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POSITIVE OPERATORS ON THE BERGMAN SPACE AND BEREZIN TRANSFORM

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Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $L^2_a(\mathbb{D})$ be the Bergman space of the disk. In this paper we characterize the class $\mathcal{A} \subset \mathrm{L}^\infty(\mathbb{D})$ such that if $\phi, \psi \in \mathcal{A}, \alpha \geq 0$ and $0 \leq \phi \leq \alpha\psi$ then there exist positive operators $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\phi(z) = \widetilde{S}(z) \leq \alpha\widetilde{T}(z) = \alpha\psi(z)$ for all $z \in \mathbb{D}$. Further, we have shown that if S and T are two positive operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ and T is invertible then there exists a constant $\alpha \geq 0$ such that $\widetilde{S}(z) \leq \alpha\widetilde{T}(z)$ for all $z \in \mathbb{D}$ and $\widetilde{S}, \widetilde{T} \in \mathcal{A}$. Here $\mathcal{L}(L^2_a(\mathbb{D}))$ is the space of all bounded linear operators from $L^2_a(\mathbb{D})$ into $L^2_a(\mathbb{D})$ and $\widetilde{A}(z) = \langle Ak_z, k_z\rangle$ is the Berezin transform of $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ and k_z is the normalized reproducing kernel of $L^2_a(\mathbb{D})$. Applications of these results are also obtained.

1. Introduction

The Bergman space $L^2_a(\mathbb{D})$ is the space of all analytic functions defined on the open unit disk $\mathbb{D}$, that are square integrable with respect to the area measure $dA(z) = \frac{1}{\pi}dxdy$. It is known that $L^2_a(\mathbb{D})$ is a closed subspace [2] of $L^2(\mathbb{D}, dA)$ and hence it is a Hilbert space with an orthonormal basis $\{\sqrt{n+1}z^n\}_{n=0}^\infty$. From this it follows that the analytic polynomials are dense in $L^2_a(\mathbb{D})$, and consequently, the space of all bounded analytic functions on $\mathbb{D}, H^\infty(\mathbb{D})$ is dense in $L^2_a(\mathbb{D})$. Moreover, if $f(z) = \sum_{n=0}^\infty a_n z^n$ is in $L^2_a(\mathbb{D})$, then it can be checked easily that $\|f\|^2 = \sum_{n=0}^\infty \frac{|a_n|^2}{n+1}$. The Toeplitz operator T_ψ with symbol ψ in $L^\infty(\mathbb{D})$ is defined on $L^2_a(\mathbb{D})$ by $T_\psi f = P(\psi f)$, where P is the orthogonal projection of $L^2(\mathbb{D})$ onto $L^2_a(\mathbb{D})$. The Hankel operator H_ψ is the operator $H_\psi : L^2_a(\mathbb{D}) \longrightarrow (L^2_a(\mathbb{D}))^\perp$, defined by $H_\psi f = (I - P)(\psi f)$.

Let $K(z, \bar{w})$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L^2_a(\mathbb{D})$ because the formula

$$f(z) = \int_{\mathbb{D}} f(w)K(z, \bar{w})\, dA(w)$$

reproduces each f in $L^2_a(\mathbb{D})$ and $K(z, \bar{w}) = \sum_{n=0}^\infty e_n(z)\overline{e_n(w)}$ where $e_n(z) = \sqrt{n+1}z^n$, $n \geq 0$, $n \in \mathbb{Z}$. Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L^2_a(\mathbb{D})$; it is clear that they are unit vectors in $L^2_a(\mathbb{D})$. For any $a \in \mathbb{D}$, let ϕ_a be the analytic mapping on $\mathbb{D}$ defined by $\phi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4}.$$

Let $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators from the Hilbert space H into itself. Let $\mathcal{LC}(H)$

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denote the ideal of compact operators in $\mathcal{L}(H)$. An operator $A \in \mathcal{L}(H)$ is called positive if $\langle Ax, x\rangle \geq 0$ holds for every $x \in H$ in which case we write $A \geq 0$. Define the Berezin transform for operators $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ by the formula

$$\sigma(T)(z) = \widetilde{T}(z) = \langle Tk_z, k_z\rangle, \quad z \in \mathbb{D}.$$

The function $\widetilde{T}$ is called the Berezin transform of T. Let $V(\mathbb{D}) = \{\phi \in L^\infty(\mathbb{D}) : \operatorname{ess} \lim_{|z| \longrightarrow 1^-} \phi(z) = 0\}$. If $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ then $\widetilde{T} = \sigma(T) \in L^\infty(\mathbb{D})$ and $\|\sigma(T)\|_\infty \leq \|T\|$ as $|\sigma(T)(z)| = |\langle Tk_z, k_z\rangle| \leq \|T\|$ for all $z \in \mathbb{D}$. Further, if $T \in \mathcal{LC}(L^2_a(\mathbb{D}))$, then as $k_z \longrightarrow 0$ weakly, hence $\sigma(T) \in V(\mathbb{D})$. One may also notice that if $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is diagonal with respect to the basis $\{e_n\}_{n=0}^\infty$, then $\sigma(T)$ is radial. For $\phi \in L^2(\mathbb{D}, dA)$ and $\lambda \in \mathbb{D}$, let

$$\widetilde{\phi}(\lambda) = \langle \phi k_\lambda, k_\lambda\rangle = \int_{\mathbb{D}} \phi(z) \frac{(1-|\lambda|^2)^2}{|1-\overline{\lambda}z|^4}\, dA(z).$$

If T_ϕ is a Toeplitz operator with symbol $\phi \in L^2$, then $\sigma(T_\phi) = \widetilde{\phi}$. In section 2, we establish our main result and some applications of the result are also discussed. In section 3, we discuss about Berezin transform and hyponormal operators.

2. Berezin transform

In this section we characterize the class $\mathcal{A} \subset L^\infty(\mathbb{D})$ such that if $\phi, \psi \in \mathcal{A}, \alpha \geq 0$ and $0 \leq \phi \leq \alpha\psi$ then there exist positive operators $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\phi(z) = \widetilde{S}(z) \leq \alpha\widetilde{T}(z) = \alpha\psi(z)$ for all $z \in \mathbb{D}$. Further, we establish that if S and T are two positive operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ and T is invertible then there exists a constant $\alpha \geq 0$ such that $\widetilde{S}(z) \leq \alpha\widetilde{T}(z)$ for all $z \in \mathbb{D}$ and $\widetilde{S}, \widetilde{T} \in \mathcal{A}$.

Notice that if $S \geq T \geq 0,\ S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ and A is any invertible positive operator in $\mathcal{L}(L^2_a(\mathbb{D}))$, then $ASA \geq ATA$. Conversely, if there exist a positive invertible operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $ASA \geq ATA$, then $S \geq T \geq 0$ and therefore $\widetilde{S} \geq \widetilde{T}$. Further if S, T are two positive invertible operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ such that $S \geq T$ then $T^{-1} \geq S^{-1}$. This can be verified as follows: If $S \geq T$ then $T^{-\frac{1}{2}}ST^{-\frac{1}{2}} \geq I$. This implies $I \geq (T^{-\frac{1}{2}}ST^{-\frac{1}{2}})^{-1} = T^{\frac{1}{2}}S^{-1}T^{\frac{1}{2}}$. Hence $T^{-1} \geq S^{-1}$. Thus if $S \geq T \geq 0$ and S, T are invertible then $\widetilde{T^{-1}} \geq \widetilde{S^{-1}}$. The following is also valid.

Proposition 2.1. *Suppose S, T are two positive operators in $\mathcal{L}(L^2_a(\mathbb{D}))$.*

(1) *If T_ϕ is a nonsingular positive Toeplitz operator in $\mathcal{L}(L^2_a(\mathbb{D}))$ and $ST_\phi S = TT_\phi T$ then $\widetilde{S} = \widetilde{T}$.*

(2) *If T_ϕ, T_ψ are two positive, invertible Toeplitz operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ and $T_\phi \geq T_\psi$ and $ST_\psi S \geq TT_\phi T$ then $\widetilde{S} \geq \widetilde{T}$.*

Proof. (1) If $ST_\phi S = TT_\phi T$ holds, then we have $(T_\phi^{\frac{1}{2}}ST_\phi^{\frac{1}{2}})^2 = (T_\phi^{\frac{1}{2}}TT_\phi^{\frac{1}{2}})^2$, so that $T_\phi^{\frac{1}{2}}ST_\phi^{\frac{1}{2}} = T_\phi^{\frac{1}{2}}TT_\phi^{\frac{1}{2}}$ holds and the nonsingularity of T_ϕ ensures $S = T$ and hence $\widetilde{S} = \widetilde{T}$.

(2) Since $T_\phi \geq T_\psi$ and S is invertible hence $ST_\phi S \geq ST_\psi S$. Thus $ST_\phi S \geq TT_\phi T$ and so $(T_\phi^{\frac{1}{2}}ST_\phi^{\frac{1}{2}})(T_\phi^{\frac{1}{2}}ST_\phi^{\frac{1}{2}}) \geq (T_\phi^{\frac{1}{2}}TT_\phi^{\frac{1}{2}})(T_\phi^{\frac{1}{2}}TT_\phi^{\frac{1}{2}})$. This implies $(T_\phi^{\frac{1}{2}}ST_\phi^{\frac{1}{2}})^2 \geq (T_\phi^{\frac{1}{2}}TT_\phi^{\frac{1}{2}})^2$. By [6], it follows that $T_\phi^{\frac{1}{2}}ST_\phi^{\frac{1}{2}} \geq T_\phi^{\frac{1}{2}}TT_\phi^{\frac{1}{2}}$. Hence $S \geq T$ and therefore $\widetilde{S} \geq \widetilde{T}$. □

Definition 2.2. A function $g(x, \bar{y})$ on $\mathbb{D} \times \mathbb{D}$ is called of positive type (or positive definite), written $g \gg 0$, if

$$\sum_{j,k=1}^{n} c_j\overline{c_k}g(x_j, \overline{x_k}) \geq 0 \tag{1}$$

for any n-tuple of complex numbers $c_1, \dots, c_n$ and points $x_1, \dots, x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$. We shall say $\gamma \in \mathcal{A}$ if $\gamma \in L^\infty(\mathbb{D})$ and is such that

$$\gamma(z) = \Theta(z, \bar{z}) \tag{2}$$

where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y and if there exists a constant $c > 0$ such that

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0 \quad \text{for all} \quad x, y \in \mathbb{D}.$$

It is a fact that (see [3], [4]) Θ as in (2), if it exists, is uniquely determined by γ.

Theorem 2.3. *If $\phi, \psi \in \mathcal{A}, \alpha \geq 0$ and $0 \leq \phi \leq \alpha\psi$ then there exist positive operators $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\phi(z) = \widetilde{S}(z) \leq \alpha\widetilde{T}(z) = \alpha\psi(z)$ for all $z \in \mathbb{D}$. Further, if S and T are two positive operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ and T is invertible then there exists a constant $\alpha \geq 0$ such that $\widetilde{S}(z) \leq \alpha\widetilde{T}(z)$ for all $z \in \mathbb{D}$ and $\widetilde{S}, \widetilde{T} \in \mathcal{A}$.*

Proof. For the first part it suffices to show that $0 \leq \phi \in \mathcal{A}$ if and only if there exists a positive operator $S \in \mathcal{L}(L^2_a)$ such that $\phi(z) = \langle Sk_z, k_z \rangle$ for all $z \in \mathbb{D}$. So let $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ be a positive operator. Let $\Theta(x, \bar{y}) = \frac{\langle SK_y, K_x \rangle}{\langle K_y, K_x \rangle}$ where $K_x = K(., \bar{x})$ is the unnormalized reproducing kernel at x. Then $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. Let $\phi(z) = \Theta(z, \bar{z})$.

Then $\phi(z) = \langle Sk_z, k_z \rangle$ for all $z \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{D})$ as S is bounded. Now let $f = \sum_{j=1}^n c_j K_{x_j}$ where c_j's are constants, $x_j \in \mathbb{D}$ for $j = 1, 2, \dots, n$. Since S is bounded and positive there exists a constant $c > 0$ such that $0 \leq \langle Sf, f \rangle \leq c\|f\|^2$. But

$$\begin{aligned}
\langle Sf, f \rangle = \Big\langle S\Big(\sum_{j=1}^n c_j K_{x_j}\Big), \sum_{j=1}^n c_j K_{x_j} \Big\rangle &= \sum_{j,k=1}^n c_j \bar{c_k} \langle SK_{x_j}, K_{x_k} \rangle \\
&= \sum_{j,k=1}^n c_j \bar{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j})
\end{aligned}$$

and $c\|f\|^2 = c\langle f, f \rangle = c\sum_{j,k=1}^n c_j \bar{c_k} K(x_k, \bar{x_j})$.

Hence we obtain that $cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0$. Thus $\phi \in \mathcal{A}$.

Now let $\phi \in \mathcal{A}$ and $\phi(z) = \Theta(z, \bar{z})$ where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. We shall prove the existence of a positive, bounded operator $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\phi(z) = \langle Sk_z, k_z \rangle$. Let

$$Sf(x) = \int_{\mathbb{D}} f(z)\Theta(x, \bar{z})K(x, \bar{z})\, dA(z). \tag{3}$$

Indeed,

$$\begin{aligned}
Sf(x) = \langle Sf, K_x \rangle = \langle f, S^*K_x \rangle &= \int_{\mathbb{D}} f(z)\overline{\langle S^*K_x, K_z \rangle}\, dA(z) \\
&= \int_{\mathbb{D}} f(z)\langle SK_z, K_x \rangle\, dA(z) = \int_{\mathbb{D}} f(z)\Theta(x, \bar{z})K(x, \bar{z})\, dA(z).
\end{aligned}$$

Then

$$\begin{aligned}
\langle SK_y, K_x \rangle &= \int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z})K(x, \bar{z})\, dA(z) = \int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z})\overline{K_x(z)}\, dA(z) \\
&= \overline{\langle \overline{\Theta(x, \bar{z})}K_x, K_y \rangle} = \overline{\overline{\Theta(x, \bar{y})}\langle K_x, K_y \rangle} = \Theta(x, \bar{y})\langle K_y, K_x \rangle.
\end{aligned}$$

Hence $\Theta(x, \bar{y}) = \frac{\langle SK_y, K_x \rangle}{\langle K_y, K_x \rangle}$ and $\phi(z) = \Theta(z, \bar{z}) = \langle Sk_z, k_z \rangle$. We shall now prove that S is positive, bounded. That is, there exists a constant $c > 0$ such that $0 \leq \langle Sf, f \rangle \leq c\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$. Since $\phi \in \mathcal{A}$, there exists a constant $c > 0$ such that for all $x, y \in \mathbb{D}$,

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0. \tag{4}$$

Let $f = \sum_{j=1}^{n} c_j K_{x_j}$ where c_j are constants, $x_j \in \mathbb{D}$ for $j = 1, 2, \ldots, n$. Then from (4) it follows that $\langle Sf, f\rangle = \sum_{j,k=1}^{n} c_j \bar{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j}) \geq 0$ and

$$\langle Sf, f\rangle = \sum_{j,k=1}^{n} c_j \bar{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j}) \leq c \sum_{j,k=1}^{n} c_j \bar{c_k} K(x_k, \bar{x_j}) = c\|f\|^2.$$

Since the set of vectors $\Big\{ \sum_{j=1}^{n} c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2, \ldots, n \Big\}$ is dense in $L_a^2(\mathbb{D})$, hence $0 \leq \langle Sf, f\rangle \leq c\|f\|^2$ for all $f \in L_a^2(\mathbb{D})$ and thus S is bounded and positive. To prove the second part, assume S, T are two positive linear operators in $\mathcal{L}(L_a^2(\mathbb{D}))$ and T is invertible. Let $\alpha = \sup\Big\{ \frac{\|Sk_z\|}{\|Tk_z\|} : z \in \mathbb{D} \Big\}$. We shall show that $\widetilde{S}(z) \leq \alpha \widetilde{T}(z)$ for all $z \in \mathbb{D}$ and $\widetilde{S}, \widetilde{T} \in \mathcal{A}$. Since T is invertible and positive, hence $T^{\frac{1}{2}}$ is also positive and invertible. Let $A = T^{-\frac{1}{2}}$ and $B = ASA$. Then B is positive and there is a spectral decomposition associated with B, say $\{E_\lambda\}$ for $-\infty < \lambda < \infty$ and $B = \int_{0^-}^{k} \lambda dE_\lambda$ where $k = \|B\|$. Hence $\langle Bf, f\rangle = \int_{0^-}^{k} \lambda d\|E_\lambda\|^2$ and $\sup\big\{ \langle Bf, f\rangle : f \in L_a^2(\mathbb{D}), \|f\| = 1 \big\} = k$. Further $\sup\Big\{ \frac{\langle Sf,f\rangle}{\langle Tf,f\rangle} : f \in L_a^2(\mathbb{D}), f \neq 0 \Big\} = \sup\big\{ \langle Bf, f\rangle : f \in L_a^2(\mathbb{D}), \|f\| = 1 \big\}$. We shall now establish that given $\epsilon > 0$, we can find a $g \in L_a^2(\mathbb{D})$ such that $\frac{\|Sg\|}{\|Tg\|} \geq k - \epsilon$. Let $r = \frac{\epsilon}{2} \frac{\|T^{-1}\|}{\|T^{\frac{1}{2}}\|^2}$ and $s = k - r$. We shall choose ϵ small enough such that $s > 0$. Let $f \in L_a^2(\mathbb{D}), f \neq 0$ such that $E_\lambda f = 0$ for some λ such that $s < \lambda < k$. Let $g = Af$. Now

$$Sg = T^{\frac{1}{2}} \int_s^k \lambda \, dE_\lambda (T^{\frac{1}{2}} g)$$

and

$$Tg = T^{\frac{1}{2}} f = T^{\frac{1}{2}} \int_s^k dE_\lambda (T^{\frac{1}{2}} g).$$

Thus

$$\|Sg - sTg\| = \|T^{\frac{1}{2}} \int_s^k (\lambda - s) \, dE_\lambda (T^{\frac{1}{2}} g)\| \leq \|T^{\frac{1}{2}}\| r \|T^{\frac{1}{2}} g\| \leq r \|T^{\frac{1}{2}}\|^2 \|g\|.$$

Hence

$$\begin{aligned} \frac{\|Sg\|}{\|Tg\|} &= \frac{\|sTg + Sg - sTg\|}{\|Tg\|} \geq \frac{(\|sTg\| - \|Sg - sTg\|)}{\|Tg\|} \\ &\geq s - \frac{r\|T^{\frac{1}{2}}\|^2 \|g\|}{\|T^{-1}\|} \|g\| \geq k - \epsilon. \end{aligned}$$

Now

$$\begin{aligned} \sup\left\{ \frac{\|Sf\|}{\|Tf\|} : f \in L_a^2(\mathbb{D}), f \neq 0 \right\} &= \sup\left\{ \frac{\langle Sf, f\rangle}{\langle Tf, f\rangle} : f \in L_a^2(\mathbb{D}), f \neq 0 \right\} \\ &= \sup\big\{ \langle Bf, f\rangle : f \in L_a^2(\mathbb{D}), \|f\| = 1 \big\} = k. \end{aligned}$$

Thus $\langle Sk_z, k_z\rangle \leq k\langle Tk_z, k_z\rangle$ and k is our required constant α. Further since S and T are positive and bounded, hence $\widetilde{S}, \widetilde{T} \in \mathcal{A}$ as we have proved in first part and the theorem is proved. □

Corollary 2.4. *The function $\phi \in \mathcal{A}$ and satisfies*

$$CK(x, \bar{y}) \gg \Theta(x, \bar{y}) K(x, \bar{y}) \gg mK(x, \bar{y}) \gg 0 \tag{5}$$

for all $x, y \in \mathbb{D}$ and some constants $C, m > 0$ if and only if there exists a positive, invertible operator $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\phi(z) = \langle Tk_z, k_z\rangle$ for all $z \in \mathbb{D}$.

Proof. Suppose $\phi \in \mathcal{A}$ and (5) holds. Then from theorem 2.3 it follows that there exists a positive linear operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\phi(z) = \langle Tk_z, k_z\rangle$. Now let $f = \sum_{j=1}^n c_j K_{x_j}$ where c_j's are constants, $x_j \in \mathbb{D}$ for $j = 1, 2, \ldots, n$. Since

$$\langle Tf, f\rangle = \Big\langle T\Big(\sum_{j=1}^n c_j K_{x_j}\Big), \sum_{j=1}^n c_j K_{x_j}\Big\rangle = \sum_{j,k=1}^n c_j \bar{c_k}\Theta(x_k, \bar{x_j})K(x_k, , \bar{x_j})$$

and

$$m\|f\|^2 = m\langle f, f\rangle = m \sum_{j,k=1}^n c_j \bar{c_k} K(x_k, \bar{x_j}),$$

it follows from (5) that $\langle Tf, f\rangle \geq m\|f\|^2$. As the set of vectors

$$\Big\{ \sum_{j=1}^n c_j K_{x_j}, x_j \in \mathbb{D}, \quad j = 1, 2, \ldots, n\Big\}$$

is dense in $L^2_a(\mathbb{D})$, hence $0 \leq \langle Tf, f\rangle \geq m\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$. That is, $T \geq mI$ where I is the identity operator in $\mathcal{L}(L^2_a(\mathbb{D}))$. Hence T is invertible. Conversely, suppose T is a bounded, positive operator in $\mathcal{L}(L^2_a(\mathbb{D}))$ which is also invertible. Then from theorem 2.3, it follows that $\phi(z) = \langle Tk_z, k_z\rangle \in \mathcal{A}$ and there exists a constant $m > 0$ such that $T \geq mI$. Hence if $f = \sum_{j=1}^n c_j K_{x_j}$ where c_j's are constants, $x_j \in \mathbb{D}$, $j = 1, 2, \ldots, n$, then $\langle Tf, f\rangle = \sum_{j,k=1}^n c_j \bar{c_k}\Theta(x_k, \bar{x_j})K(x_k, \bar{x_j})$ and $m\|f\|^2 = m\langle f, f\rangle = m\sum_{j,k=1}^n c_j\bar{c_k}K(x_k, \bar{x_j})$. As $\langle Tf, f\rangle \geq m\|f\|^2$, hence $\Theta(x, \bar{y})K(x, \bar{y}) \gg mK(x, \bar{y})$ for all $x, y \in \mathbb{D}$. The corollary follows. □

If $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ and S is positive, then let

$$\Theta_S(x, \bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle} \quad \text{for all} \quad x, y \in \mathbb{D}.$$

Corollary 2.5. *Let S and T be two positive, invertible operators in $\mathcal{L}(L^2_a(\mathbb{D}))$. If*

$$\Theta_S(x, \bar{y})K(x, \bar{y})\Theta_{T^{-1}}(x, \bar{y})K(x, \bar{y}) \gg (K(x, \bar{y}))^2 \tag{6}$$

and

$$\Theta_T(x, \bar{y})K(x, \bar{y})\Theta_{S^{-1}}(x, \bar{y})K(x, \bar{y}) \gg (K(x, \bar{y}))^2 \tag{7}$$

for all $x, y \in \mathbb{D}$ then $S = T$.

Proof. Suppose (6) holds. Let $f = \sum_{j=1}^n c_j K_{x_j}$ where c_j's are constants, $x_j \in \mathbb{D}$, $j = 1, 2, \ldots, n$. Then

$$\langle Sf, f\rangle\langle T^{-1}f, f\rangle = \Big(\sum_{j,k=1}^n c_j\bar{c_k}\Big)^2 \Theta_S(x_k, \bar{x_j})K(x_k, \bar{x_j})\Theta_{T^{-1}}(x_k, \bar{x_j})K(x_k, \bar{x_j})$$

and $\|f\|^4 = \langle f, f\rangle\langle f, f\rangle = \left(\sum_{j,k=1}^n c_j\bar{c_k}\right)^2 (K(x_k, \bar{x_j}))^2$. Then from (6) it follows that $\langle Sf, f\rangle\langle T^{-1}f, f\rangle \geq \|f\|^4$. Since the set of vectors $\left\{\sum_{j=1}^n c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2, \ldots, n\right\}$ is dense in $L^2_a(\mathbb{D})$, hence for all $f \in L^2_a(\mathbb{D}), \langle Sf, f\rangle\langle T^{-1}f, f\rangle \geq \|f\|^4$. That is, $\langle S\xi, \xi\rangle\langle T^{-1}\xi, \xi\rangle \geq 1$ for any unit vector $\xi \in L^2_a(\mathbb{D})$. Now suppose (7) holds. Proceeding similarly one can show that $\langle T\xi, \xi\rangle\langle S^{-1}\xi, \xi\rangle \geq 1$ for any unit vector $\xi \in L^2_a(\mathbb{D})$. From [5], it follows that $S = T$. □

If S, T are two positive, invertible operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ then $\langle S\xi, \xi\rangle\langle T^{-1}\xi, \xi\rangle \geq 1$ for any unit vector $\xi \in L^2_a(\mathbb{D})$ if and only if $\Theta_{tS+(tT)^{-1}}(x, \bar{y})K(x, \bar{y}) \gg 2K(x, \bar{y})$ for all $x, y \in \mathbb{D}$ and for any $t > 0$. This can be verified as follows:

Suppose $\langle S\xi, \xi\rangle\langle T^{-1}\xi, \xi\rangle \geq 1$ for any unit vector $\xi \in L^2_a(\mathbb{D})$. Let $g \in L^2_a(\mathbb{D})$ be a unit vector. Let $\Omega(t) = t\langle Sg, g\rangle + t^{-1}\langle T^{-1}g, g\rangle, t > 0$. The minimum value of $\Omega(t)$ is equal to $2\sqrt{\langle Sg, g\rangle\langle T^{-1}g, g\rangle}$. Thus $\langle S\xi, \xi\rangle\langle T^{-1}\xi, \xi\rangle \geq 1$ for any unit vector $\xi \in L^2_a(\mathbb{D})$ if and only if $tS + (tT)^{-1} \geq 2$ for any $t > 0$. This is true if and only if $\Theta_{tS+(tT)^{-1}}(x, \bar{y})K(x, \bar{y}) \gg 2K(x, \bar{y})$ for all $x, y \in \mathbb{D}$.

Corollary 2.6. *Suppose the function $\phi \in \mathcal{A}$ and satisfies (5) for all $x, y \in \mathbb{D}$ and for some constants $C, m > 0$. Then*

$$|\Theta_T(x, \bar{y})|\,|\Theta_{T^{-1}}(y, \bar{x})|\,|K(x, \bar{y})|^2 \leq LK(y, \bar{y})K(x, \bar{x}).$$

If $\|T\| = R$ and $\|T^{-1}\| = \frac{1}{r}$, then $L = \frac{(R+r)^2}{4Rr}$.

Proof. Suppose the function $\phi \in \mathcal{A}$ satisfies (5) for all $x, y \in \mathbb{D}$ and for some constants $C, m > 0$. Then it follows from corollary 2.4 that T is invertible and $mI \leq T \leq CI$. Hence for all $x, y \in \mathbb{D}$,

$$\begin{aligned}|\Theta_T(x, \bar{y})|\,|\Theta_{T^{-1}}(y, \bar{x})|\,|K(x, \bar{y})|^2 &= \frac{\langle|TK_y, K_x\rangle|}{|\langle K_y, K_x\rangle|}\frac{\left|\left\langle T^{-1}K_x, K_y\right\rangle\right|}{|\langle K_x, K_y\rangle|}\,|\langle K_y, K_x\rangle|^2 \\ &= |\langle TK_y, K_x\rangle|\left|\left\langle T^{-1}K_x, K_y\right\rangle\right| \leq L\left\langle K_y, K_y\right\rangle\left\langle K_x, K_x\right\rangle,\end{aligned}$$

where $L = \frac{(R+r)^2}{4Rr}$. The last inequality follows from [7] Kantorovich's inequality. □

3. Hyponormal operators

It is well known [1] that if $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $\widetilde{S}(z) = \widetilde{T}(z)$ for all $z \in \mathbb{D}$ then $S = T$. In this section we show that if $\widetilde{T}(\phi_a(z)) = \widetilde{S}(z)$ for some $a \in \mathbb{D}$ and for all $z \in \mathbb{D}$, T^* is p-hyponormal and S is a dominant operator then $S = T$.

An operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ is called hyponormal if $A^*A \geq AA^*$. For $0 < p \leq 1$, the operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ is called p-hyponormal if $|A|^{2p} \geq |A^*|^{2p}$ where $|A|$ is the square root of A^*A, that is, $|A| = (A^*A)^{\frac{1}{2}}$. The operator A is called dominant if Range$(A-\lambda I) \subset$ Range$(A - \lambda I)^*$ for all $\lambda \in \mathbb{C}$.

Theorem 3.1. *Let $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$. Assume T^* is a p-hyponormal operator, $0 < p \leq 1$ and S is a dominant operator. If $\widetilde{T}(\phi_a(z)) = \widetilde{S}(z)$ for all $z \in \mathbb{D}$ and for some $a \in \mathbb{D}$ then $S = T$.*

Proof. Suppose for some $a \in \mathbb{D}$, $\widetilde{T}(\phi_a(z)) = \widetilde{S}(z)$ for all $z \in \mathbb{D}$. Then $\left\langle Tk_{\phi_a(z)}, k_{\phi_a(z)}\right\rangle = \langle Sk_z, k_z\rangle$ for all $z \in \mathbb{D}$. For $a \in \mathbb{D}$, define the operator U_a from $L^2_a(\mathbb{D})$ into itself as $U_af = (fo\phi_a)k_a$. The operator U_a is a bounded linear operator and $U_ak_z = k_{\phi_a(z)}$. Further $U_a^* = U_a$ and $U_a^2 = I$. Hence $\langle U_aTU_ak_z, k_z\rangle = \langle Sk_z, k_z\rangle$ for all $z \in \mathbb{D}$. Thus $U_aTU_a = S$. It also follows that $S^*U_a = U_aT^*$ and $U_aS = TU_a$. Since U_a is unitary, the operators S and T are unitarily equivalent. So T is dominant and S^* is p-hyponormal. Thus S and T are normal. As U_a is invertible, it follows that $S = T$. □

Theorem 3.2. *Let $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$. Assume that T^* is a p-hyponormal operator, $0 < p < 1$ and S is an isometry in $\mathcal{L}(L^2_a(\mathbb{D}))$. If $\widetilde{T}(\phi_a(z)) = \widetilde{S}(z)$ for all $z \in \mathbb{D}$ and for some $a \in \mathbb{D}$, then T is unitary.*

Proof. If for some $a \in \mathbb{D}$, $\widetilde{T}(\phi_a(z)) = \widetilde{S}(z)$ for all $z \in \mathbb{D}$, then it follows that $\langle Tk_{\phi_a(z)}, k_{\phi_a(z)}\rangle = \langle Sk_z, k_z\rangle$ for all $z \in \mathbb{D}$. That is, $U_aTU_a = S$ for some $a \in \mathbb{D}$. Thus $U_aT = SU_a$ and $S^*U_a = U_aT^*$. It follows therefore that $U_a = S^*U_aT = U_aT^*T$. Hence $U_a(I - T^*T) = 0$. This implies $T^*T = I$ and T is an isometry. Thus T and T^* are

p-hyponormal. Hence T is a normal operator which is also an isometry. Thus T is unitary. □

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