

NON-ARCHIMEDEAN UNITARY OPERATORS

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ABSTRACT. We describe a subclass of the class of normal operators on Banach spaces over non-Archimedean fields (A. N. Kochubei, J. Math. Phys. 51 (2010), article 023526) consisting of operators whose properties resemble those of unitary operators. In particular, an analog of Stone’s theorem about one-parameter groups of unitary operators is proved.

1. INTRODUCTION

1.1. In a previous paper [4], we found a class of non-Archimedean normal operators, bounded linear operators on Banach spaces over non-Archimedean fields possessing orthogonal, in the non-Archimedean sense, spectral decompositions. It is a natural problem now to find out what operators in the non-Archimedean setting should be seen as unitary ones. Classically, the correspondence between selfadjoint and unitary operators extends, via the well-known functional calculus, the correspondence $\lambda \mapsto e^{i\lambda}$ between real numbers and complex numbers from the unit circle.

There is no direct analog of the function $e^{i\lambda}$ in the non-Archimedean case. However we will see that its natural counterpart in the context of non-Archimedean operator theory is the function $\lambda \mapsto z^\lambda$ where λ runs the ring \mathbb{Z}_p of p -adic integers, z belongs to the group of principal units of a non-Archimedean field K (in another language, z is a positive element of K [8]). The image of this function also belongs to the group of principal units. This prompts to define a non-Archimedean unitary operator as an operator of the form $I + V$ where I is the unit operator, V is a normal operator in the sense of [4], $\|V\| < 1$. The normality assumption is essential – otherwise $I + V$ can be non-diagonalizable together with V . This shows (see also [11]) that the isometricity is not a substitute of unitarity in the non-Archimedean case. In fact we will use a refined version of the above definition; see Section 2.

In classical operator theory, the main result about unitary operators is Stone’s theorem about the representation of a one-parameter unitary group in the form $t \mapsto e^{itA}$ where $t \in \mathbb{R}$, A is a selfadjoint operator. We find its non-Archimedean analog – a one-parameter group parametrized by the group of principal units of \mathbb{Q}_p has the form s^A where A belongs to the class of normal operators in the sense of [4], $\|A\| \leq 1$. This result can be reformulated from the setting with the parameter from the group of principal units to the case of the parameter from \mathbb{Z}_p .

1.2. Let us recall principal notions and results from [4]. We will not explain the basic notions of non-Archimedean analysis; see [5, 6, 7, 8].

Let A be a bounded linear operator on a Banach space \mathcal{B} over a complete non-Archimedean valued field K with a nontrivial valuation; $|\cdot|$ will denote the absolute

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value in K . Denote by \mathcal{L}_A the commutative Banach algebra generated by the operators A and I . \mathcal{L}_A is a closure of the algebra $K[A]$ of polynomials in A , with respect to the norm of operators; thus \mathcal{L}_A is a Banach subalgebra of the algebra $L(\mathcal{B})$ of all bounded linear operators. Elements $\lambda \in K$ are identified with the operators λI .

The spectrum $\mathcal{M}(\mathcal{L}_A)$ is defined (see [2]) as the set of all bounded multiplicative seminorms on \mathcal{L}_A . In a natural topology, it is a nonempty Hausdorff compact topological space. If the algebra \mathcal{L}_A is uniform, that is $\|T^2\| = \|T\|^2$ for any $T \in \mathcal{L}_A$, and all its characters take their values in K , then [2] the space $\mathcal{M}(\mathcal{L}_A)$ is totally disconnected and \mathcal{L}_A is isomorphic to the algebra $C(\mathcal{M}(\mathcal{L}_A), K)$ of continuous functions on $\mathcal{M}(\mathcal{L}_A)$ with values from K . In this case, under the above isomorphism, the characteristic functions η_Λ of nonempty open-closed subsets $\Lambda \subset \mathcal{M}(\mathcal{L}_A)$ correspond to idempotent operators $E(\Lambda) \in \mathcal{L}_A$, $\|E(\Lambda)\| = 1$. These operators form a finitely additive norm-bounded projection-valued measure on the algebra of open-closed sets, with the non-Archimedean orthogonality property

$$\|f\| = \sup_{\Lambda} \|E(\Lambda)f\|, \quad f \in \mathcal{B}.$$

An operator A , for which the above picture takes place, is called *normal*. We will call a normal operator *strongly normal*, if its spectrum $\sigma(A)$ is a nonempty totally disconnected compact subset of K , and $\mathcal{M}(\mathcal{L}_A) = \sigma(A)$. A strongly normal operator admits the spectral decomposition

$$A = \int_{\sigma(A)} \lambda E(d\lambda).$$

More generally, we get a functional calculus assigning to any K -valued continuous function φ the operator

$$\varphi(A) = \int_{\sigma(A)} \varphi(\lambda) E(d\lambda),$$

such that

$$\|\varphi(A)\| \leq \sup_{\lambda \in \sigma(A)} |\varphi(\lambda)|.$$

Some sufficient conditions of strong normality were found in [4]. Let $\dim \mathcal{B} < \infty$, and $\mathcal{B} = K^n$, with the norm $\|(x_1, \dots, x_n)\| = \max_{1 \leq i \leq n} |x_i|$. An operator A is represented, with respect to the standard basis in K^n , by a matrix $(a_{ij})_{i,j=1}^n$. Its operator norm coincides with $\|A\| = \max_{i,j} |a_{ij}|$ (see [10]). It is sufficient to consider the case where $\|A\| = 1$.

Let \widehat{K} be the residue field of the field K . Together with the operator A , we consider its reduction, the operator \mathfrak{A} on the \widehat{K} -vector space $\widehat{\mathcal{B}} = \widehat{K}^n$ corresponding to the matrix (\widehat{a}_{ij}) where \widehat{a}_{ij} is the image of a_{ij} under the canonical mapping $O \rightarrow \widehat{K}$ (O is the ring of integers of the field K). An operator A is called nondegenerate, if $\mathfrak{A} \neq \nu I$ for any $\nu \in \widehat{K}$.

It was proved in [4] that A is strongly normal, if it is nondegenerate, all its eigenvalues belong to K , and its reduction \mathfrak{A} is diagonalizable. These conditions are satisfied, for example, if \mathfrak{A} has n different eigenvalues from \widehat{K} .

In the infinite-dimensional situation, a similar result holds [4] (with the representation of operators by infinite matrices), if we assume in addition that K is algebraically closed, \mathcal{B} is the space of sequences tending to zero, A is a bounded operator with a compact spectrum, and the resolvent of A belongs, in a weak sense, to the space of Krasner analytic functions outside the spectrum. For example, if a compact operator (that is a norm limit of a sequence of finite rank operators) is such that its reduction is diagonalizable, then it is strongly normal.

Note that for a strongly normal operator A and any continuous K -valued function φ on $\sigma(A)$, the operator $B = \varphi(A)$ is strongly normal. Indeed, considering the functional model of the algebra \mathcal{L}_A we see that the spectrum of the operator B coincides with

the set $f(\sigma(A))$. The Banach algebra \mathcal{L}_B is a subalgebra of \mathcal{L}_A , and its functional model coincides with the closure in $C(\sigma(A), K)$ of the set of functions $\pi \circ f$ where π is an arbitrary polynomial. The convergence of the sequence $\pi_n \circ f$ in $C(\sigma(A), K)$ is equivalent to the convergence of the sequence of polynomials π_n in the space $C(\sigma(B), K)$. By Kaplansky's theorem (see Theorem 43.3 in [8]), \mathcal{L}_B is isometrically isomorphic to $C(\sigma(B), K)$.

2. UNITARY OPERATORS

2.1. An operator U on a Banach space \mathcal{B} over a complete non-Archimedean valued field K with a nontrivial valuation will be called *unitary*, if $U = I + V$ where $\|V\| < 1$ and V is strongly normal. A unitary operator admits a spectral decomposition

$$U = \int_{\sigma(V)} (1 + \lambda) E_V(d\lambda) = \int_{\sigma(U)} \mu E_U(d\mu).$$

Here E_V is the spectral measure of the operator V , the mapping $\varphi(\lambda) = 1 + \lambda$ transforms the spectrum of V into that of U , $E_U(M) = E_V(\varphi^{-1}(M))$ for any open-closed subset of $\sigma(U)$.

Below we assume that the field K is an extension of the field \mathbb{Q}_p of p -adic numbers, and the absolute value $|\cdot|$ is an extension of the p -adic absolute value.

Denote by $\mathfrak{U}_1(K)$ the group of principal units of the field K , that is $\mathfrak{U}_1(K) = \{1 + \lambda : \lambda \in K, |\lambda| < 1\}$. We will consider *one-parameter groups* $U(s)$, $s \in \mathfrak{U}_1(K)$, of unitary operators, that is families of unitary operators, continuous with respect to the norm of operators, such that

$$(1) \quad U(s_1 s_2) = U(s_1) U(s_2), \quad s_1, s_2 \in \mathfrak{U}_1(K).$$

A one-parameter group of unitary operators can be constructed as follows. Let A be a strongly normal operator, $\|A\| \leq 1$, $\sigma(A) \subseteq \mathbb{Z}_p$. Consider the function $f_s(\lambda) = (1 + z)^\lambda$ where $s = 1 + z$, $z \in K$, $|z| < 1$, $\lambda \in \mathbb{Z}_p$. This function can be defined by its Mahler expansion [6, 8]:

$$f_s(\lambda) = \sum_{n=0}^{\infty} z^n P_n(\lambda),$$

where

$$P_n(\lambda) = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}, \quad n \geq 1; \quad P_0(\lambda) \equiv 1.$$

An equivalent definition [6, 8] can be made in terms of the approximation of a p -adic integer λ by a sequence of nonnegative integers, for which the function is defined in the straightforward way.

Set

$$(2) \quad U(s) = (1 + z)^A = \int_{\sigma(A)} (1 + z)^\lambda E_A(d\lambda), \quad s = 1 + z \in \mathfrak{U}_1(K).$$

Due to the non-Archimedean orthonormality of the Mahler basis, we have

$$(1 + z)^\lambda = 1 + v_z(\lambda), \quad v_z(\lambda) = \sum_{n=1}^{\infty} z^n P_n(\lambda),$$

$$\sup_{\lambda \in \mathbb{Z}_p} \left| \sum_{n=1}^{\infty} z^n P_n(\lambda) \right| = |s| < 1,$$

so that $U(s)$ is a unitary operator, that is $U(s) = I + V(s)$ where

$$V(s) = \int_{\sigma(A)} v_z(\lambda) E_A(d\lambda) = \int_{\sigma(V(s))} \mu E_{V(s)}(d\mu), \quad E_{V(s)}(M) = E_A(v_z^{-1}(M)).$$

It follows from the approximative description of the function f_s that $U(s)$ possesses the required group property. Next, let $s_1 = 1 + z_1$, $s_2 = 1 + z_2$, $|z_1| < 1$, $|z_2| < 1$. Using (2) we find that

$$U(s_1) - U(s_2) = \int_{\sigma(A)} \left[\sum_{n=1}^{\infty} (z_1^n - z_2^n) P_n(\lambda) \right] E_A(d\lambda),$$

so that

$$\|U(s_1) - U(s_2)\| \leq \sup_{\lambda \in \mathbb{Z}_p} \left| \sum_{n=1}^{\infty} (z_1^n - z_2^n) P_n(\lambda) \right| \leq \sup_{n \geq 1} |z_1^n - z_2^n| = |s_1 - s_2|.$$

Therefore the function $s \mapsto U(s)$ is continuous with respect to the operator norm.

2.2. In particular, the above construction makes sense for $s \in \mathfrak{U}_1(\mathbb{Q}_p)$, and the formula (2) defines a norm-continuous one-parameter group of unitary operators $\mathfrak{U}_1(\mathbb{Q}_p) \ni s \mapsto U(s)$. The next result is a converse statement, an analog of Stone's theorem.

Theorem. *Let $U(s)$, $s \in \mathfrak{U}_1(\mathbb{Q}_p)$, $p \neq 2$, be a norm continuous one-parameter group of unitary operators, such that the spectrum of the strongly normal operator $U(1+p) - I$ is contained in $p\mathbb{Z}_p$. Then there exist such a strongly normal operator A , $\sigma(A) \subseteq \mathbb{Z}_p$, that $U(s) = s^A$, $s \in \mathfrak{U}_1(\mathbb{Q}_p)$.*

Proof. Each element $s \in \mathfrak{U}_1(\mathbb{Q}_p)$ can be represented, in a unique way, as

$$(3) \quad s = (1+p)^\zeta, \quad \zeta \in \mathbb{Z}_p.$$

Indeed, set $\zeta = \frac{\log s}{\log(1+p)}$ (see [6, 8] regarding properties of the p -adic logarithm).

We have $\zeta \in \mathbb{Q}_p$, $|\log s| \leq p^{-1}$, $|\log(1+p)| = |p| = p^{-1}$, so that $\zeta \in \mathbb{Z}_p$. On the other hand, $\exp(\zeta \log(1+p)) = (1+p)^\zeta$ ([8], Theorem 47.10), which implies (3).

Let us write the canonical representation

$$\zeta = \zeta_0 + \zeta_1 p + \zeta_2 p^2 + \dots, \quad \zeta_j \in \{0, 1, \dots, p-1\}.$$

The series $(1+p)^\zeta = \sum_{n=0}^{\infty} p^n P_n(\zeta)$ converges uniformly with respect to $\zeta \in \mathbb{Z}_p$, so that the function $\zeta \mapsto (1+p)^\zeta$ is continuous. Due to the norm continuity of U ,

$$(4) \quad U(s) = \lim_{n \rightarrow \infty} [U(1+p)]^{\zeta_0 + \zeta_1 p + \dots + \zeta_n p^n}.$$

Denote

$$a_n(\lambda) = (1+p)^{(\zeta_0 + \zeta_1 p + \dots + \zeta_n p^n)\lambda}, \quad \lambda \in \mathbb{Z}_p.$$

Let us prove that

$$(5) \quad a_n(\lambda) \longrightarrow (1+p)^{\zeta\lambda}, \quad \text{as } n \rightarrow \infty,$$

uniformly with respect to λ .

Indeed, we have the estimate

$$\begin{aligned} |a_n(\lambda) - (1+p)^{\zeta\lambda}| &= \left| (1+p)^{(\zeta_{n+1} p^{n+1} + \zeta_{n+2} p^{n+2} + \dots)\lambda} - 1 \right| \\ &= \left| \sum_{k=1}^{\infty} p^k P_k((\zeta_{n+1} p^{n+1} + \zeta_{n+2} p^{n+2} + \dots)\lambda) \right| \\ &\leq \sup_{k \geq 1} p^{-k} |P_k((\zeta_{n+1} p^{n+1} + \zeta_{n+2} p^{n+2} + \dots)\lambda)|, \end{aligned}$$

where

$$|P_k((\zeta_{n+1} p^{n+1} + \zeta_{n+2} p^{n+2} + \dots)\lambda)| \leq p^{-n-1} |k!|^{-1} \leq p^{-n-1 + \frac{k-1}{p-1}},$$

so that, uniformly with respect to $\lambda \in \mathbb{Z}_p$,

$$|a_n(\lambda) - (1 + p)^{\zeta\lambda}| \leq p^{-n-1} \sup_{k \geq 1} p^{-k + \frac{k-1}{p-1}} \rightarrow 0,$$

as $n \rightarrow \infty$.

Now we return to the expression (4). By our assumption, $U(1 + p) = I + V$ where V is a strongly normal operator, $\sigma(V) \subseteq p\mathbb{Z}_p$. We have

$$U(1 + p) = \int_{\sigma(V)} (1 + \lambda)E_V(d\lambda).$$

The Banach algebra \mathcal{L}_A generated by the operator V contains the strongly normal operator

$$A = \frac{1}{\log(1 + p)} \log(I + V) = \frac{1}{\log(1 + p)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} V^k = \int_{\sigma(V)} \frac{\log(1 + \lambda)}{\log(1 + p)} E_V(d\lambda).$$

Obviously, $\sigma(A) \subseteq \mathbb{Z}_p$ and

$$(1 + p)^{\frac{\log(1+\lambda)}{\log(1+p)}} = 1 + \lambda,$$

so that $U(1 + p) = (1 + p)^A$, and it follows from (4) that

$$U(s) = \lim_{n \rightarrow \infty} [(1 + p)^A]^{\zeta_0 + \zeta_1 p + \dots + \zeta_n p^n}.$$

Switching to the functional model and using (5) and (3) we obtain the required formula for the operators $U(s)$. □

Note that the condition regarding the operator $U(1 + p) - I$ is satisfied automatically, if $U(1 + p) = I + V$, $\|V\| < 1$, and $K = \mathbb{Q}_p$.

2.3. Let $W(z)$, $z \in \mathbb{Z}_p$, $p \neq 2$, be a norm continuous unitary representation of the additive group \mathbb{Z}_p , and the spectrum of the operator $W(p^{-1} \log(1 + p)) - I$ lies in $p\mathbb{Z}_p$. Denote $s = e^{pz}$, $U(s) = W(z)$. Then $s \in \mathfrak{U}_1(\mathbb{Q}_p)$, and $s \mapsto U(s)$ is a one-parameter group satisfying the conditions of the above Theorem. We obtain the expression

$$W(z) = e^{pzA}, \quad z \in \mathbb{Z}_p,$$

where A is a strongly normal operator, $\sigma(A) \subseteq \mathbb{Z}_p$.

3. EXAMPLE. GALOIS REPRESENTATIONS

In this section we follow [1, 3, 9].

Let K be a finite extension of \mathbb{Q}_p , and $\varepsilon^{(n)} \in \bar{K}$ (\bar{K} is an algebraic closure of K) is a sequence of primitive roots of unity of orders p^n , such that

$$\varepsilon^{(0)} = 1, \quad \varepsilon^{(1)} \neq 1, \quad (\varepsilon^{(n+1)})^p = \varepsilon^{(n)}, \quad n = 0, 1, \dots$$

Denote $K_n = K(\varepsilon^{(n)})$, $K_\infty = \bigcup_{n=0}^{\infty} K_n$, $G_K = \text{Gal}(\bar{K}/K)$. Let μ_{p^n} be the set of roots of unity of order p^n ; thus $\varepsilon^{(n)} \in \mu_{p^n}$, $n \geq 0$. The cyclotomic character $\chi: G_K \rightarrow \mathbb{Z}_p^* = \{x \in \mathbb{Z}_p : |x| = 1\}$ is defined via the equality

$$\sigma(\zeta) = \zeta^{\chi(\sigma)}, \quad \text{for all } \sigma \in G_K, \quad \zeta \in \mu_{p^\infty} = \bigcup_{n=0}^{\infty} \mu_{p^n}.$$

χ is continuous with respect to the standard topology of G_K as a profinite group.

The kernel of the cyclotomic character coincides with $H_K = \text{Gal}(\bar{K}/K_\infty)$. Therefore χ identifies $\Gamma_K = \text{Gal}(K_\infty/K) = G_K/H_K$ with an open subgroup of the multiplicative group \mathbb{Z}_p^* .

By definition, a *p-adic representation* V of the group G_K is a finite-dimensional vector space over \mathbb{Q}_p with a continuous linear action of G_K .

Let \widehat{K}_∞ be the p -adic completion of K_∞ . Let us consider the action of Γ_K on the \widehat{K}_∞ -vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ of elements from $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ fixed under the action of H_K . If $d = \dim_{\mathbb{Q}_p} V$, then $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ is a \widehat{K}_∞ -vector space of dimension d . The group Γ_K acts on the union $\mathbb{D}_{\text{Sen}}(V)$ of finite-dimensional subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ invariant with respect to Γ_K , and $\dim_{K_\infty} \mathbb{D}_{\text{Sen}}(V) = d$.

By Sen's theorem [9], there is a unique K_∞ -linear operator Θ_V on $\mathbb{D}_{\text{Sen}}(V)$, such that for any $\omega \in \mathbb{D}_{\text{Sen}}(V)$, there exists such an open subgroup $\Gamma_\omega \subset \Gamma_K$ that

$$(6) \quad \sigma(\omega) = [\exp(\Theta_V \log \chi(\sigma))] \omega$$

for all $\sigma \in \Gamma_\omega$.

A representation V is called a *Hodge-Tate representation* if, for a certain basis $e_1, \dots, e_d \in \mathbb{D}_{\text{Sen}}(V)$, the operator Θ_V is diagonal, with eigenvalues from \mathbb{Z} . In this case, we can introduce a norm in $\mathbb{D}_{\text{Sen}}(V)$ setting

$$\|x_1 e_1 + \dots + x_d e_d\| = \max(|x_1|, \dots, |x_d|), \quad x_1, \dots, x_d \in K_\infty.$$

Then Θ_V is obviously strongly normal, $\|\Theta_V\| \leq 1$.

Taking into account the continuity of χ we can choose a so small open subgroup $\Lambda \subset \Gamma_K$ that $|\chi(\sigma) - 1| \leq p^{-1}$ for all $\sigma \in \Lambda$. For every $\sigma \in \Lambda$, the right-hand side of (6) defines a unitary operator on $\mathbb{D}_{\text{Sen}}(V)$.

REFERENCES

1. L. Berger, *An introduction to the theory of p -adic representations*, A. Adolphson et al. (eds), Geometric Aspects of Dwork Theory, de Gruyter, Berlin, 2004, pp. 255–292.
2. V. G. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Amer. Math. Soc., Providence, RI, 1990.
3. J.-M. Fontaine, *Arithmétique des représentations galoisiennes p -adiques*, Astérisque **295** (2004), 1–115.
4. A. N. Kochubei, *Non-Archimedean normal operators*, J. Math. Phys. **51** (2010), article 023526.
5. C. Perez-García and W. H. Schikhof, *Locally Convex Spaces over Non-Archimedean Valued Fields*, Cambridge University Press, Cambridge, 2010.
6. A. M. Robert, *A Course in p -Adic Analysis*, Springer, New York, 2000.
7. A. van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, New York, 1978.
8. W. H. Schikhof, *Ultrametric Calculus*, Cambridge University Press, Cambridge, 1984.
9. S. Sen, *Continuous cohomology and p -adic Galois representations*, Invent. Math. **62** (1980), 89–116.
10. J.-P. Serre, *Endomorphismes complètement continus des espaces de Banach p -adiques*, Publ. Math. IHES **12** (1962), 69–85.
11. N. Shilkret, *Orthogonal transformations in non-Archimedean spaces*, Arch. Math. **23** (1972), 285–291.

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