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HARDY TYPE SPACES ON INFINITE DIMENSIONAL GROUP ORBITS

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Abstract. In Hilbert Hardy spaces of complex analytic functions with infinitely many variables, defined on unitary orbits of locally compact second countable group, the Cauchy type integral formulas are established. Existence of radial boundary values is proved. Results are illustrated for a reduced Heisenberg group.

1. Motivation and main results

The classical theory of the Hardy space $\mathcal{H}^2_\chi(\mathsf{B})$ of complex analytic functions on the 1-dimensional complex ball B, $\Big\{f\colon \sup_{r\in[0,1)}\big(\int_{\mathbb{T}}|f(re^{\mathrm{i}\vartheta})|^2 d\chi(e^{\mathrm{i}\vartheta})\big)^{1/2}<\infty\Big\}$, essentially uses invariant properties of the Haar measure χ on the cyclic group $\mathbb{T}=\big\{e^{\mathrm{i}\vartheta}\colon \vartheta\in[0,2\pi)\big\}$. The $\mathbb{T}$-invariancy provides that $\mathcal{H}^2_\chi(\mathsf{B})$ is unitary equivalent to the Hermitian dual of the ℓ^2-space of Taylor coefficients $(d_0^n f/n!)$. Moreover, any function $f\in\mathcal{H}^2_\chi(\mathsf{B})$ can be uniquely defined by the integral Cauchy formula through its radial boundary values on $\mathbb{T}$. There is a natural question: is it possible in this theory to replace $\mathbb{T}$ with a general locally compact group?

The Hardy type space $\mathcal{H}^2_\chi(\mathsf{B})$ with a Haar measure χ on an abstract locally compact group $\mathcal{G}$ were considered in [6], where some of its properties were described. In the present work we analyze a more general case, when a $\mathcal{G}$-invariant measure χ is defined on a unitary orbit G of a locally compact second countable group $\mathcal{G}$ acting in an infinite dimensional Hilbert space E. We establish the Cauchy type formula

$$\mathfrak{C}[f](\xi)=\int_{\mathsf{G}}\mathfrak{C}(\xi,\zeta)f(\zeta)\,d\chi(\zeta),\quad \xi\in\mathsf{B}, \tag{1.1}$$

which for every function $f\in\mathcal{H}^2_\chi$ produces its unique analytic extensions $\mathfrak{C}[f]$ on the open unit ball B in E, where $\mathcal{H}^2_\chi$ denotes the closure in the space L^2_χ of all Hilbert-Schmidt polynomials over E. We also describe the space of Taylor coefficients for $\mathcal{H}^2_\chi$. Moreover, in the case of Hardy spaces, the boundary values problem, which is defined on orbits, for analytic functions becomes substantial. Namely, we establish that the radial boundary values of $\mathfrak{C}[f]$ on the orbit G are equal to f for every function $f\in\mathcal{H}^2_\chi$. As an example, we consider a reduced Heisenberg group.

Note that integral representations of Hardy spaces $\mathcal{H}^p$ $(p\geq 1)$ with infinitely many variables were an object of research in [5, 7]. The Hardy spaces $\mathcal{H}^\infty$ were investigated in [1] and in many other publications.

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2. Preliminaries

Let E stand for a complex separable Hilbert space and let $\mathcal{G}$ stand for a locally compact second countable group. Suppose that there exists a unitary representation

$$U\colon \mathcal{G} \ni x \longmapsto U_x \in \mathcal{L}(\mathsf{E}),$$

which is weakly continuous. Hereafter $\mathcal{L}(\cdot)$ denotes the algebra of all bounded linear operators.

Fix an element $\hbar \in \mathsf{E}$ with $\|\hbar\|_{\mathsf{E}} = 1$ and consider its orbit

$$\mathsf{G} := \big\{U_x\hbar = \zeta \in \mathsf{E}\colon x \in \mathcal{G}\big\},$$

which as a topological space we identify with the factor-space $\mathcal{G}/\mathcal{G}_\hbar$, where $\mathcal{G}_\hbar := \big\{v \in \mathcal{G}\colon U_v\hbar = \hbar\big\}$. The closed unit ball in E endowed with the weak topology, we will denote by K. The weak continuity of U implies that the embedding $\mathsf{G} \hookrightarrow \mathsf{K}$ is continuous. Further we denote by $C(\mathsf{K})$ the uniform algebra of continuous complex functions on K.

Recall that a measure χ on the orbit G is $\mathcal{G}$-invariant if for any $x \in \mathcal{G}$ its shift $\chi \circ U_{x^{-1}}$ is equal to χ, i.e., if

$$\int_{\mathsf{G}} f(\zeta)\, d\chi(U_{x^{-1}}\zeta) = \int_{\mathsf{G}} (f \circ U_x)(\zeta)\, d\chi(\zeta) = \int_{\mathsf{G}} f(\zeta)\, d\chi(\zeta) \tag{2.1}$$

for all $x \in \mathcal{G}$ and χ-integrable complex function f on G. As is well known (see e.g., [9]), for any locally compact second countable group $\mathcal{G}$ an invariant measure χ on an orbit G exists and the equality

$$\int_{\mathsf{G}} d\chi(U_x\hbar) \int_{\mathcal{G}_\hbar} \varphi(xv)\, dv = \int_{\mathcal{G}} \varphi(x)\, dx \tag{2.2}$$

uniquely connects it with a Haar measure dx on $\mathcal{G}$. Here φ is any integrable complex function on $\mathcal{G}$ and dv denotes a Haar measure on $\mathcal{G}_\hbar$. Clearly, the choice of a $\mathcal{G}$-invariant measure χ on G depends on the element $\hbar \in \mathsf{S}$.

In what follows we suppose that an element $\hbar \in \mathsf{S}$ and the corresponding $\mathcal{G}$-invariant measure χ on its orbit G are fixed, and that the representation codomain $U_{\mathcal{G}}$ of a group $\mathcal{G}$ contains the complex cyclic subgroup $\mathbb{T}$. Let L^2_χ stand for the Hilbert space of all quadratically χ-integrable complex functions on the orbit G.

First we give an auxiliary result which at once follows from our assumptions about the group.

Proposition 2.1. *If χ is a $\mathcal{G}$-invariant measure on G then the decomposition*

$$\int_{\mathsf{G}} f\, d\chi = \frac{1}{2\pi} \int_{\mathsf{G}} d\chi(\zeta) \int_0^{2\pi} f(e^{\mathrm{i}\vartheta}\zeta)\, d\vartheta \tag{2.3}$$

with χ-integrable complex function f on G holds and the condition $\int_{\mathsf{G}} \zeta\, d\chi(\zeta) = 0$ is valid.

Proof. By Fubini's theorem

$$\int_{\mathsf{G}} d\chi(\zeta) \int_0^{2\pi} f(e^{\mathrm{i}\vartheta}\zeta) d\vartheta = \int_0^{2\pi} d\vartheta \int_{\mathsf{G}} f(e^{\mathrm{i}\vartheta}\zeta) d\chi(\zeta)$$

for all f, since the integrand $f \circ e^{\mathrm{i}\vartheta}$ is integrable vector-valued function on $\vartheta \in [0, 2\pi)$. The internal integrals on the right side do not depend on $\vartheta \in [0, 2\pi)$ in view of the $\mathbb{T}$-invariance of χ. As $\int_0^{2\pi} d\vartheta = 2\pi$, we obtain (2.3). Using (2.3) for any linear continuous functional f we have

$$\int_{\mathsf{G}} f d\chi = \frac{1}{2\pi} \int_{\mathsf{G}} f d\chi \int_0^{2\pi} \exp(\mathrm{i}\vartheta) d\vartheta = 0$$

and the proposition is proved. □

3. Polynomial orthogonal systems on irreducible orbits

Let $\otimes_{\mathfrak{h}}^n \mathsf{E}$, $(n \in \mathbb{N})$ denote the complete nth tensor Hilbert power of E, and $\otimes_{\mathfrak{h}}^0 \mathsf{E} = \mathbb{C}$. If $\mathfrak{s}\colon \{1,\dots,n\} \longmapsto \{\mathfrak{s}(1),\dots,\mathfrak{s}(n)\}$ runs all n-elements permutations then the codomain of the corresponding orthogonal projector

$$\mathfrak{s}_n\colon \otimes_{\mathfrak{h}}^n \mathsf{E} \ni \xi_1 \otimes \ldots \otimes \xi_n \longmapsto \xi_1 \odot \ldots \odot \xi_n := \frac{1}{n!} \sum \xi_{\mathfrak{s}(1)} \otimes \ldots \otimes \xi_{\mathfrak{s}(n)},$$

which means the symmetric Hilbert nth tensor power of E, we denote by $\odot_{\mathfrak{h}}^n \mathsf{E}$. Thus, $\odot_{\mathfrak{h}}^n \mathsf{E} = \otimes_{\mathfrak{h}}^n \mathsf{E} \ominus \ker \mathfrak{s}_n$. Recall that the symmetric Fock space is defined as the Hilbert orthogonal sum $\mathsf{F} = \mathbb{C} \oplus \mathsf{E} \oplus \big(\odot_{\mathfrak{h}}^2 \mathsf{E}\big) \oplus \big(\odot_{\mathfrak{h}}^3 \mathsf{E}\big) \oplus \ldots$

We use $\mathsf{E}^* = \big\{\zeta^* := \langle \cdot \mid \zeta\rangle_{\mathsf{E}} \colon \zeta \in \mathsf{E}\big\}$ to denote the Hermitian dual space for E. The isometries $(\otimes_{\mathfrak{h}}^n \mathsf{E})^* = \otimes_{\mathfrak{h}}^n \mathsf{E}^*$ and $(\odot_{\mathfrak{h}}^n \mathsf{E})^* = \odot_{\mathfrak{h}}^n \mathsf{E}^*$ hold. For any element $\psi_n \in \odot_{\mathfrak{h}}^n \mathsf{E}$ uniquely assists the form $\psi_n^* := \langle \cdot \mid \psi_n\rangle_{\mathsf{F}}$ belonging to $\odot_{\mathfrak{h}}^n \mathsf{E}^*$, which further we will identify with the n-homogeneous Hilbert-Schmidt polynomial

$$\psi_n^*\colon \mathsf{E} \ni \xi \longmapsto \psi_n^*(\xi) := \big\langle \xi^{\otimes n} \mid \psi_n \big\rangle_{\mathsf{F}},$$

where is denoted

$$\xi^{\otimes n} := \xi \otimes \ldots \otimes \xi \in \odot_{\mathfrak{h}}^n \mathsf{E}, \quad \xi \in \mathsf{E}.$$

For each n-homogeneous polynomial ψ_n^* with $\psi_n \in \odot_{\mathfrak{h}}^n \mathsf{E}$ we assign the polynomial function

$$\big[\hbar_n(\psi_n)\big](\zeta) = \delta_\zeta(\psi_n^*)$$

of the variable $\zeta = U_x \hbar$ with $x \in \mathcal{G}$, generating on the orbit G by all $\mathcal{G}$-shifts of the point evaluation character

$$\delta_\hbar(\psi_n^*) = \psi_n^*(\hbar).$$

Theorem 3.1. *Let an element $\hbar \in \mathsf{S}$ be fixed in such way that the antilinear operators*

$$\hbar_n\colon \odot_{\mathfrak{h}}^n \mathsf{E} \ni \psi_n \longmapsto \hbar_n(\psi_n) \in L^2_\chi \quad (n \in \mathbb{N})$$

are well defined and have the bounded norm $\|\hbar_n\| = \|\hbar_n\|_{\mathcal{L}(\odot_{\mathfrak{h}}^n \mathsf{E},\, L^2_\chi)}$, and let

$$\mathsf{E}_{\mathfrak{h}}^n := \odot_{\mathfrak{h}}^n \mathsf{E} \ominus \ker \hbar_n, \quad \mathsf{F}_{\mathfrak{h}} := \mathbb{C} \oplus \mathsf{E}_{\mathfrak{h}}^1 \oplus \mathsf{E}_{\mathfrak{h}}^2 \oplus \mathsf{E}_{\mathfrak{h}}^3 \oplus \ldots$$

Then

(i) *the corresponding restricted mapping*

$$\widehat{\hbar}_n\colon \mathsf{E}_{\mathfrak{h}}^n \ni \psi_n \longmapsto \widehat{\psi}_n := \widehat{\hbar}_n(\psi_n) \in L^2_\chi, \quad \widehat{\hbar}_n := \frac{\hbar_n}{\|\hbar_n\|}$$

is an isometry between the subspace $\mathsf{E}_{\mathfrak{h}}^n$ and its image $\mathcal{H}_n^2 := \widehat{\hbar}_n(\mathsf{E}_{\mathfrak{h}}^n)$, so

$$\int_{\mathsf{G}} \widehat{\psi}_n \,\overline{\widehat{\omega}}_n \, d\chi = \big\langle \omega_n \bigm| \psi_n \big\rangle_{\mathsf{F}}, \quad \psi_n, \omega_n \in \mathsf{E}_{\mathfrak{h}}^n; \tag{3.1}$$

(ii) *the antilinear mapping*

$$\widehat{\hbar}\colon \mathsf{F}_{\mathfrak{h}} \ni \psi = \sum_{n \in \mathbb{Z}_+} \psi_n \longmapsto \widehat{\psi} := \sum_{n \in \mathbb{Z}_+} \widehat{\hbar}_n(\psi_n) \in \mathcal{H}^2_\chi, \quad \widehat{\hbar} := \big(\widehat{\hbar}_n\big), \tag{3.2}$$

where $\widehat{\psi}_0 = \bar{\psi}_0$ with $\psi_0 \in \mathbb{C}$, is an isometry between the subspace $\mathsf{F}_{\mathfrak{h}}$ and its image $\mathcal{H}^2_\chi := \widehat{\hbar}(\mathsf{F}_{\mathfrak{h}})$, so

$$\int_{\mathsf{G}} \widehat{\psi}\,\overline{\widehat{\omega}} \, d\chi = \big\langle \omega \mid \psi \big\rangle_{\mathsf{F}}, \quad \psi, \omega \in \mathsf{F}_{\mathfrak{h}}; \tag{3.3}$$

(iii) *the following orthogonal decomposition holds:*

$$\mathcal{H}^2_\chi = \mathbb{C} \oplus \mathcal{H}_1^2 \oplus \mathcal{H}_2^2 \oplus \mathcal{H}_3^2 \oplus \ldots$$

Proof. Due to boundedness of $\hbar_n$ the following integral

$$\int_{\mathsf{G}} \hbar_n(\psi_n)\overline{\hbar_n(\omega_n)}\, d\chi = \int_{\mathsf{G}} (\psi_n^* \circ U_x)(\hbar)\overline{(\omega_n^* \circ U_x)}(\hbar)\, d\chi(U_x\hbar)$$

is an Hermitian continuous form on the Hilbert space $\odot_{\mathfrak{h}}^n \mathsf{E}$, which is antilinear by $\psi_n \in \odot_{\mathfrak{h}}^n \mathsf{E}$ and linear by $\omega_n \in \odot_{\mathfrak{h}}^n \mathsf{E}$. Therefore, there exists a bounded positive linear operator $A_n \in \mathcal{L}\big(\odot_{\mathfrak{h}}^n \mathsf{E}\big)$ for which

$$\big\langle \omega_n \mid A_n\psi_n \big\rangle_{\mathsf{F}} = \int_{\mathsf{G}} \hbar_n(\psi_n)\overline{\hbar_n(\omega_n)}\, d\chi. \tag{3.4}$$

Similarly as in the proof of [6, Theorem 2.4] from the $\mathcal{G}$-invariance properties (2.1) it follows that A_n commutes with all diagonal unitary representations of the form $\big\{U_y^{\otimes n} \in \mathcal{L}\big(\odot_{\mathfrak{h}}^n \mathsf{E}\big) \colon y \in \mathcal{G}\big\}$, i.e., the equality

$$A_n \circ U_y^{\otimes n} = U_y^{\otimes n} \circ A_n, \quad y \in \mathcal{G}$$

holds, where $U_y^{\otimes n} = U_y \otimes \ldots \otimes U_y$ denotes the nth tensor power. In fact, we have

$$\big\langle \omega_n \mid (A_n \circ U_y^{\otimes n})\psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = \int_{\mathsf{G}} \big\langle (U_x\hbar)^{\otimes n} \mid U_y^{\otimes n}\psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \overline{\big\langle (U_x\hbar)^{\otimes n} \mid \omega_n \big\rangle}_{\otimes_{\mathfrak{h}}^n \mathsf{E}}\, d\chi(U_x\hbar).$$

The $\mathcal{G}$-invariancy of the measure χ on G implies that

$$\begin{aligned}
&\int_{\mathsf{G}} \big\langle (U_x\hbar)^{\otimes n} \mid U_y^{\otimes n}\psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \overline{\big\langle (U_x\hbar)^{\otimes n} \mid \omega_n \big\rangle}_{\otimes_{\mathfrak{h}}^n \mathsf{E}}\, d\chi(U_x\hbar) \\
&\quad = \int_{\mathsf{G}} \big\langle (U_{y^{-1}x}\hbar)^{\otimes n} \mid \psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \overline{\big\langle (U_{y^{-1}x}\hbar)^{\otimes n} \mid U_{y^{-1}}^{\otimes n}\omega_n \big\rangle}_{\otimes_{\mathfrak{h}}^n \mathsf{E}} d\chi(U_x\hbar) \\
&\quad = \int_{\mathsf{G}} \big\langle (U_x\hbar)^{\otimes n} \mid \psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} \overline{\big\langle (U_x\hbar)^{\otimes n} \mid U_{y^{-1}}^{\otimes n}\omega_n \big\rangle}_{\otimes_{\mathfrak{h}}^n \mathsf{E}} d\chi(U_x\hbar).
\end{aligned}$$

As a result, we obtain

$$\big\langle \omega_n \mid (A_n \circ U_y^{\otimes n})\psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = \big\langle U_{y^{-1}}^{\otimes n}\omega_n \mid A_n\psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}} = \big\langle \omega_n \mid (U_y^{\otimes n} \circ A_n)\psi_n \big\rangle_{\otimes_{\mathfrak{h}}^n \mathsf{E}}.$$

On the other hand, for any $n \in \mathbb{N}$ the set $\big\{(U_x\hbar)^{\otimes n} \colon x \in \mathcal{G}\big\}$ is total in the subspace $\mathsf{E}_{\mathfrak{h}}^n$ under its definition. Hence, the corresponding representations $U_y^{\otimes n}$ are irreducible over $\mathsf{E}_{\mathfrak{h}}^n$. Via the well-known property [4, Theorem 21.30] the restriction $A_n \,|_{\mathsf{E}_{\mathfrak{h}}^n}$ is proportional to the identity operator $\mathbf{1}_{\mathsf{E}_{\mathfrak{h}}^n}$ on $\mathsf{E}_{\mathfrak{h}}^n$, i.e.,

$$A_n \,|_{\mathsf{E}_{\mathfrak{h}}^n} = \aleph_n^{-2}\mathbf{1}_{\mathsf{E}_{\mathfrak{h}}^n}$$

for a constant $\aleph_n^2 \in \mathbb{C}$. Hence, we can rewrite (3.4) as follows

$$\big\langle \omega_n \mid \psi_n \big\rangle_{\mathsf{F}} = \aleph_n^2 \int_{\mathsf{G}} \hbar_n(\psi_n)\overline{\hbar_n(\omega_n)}\, d\chi, \quad \psi_n, \omega_n \in \mathsf{E}_{\mathfrak{h}}^n. \tag{3.5}$$

In particular, it follows that

$$\|\hbar_n\| = \sup_{\|\psi_n\|_{\odot_{\mathfrak{h}}^n \mathsf{E}}=1} \|\hbar_n(\psi_n)\|_{L^2_\chi} = \frac{1}{\aleph_n}.$$

Via Proposition 2.1 for any $\psi_n \in \mathsf{E}_{\mathfrak{h}}^n$ and $\omega_m \in \mathsf{E}_{\mathfrak{h}}^m$ we obtain

$$\int_{\mathsf{G}} \widehat{\psi}_n\, \overline{\widehat{\omega}}_m\, d\chi = \frac{1}{2\pi} \int_{\mathsf{G}} \widehat{\psi}_n\, \overline{\widehat{\omega}}_m\, d\chi \int_0^{2\pi} e^{\mathrm{i}(n-m)\vartheta}\, d\vartheta = \begin{cases} 0 & : \quad n \neq m \\ \langle \omega_n \mid \psi_n \rangle_{\mathsf{F}} & : \quad n = m. \end{cases}$$

Hence, $\widehat{\psi}_n \perp \widehat{\omega}_m$ in L^2_χ if $n \neq m$. Thus, the orthogonal decomposition (iii) holds. □

Remark 3.2. In [6] it was proved that in the case if χ is a Haar measure on the group $\mathcal{G}$ and U is its regular irreducible representation over L^2_χ then the assumptions of Theorem 3.1 are satisfied for any

$$\hbar \in L^\infty_\chi \cap \mathsf{S}.$$

4. Cauchy type formula and radial boundary values

In [6] it was proved that if $\mathcal{G}$ is the full unitary group of linear operators over the m-dimensional complex space $\mathbb{C}^m$ ($m \in \mathbb{N}$), endowed with the probability Haar measure, then for any $\hbar \in \mathbb{C}^m$ such that $\|\hbar\|_{\mathbb{C}^m} = 1$ the Cauchy kernel with the variables $\xi \in \mathbb{C}^m$, $\|\xi\|_{\mathbb{C}^m} < 1$ and $\zeta \in \mathbb{C}^m$, $\|\zeta\|_{\mathbb{C}^m} = 1$ has the form (see [8, 1.4.9])

$$\mathfrak{C}(\xi,\zeta) = \sum_{n\in\mathbb{Z}_+} \frac{(m-1+n)!}{(m-1)!n!} \langle \xi \mid \zeta \rangle^n_{\mathbb{C}^m} \quad \text{with} \quad \aleph_n^2 = \frac{(m-1+n)!}{(m-1)!n!},$$

where the condition $\lim_{n\to\infty} \sqrt[n]{\aleph_n^2} = 1$ is satisfied.

This fact justifies that the following kernel

$$\mathfrak{C}\,(\xi,\zeta) = \sum_{n\in\mathbb{Z}_+} \aleph_n^2 \, \langle \alpha\xi \mid \zeta\rangle^n_{\mathsf{E}}\,, \quad \aleph_n^2 = \frac{1}{\|\hbar_n\|^2} \tag{4.1}$$

with $\|\xi\|_{\mathsf{E}} < 1$ and $\|\zeta\|_{\mathsf{E}} \le 1$, for which there exists the limit

$$\lim_{n\to\infty} \sqrt[n]{\aleph_n^2} = 1/\alpha \quad \text{for some constant} \quad \alpha \ge 0, \tag{4.2}$$

we can mean the Cauchy type kernel in more general cases. Now we are going to consider this more carefully.

Recall (see e.g. [3]) that a function defined on an open ball in a normed space is Gâteaux analytic if its restrictions to all finite dimensional affine subsets are analytic. If a Gâteaux analytic function is, in addition, norm continuous then it is called analytic.

Put for simplicity $\mathsf{E} = \mathsf{E}^1_{\mathfrak{h}}$ and denote

$$\mathsf{B} := \{\xi \in \mathsf{E}\colon \|\xi\|_{\mathsf{E}} < 1\}, \quad \mathsf{S} := \{\xi \in \mathsf{E}\colon \|\xi\|_{\mathsf{E}} = 1\}.$$

Proposition 4.1. *If the condition* (4.2) *is satisfied then the kernel* $\mathfrak{C}\,(\xi,\zeta)$ *with* $\zeta \in \mathsf{K}$ *is an analytic* $C(\mathsf{K})$*-valued function by the variable* $\xi \in \mathsf{B}$*.*

Proof. Calculating a uniform norm by $\zeta \in \mathsf{K}$ of the power series (4.1), we obtain

$$\|\mathfrak{C}\,(\xi,\cdot)\|_{C(\mathsf{K})} \le \sum_{n\in\mathbb{Z}_+} \aleph_n^2 \, \|\alpha\xi\|^n_{\mathsf{E}} < \infty$$

for all $\xi \in \mathsf{B}$. Hence, $\mathfrak{C}\,(\xi,\cdot)$ is an analytic $C(\mathsf{K})$-valued function by $\xi \in \mathsf{B}$. □

Proposition 4.2. *Let the assumptions of Theorem 3.1 be satisfied. Then for any fixed* $r \in [0, 1/\alpha)$ *the integral operator*

$$\mathfrak{C}[f]\,(\xi) = \int_{\mathsf{G}} \mathfrak{C}\,(\xi,\zeta)\, f(\zeta)\, d\chi(\zeta), \quad f \in \mathcal{H}^2_\chi \tag{4.3}$$

with $\xi = r\lambda$, $(\lambda \in \mathsf{G})$ *belongs to the algebra* $\mathcal{L}(\mathcal{H}^2_\chi)$*. The function*

$$\mathfrak{C}[f]_r\colon \mathsf{G} \ni \lambda \longmapsto \mathfrak{C}[f]\,(r\lambda)$$

with $r \in [0, 1/\alpha)$ *belongs to* $\mathcal{H}^2_\chi$ *and*

$$\|f\|_{L^2_\chi} = \sup_{r\in[0,1/\alpha)} \Big(\int_{\mathsf{G}} |\mathfrak{C}[f]\,(r\lambda)|^2 \, d\chi(\lambda) \Big)^{1/2}.$$

Proof. Let $(\varphi_{j,n})_{j\in\mathbb{N}}$ be an orthonormal basis in the space $\mathsf{E}^n_{\mathfrak{h}}$ with $n \in \mathbb{N}$. Then the system $(\widehat{\varphi}_{j,n})_{j\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}^2_n$. Indeed, substituting in (3.5) $\omega_n = \varphi_{j,n}$ and $\psi_n = \varphi_{i,n}$ with $j \ne i$ we have

$$\int_{\mathsf{G}} \widehat{\varphi}_{j,n}\, \overline{\widehat{\varphi}}_{i,n}\, d\chi = \langle \varphi_{i,n} \mid \varphi_{j,n} \rangle_{\mathsf{F}} = 0,$$

i.e., $\widehat{\varphi}_{j,n} \perp \widehat{\varphi}_{i,n}$ in L^2_χ. So, the system $\big(\widehat{\varphi}_{j,n}\big)$ is orthonormal in the space L^2_χ. If $\xi^{\otimes n} = \sum_j \langle \xi^{\otimes n} \mid \varphi_{j,n}\rangle_{\mathsf{F}} \varphi_{j,n}$ denotes the Fourier expansions under $\big(\varphi_{j,n}\big)$ of an element $\xi \in \mathsf{E}$ then we have

$$\mathfrak{C}_n(\xi,\zeta) := \aleph_n^2 \big\langle \alpha^n \xi^{\otimes n} \mid \zeta^{\otimes n}\big\rangle_{\mathsf{F}} = (r\alpha)^n \sum_{j\in\mathbb{N}} \widehat{\varphi}_{j,n}(\lambda)\overline{\widehat{\varphi}}_{j,n}(\zeta),$$

i.e., $\mathfrak{C}_n(\xi,\zeta) = (r\alpha)^n \mathfrak{C}_n(\lambda,\zeta)$ with $\zeta = U_y\hbar$, $\lambda = U_x\hbar \in \mathsf{G}$ for all $x,y \in \mathcal{G}$ and $\xi = r\alpha\lambda$, $(r\alpha = \|\xi\|_{\mathsf{E}})$. So,

$$\mathfrak{C}\big(\xi,\zeta\big) = \sum_{n\in\mathbb{Z}_+} (r\alpha)^n \sum_{j\in\mathbb{N}} \widehat{\varphi}_{j,n}(\lambda)\overline{\widehat{\varphi}}_{j,n}(\zeta) = \sum_{n\in\mathbb{Z}_+} (r\alpha)^n \mathfrak{C}_n(\zeta,\lambda).$$

Theorem 3.1 implies that

$$\int_{\mathsf{G}} \widehat{\varphi}_{j,n}(\zeta)\mathfrak{C}_n(\xi,\zeta)\,d\chi(\zeta) = \widehat{\varphi}_{j,n}(\xi)\int_{\mathsf{G}} \widehat{\varphi}_{j,n}(\zeta)\,\overline{\widehat{\varphi}}_{j,n}(\zeta)\,d\chi(\zeta) = \widehat{\varphi}_{j,n}(\xi)$$

for all $\varphi_{j,n}$ and $\xi \in \mathsf{G}$. Since $\big(\widehat{\varphi}_{j,n}\big)$ is an orthonormal basis in $\mathcal{H}^2_n$, the integral operator with kernel $\mathfrak{C}_n$ produces the identity mapping on $\mathcal{H}^2_n$.

Let $f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2_\chi$ with $f_n \in \mathcal{H}^2_n$. Using that $f_n \perp \mathfrak{C}_m$ at $n \neq m$ in L^2_χ, we obtain

$$f(\xi) = \sum_{n\in\mathbb{Z}_+} \int_{\mathsf{G}} \mathfrak{C}_n(\xi,\zeta) f_n(\zeta)\,d\chi(\zeta) = \int_{\mathsf{G}} \mathfrak{C}(\xi,\zeta) f(\zeta)\,d\chi(\zeta)$$

for all $\xi \in \mathsf{G}$. It follows that the series $\mathfrak{C}[f](\xi) = \sum_{n\in\mathbb{Z}_+} \mathfrak{C}[f_n](\xi)$ with

$$\begin{aligned}\mathfrak{C}[f_n](\xi) &= \int_{\mathsf{G}} \mathfrak{C}_n(\xi,\zeta) f_n(\zeta)\,d\chi(\zeta) = \aleph_n^2 \int_{\mathsf{G}} \big\langle \alpha\xi \mid \zeta\big\rangle^n_{\mathsf{E}} f_n(\zeta)\,d\chi(\zeta)\\ &= (r\alpha)^n \int_{\mathsf{G}} \mathfrak{C}_n(\lambda,\zeta) f_n(\zeta)\,d\chi(\zeta) = (r\alpha)^n f_n(\lambda) = f_n(\xi)\end{aligned}$$

is convergent in $\mathcal{H}^2_\chi$ by the variable $\lambda \in \mathsf{G}$, uniformly by $r \in [0,\varepsilon]$ with $0 < \varepsilon < 1/\alpha$. Applying that $\mathfrak{C}_n \perp f_m$ and $f_n \perp f_m$ at $n \neq m$ in L^2_χ, we have

$$\begin{aligned}\big\|\mathfrak{C}_r[f]\big\|^2_{L^2_\chi} &= \int_{\mathsf{G}} \Big| \sum_{n\in\mathbb{Z}_+} (r\alpha)^n \int_{\mathsf{G}} \mathfrak{C}_n(\lambda,\zeta) f_n(\zeta)\,d\chi(\zeta)\Big|^2 d\chi(\lambda)\\ &= \int_{\mathsf{G}} \Big| \sum_{n\in\mathbb{Z}_+} (r\alpha)^n f_n(\lambda)\Big|^2 d\chi(\lambda) = \Big\| \sum (r\alpha)^n f_n \Big\|^2_{L^2_\chi} = \sum_{n\in\mathbb{Z}_+} (r\alpha)^{2n} \|f_n\|^2_{L^2_\chi}\end{aligned}$$

for any $r < 1/\alpha$. It follows that

$$\sup_{r\in[0,1/\alpha)} \sum (r\alpha)^{2n} \|f_n\|^2_{L^2_\chi} = \sum \|f_n\|^2_{L^2_\chi} = \|f\|^2_{L^2_\chi}.$$

Via the Cauchy-Schwarz inequality, we have

$$\big\|\mathfrak{C}_r[f]\big\|_{L^2_\chi} \le \frac{1}{(1-r^2\alpha^2)^{1/2}} \Big(\sum_{n\in\mathbb{Z}_+} \|f_n\|^2_{L^2_\chi}\Big)^{1/2} = \frac{\|f\|_{L^2_\chi}}{(1-r^2\alpha^2)^{1/2}}$$

for all $f \in \mathcal{H}^2_\chi$. Hence, the operator (4.3) belongs to $\mathcal{L}(\mathcal{H}^2_\chi)$. □

Theorem 4.3. *Let the assumptions of Theorem 3.1 and the condition* (4.2) *are simultaneously satisfied. Then for any* $f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2_\chi$ *with* $f_n \in \mathcal{H}^2_n$ *the integral transform* (1.1) *with the Cauchy type kernel* (4.1) *is a unique analytic extension* $\mathfrak{C}[f]$ *of the function* f *on the open ball* B *with the Taylor coefficients at the origin*

$$\frac{d_0^n \mathfrak{C}[f](\xi)}{n!} = \aleph_n^2 \int_{\mathsf{G}} \langle \alpha\xi \mid \zeta\rangle^n_{\mathsf{E}} f_n(\zeta)\,d\chi(\zeta), \quad \xi \in \mathsf{E}. \tag{4.4}$$

For each analytic function $\mathfrak{C}[f]$ its radial boundary values on the orbit G *are equal to* f *in the following sense*

$$\lim_{r\to 1/\alpha}\int_{\mathsf{G}}\left|\mathfrak{C}_r[f]-f\right|^2 d\chi=0,\quad r\in[0,1/\alpha)\,. \tag{4.5}$$

Proof. Via Proposition 4.1 $\mathfrak{C}(\xi,\cdot)$ is an analytic $C(\mathsf{K})$-valued function by $\xi\in\mathsf{B}$. Hence, the function $\mathfrak{C}[f]$ determined by (1.1) is also analytic by $\xi\in\mathsf{B}$ in view of [3, 3.1.2]. Differentiating at the origin, we obtain

$$\frac{d_0^n\mathfrak{C}[f](\xi)}{n!}=\aleph_n^2\int_{\mathsf{G}}\langle\alpha\xi\mid\zeta\rangle_{\mathsf{E}}^n\,f_n(\zeta)\,d\chi(\zeta)=\mathfrak{C}[f_n](\xi),\quad \xi\in\mathsf{B}.$$

By the Cauchy-Schwarz inequality,

$$|\mathfrak{C}[f_n](\xi)|\le\aleph_n^2\int_{\mathsf{G}}\left|\langle\alpha\xi\mid\zeta\rangle_{\mathsf{E}}^n\,f_n(\zeta)\right|d\chi(\zeta)\le\aleph_n^2\|\alpha\xi\|_{\mathsf{E}}^n\|f_n\|_{L^2_\chi}$$

for all $\xi\in\mathsf{E}$. Hence, any $\mathfrak{C}[f_n]$ is a n-homogeneous polynomial on E, which takes the form (4.4). As it is well known [3, 2.4.2], continuous Taylor coefficients uniquely define the analytic function $\mathfrak{C}[f]$ on B. So, the uniqueness of the analytic extension $\mathfrak{C}[f]$ is proved. Finally, using the orthogonal property we have

$$\int_{\mathsf{G}}\left|\mathfrak{C}_r[f]-f\right|^2\,d\chi=\sum_{n\in\mathbb{Z}_+}\left(r^{2n}\alpha^{2n}-1\right)\|f_n\|^2_{L^2_\chi}\longrightarrow 0$$

if $r\to 1/\alpha$ and the theorem is proved. ☐

Following to [6], by the Hardy space associated with $\mathcal{G}$ we mean the space of analytic functions $\mathcal{H}^2_\chi(\mathsf{B}):=\left\{\mathfrak{C}[f]\colon f\in\mathcal{H}^2_\chi\right\}$, defined by the formula (1.1), with the finite norm

$$\left\|\mathfrak{C}[f]\right\|_{\mathcal{H}^2_\chi}=\sup_{r\in[0,1/\alpha)}\left(\int_{\mathsf{G}}\left|\mathfrak{C}[f](r\lambda)\right|^2d\chi(\lambda)\right)^{1/2}.$$

Corollary 4.4. *The following antilinear isometry is valid:*

$$\mathcal{H}^2_\chi(\mathsf{B})\simeq\mathsf{F}_{\mathfrak{h}}.$$

Proof. Since $\left\|\mathfrak{C}[f]\right\|_{\mathcal{H}^2_\chi}=\|f\|_{L^2_\chi}$ for all $f\in\mathcal{H}^2_\chi$, the isometry $\mathcal{H}^2_\chi(\mathsf{B})\simeq\mathcal{H}^2_\chi$ holds. Now the desired isometry $\mathcal{H}^2_\chi(\mathsf{B})\simeq\mathsf{F}_{\mathfrak{h}}$ at once follows from Theorem 3.1. ☐

5. The case of reduced Heisenberg group

In what follows we put $\mathcal{G}=\mathbb{H}$, where the Cartesian product $\mathbb{H}=\mathbb{R}^2\times\mathbb{T}$ stands for the reduced Heisenberg group with the multiplication

$$(x,y,e^{\mathrm{i}\vartheta})\cdot(u,v,e^{\mathrm{i}\eta})=\left(x+u,y+v,e^{\mathrm{i}(\vartheta+\eta)}e^{\mathrm{i}(xv-yu)/2}\right),$$

having the Haar measure $dx\,dy\,d\tau$ with $\tau=e^{\mathrm{i}\vartheta}\in\mathbb{T}$ and $d\tau=\dfrac{d\vartheta}{2\pi}$. We refer to [10] about Heisenberg groups.

Let $\mathsf{E}=L^2_{\mathbb{R}}$ be the Hilbert space of quadratically integrable complex functions f on $\mathbb{R}$ with the norm $\|f\|_{L^2_{\mathbb{R}}}=\left(\int_{\mathbb{R}}|f(x)|^2\,dx\right)^{1/2}$. Consider in $L^2_{\mathbb{R}}$ the orthonormal basis

$$\varphi_j\colon\mathbb{R}\ni t\longmapsto\frac{e^{-t^2/2}}{\sqrt[4]{\pi}}\frac{\phi_{j-1}(t)}{\sqrt{2^{j-1}(j-1)!}},\quad \phi_{j-1}(t)=(-1)^{j-1}e^{t^2}\frac{d^{j-1}}{dt^{j-1}}e^{-t^2},$$

where $j\in\mathbb{N}$ and ϕ_{j-1} is the Hermite $(j-1)$-degree polynomial. Note that the space $L^2_{\mathbb{R}^n}=\otimes^n_{\mathfrak{h}}L^2_{\mathbb{R}}$ coincides with the closure of complex linear span of functions

$$\left\{\xi_1(t_1)\ldots\xi_n(t_n)\colon\xi_1,\ldots,\xi_n\in L^2_{\mathbb{R}},\,(t_1,\ldots,t_n)\in\mathbb{R}^n\right\}.$$

Therefore, $\odot_{\mathfrak{h}}^n L^2_{\mathbb{R}}$ is the closed subspace in $L^2_{\mathbb{R}^n}$ of symmetric functions with respect to the permutations of n scalar variables. The following system

$$\varphi_{(j)}^{\otimes(k)} := \varphi_{j_1}^{\otimes k_1} \odot \ldots \odot \varphi_{j_n}^{\otimes k_n}$$

with all $(j) = (j_1, \ldots, j_n) \in \mathbb{N}^n$, $j_1 < \ldots < j_n$ and $(k) = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ such that $k_1 + \cdots + k_n = n$ forms an orthogonal basis in $\odot_{\mathfrak{h}}^n L^2_{\mathbb{R}}$, which is non-orthonormal and $\|\varphi_{(j)}^{\otimes(k)}\|_{\odot_{\mathfrak{h}}^n L^2_{\mathbb{R}}} = \sqrt{(k)!/n!}$, where $(k)! := k_1! \ldots k_n!$. (see [2, 2.2.2]).

The Schrödinger representation U of the group $\mathbb{H}$ into $\mathcal{L}\left(L^2_{\mathbb{R}}\right)$ is given by

$$U_{x,y,\tau}\xi(t) = \tau e^{\mathrm{i}xy/2} e^{\mathrm{i}yt} \xi(t+x), \quad x, y, t \in \mathbb{R}, \quad \tau \in \mathbb{T}, \quad \xi \in L^2_{\mathbb{R}},$$

which is unitary and irreducible. It is easy to see that the codomain of U contains the complex cyclic group, since $\mathbb{T} = \left\{U_{0,0,\tau} \colon (0,0,\tau) \in \mathbb{H}\right\}$. Via the Stone-von Neumann Theorem every irreducible unitary representation V of $\mathbb{H}$ over any Hilbert space E, satisfying the condition $V(0,0,\tau)\xi = \tau\xi$ for all $\tau \in \mathbb{T}$ and $\xi \in \mathsf{E}$, is unitarily equivalent to the Schrödinger representation U.

The Gauss density function

$$\hbar \colon \mathbb{R} \ni t \longmapsto \pi^{-1/4} e^{-t^2/2} \quad (\text{i.e.} \quad \hbar = \varphi_1)$$

belongs to the unit sphere $\mathsf{S} \subset L^2_{\mathbb{R}}$ and the $\mathbb{H}$-orbit of $\hbar$

$$\begin{aligned} \mathsf{G} &= \left\{U_{x,y,\tau}\hbar \in L^2_{\mathbb{R}} \colon (x,y,\tau) \in \mathbb{H}\right\} \\ &= \left\{g_{x,y,\tau}(t) = \pi^{-1/4} \tau e^{\mathrm{i}xy/2} e^{\mathrm{i}yt} e^{-(t+x)^2/2} \colon (x,y,\tau) \in \mathbb{H},\ t \in \mathbb{R}\right\} \end{aligned}$$

also contains in S, as a function of the variable t for any fixed $(x,y,\tau) \in \mathbb{H}$. In fact, for any fixed $(x,y,\tau) \in \mathbb{H}$ we have

$$\|U_{x,y,\tau}\hbar\|_{L^2_{\mathbb{R}}} = \left(\int_{\mathbb{R}} \left|\pi^{-1/4} e^{-t^2/2}\right|^2 dt\right)^{1/2} = 1.$$

The stationary subgroup $\left\{(x,y,\tau) \in \mathbb{H} \colon U_{x,y,\tau}\hbar = \hbar\right\}$ coincides with the group unit $(0,0,1) \in \mathbb{H}$, hence the equality (2.2) has the form

$$\int_{\mathsf{G}} f \, d\chi = \int_{\mathbb{H}} (f \circ U_{x,y,\tau})(\hbar) \, dx \, dy \, d\tau,$$

where the $\mathbb{H}$-invariant measure χ on G is defined by the Haar measure $dx\,dy\,d\tau$ on $\mathbb{H}$ and $f \circ U$ is an integrable complex function on $\mathbb{H}$.

Consider the diagonal nth tensor power of Schrödinger's representation

$$\mathbb{H} \ni (x,y,\tau) \longmapsto U_{x,y,\tau}^{\otimes n} \in \mathcal{L}\left(\odot_{\mathfrak{h}}^n L^2_{\mathbb{R}}\right), \quad n \in \mathbb{N}$$

and put $U_{x,y,\tau}^{\otimes 0} = 1$. Let L^2_χ be the corresponding Hilbert space of quadratically χ-integrable complex functions on the orbit G. Each function

$$\hbar_n\big(\varphi_{(j)}^{\otimes(k)}\big) \colon \mathbb{H} \ni (x,y,\tau) \longmapsto \left\langle (U_{x,y,\tau}\hbar)^{\otimes n} \mid \varphi_{(j)}^{\otimes(k)} \right\rangle_{L^2_{\mathbb{R}^n}}$$

with $k_1 + \cdots + k_n = n$, belongs to L^2_χ and the following operator

$$\hbar_n \colon \odot_{\mathfrak{h}}^n L^2_{\mathbb{R}} \ni \varphi_{(j)}^{\otimes(k)} \longrightarrow \hbar_n\big(\varphi_{(j)}^{\otimes(k)}\big) \in L^2_\chi, \quad (j) \in \mathbb{N}^n$$

is well-defined. In fact, calculating the Fourier transformation by the variable $t \in \mathbb{R}$, we have

$$\begin{aligned} \hbar_1\big(\varphi_j\big)(x,y,\tau) &= \frac{\tau e^{\mathrm{i}xy/2}(-1)^{j-1}}{\sqrt{2^{j-1}\pi(j-1)!}} \int_{\mathbb{R}} e^{\mathrm{i}yt} e^{-(x+t)^2/2} e^{t^2/2} \frac{d^{j-1}}{dt^{j-1}} e^{-t^2} \, dt \\ &= \frac{\tau e^{\mathrm{i}xy/2}(-1)^{j-1}(x-\mathrm{i}y)^{j-1}}{\sqrt{2^{j-1}(j-1)!}} e^{(-x^2-2\mathrm{i}xy-y^2)/4} \end{aligned}$$

for any φ_j. For all (k) such that $k_1+\dots+k_n=n$ it follows

$$\Big|\hbar_n\big(\varphi_{(j)}^{\otimes(k)}\big)\Big|=\prod_{l=1}^{n}\big|\hbar_1\big(\varphi_{j_l}\big)(x,y,\tau)\big|^{k_l}=e^{-\frac{n}{4}(x^2+y^2)}\prod_{l=1}^{n}\left(\frac{(x^2+y^2)^{j_l-1}}{2^{j_l-1}(j_l-1)!}\right)^{k_l/2}.$$

Since,

$$\begin{aligned}\int_0^\infty e^{-nu}\prod_{l=1}^{n}\left(\frac{u^{j_l-1}}{(j_l-1)!}\right)^{k_l}du&=\prod_{l=1}^{n}\frac{m!}{(j_l-1)!^{k_l}}\int_0^\infty e^{-nu}\frac{u^m}{m!}\,du\\&=\prod_{l=1}^{n}\frac{m!}{(j_l-1)!^{k_l}}\frac{1}{n^m}\int_0^\infty e^{-nu}\frac{(un)^m}{m!}\,du\le\frac{1}{n}\end{aligned}$$

with $m=\sum\limits_{l=1}^{n}(j_l-1)k_l$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(\frac{x^2+y^2}{2}\right)dx\,dy=4\int_0^\infty\int_0^{\pi/2}f(u)\,du\,d\vartheta=2\pi\int_0^\infty f(u)\,du,$$

where $x^2=2u\cos^2\vartheta$ and $y^2=2u\sin^2\vartheta$, we obtain that each such function $\hbar_n\big(\varphi_{(j)}^{\otimes(k)}\big)$ belongs to L^2_χ and the following estimation holds:

$$\int_{\mathbb{H}}\Big|\hbar_n\big(\varphi_{(j)}^{\otimes(k)}\big)\Big|^2dx\,dy\,d\tau\le\frac{2\pi}{n}. \tag{5.1}$$

Any element $\psi_n\in\odot_{\mathfrak{h}}^nL^2_{\mathbb{R}}$ with $\|\psi_n\|_{\odot_{\mathfrak{h}}^nL^2_{\mathbb{R}}}\le1$ may be presented in the form of its Fourier decomposition

$$\psi_n=\sum_{(k),(j)}\alpha_{(j)}^{(k)}\varphi_{(j)}^{\otimes(k)}\sqrt{\frac{n!}{(k)!}},\quad k_1+\dots+k_n=n,\quad\sum_{(k),(j)}|\alpha_{(j)}^{(k)}|^2\le1.$$

Applying the inequality (5.1), we have

$$\Big\|\sum_{(k),(j)}\alpha_{(j)}^{(k)}\hbar_n\big(\varphi_{(j)}^{\otimes(k)}\big)\sqrt{\frac{n!}{(k)!}}\Big\|_{L^2_\chi}\le\sum_{(k),(j)}\alpha_{(j)}^{(k)}\sqrt{\frac{n!}{(k)!}}\,\Big\|\hbar_n\big(\varphi_{(j)}^{\otimes(k)}\big)\Big\|_{L^2_\chi}\le\sum_{(k),(j)}\alpha_{(j)}^{(k)}\sqrt{\frac{n!}{(k)!}\frac{2\pi}{n}}.$$

It follows that

$$\|\hbar_n(\psi_n)\|^2_{L^2_\chi}\le2\pi(n-1)!\|\psi_n\|^2_{\odot_{\mathfrak{h}}^nL^2_{\mathbb{R}}}\quad\text{or}\quad\|\hbar_n\|\le\sqrt{2\pi(n-1)!}.$$

If $(j)=(1,j_2,\dots,j_n)$ and $(k)=(n,0,\dots,0)$ we have that $\varphi_{(j)}^{\otimes(k)}=\varphi_1^{\otimes n}$ and

$$\begin{aligned}\int_{\mathbb{H}}\big|\hbar_n\big(\varphi_1^{\otimes n}\big)\big|^2dx\,dy\,d\tau&=\int_{\mathbb{H}}\big|\hbar_1\big(\varphi_1\big)(x,y,\tau)\big|^{2n}dx\,dy\,d\tau\\&=\int_{\mathbb{R}^2}\Big|e^{-(x^2+2ixy+y^2)/4}\Big|^{2n}dx\,dy=\frac{2\pi}{n}.\end{aligned}$$

Since $1=\|\varphi_1^{\otimes n}\|_{\odot_{\mathfrak{h}}^nL^2_{\mathbb{R}}}=\sqrt{\dfrac{n}{2\pi}}\,\big\|\hbar_n\big(\varphi_1^{\otimes n}\big)\big\|_{L^2_\chi}$, we have $\varphi_1^{\otimes n}\notin\ker\hbar_n$. It follows that

$$\|\hbar_n\|=\sup_{\|\psi_n\|_{\odot_{\mathfrak{h}}^nL^2_{\mathbb{R}}\le1}}\|\hbar_n(\psi_n)\|_{L^2_\chi}\ge\big\|\hbar_n\big(\varphi_1^{\otimes n}\big)\big\|_{L^2_\chi}=\sqrt{\frac{2\pi}{n}}.$$

Hence, $\lim_{n\to\infty} \sqrt[n]{\aleph_n^2} \le \lim_{n\to\infty} \sqrt[n]{\dfrac{n}{2\pi}} = 1$ and the Cauchy type kernel has the form

$$\begin{aligned}\widehat{\mathfrak{C}}\,(\xi, U_{x,y,\tau}\hbar) &= 1 + \sum_{n=1}^{\infty} \aleph_n^2 \left\langle \xi \mid U_{x,y,\tau}\hbar \right\rangle_{L^2_{\mathbb{R}}}^n \\ &= 1 + \sum_{n=1}^{\infty} \aleph_n^2 \left(\frac{\tau e^{\mathrm{i}xy/2}}{\sqrt[4]{\pi}} \int_{\mathbb{R}} \xi(t) e^{\mathrm{i}yt-(t+x)^2/2}\, dt \right)^n,\end{aligned}$$

which is a L^2_χ-valued analytic function by the variable $\xi \in \mathsf{B}$, where $\mathsf{B} \subset L^2_{\mathbb{R}}$. Thus, for any $f \in \mathcal{H}^2_\chi$ and $\xi \in \mathsf{B}$ we have

$$\widehat{\mathfrak{C}}[f](\xi) = \int_{\mathbb{H}} \widehat{\mathfrak{C}}\,(\xi, U_{x,y,\tau}\hbar)\,(f \circ U_{x,y,\tau})(\hbar)\, dx\, dy\, d\tau,$$

$$\lim_{r\to 1} \int_{\mathbb{H}} \left| \widehat{\mathfrak{C}}_r[f](x,y,\tau) - (f \circ U_{x,y,\tau})(\hbar) \right|^2 dx\, dy\, d\tau = 0,$$

where the functions $\widehat{\mathfrak{C}}[f]_r \colon \mathbb{H} \ni (x,y,\tau) \longmapsto \widehat{\mathfrak{C}}[f](rU_{x,y,\tau}\lambda)$ with $r \in [0,1)$ belong to $\mathcal{H}^2_\chi$ for any $\lambda \in \mathsf{S}$.

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