

## ON GENERALIZED SELFADJOINT OPERATORS ON SCALES OF HILBERT SPACES

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ABSTRACT. We consider examples of generalized selfadjoint operators that act from a positive Hilbert space to a negative space. Such operators were introduced and studied in [1]. We give examples of selfadjoint operators on the principal Hilbert space  $H_0$  that, being considered as operators from the positive space  $H_+ \subset H_0$  into the negative space  $H_- \supset H_0$ , are not essentially selfadjoint in the generalized sense.

### 1. INTRODUCTION

Let us first give a definition of a generalized selfadjoint operator and some results from [1]. Consider a chain of Hilbert spaces (rigging)

$$(1) \quad H_- \supset H_0 \supset H_+, \quad \|\cdot\|_{H_-} \leq \|\cdot\|_{H_0} \leq \|\cdot\|_{H_+}.$$

We will assume that the positive Hilbert space  $H_+$  is dense in the principal space  $H_0$ . The space  $H_-$  is negative and consists of antilinear functionals defined on  $H_+$ . Here, the scalar product  $(\cdot, \cdot)_{H_0}$  on  $H_0$  is a continuous sesquilinear form on  $H_- \times H_+$  that, being extended, defines a pairing  $\langle \cdot, \cdot \rangle$  between elements of  $H_-$  and  $H_+$  [2, 3],

$$(2) \quad \langle \alpha, \varphi \rangle = \overline{\langle \varphi, \alpha \rangle}, \quad \alpha \in H_-, \quad \varphi \in H_+.$$

There is a standard procedure [2, 3] that relates chain (1) to an operator  $\mathbb{I} : H_- \rightarrow H_+$  that is an isometry from  $H_-$  onto  $H_+$ ,

$$(3) \quad \forall \alpha, \beta \in H_- \quad (\alpha, \beta)_{H_-} = \langle \alpha, \mathbb{I}\beta \rangle = \langle \mathbb{I}\alpha, \beta \rangle.$$

Let  $A$  be an operator that acts from  $H_+$  into  $H_-$  with dense domain  $\mathfrak{D}(A)$  in  $H_+$ . The operator  $A^+$  that acts from  $H_+$  into  $H_-$  is called adjoint to the operator  $A$  in the generalized sense (in the sense of chain (1)) if its domain  $\mathfrak{D}(A^+)$  consists of all elements  $\psi \in H_+$  such that there exists  $\psi_- \in H_-$  satisfying  $\langle A\varphi, \psi \rangle = \langle \varphi, \psi_- \rangle \forall \varphi \in \mathfrak{D}(A)$  and  $A^+\psi = \psi_-$  [1].

In the case when  $H_+ = H_0 = H_-$ , this definition coincides with the usual definition of an adjoint operator on a Hilbert space [4, 3]. By using the operator  $\mathbb{I}$  defined in (3), it is easy to see that  $\mathbb{I}A$  acts from  $H_+$  into  $H_+$  and its adjoint on  $H_+$  is  $(\mathbb{I}A)^*$  that coincides with  $\mathbb{I}A^+$ .

If the operator  $A^+$  coincides with  $A$ , then such an operator will be called generalized selfadjoint [1]. In this case, the operator  $A_+ = \mathbb{I}A$  is selfadjoint on the space  $H_+$ . Conversely, every operator  $A_+$  that is selfadjoint on the space  $H_+$  gives rise to a generalized selfadjoint operator,  $A = (\mathbb{I})^{-1}A_+$ , from the space  $H_+$  into the space  $H_-$ . Note that there is a number of problems in physics connected with selfadjoint operators on the Sobolev spaces  $W_2^s(\mathbb{R}^n)$  [5, 6].

If  $\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle$  for all  $\varphi, \psi \in \mathfrak{D}(A) \subset H_+$ , then the operator  $A$  is called generalized Hermitian. The closure  $\bar{A}$  of a generalized Hermitian operator  $A$  always exists and is a generalized Hermitian operator. If the closure  $\bar{A}$  of an operator  $A$  that

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acts from  $H_+$  into  $H_-$  is generalized selfadjoint,  $\bar{A}^+ = \bar{A}$ , then the operator  $A$  is called generalized essentially selfadjoint. It is clear that a bounded generalized Hermitian operator  $A$  acting from  $H_+$  into  $H_-$  and having dense domain  $\mathfrak{D}(A)$  in  $H_+$  is essentially selfadjoint. Hence, it is also true that a generalized Hermitian operator defined on the whole space  $H_+$  is a bounded selfadjoint operator acting from  $H_+$  into  $H_-$ .

2. SELFADJOINT DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

Consider rigging (1), where  $H_+ = W_2^s(\mathbb{R}^n)$  is the Sobolev space of functions  $\varphi(x)$  defined on  $x \in \mathbb{R}^n$  by the following norm for the integer  $s$ :

$$(4) \quad \|\varphi\|_{W_2^s(\mathbb{R}^n)}^2 = \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} |\varphi^{(\alpha)}(x)|^2 dx, \quad \varphi^{(\alpha)}(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \varphi(x).$$

The principal space  $H_0$  is the space  $L_2(\mathbb{R}^n)$ . The Fourier transform of a function  $\varphi$

$$(5) \quad \tilde{\varphi}(\xi) = \int e^{i\xi \cdot x} \varphi(x) dx$$

belongs to the space  $L_2(\mathbb{R}^n, \rho_s(\xi))$ , where the weight is  $\rho_s(\xi) = (1 + |\xi|^2)^s$  and the norm is defined by

$$(6) \quad \|\tilde{\varphi}\|_{L_2(\mathbb{R}^n, \rho_s(\xi))}^2 = \int_{\mathbb{R}^n} |\tilde{\varphi}(\xi)(x)|^2 \rho_s(\xi) d\xi.$$

By the Parceval identity, the  $W_2^s$ -norm of functions  $\varphi$  is equivalent to the  $L_2(\mathbb{R}^n, \rho_s(\xi))$ -norm of  $\tilde{\varphi}$ , the Fourier transform of the functions  $\varphi$ . This permits to define the Sobolev spaces  $W_2^s(\mathbb{R}^n)$  for arbitrary real  $s$ . Note that, for  $s > 0$ ,  $W_2^{-s}(\mathbb{R}^n)$  is a negative space relatively to the positive space  $W_2^s(\mathbb{R}^n)$  and the space  $H_0 = L(\mathbb{R}^n)$ .

Consider a differential expression with constant coefficients,

$$(7) \quad \mathcal{P}(\mathcal{D}) = \sum_{|\alpha| \leq m} a_\alpha \mathcal{D}^\alpha,$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is an integer multi-index,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_n^{\alpha_n}$ ,  $\mathcal{D}_k = \frac{1}{i} \frac{\partial}{\partial x_k}$ , and  $a_\alpha$  are constant complex coefficients. Restricted to compactly supported infinitely differentiable functions in  $C_0^\infty(\mathbb{R}^n)$ , the differential expression  $\mathcal{P}(\mathcal{D})$  defines a minimal operator  $P$ ,  $P\varphi = \mathcal{P}(\mathcal{D})\varphi(x)$ , on the space  $L_2(\mathbb{R}^n)$ . The operator  $P$  is Hermitian if and only if all the coefficients  $a_\alpha$  in the differential expression (7) are real.

**Theorem 1.** *Let  $\mathcal{P}(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$  be a polynomial of  $n$  variables  $\xi = (\xi_1, \dots, \xi_n)$  with real coefficients. Then the minimal operator  $P$  defined on  $C_0^\infty(\mathbb{R}^n)$  by the differential expression  $\mathcal{P}(\mathcal{D}) = \sum_{|\alpha| \leq m} a_\alpha \mathcal{D}^\alpha$  is a generalized essentially selfadjoint operator from the Sobolev space  $W_2^s(\mathbb{R}^n)$  into  $W_2^{-s}(\mathbb{R}^n)$  for arbitrary  $s \geq 0$ .*

*Proof.* Passing to the Fourier transform, we need to prove that the operator of multiplication by the polynomial  $\mathcal{P}(\xi)$  is generalized essentially selfadjoint acting from the space  $L_2(\mathbb{R}^n, \rho_s)$  to the space  $L_2(\mathbb{R}^n, \rho_{-s})$ . Since the multiplication operator commutes with a weight, the claim follows, because the operator of multiplication by  $\mathcal{P}(\xi)$  in the space  $L_2(\mathbb{R}^n)$  is essentially selfadjoint if its domain is  $\mathfrak{D}$ , the space of all functions that are images of the Fourier transforms of functions in  $C_0^\infty(\mathbb{R}^n)$ , and the set  $\mathfrak{D}$  is dense in  $L_2(\mathbb{R}^n)$ . □

*Remark 1.* Theorem 1 shows that example 3.4 in [1] is erroneous (the error of this example was indicated by the authors of [1] earlier).

*Remark 2.* If  $s \geq \frac{m}{2}$ , where  $m$  is the order of the differential operator, then the closure of the differential operator  $P$  is defined on the whole space  $W_2^s(\mathbb{R}^n)$ , and is a bounded selfadjoint operator from  $W_2^s(\mathbb{R}^n)$  into  $W_2^{-s}(\mathbb{R}^n)$ .

*Remark 3.* If  $\mathcal{P}(\mathcal{D})$  is an elliptic differential expression, then the closure  $\bar{P}$  of the differential operator from  $W_2^s(\mathbb{R}^n)$  into  $W_2^{-s}(\mathbb{R}^n)$  is a generalized essentially selfadjoint operator with the domain  $\mathfrak{D}(P) = W_2^{m-s}(\mathbb{R}^n)$  for  $s \leq \frac{m}{2}$ .

### 3. A MULTIPLICATION OPERATOR IN A CHAIN OF SOBOLEV SPACES

If an operator  $A$  acts on a space  $H_0$ , then it can be considered, with respect to scale (1), as an operator  $\hat{A}$  from the space  $H_+$  into  $H_-$  by setting  $\mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \cap H_+$  and  $\hat{A}\varphi = A\varphi \in H_0 \subset H_-$ .

It was claimed in [1] that an operator  $A$ , which is selfadjoint on  $H_0$ , being considered as an operator from  $H_+$  into  $H_-$ , i.e., the operator  $\hat{A}$ , may or may not be essentially selfadjoint. Consider an example.

*Example 1.* Let  $H_0 = L_2(\mathbb{R}^1)$ ,  $H_+ = W_2^1(\mathbb{R}^1)$ ,  $H_- = W_2^{-1}(\mathbb{R}^1)$ . The operator  $A$  of multiplication by the function  $\frac{1}{x}$  will be selfadjoint on  $L_2(\mathbb{R}^1)$ , with the domain  $\mathfrak{D}(A) = \{u : u \in L_2, \frac{1}{x}u(x) \in L_2\}$ . It is easy to see that domain of the operator  $\hat{A}$  is

$$(8) \quad \mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \cap W_2^1(\mathbb{R}^1) = \{u : u \in W_2^1(\mathbb{R}^1), u(0) = 0\}.$$

The set  $\mathfrak{D}(\hat{A})$  is not dense in the space  $W_2^1(\mathbb{R}^1)$ . Indeed, we fix  $\varphi_0 = \frac{1}{2}e^{-|x|} \in W_2^1(\mathbb{R}^1)$  and  $(u, \varphi_0)_{W_2^1(\mathbb{R}^1)} = u(0)$ ,  $\forall u \in W_2^1(\mathbb{R}^1)$ . Hence,  $\varphi_0$  is orthogonal to  $\mathfrak{D}(\hat{A})$  in the space  $W_2^1(\mathbb{R}^1)$ . Since  $\mathfrak{D}(\hat{A})$  is not dense in the space  $H_+ = W_2^1(\mathbb{R}^1)$ , the operator  $\hat{A}$  can not be selfadjoint, if considered as an operator from  $H_+$  into  $H_-$ .

**Theorem 2.** *An operator  $\hat{A}$  defined by  $\hat{A}u = \frac{1}{x}u(x)$  on functions  $u \in W_2^1(\mathbb{R}^1)$  such that  $u(0) = 0$  is a generalized Hermitian operator from  $H_+ = W_2^1(\mathbb{R}^1)$  into  $H_- = W_2^{-1}(\mathbb{R}^1)$  and admits a selfadjoint extension  $\tilde{A}$  to the whole space  $H_+ = W_2^1(\mathbb{R}^1)$ ,*

$$(9) \quad \tilde{A}u(x) = (\mathcal{P})\frac{u(x)}{x},$$

where the functional  $(\mathcal{P})\frac{u(x)}{x} \in H_- = W_2^{-1}(\mathbb{R}^1)$  is the Cauchy principal value, that is,

$$(10) \quad \begin{aligned} < (\mathcal{P})\frac{u(x)}{x}, \varphi(x) > = (\mathcal{P}) \int \frac{u(x)\overline{\varphi(x)}}{x} dx = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^1 \setminus [-\varepsilon, \varepsilon]} \frac{u(x)\overline{\varphi(x)}}{x} dx \\ &= \int_{\mathbb{R}^1 \setminus [-1, 1]} \frac{u(x)\overline{\varphi(x)}}{x} dx + \int_{-1}^1 \frac{u(x)\overline{\varphi(x)} - u(0)\overline{\varphi(0)}}{x} dx, \quad \varphi \in W_2^1(\mathbb{R}^1). \end{aligned}$$

All the generalized selfadjoint extensions of the operator  $\hat{A}$ , which act from  $W_2^1(\mathbb{R}^1)$  into  $W_2^{-1}(\mathbb{R}^1)$  are given by the identity

$$(11) \quad \tilde{A}_\varepsilon = \tilde{A} + \varepsilon\delta < \cdot, \delta > = \tilde{A} + \varepsilon\delta(\cdot, \varphi_0)_{W_2^1(\mathbb{R}^1)},$$

where  $\varepsilon$  is an arbitrary real number and  $\delta \in W_2^{-1}(\mathbb{R}^1)$  is the Dirac  $\delta$ -functional,  $< \varphi, \delta > = \varphi(0)$ .

*Proof.* It is clear that the operator  $\tilde{A}$  is an extension of the bounded operator  $\hat{A}$  if considered as an operator from  $H_+ = W_2^1(\mathbb{R}^1)$  to  $H_- = W_2^{-1}(\mathbb{R}^1)$ . It immediately follows from definition (10) of the operator  $\tilde{A}$  that

$$| < \hat{A}u, \varphi > | \leq c\|u\|_{H_+}\|\varphi\|_{H_+},$$

this operator  $\hat{A}$  is generalized Hermitian and defined on the whole space  $H_+ = W_2^1(\mathbb{R}^1)$ . Hence,  $\tilde{A}$  is generalized essentially selfadjoint.

If  $\tilde{A}$  is an arbitrary selfadjoint extension of the operator  $\hat{A}$  from  $H_+ = W_2^1(\mathbb{R}^1)$  into  $H_- = W_2^{-1}(\mathbb{R}^1)$ , then the operator  $\tilde{A} - \tilde{A}$  is distinct from zero only on a one-dimensional subspace that contains  $\varphi_0 = \frac{1}{2}e^{-|x|} \in W_2^1(\mathbb{R}^1)$ , and it is generalized Hermitian. But then there exists a real number  $\varepsilon$  such that  $\mathbb{I}(\tilde{A} - \tilde{A}) = \varepsilon\varphi_0(\cdot, \varphi_0)_{W_2^1}$ . This implies (11), since  $\varphi_0 = \mathbb{I}\delta$ .  $\square$

4. DIFFERENTIAL OPERATORS WITH BOUNDARY-VALUE CONDITIONS

*Example 2.* On the space  $L_2(\mathbb{R}^1)$ , consider a selfadjoint differential operator  $A$  given by the differential expression  $-\frac{d^2}{dx^2}$  on functions such that their restrictions to the positive and the negative semiaxes belong to the Sobolev spaces  $W_2^2(0, \infty)$  and  $W_2^2(-\infty, 0)$ , correspondingly. The functions in the domain of the operator  $A$  satisfy, in the point  $x = 0$ , the adjacency conditions

$$(12) \quad u(+0) = u(-0) \equiv u(0), \quad u'(+0) - u'(-0) = \alpha u(0),$$

where  $\alpha \neq 0$  is a real number. We have  $Au(x) = -u''(x)$  for  $x \neq 0$ . Such an operator is used in quantum mechanics [7] and corresponds to a one-dimensional Schrödinger operator with the intensity  $\alpha$  point interaction in the point  $x = 0$ .

Consider chain (1) with  $H_+ = W_2^2(\mathbb{R}^1)$ ,  $H_- = W_2^{-2}(\mathbb{R}^1)$ ,  $H_0 = L_2(\mathbb{R}^1)$ . Let the operator  $A$  be considered as acting from  $H_+$  in  $H_-$ , that is, the operator  $\hat{A}$  is defined on the set

$$(13) \quad \mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \cap H_+ = \{u : u \in W_2^2(\mathbb{R}^1), u(0) = 0\}.$$

It is easy to see that  $\mathfrak{D}(\hat{A})$  is not dense in the space  $H_+ = W_2^2(\mathbb{R}^1)$ . This follows, in particular, from the fact proved in Section 3 that  $\{u : u \in W_2^1(\mathbb{R}^1), u(0) = 0\}$  is not dense in  $W_2^1(\mathbb{R}^1)$  since  $W_2^2(\mathbb{R}^1)$  is dense in  $W_2^1(\mathbb{R}^1)$ . Hence, the operator  $\hat{A}$  is not essentially selfadjoint.

**Theorem 3.** *Let  $A$  be the selfadjoint operator on  $H_0 = L_2(\mathbb{R}^1)$  considered in Example 2. Let  $H_+ = W_2^1(\mathbb{R}^1)$  and  $H_- = W_2^{-1}(\mathbb{R}^1)$ . Then the operator  $\hat{A}$  is a generalized essentially selfadjoint bounded operator from  $H_+$  into  $H_-$ . The closure  $\bar{A}$  of this operator  $\hat{A}$  admits the representation*

$$\bar{A} = -\frac{d^2}{dx^2} + \alpha\delta \langle \cdot, \delta \rangle,$$

where  $\delta$  is the Dirac delta-function.

*Proof.* This operator  $\bar{A}$  is a sum of two bounded operators from  $H_+$  into  $H_-$ . Let us show that it is an extension of the operator  $\hat{A}$ . Let  $u \in \mathfrak{D}(\hat{A})$ . Then  $\hat{A}u = -u''(x)$  for  $x \neq 0$ . However, for an arbitrary function  $\varphi \in C_0^\infty$ , we have  $\langle \bar{A}u, \varphi \rangle = \langle u, \bar{A}\varphi \rangle = \langle u, -\varphi'' \rangle + \alpha u(0)\varphi(0) = (u, -\varphi'') + \alpha u(0)\varphi(0) = (-u'', \varphi)$ . Here we have used the integration by parts formula and the condition  $u'(+0) - u'(-0) = \alpha u(0)$ . Hence,  $\bar{A}u = \hat{A}u$ . Since  $\mathfrak{D}(\hat{A})$  is a dense subset of  $H_+$ , the operator  $\hat{A}$  coincides on  $\mathfrak{D}(\hat{A})$  with the bounded operator  $\bar{A}$ .  $\square$

In Example 1, the operator  $\hat{A}$  was not generalized essentially selfadjoint, since its domain  $\mathfrak{D}(\hat{A})$  is not dense in  $H_+$ . However, such an operator has selfadjoint extensions given in Theorem 2. There is a question of whether the operator  $\hat{A}$  would always have selfadjoint extensions if the set  $\mathfrak{D}(\hat{A})$  is dense in  $H_+$  [1]. A negative answer to this question is given by the example of [8].

**Theorem 4.** Let  $A$  be the selfadjoint differential operator  $i\frac{d}{dx}$  defined in the space  $H_0 = L_2(0,1)$  on the set  $W_2^1(0,1)$  of functions that satisfy the boundary condition  $u(0) = u(1)$ . Let  $H_+ = L_2((0,1),\rho)$  be the space of functions that are square integrable on the interval  $(0,1)$  with the weight  $\rho \geq 1$ , where  $\rho(x)$  is a continuous unbounded function on the interval  $(0,1)$ , and  $H_- = L_2((0,1),\rho^{-1})$ . Then the operator  $\hat{A}$  is densely defined on the space  $H_+$ . We have the following statements:

- 1) If  $\int_0^1 \rho(x) dx = \infty$  then  $\mathfrak{D}(\hat{A}) \equiv \mathfrak{D}(A) \cap H_+ \subset \overset{\circ}{W}_2^1 \equiv \{u : u \in W_2^1(0,1), u(0) = u(1) = 0\}$ .
- 2) If  $\rho = [x(1-x)]^{-1}$  then operator  $\hat{A}$  is generalized essentially selfadjoint.
- 3) If  $\rho = x^{-1}$  then operator  $\hat{A}$  is not generalized essentially selfadjoint and, moreover, does not have generalized selfadjoint extensions as the operator from  $H_+$  into  $H_-$ .

*Proof.* For  $\varphi_+ \in H_+$ , it is necessary and sufficient that there would exist a function  $\varphi \in L_2(0,1)$  and  $\varphi_+(x) = \rho^{-\frac{1}{2}}(x)\varphi(x)$ . For functions  $\varphi_- \in H_-$ , we have the representation  $\varphi_- = \rho^{\frac{1}{2}}(x)\varphi(x)$ . Here,  $\|\varphi_+\|_{H_+} = \|\varphi\|_{L_2}$ ,  $\|\varphi_-\|_{H_-} = \|\varphi\|_{L_2}$ . The domain of the operator  $\hat{A}$  is  $\mathfrak{D}(A) \cap H_+$ . Since  $\mathfrak{D}(\hat{A}) \supset C_0^\infty(0,1)$  and the space  $C_0^\infty(0,1)$  is dense in  $H_+$ , the set  $\mathfrak{D}(\hat{A})$  is dense in  $H_+$ .

Statement 1) is obvious since, from the condition  $u \in \mathfrak{D}(\hat{A})$ , it follows that function  $u(x)$  is continuous on the interval  $[0,1]$  and  $u(0) = u(1)$ . However, if this function belongs to  $H_+$  then  $u(0) = u(1) = 0$  that is  $\mathfrak{D}(\hat{A}) \subset \overset{\circ}{W}_2^1$ . It is easy to prove that in the cases 2) and 3) we have  $\mathfrak{D}(\hat{A}) = \overset{\circ}{W}_2^1$ .

Existence of generalized selfadjoint extensions of the operator  $\hat{A}$  is equivalent to existence of selfadjoint extensions of the operator  $\mathbb{I}\hat{A}$  on  $H_+$ . Let us find deficiency indices of this symmetric operator. To this end, find functions  $\eta_\pm(x) \in H_+$  that satisfy the identities

$$(14) \quad (\mathbb{I}\hat{A}u \pm iu, \eta_\pm)_{H_+} = 0$$

for arbitrary  $u \in \mathfrak{D}(\hat{A})$ . Note that the operator  $\mathbb{I}$  is a multiplication operator,  $\mathbb{I}\varphi_-(x) = \rho^{-1}(x)\varphi_-(x)$ . The identity (14) can be written in an equivalent form,

$$(15) \quad \int [i\frac{d}{dx} \pm i\rho(x)]u(x)\bar{\eta}_\pm(x) dx = 0.$$

Since  $u \in \mathfrak{D}(\hat{A}) \supset C_0^\infty(0,1)$ , it follows from (15) that the functions  $\eta_\pm$  are generalized solutions of the equations

$$(16) \quad [i\frac{d}{dx} \mp i\rho(x)]\eta_\pm(x) = 0.$$

In the case where  $\rho = [x(1-x)]^{-1}$ , equation (16) has solutions  $\eta_+ = x(1-x)^{-1}$  and  $\eta_- = x^{-1}(1-x)$  that do not belong to the space  $H_+$ . Hence, the deficiency indices of the symmetric operator  $\mathbb{I}\hat{A}$  are zero then the operator is essentially selfadjoint [4, 3].

If  $\rho = x^{-1}$ , equation (16) has solutions  $\eta_+ = x$  and  $\eta_- = x^{-1}$ . The function  $\eta_+ \in H_+$ , and the function  $\eta_-$  does not belong to the space  $H_+$ . Hence, the deficiency indices of the operator  $\mathbb{I}\hat{A}$  are  $(1,0)$ , and since they are different, the operator  $\mathbb{I}\hat{A}$  does not admit selfadjoint extensions on  $H_+$ .  $\square$

*Remark 4.* As it has been shown in examples of selfadjoint operators  $A$  on the principal Hilbert space  $H_0$  of theorems 1–4, the operator  $\hat{A}$  may be essentially selfadjoint, if considered as an operator from  $H_+$  into  $H_-$ , or may be not. This fact does not depend on essential selfadjointness of  $A_+ = A \upharpoonright_{\mathfrak{D}(A) \cap H_+}$  in  $H_0$ . All four cases are possible.

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