{H}$, the operators X_μ, Y_μ, in such a way that $\|X_\mu\xi\|$ remains sufficiently close to 1. It follows that there is some $\alpha > 0$, such that conditions $C1$ and $C2$ are fulfilled (with the same operators X_μ, Y_μ, and a slightly larger ϵ), for every vector in the open ball of radius α around each of the vectors $\xi \in S$. We shall cut the support of ξ into suitable parts, and construct from 4.2.1 partial operators X_μ and Y_μ corresponding to each of these parts. Our operators will be a combination of these partial operators, using a partition of unity.

We now assume that a unit vector $\xi = \xi_1 \otimes \xi_2 \in \mathcal{K}_\mu$ and a real $\epsilon > 0$ are given. The vector ξ_1 is supposed to be a smooth function of rapid decrease, whose zeroes are isolated. We choose a function ξ_1^c, such that

- $\|\xi_1 - \xi_1^c\| < \epsilon$.
- The support of ξ_1^c is a finite union of compact intervals.
- For any element ν of the support of ξ_1^c, $\xi_1^c(\nu) \neq 0$.
- For any element ν of the support of ξ_1^c, $\xi_1^c(\nu) = \xi_1(\nu)$.

Such a function ξ_1^c is obtained by cutting the support of ξ_1 near infinity, and removing (small) open intervals around each of the zeroes of ξ_1. We shall call parts of ξ_1^c, the restriction of ξ_1^c to each of the components of its support. We denote these parts by $\xi_1^{c;j}$, with j running from 1 to some integer n. We denote by I_j the smallest open interval containing the support of $\xi_1^{c;j}$, such that ξ_1 vanishes at the boundary of I_j. Let $j \in \{1, 2, \ldots, n\}$ be given. For each point ν_1^j, of the support of $\xi_1^{c;j}$, there exists a point $\nu_2^j(\nu_1^j) \in I_j$, and a strictly positive real $\rho(\nu_1^j)$, such that conditions $A1$, and $A2$ in 4.2.1 are fulfilled, with ν ranging in the support of $\xi_1^{c;j}$. We now choose a chain of open intervals $\mathcal{C} = \{]\nu^j_{1;l} - \rho(\nu^j_{1;l}),\, \nu^j_{1;l} + \rho(\nu^j_{1;l})[,\ l = 1, 2, \ldots, l(j)\}$, covering the support of $\xi_1^{c;j}$, and call $\mathcal{U}_j$ the union of these open intervals. To summarize, we have: *Support* $(\xi^c_{1;j}) \subset \mathcal{U}_j \subset I_j$. We now choose a partition of unity $\{y_1^{j;1}, y_1^{j;2}, \ldots, y_1^{j;l(j)}\}$, subject to the cover $\mathcal{C}$, such that each of the functions $y_1^{j;l}$ is the characteristic function of some interval. Hence, the $y_1^{j;l}$'s are pairwise orthogonal. Given a real number $\mu > 0$, we repeat the same construction for each $j \in \{1, 2, \ldots, n\}$, and define $X_\mu^{j;l}$, $Y_\mu^{j;l}$, $(X_\mu^{j;l}\xi)_{\rm app}$, $(X_\mu^{*j;l}\xi)_{\rm app}$ as in 4.5, with $(j,l) \in \{(j,1), (j,2), \ldots, (j, l(j))\}$, and $j \in \{1, 2, \ldots, n\}$. For each pair (j,l), the operators $X_\mu^{j;l}$, $Y_\mu^{j;l}$ fulfill condition $C_{\rm app}$. Set

$$X_\mu = \sum_{j=1}^{n} \sum_{l=1}^{l(j)} X_\mu^{j;l}, \quad Y_\mu = \sum_{j=1}^{n} \sum_{l=1}^{l(j)} Y_\mu^{j;l}.$$

Observe $(X_\mu \xi)_{\mathrm{app}}$, and $Y_\mu^* \xi$ are sums of pairwise orthogonal vectors, indexed by the couples (j,l). Since C_{app} is fulfilled for each (j,l), it follows that X_μ and Y_μ fulfill C_{app}. Now, since there are finitely many indices (j,l), we may find a $\mu_0 > 0$, such that X_μ and Y_μ fulfill $C1$, and $C2$, for every $|\mu| < |\mu_0|$. We summarize these results in the following lemma:

Lemma 4.8. *Let μ be a non zero real number. Let $\eta = \eta_1 \otimes \eta_2$ be a unit vector in $\mathcal{K}_\mu$, with η_1 a smooth function of rapid decrease, and isolated zeroes. Let $\epsilon > 0$, and $\lambda > 0$ be given. There exists a constant $C > 0$, depending only on η, and ϵ, such that:*

1. *There exist operators X_μ, and Y_μ, whose norm is less or equal to C, such that: $\|(X_\mu \eta)_{\mathrm{app}}\| > \|\eta\| - \epsilon$, and C_{app} is fulfilled.*
2. *There exists $\mu_0 > 0$, such that, if $|\mu| < |\mu_0|$, there are operators X_μ, and Y_μ, whose norm is less or equal to C, such that: $\|(X_\mu \eta)\| > \|\eta\| - \epsilon$, and $C1$ and $C2$ are fulfilled.*

4.2.3. *Proof of proposition 4.4.* Proposition 4.4 follows from the remark:

Lemma 4.9. *Let $\mathcal{K}$ be a Hilbert space. Let $\epsilon > 0$, $C > 0$, and $\lambda > 0$ be given. Suppose we are given a unit vector $\xi_0 \in \mathcal{K}$, and operators $A \in \mathcal{L}(\mathcal{K})$, $B \in \mathcal{L}(\mathcal{K})$, such that*

1. *$\|A\| \leq C$, and $\|B\| \leq C$.*
2. *$\|A\xi_0\| > 1/2$.*
3. *$\|\sqrt{\lambda} A\xi_0 - B\xi_0\| < \epsilon \|A\xi_0\|$, and $\|A^*\xi_0 - \sqrt{\lambda} B^* \xi_0\| < \epsilon \|A\xi_0\|$.*

There exists $\alpha > 0$, which depends only on ϵ and λ, such that, for every $\xi \in \mathcal{K}$, with $\|\xi - \xi_0\| < \alpha / C$, we have $\|\sqrt{\lambda} A\xi - B\xi\| < 2\epsilon \|A\xi\|$, and $\|A^*\xi - \sqrt{\lambda} B^*\xi\| < 2\epsilon \|A\xi\|$.

Proof. The triangle inequality shows that: $\|\sqrt{\lambda} A\xi - B\xi\| < \epsilon \|A\xi_0\| + \alpha(1 + \sqrt{\lambda}) < \epsilon \|A\xi\| + \alpha(\epsilon + 1 + \sqrt{\lambda})$. We may assume that $\alpha < 1/4$, hence, $\|A\xi\| > 1/4$, and $\|\sqrt{\lambda} A\xi - B\xi\| < \|A\xi\|(\epsilon + 4\alpha(\epsilon + 1 + \sqrt{\lambda}))$. The proof of the second inequality is the same. □

Lemma 4.9 shows that, given ϵ, and λ, to prove the existence of the operators X_μ, and Y_μ for every vector in some subset S of $\mathcal{H}$, it suffices to construct X_μ and Y_μ for every vector in some dense subset of S. In particular, lemma 4.8 extends to any elementary tensor, $\xi = \xi_1 \otimes \xi_2$, with $\xi_1 \in L^2(\widehat{\mathbf{R}})$, $\xi_2 \in L^2(\mathbf{R})$.

4.3. **Computing the Connes spectrum of $\mathcal{M}$.** In this section, we adapt the construction of the preceding section to the algebras $\mathcal{M}$ and its commutant $\mathcal{N}$. The vector ξ_μ of proposition 4.4 will be replaced by $\widehat{\Omega}_{1,k-1} \otimes \Omega_{k-1,k}$, with k ranging in the set of strictly positive integers. For each k, the operators X_μ, and Y_μ are obtained from a generic construction, using a simple change of variables (see definition 4.14). Since any vector in $\mathcal{H}$ can be approximated by some vector of the form $\eta \bigotimes \otimes_{j \geq k} \Omega_j$, we can use an argument similar to lemma 4.9, and extend our results to the whole of $\mathcal{H}$. The difference is that we shall need some analog of proposition 4.4 to hold globally, for sections of the field $(\mathcal{K}_\mu)_\mu$. The set of admissible values for μ will be altered during the change of variables. So, we shall also need to check the change of the support of $(X_\mu)_\mu$, with respect to k . This will, of course, depend on the coefficients $b_{j,k}$. Finally, for each sufficiently large k, using results from section 2.2.2, we shall find in $\mathcal{M}$ and $\mathcal{N}$, two operators, close enough (in some sense that will be made precise) to $(X_\mu)_\mu$, and $(Y_\mu)_\mu$, to fulfill conditions $C1$ and $C2$. We begin with some definitions.

4.3.1. *Change of variables.*

Definition 4.10. Given a function y on the reals, and $\mu,\ \nu_1, \nu_2, \in \mathbf{R}$, denote by $X_\mu[y;\nu_1, \nu_2]$ the operator constructed as in 4.5, and 4.2.2. Similar notation is used for X_μ^*, Y_μ, Y_μ^*. Let $\alpha > 0$, $\lambda > 0$, and $\xi \in L^2(\widehat{\mathbf{R}}) \otimes L^2(\mathbf{R})$ be given. We shall say that $(y;\nu_1,\nu_2)$ is $\alpha-$convenient for (ξ,μ_0,λ), if the operators $X_\mu[y;\nu_1,\nu_2]$ and $Y_\mu[y;\nu_1,\nu_2]$ fulfill conditions $C1$ and $C2$ in proposition 4.4, for every $0 < |\mu| < \mu_0$. We denote by $\mu_0(\xi,\alpha,\lambda)$ the greatest real such that there exists an $\alpha-$convenient triple for $(\xi,\mu_0(\xi,\alpha,\lambda),\lambda)$.

Definition 4.11. Let $k > 0$, be given. We define the unitary

$$\begin{cases} U(k) : L^2(\widehat{\mathbf{R}}) \otimes L^2(\mathbf{R}) \to L^2(\widehat{\mathbf{R}}) \otimes L^2(\mathbf{R}) \\ \\ \big(U(k)\xi\big)(\nu,t) = b_{1,k-1}^{-1/4} b_{k-1,k}^{1/4} \xi(\nu b_{1,k-1}^{-1/2},\ t b_{k-1,k}^{1/2}\big). \end{cases}$$

Lemma 4.12. *For every integer $k > 0$, and every real $\alpha > 0$, we have: $\mu_0(U(k)\xi,\alpha,\lambda) \geq (b_{1,k-1}b_{k-1,k})^{1/2}\mu_0(\xi,\alpha,\lambda)$.*

Proof. We have: $U(k)X_\mu[y(\nu);\nu_1,\nu_2]U(k)^* = X_{\mu'}[y(b_{1,k-1}^{-1/2}\nu);\nu_1\sqrt{b_{1,k-1}},\nu_2\sqrt{b_{1,k-1}}]$, with $\mu' = \mu(b_{1,k-1}b_{k-1,k})^{1/2}$, and similar result for Y_μ. It follows that there is an $\alpha-$ convenient triple for $(U(k)\xi,\ \mu',\lambda)$, provided that $|\mu'| < \mu_0(\xi,\ \alpha,\lambda)(b_{1,k-1}b_{k-1,k})^{1/2}$. □

Definition 4.13. Set $b(k) = (b_{1,k-1}b_{k-1,k})^{1/2}$. In view of lemma 4.12, it is natural to define, for each μ, the unitary $U_\mu(k)$, from $\mathcal{K}_\mu$ to $\mathcal{K}_{\mu b(k)}$, with the same formula as in 4.11.

4.3.2. *Operators on $\widehat{\mathcal{H}_{1,k-1}}\otimes\mathcal{H}_{k-1,k}\otimes\widehat{\mathcal{H}_{1,k}}$.* We now consider the space $\widehat{\mathcal{H}_{1,k-1}}\otimes\mathcal{H}_{k-1,k}\otimes\widehat{\mathcal{H}_{1,k}}$ as the field of Hilbert spaces $((\widehat{\mathcal{H}_{1,k-1}} \otimes \mathcal{H}_{k-1,k})_\mu)_{\mu\in\mathbf{R}}$, with parameter μ. We denote by Γ_ω the constant cross-section $\mu \to \widehat{\omega}\otimes\omega$. The fiber over μ is isomorphic, as an A_μ-bimodule, to $\mathcal{K}_\mu$. The vector $\Omega_{1,k-1}\otimes\Omega_{k-1,k}\otimes\Omega_{1,k}$ is the cross-section $\xi_{\mu b(k)} = \Omega_{1,k}(\mu(b(k))U_\mu(k)(\Gamma_\omega)$.

Definition 4.14.
1. Let $\epsilon > 0$ be given. Choose an $\epsilon-$ convenient triple $(y;\nu_1,\nu_2)$ for $(\widehat{\omega}\otimes\omega,\mu_0(\widehat{\omega}\otimes\omega,\epsilon,\lambda),\lambda)$. Set $X_\mu = X[y;\nu_1,\nu_2]$, $Y_\mu = Y[y;\nu_1,\nu_2]$, if $0 < |\mu| < \mu_0(\widehat{\omega}\otimes\omega,\epsilon,\lambda)$, and $X_\mu = Y_\mu = 0$ otherwise. Define, on the space $\mathcal{K}_{\mu b(k)}$, the operators $X_\mu(k) = U(k)X_{(\mu/b(k))}U^*(k)$, and $Y_\mu(k) = U(k)Y_{(\mu/b(k))}U^*(k)$.
2. With $X_\mu(k)$ as in 1, we shall denote by $X(k)$ the operator on $\widehat{\mathcal{H}_{1,k-1}}\otimes\mathcal{H}_{k-1,1}\otimes\widehat{\mathcal{H}_{1,k}}$, defined by $[X(k)(\xi_1\otimes\xi_2\otimes\eta)](\mu) = \eta(\mu)X_\mu(k)(\xi_1\otimes\xi_2)$. We define the operator $Y(k)$ similarly.

Proposition 4.15. *Let $\epsilon > 0$, and $k > 1$ be given. For every real μ, with $\mu_0\sqrt{b_{1,k}} < |\mu| < \mu_0(\widehat{\omega}\otimes\omega,\epsilon,\lambda)b(k)$, the operators $X_\mu(k)$, and $Y_\mu(k)$, defined in 4.14, fulfill inequalities $C1$ and $C2$ in 4.4, with $\xi_\mu = (\widehat{\Omega}_{1,k-1}\otimes\Omega_{k-1,k})$. Moreover, we may choose y such that, for every sufficiently large k, $\|X(k)(\Omega_{1,k-1}\otimes\Omega_{k-1,k}\otimes\Omega_{1,k})\| \geq 1-2\epsilon$.*

Proof. The first assertion follows from lemma 4.12. Recall that we may choose the function y such that $\|X_\mu(k)(\Omega_{1,k-1}\otimes\Omega_{k-1,k})\| > 1-\epsilon$, for every μ in the support of X_μ (see lemma 4.8). The second assertion then follows from the limit:

$$\int_{|\mu|<\mu_0 b(k)} |\widehat{\Omega}_{1,k}(\mu)|^2 d\mu \ \to\ 1,$$

when k goes to infinity. □

It will be convenient to restrict the support of our fields of operators $X(k)$, $Y(k)$ to the set $\{\mu_0\sqrt{b_{1,k}} < |\mu| < \mu_0 b(k)\}$. The conclusions of proposition 4.15 remain valid provided that μ_0 is chosen to be sufficiently small. To summarize, the parameters for $X_\mu(k)$, and $Y_\mu(k)$, are: $y(\nu/\sqrt{b_{1,k-1}})$, $\kappa = \mu^{-1}(\nu_1 - \nu_2)\sqrt{b_{1,k-1}}$, $\mu_0\sqrt{b_{1,k}} < |\mu| < \mu_0\sqrt{b_{1,k-1}}\sqrt{b_{k-1,k}}$.

4.4. **Approximating operators in $\mathcal{M}$ and $\mathcal{N}$.** We now need to replace our operators $X(k)$, and $Y(k)$ of the preceding section by operators $\widetilde{X}(k)$ in $\mathcal{M}$, and $\widetilde{Y}(k)$ in $\mathcal{N}$, and check that the hypothesis of theorem 4.1 are fulfilled, for every unit vector $\xi \in \mathcal{H}$.

Definition 4.16. (1) Let $k > 2$ be given. Let $H_3(k)$ be the subgroup of G generated by $\{\mathbf{1} + e_{1,k-1},\ \mathbf{1} + e_{1,k},\ \mathbf{1} + e_{k-1,k}\}$. The left and right regular representations of G restrict to unitary representations of $H_3(k)$ on $\mathcal{H}$. We shall denote by $\widetilde{\pi}_L^{\mu,k}$, and $\widetilde{\pi}_R^{\mu,k}$ the corresponding representations of A_μ.

(2) Given $\epsilon > 0$, choose k, $X(k)$, $Y(k)$, as in proposition 4.15, with defining function y. We define $\widetilde{X}(k)$ (resp. $\widetilde{Y}(k)$) as the field with the same support as $X(k)$, with $\widetilde{X}_\mu(k) = \widetilde{\pi}_L^{\mu,k}(y)$ (resp. $\widetilde{Y}_\mu(k) = \widetilde{\pi}_R^{\mu,k}(y)$).

(3) We shall denote in the same way the operator $X(k) \in \mathcal{L}(\widehat{\mathcal{H}_{1,k-1}} \otimes \mathcal{H}_{k-1,1} \otimes \widehat{\mathcal{H}_{1,k}})$, and its amplification, $X(k) \otimes \mathrm{Id} \in \mathcal{L}(\mathcal{H})$. The same holds for $Y(k)$.

Proposition 4.17. *The operators: $X(k) - \widetilde{X}(k)$, $Y(k) - \widetilde{Y}(k)$, $X^*(k) - \widetilde{X}^*(k)$, $Y^*(k) - \widetilde{Y}^*(k)$ converge weakly to 0, in $\mathcal{L}(\mathcal{H})$, when k goes to infinity.*

Proof. This follows from lemmas 2.10 and 2.11, and the observation that π_L^μ is the $k-cut$ of $\widetilde{\pi}_L^{\mu,k}$, and π_R^μ is the cut of $\widetilde{\pi}_R^{\mu,k}$ (under the identification $L^2(\widehat{\mathbf{R}}) \otimes L^2(\mathbf{R}) \simeq \widehat{\mathcal{H}_{1,k-1}} \otimes \mathcal{H}_{k-1,k}$). □

Corollary 4.18. *There is an integer k_0, such that the assumptions of lemma 4.9 are fulfilled, with $\xi_0 = \Omega$, $A = \widetilde{X}(k)$, $B = \widetilde{Y}^*(k)$, for every $k \geq k_0$.*

Proof. Using the bounds on μ, and an argument similar to the proof of lemma 2.10, one can check that $\|(X(k) - {\mathcal{V}^{k+1}}^* X(k))\Omega\|$ converge to 0, when k goes to infinity. Since the $\mathcal{V}^k$'s converge strongly to 1, when k goes to infinity, the sequence $((X(k) - \widetilde{X}(k))\Omega)_{k\in\mathbf{N}}$ is norm convergent to 0. The same holds for $Y(k)$, and $\widetilde{Y}(k)$. Using proposition 4.15, we find an integer k_0', such that the assumptions of lemma 4.9 are fulfilled, with $\xi_0 = \Omega$, $A = X(k)$, $B = Y^*(k)$, for every $k \geq k_0'$. The existence of k_0 follows from proposition 4.17. □

Now, suppose we are given $\xi \in \mathcal{H}$, with $\|\xi\| = 1$, $\epsilon > 0$, $\alpha > 0$, and $\lambda > 0$. Choose an integer k_α, such that there exists a unit vector $\eta \in \mathcal{H}^{k_\alpha - 1}$, with $\|\xi - \eta \otimes \bigotimes_{k=k_\alpha}^\infty \Omega_k\| < \alpha$. We shall denote by ξ_α the vector $\eta \otimes \bigotimes_{k=k_\alpha+1}^\infty \Omega_k$. Construct the operators $X(k_\alpha)$, $Y(k_\alpha)$, $\widetilde{X}(k_\alpha)$, $\widetilde{Y}(k_\alpha)$ as in proposition 4.15, and definition 4.16. We may assume that k_α is greater than the integer k_0 of corollary 4.18. Since all our operators are in $Id_{\mathcal{H}^{k_\alpha-1}} \otimes \mathcal{L}(\bigotimes_{k \geq k_\alpha} \mathcal{H}_k)$, the assumptions of lemma 4.9 remain true with $\xi_0 = \xi_\alpha$. So, we have proved:

Theorem 4.19. (See also [5]). *If Ω fulfills condition $B2$ of section 2, then*

$$\Gamma(\mathcal{M}) = (\mathbf{R}).$$

It follows that $\mathcal{M}$ is isomorphic to $\mathcal{R}_\infty$, the Araki-Wood factor of type III_1([2]).

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