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HILL'S POTENTIALS IN HÖRMANDER SPACES AND THEIR SPECTRAL GAPS

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Abstract. The paper deals with the Hill–Schrödinger operators with singular periodic potentials in the space $H^{\omega}(\mathbb{T}) \subset H^{-1}(\mathbb{T})$. The authors exactly describe the classes of sequences being the lengths of spectral gaps of these operators. The functions ω may be nonmonotonic. The space $H^{\omega}(\mathbb{T})$ coincides with the Hörmander space $H_2^{\omega}(\mathbb{T})$ with the weight function $\omega(\sqrt{1+\xi^2})$ if ω is in the Avakumovich class OR.

1. Introduction

Let us consider the Hill–Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad x \in \mathbb{R}, \tag{1}$$

with 1-periodic real-valued potentials

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k) e^{ik2\pi x} \in L^2(\mathbb{T},\mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z},$$

in the complex Hilbert space $L^2(\mathbb{R})$.

It is known that the operators $S(q)$ are lower semibounded and self-adjoint. Their spectra are absolutely continuous and have a zone structure [26].

Spectra of the operators $S(q)$ are completely defined by the location of the endpoints of the spectral gaps $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ which satisfy the inequalities

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \cdots \tag{2}$$

For even/odd numbers $n \in \mathbb{Z}_+$, the endpoints of spectral gaps $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ are eigenvalues of the periodic/semiperiodic problems on the interval $[0,1]$:

$$S_{\pm}(q)u := -u'' + q(x)u = \lambda u,$$

$$\mathrm{Dom}(S_{\pm}(q)) := \left\{ u \in H^2[0,1] \,\middle|\, u^{(j)}(0) = \pm\, u^{(j)}(1),\ j = 0,1 \right\}.$$

Interiors of the spectral bands (stability or tied zones)

$$\mathcal{B}_0(q) := (\lambda_0(q), \lambda_1^-(q)), \quad \mathcal{B}_n(q) := (\lambda_n^+(q), \lambda_{n+1}^-(q)), \quad n \in \mathbb{N},$$

together with the *collapsed* gaps

$$\lambda = \lambda_n^+ = \lambda_n^-, \quad n \in \mathbb{N},$$

are characterized as a locus of those real $\lambda \in \mathbb{R}$ for which all solutions of the equation $-u'' + q(x)u = \lambda u$ are bounded. *Open* spectral gaps (instability or forbidden zones)

$$\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \quad \mathcal{G}_n(q) := (\lambda_n^-(q), \lambda_n^+(q)) \ne \emptyset, \quad n \in \mathbb{N},$$

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make a set of those real $\lambda \in \mathbb{R}$ for which any nontrivial solution of the equation $-u'' + q(x)u = \lambda u$ is unbounded.

We will study the behaviour of the lengths of the spectral gaps

$$\gamma_q(n) := \lambda_n^+(q) - \lambda_n^-(q), \quad n \in \mathbb{N},$$

of the operators $S(q)$ in terms of the behaviour of the Fourier coefficients $\{\widehat{q}(n)\}_{n\in\mathbb{N}}$ of the potentials q with respect to appropriate weight spaces, that is by means of potential regularity.

For $L^2(\mathbb{T},\mathbb{R})$-potentials, a fundamental result in this problem follows from the Marchenko and Ostrovskii paper [13] (see also [12]),

$$q \in H^s(\mathbb{T},\mathbb{R}) \Leftrightarrow \sum_{n\in\mathbb{N}} (1+2n)^{2s}\gamma_q^2(n) < \infty, \quad s \in \mathbb{Z}_+, \tag{3}$$

where $H^s(\mathbb{T},\mathbb{R})$, $s \in \mathbb{Z}_+$, denotes the Sobolev spaces of 1-periodic real-valued functions on the circle $\mathbb{T}$.

To characterize regularity of the potentials in a finer, way we will use the real Hörmander spaces

$$H^\omega(\mathbb{T},\mathbb{R}) := \Big\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ik2\pi x} \,\Big|\, \widehat{f}(k) = \overline{\widehat{f}(-k)},\ k \in \mathbb{Z},\ \sum_{k\in\mathbb{N}} \omega^2(k)|\widehat{f}(k)|^2 < \infty \Big\},$$

where $\omega(\cdot)$ is a positive weight. In the case of Sobolev spaces, the weight is a power function. Such definition of the real Hörmander spaces on the circle completely corresponds to the theory of function spaces on a smooth closed manifold [19, 20] (for more details see Appendix 5 and the monograph [21]).

Djakov, Mityagin [2], Pöschel [24, 25] extended the Marchenko–Ostrovskii Theorem (3) to the general class of weights $\Omega = \{\Omega(k)\}_{k\in\mathbb{N}}$ satisfying the following conditions:

(i) $\Omega(k) \nearrow \infty$, $k \in \mathbb{N}$ (monotonicity);
(ii) $\Omega(k+m) \le \Omega(k)\Omega(m)$ $\forall k, m \in \mathbb{N}$ (submultiplicity);
(iii) $\frac{\log \Omega(k)}{k} \searrow 0$, $k \to \infty$ (subexponentiality).

For such weights they proved that

$$q \in H^\Omega(\mathbb{T},\mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\Omega(\mathbb{N}). \tag{4}$$

Here $h^\Omega(\mathbb{N})$ is the Hilbert space of weighted sequences generated by the weight $\Omega(\cdot)$.

For the power weights

$$w_s = \{w_s(k)\}_{k\in\mathbb{N}}, \quad w_s(k) := (1+2k)^s, \quad s \in \mathbb{R},$$

it is convenient to use shorter notation $H^{\omega_s}(\mathbb{T}) \equiv H^s(\mathbb{T})$, $\quad h^{\omega_s}(\mathbb{N}) \equiv h^s(\mathbb{N})$.

After the celebrated Kronig and Penney paper [11], the Schrödinger operators with (periodic) distributions as potentials came into mathematical physics. A subsequent development of quantum mechanics stimulated an active research in area (see the bibliography in the monograph [1]).

In this paper, we study the Hill–Schrödinger operators $S(q)$ with 1-periodic real-valued distribution potentials q in the negative Sobolev space $H^{-1}(\mathbb{T},\mathbb{R})$,

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k)e^{ik2\pi x} \in H^{-1}(\mathbb{T},\mathbb{R}). \tag{5}$$

This means that

$$\sum_{k\in\mathbb{N}} (1+2k)^{2s}|\widehat{q}(k)|^2 < \infty, \quad s = -1, \quad \text{and} \quad \widehat{q}(k) = \overline{\widehat{q}(-k)}, \quad k \in \mathbb{Z}. \tag{6_s}$$

All real-valued pseudo-functions, measures, pseudo-measures and some more singular distributions on the circle satisfy this condition.

Under the assumption (5) the operators (1) can be well defined in the complex Hilbert space $L^2(\mathbb{R})$ in the following basic ways:

- as form-sum operators;
- as quasi-differential operators (minimal operators, maximal operators, the Friedrichs extensions of the minimal operators);
- as limits of operators with smooth 1-periodic potentials in the norm resolvent sense.

Equivalence of all these definitions was proved in the paper [16].

The Hill–Schrödinger operators $S(q)$ are lower semibounded and self-adjoint, their spectra are absolutely continuous and have a band and gap structure as in the classical case [7, 10, 16, 3, 22, 5]. The endpoints of spectral gaps satisfy the inequalities (2). For even/odd numbers $n \in \mathbb{Z}_+$, they are eigenvalues of the periodic/semiperiodic problems on the interval $[0,1]$ [16, Theorem C].

In the paper [17], we extended the Marchenko–Ostrovskii Theorem (3) to the case of singular potentials $q \in H^{-1+}(\mathbb{T},\mathbb{R})$. This means that q satisfies (6_s) with some $s > -1$. We proved that

$$q \in H^s(\mathbb{T},\mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^s(\mathbb{N}), \quad s \in (-1,\infty).$$

Djakov, Mityagin [4] extended the latter statement to the limiting case $s = -1$,

$$q \in H^s(\mathbb{T},\mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^s(\mathbb{N}), \quad s \in [-1,\infty), \tag{7}$$

under the *a priori* assumption $q \in H^{-1}(\mathbb{T},\mathbb{R})$. Moreover, they extended the result (4) to the case of potentials $q \in H^{-1}(\mathbb{T},\mathbb{R})$ and the more general weights

$$q \in H^{\Omega^*}(\mathbb{T},\mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^{\Omega^*}(\mathbb{N}), \quad \Omega^* := \left\{\frac{\Omega(k)}{1+2k}\right\}_{k\in\mathbb{N}}, \tag{8}$$

where the weights $\Omega = \{\Omega(k)\}_{k\in\mathbb{N}}$ are supposed to be monotonic, submultiplicative and subexponential.

2. Main results

The aim of this paper is to extend the result (7) to a more extensive class of weights, for which the conditions of monotonicity and regularity of the weight behaviour may not hold, and to supplement the result (8).

For convenience of formulation of the results we introduce the following definition.

Definition. Let the set $X \subset H^{-1}(\mathbb{T},\mathbb{R})$. We write $\omega \in \mathrm{MO}(X)$ if

$$q \in H^\omega(\mathbb{T},\mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N}) \quad \forall q \in X.$$

It is easy to see that

$$X \subset Y \Rightarrow \mathrm{MO}(X) \subset \mathrm{MO}(Y),$$

and

$$\begin{aligned}
&(3) \Leftrightarrow \omega_s \in \mathrm{MO}(L^2(\mathbb{T})), \quad s \in \mathbb{Z}_+,\\
&(4) \Leftrightarrow \Omega \in \mathrm{MO}(L^2(\mathbb{T})),\\
&(7) \Leftrightarrow \omega_s \in \mathrm{MO}(H^{-1}(\mathbb{T})), \quad s \in [-1,\infty),\\
&(8) \Leftrightarrow \Omega^* \in \mathrm{MO}(H^{-1}(\mathbb{T})).
\end{aligned}$$

Further, let us recall that *lower order* $\mu(\omega)$ and *upper order* $\rho(\omega)$ of a weight sequence $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ are defined as follows:

$$\mu \equiv \mu(\omega) := \liminf_{k\to\infty} \frac{\log \omega(k)}{\log k}, \quad \rho \equiv \rho(\omega) := \limsup_{k\to\infty} \frac{\log \omega(k)}{\log k}.$$

The following statement is the main result of this paper.

Theorem 1. *Let $q \in H^{-1}(\mathbb{T},\mathbb{R})$ and a weight $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ satisfy the conditions*

$$-1 < \mu(\omega) \le \rho(\omega) < \infty, \tag{9}$$

$$\rho(\omega) < \begin{cases} 1 + 2\mu(\omega) & \text{if} \quad \mu(\omega) \in (-1, 0], \\ 1 + \mu(\omega) & \text{if} \quad \mu(\omega) \in [0, \infty). \end{cases} \tag{10}$$

Then $\omega \in \mathrm{MO}(H^{-1}(\mathbb{T}))$.

Remark 1.1. In the case of the $L^2(\mathbb{T})$-potentials, we prove a stronger result in the paper [18].

Remark 1.2. Theorem 1 shows that if the sequence $\{|\widehat{q}(n_k)|\}_{k=1}^{\infty}$ decreases/increases particularly fast on a certain subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$, then so does the sequence $\{\gamma_q(n_k)\}_{k=1}^{\infty}$ on the same subsequence. The converse statement is also true.

Corollary 1.1. *Let, for a weight $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$, there exist the order*

$$\lim_{k\to\infty} \frac{\log \omega(k)}{\log k} = s \in (-1, \infty).$$

Then $\omega \in \mathrm{MO}(H^{-1}(\mathbb{T}))$.

From Corollary 1.1 we obtain the following result.

Corollary 1.2. (cf. [17]). *Let the weight $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ be a regularly varying sequence at $+\infty$ in the Karamata sense with the index $s \in (-1,\infty)$. Then $\omega \in \mathrm{MO}(H^{-1}(\mathbb{T}))$.*

Note that the assumption of Corollary 1.2 holds, for instance, for the weight

$$\omega(k) = (1+2k)^s \,(\log(1+k))^{r_1} (\log\log(1+k))^{r_2} \cdots (\log\log\cdots\log(1+k))^{r_p},$$
$$s \in (-1,\infty), \quad \{r_1,\dots,r_p\} \subset \mathbb{R}, \quad p \in \mathbb{N},$$

see the monograph [27].

Theorem 1 extends the statement (7) to the case of non-regularly varying weights.

The following example shows that statement (8) does not cover Corollary 1.1 and Theorem 1.

Example A. Let $s \in (-1,\infty)$. Set

$$\omega(k) := \begin{cases} k^s \log(1+k) & \text{if} \quad k \in 2\mathbb{N}, \\ k^s & \text{if} \quad k \in (2\mathbb{N}-1). \end{cases}$$

Then the weight $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ satisfies the conditions of Corollary 1.1. But one can prove that the weight

$$\omega^* := \{(1+2k)\,\omega(k)\}_{k\in\mathbb{N}}$$

is not equivalent to any monotonic weight.

3. Preliminaries

Here, for convenience, we define Hilbert spaces of weighted two-sided sequences and formulate the Convolution Lemma 2.

For every positive sequence $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ there exists its unique extension on $\mathbb{Z}$, which is a two-sided sequence satisfying the conditions

(i) $\omega(0) = 1$,
(ii) $\omega(-k) = \omega(k) \quad \forall k \in \mathbb{N}$,
(iii) $\omega(k) > 0 \quad \forall k \in \mathbb{Z}$.

Let $h^\omega(\mathbb{Z}) \equiv h^\omega(\mathbb{Z},\mathbb{C})$ be the Hilbert space of two-sided sequences,

$$h^\omega(\mathbb{Z}) := \Big\{ a = \{a(k)\}_{k\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} \omega^2(k)|a(k)|^2 < \infty \Big\},$$

$$(a,b)_{h^\omega(\mathbb{Z})} := \sum_{k\in\mathbb{Z}} \omega^2(k)a(k)\overline{b(k)}, \quad a,b \in h^\omega(\mathbb{Z}),$$

$$\|a\|_{h^\omega(\mathbb{Z})} := (a,a)^{1/2}_{h^\omega(\mathbb{Z})}, \quad a \in h^\omega(\mathbb{Z}).$$

By $h^\omega(n)$, we will denote the n-th term of the sequence in the space $h^\omega(\mathbb{Z})$.

Basic weights which we use are the power ones,

$$w_s = \{w_s(k)\}_{k\in\mathbb{Z}}: \quad w_s(k) = (1+2|k|)^s, \quad s \in \mathbb{R}.$$

In this case it is convenient to use shorter notations, $h^{\omega_s}(\mathbb{Z}) \equiv h^s(\mathbb{Z})$, $s\in\mathbb{R}$.

Operation of convolution for two-sided sequences

$$a = \{a(k)\}_{k\in\mathbb{Z}} \quad \text{and} \quad b = \{b(k)\}_{k\in\mathbb{Z}}$$

is formally defined as follows:

$$(a,b) \mapsto a * b, \quad (a*b)(k) := \sum_{j\in\mathbb{Z}} a(k-j)\, b(j), \quad k \in \mathbb{Z}.$$

Sufficient conditions for the convolution to exist as a continuous map are given by the following known lemma, see for example [9].

Lemma 2. (The Convolution Lemma). *Let $s,r \geq 0$, and $t \leq \min(s,r)$, $t \in \mathbb{R}$. If $s+r-t > 1/2$ then the convolution $(a,b) \mapsto a*b$ is well defined as a continuous map acting in the spaces*

(a) $h^s(\mathbb{Z}) \times h^r(\mathbb{Z}) \to h^t(\mathbb{Z})$,

(b) $h^{-t}(\mathbb{Z}) \times h^s(\mathbb{Z}) \to h^{-r}(\mathbb{Z})$.

In the case $s+r-t < 1/2$, this statement fails to hold.

4. The Proofs

Basic point of our proof of Theorem 1 is to obtain sharp asymptotic formulae for lengths of the spectral gaps $\{\gamma_q(n)\}_{n\in\mathbb{N}}$ of the Hill–Schrödinger operators $S(q)$.

Lemma 3. *The lengths of spectral gaps $\{\gamma_q(n)\}_{n\in\mathbb{N}}$ of the Hill-Schrödinger operators $S(q)$ with $q \in H^s(\mathbb{T},\mathbb{R})$, $s \in (-1,\infty)$, uniformly on the bounded sets of potentials q in the corresponding Sobolev spaces $H^s(\mathbb{T})$ for $n \geq n_0$, $n_0 = n_0\left(\|q\|_{H^s(\mathbb{T})}\right)$, satisfy the following asymptotic formulae:*

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{1+2s-\varepsilon}(n),\ \varepsilon > 0, \quad \text{if} \quad s \in (-1,0], \tag{11}$$

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{1+s}(n) \quad \text{if} \quad s \in [0,\infty). \tag{12}$$

Proof of Lemma 3. The asymptotic estimates (11) were established by the authors in the paper [17, Theorem 1] by using the isospectral transformation method for the problem [23, 14, 15].

The asymptotic formulae (12) follow from [8, Theorem 1.2] due to the Convolution Lemma 2 (see also [8, Appendix]). Indeed, from [8, Theorem 1.2] with $q \in H^s(\mathbb{T},\mathbb{R})$, $s \in [0,\infty)$, we get

$$\sum_{n\in\mathbb{N}} (1+2n)^{2(1+s)} \Big(\min_{\pm} \Big| \gamma_q(n) \pm 2\sqrt{(\widehat{q}+\varrho)(-n)(\widehat{q}+\varrho)(n)} \Big| \Big)^2 \leq C\left(\|q\|_{H^s(\mathbb{T})}\right), \tag{13}$$

where

$$\varrho(n) := \frac{1}{\pi^2} \sum_{j\in\mathbb{Z}\setminus\{\pm n\}} \frac{\widehat{q}(n-j)\widehat{q}(n+j)}{(n-j)(n+j)}.$$

Without loos of generality, we assume that

$$\widehat{q}(0) := 0. \tag{14}$$

Taking into account that the potentials q are real-valued we have

$$\widehat{q}(k) = \overline{\widehat{q}(-k)},\ \varrho(k) = \overline{\varrho(-k)}, \quad k \in \mathbb{Z}.$$

Then from (13) we get the estimates

$$\{\gamma_n(q) - 2\,|\widehat{q}(n) + \varrho(n)|\}_{n\in\mathbb{N}} \in h^{1+s}(\mathbb{N}). \tag{15}$$

Further, since by assumption, $q \in H^s(\mathbb{T}, \mathbb{R})$, that is, $\{\widehat{q}(k)\}_{k\in\mathbb{Z}} \in h^s(\mathbb{Z})$, we have

$$\left\{\frac{\widehat{q}(k)}{k}\right\}_{k\in\mathbb{Z}} \in h^{1+s}(\mathbb{Z}), \quad s \in [0,\infty),$$

taking into account (14). Applying the Convolution Lemma 2 we obtain

$$\begin{aligned} \varrho(n) &= \frac{1}{\pi^2}\sum_{j\in\mathbb{Z}} \frac{\widehat{q}(n-j)\widehat{q}(n+j)}{(n-j)(n+j)} = \frac{1}{\pi^2}\sum_{j\in\mathbb{Z}} \frac{\widehat{q}(2n-j)}{2n-j}\cdot\frac{\widehat{q}(j)}{j} \\ &= \left(\left\{\frac{\widehat{q}(k)}{k}\right\}_{k\in\mathbb{Z}} * \left\{\frac{\widehat{q}(k)}{k}\right\}_{k\in\mathbb{Z}}\right)(2n) \in h^{1+s}(\mathbb{N}). \end{aligned} \tag{16}$$

Finally, from (15) and (16) we get the necessary estimates (12).

The proof of Lemma 3 is complete. □

Proof of Theorem 1. Let $q \in H^{-1}(\mathbb{T}, \mathbb{R})$ and $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ be a given weight satisfying conditions (9) and (10) of Theorem 1. We need to prove the statement

$$q \in H^{\omega}(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^{\omega}(\mathbb{N}). \tag{17}$$

From the condition (9) and the definition of the lower and upper orders of a weight sequence, we conclude that for the given weight $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ the following estimates are fulfilled:

$$k^{\mu-\delta} \ll \omega(k) \ll k^{\rho+\delta}, \quad -1 < \mu \le \rho < \infty,\ \delta > 0.$$

Hence, the continuous embeddings

$$H^{\rho+\delta}(\mathbb{T}) \hookrightarrow H^{\omega}(\mathbb{T}) \hookrightarrow H^{\mu-\delta}(\mathbb{T}), \tag{18}$$

$$h^{\rho+\delta}(\mathbb{N}) \hookrightarrow h^{\omega}(\mathbb{N}) \hookrightarrow h^{\mu-\delta}(\mathbb{N}), \quad -1 < \mu \le \rho < \infty, \quad \delta > 0 \tag{19}$$

are valid because of

$$H^{\omega_1}(\mathbb{T}) \hookrightarrow H^{\omega_2}(\mathbb{T}), \quad h^{\omega_1}(\mathbb{N}) \hookrightarrow h^{\omega_2}(\mathbb{N}) \quad \text{if} \quad \omega_1 \gg \omega_2. \tag{20}$$

Let $q \in H^{\omega}(\mathbb{T}, \mathbb{R})$, then from (18) we obtain that $q \in H^{\mu-\delta}(\mathbb{T}, \mathbb{R})$, $\delta > 0$. Since $\delta > 0$ is arbitrary, we may choose it so that

$$\begin{aligned} \mu - \delta > 0 &\quad \text{if} \quad \mu > 0, \\ \mu - \delta > -1 &\quad \text{otherwise}. \end{aligned}$$

Further, using Lemma 3 we get the following asymptotic formulae for the lengths of spectral gaps:

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{1+2(\mu-\delta)-\varepsilon}(n), \ \ \varepsilon > 0 \quad \text{if} \quad \mu - \delta \in (-1, 0], \tag{21}$$

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{1+\mu-\delta}(n) \quad \text{if} \quad \mu - \delta \in [0, \infty). \tag{22}$$

Now, due to the possibility to choose $\delta > 0$ and $\varepsilon > 0$ arbitrarily, we may take them so that

$$1 + 2(\mu - \delta) - \varepsilon > \rho + \delta \quad \text{if} \quad \mu - \delta \in (-1, 0], \tag{23}$$

$$1 + \mu - \delta > \rho + \delta \quad \text{if} \quad \mu - \delta \in [0, \infty). \tag{24}$$

This choice is possible since (9) and (10) hold.

Taking into account (23), (24) and using (20), it follows from (21), (22) that

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{\rho+\delta}(n).$$

From the latter formula and formula (19) we finally obtain

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{\omega}(n). \tag{25}$$

Hence, as by the assumption, $q \in H^{\omega}(\mathbb{T},\mathbb{R})$, and as a consequence of $\{\widehat{q}(\cdot)\} \in h^{\omega}(\mathbb{N})$, we get $\{\gamma_q(\cdot)\} \in h^{\omega}(\mathbb{N})$.

The implication ($\Rightarrow$) in (17) has been proved.

Conversely, let $\{\gamma_q(\cdot)\} \in h^{\omega}(\mathbb{N})$. Then applying (19) we have $\{\gamma_q(\cdot)\} \in h^{\mu-\delta}(\mathbb{N})$, $\delta > 0$,

$$\begin{aligned} \mu - \delta &> 0 \quad \text{if} \quad \mu > 0, \\ \mu - \delta &> -1 \quad \text{otherwise.} \end{aligned}$$

Now, applying (7) we conclude that $q \in H^{\mu-\delta}(\mathbb{T},\mathbb{R})$.

We have already proved the implication

$$q \in H^{\mu-\delta}(\mathbb{T},\mathbb{R}) \Rightarrow (25).$$

So we have

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{\omega}(n),$$

and hence $\{\widehat{q}(\cdot)\} \in h^{\omega}(\mathbb{N})$, i.e., $q \in H^{\omega}(\mathbb{T},\mathbb{R})$.

The implication ($\Leftarrow$) in (17) has been proved.

The proof of Theorem 1 is complete. □

5. Hörmander spaces on the circle

Let OR be a class of all Borel measurable functions $\omega : \ (0,\infty) \to (0,\infty)$, for which there exist real numbers $a, c > 1$ such that

$$c^{-1} \le \frac{\omega(\lambda t)}{\omega(t)} \le c, \quad t \ge 1, \quad \lambda \in [1, a].$$

The space $H_2^{\omega}(\mathbb{R}^n)$, $n \in \mathbb{N}$, consists of all complex-valued distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that their Fourier transformations $\widehat{u}$ are locally Lebesgue integrable on $\mathbb{R}^n$ and $\omega(\langle\xi\rangle)|\widehat{u}(\xi)| \in L^2(\mathbb{R}^n)$ with $\langle\xi\rangle := (1+\xi^2)^{1/2}$. This space is a Hilbert space with respect to the inner product

$$(u_1, u_2)_{H_2^{\omega}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \omega^2(\langle\xi\rangle)\widehat{u}_1(\xi)\overline{\widehat{u}_2(\xi)}\, d\xi.$$

It is a special case of the isotropic Hörmander spaces [6]. If Ω is a domain in $\mathbb{R}^n$ with a smooth boundary, then the spaces $H_2^{\omega}(\Omega)$ are defined in a standard way.

Let Γ be an infinitely smooth closed oriented manifold of dimension $n \ge 1$ with a given on it density dx. Let $\mathfrak{D}'(\Gamma)$ be the topological vector space of distributions on Γ dual to $C^{\infty}(\Gamma)$ with respect to the extension by continuity of the inner product in the space $L^2(\Gamma) := L^2(\Gamma, dx)$.

Now, let us define the Hörmander spaces on the manifold Γ. Choose a finite atlas from the C^{∞}-structure on Γ formed by the local charts $\alpha_j : \mathbb{R}^n \leftrightarrow U_j$, $j = 1, \ldots, r$, where the open sets U_j form a finite covering of the manifold Γ. Let functions $\chi_j \in C^{\infty}(\Gamma)$, $j = 1, \ldots, r$, satisfying the condition $\operatorname{supp}\chi_j \subset U_j$ form a partition of unity on Γ . By definition, the linear space $H_2^{\omega}(\Gamma)$ consists of all distributions $f \in \mathfrak{D}'(\Gamma)$ such that $(\chi_j f)\circ\alpha_j \in H_2^{\omega}(\mathbb{R}^n)$ for every j, where $(\chi_j f)\circ\alpha_j$ is a representation of the distribution $\chi_j f$ in the local chart α_j. In the space $H_2^{\omega}(\Gamma)$, the inner product is defined by the formula

$$(f_1, f_2)_{H_2^{\omega}(\Gamma)} := \sum_{j=1}^{r} \left((\chi_j f_1)\circ\alpha_j, (\chi_j f_2)\circ\alpha_j\right)_{H_2^{\omega}(\mathbb{R}^n)},$$

and induces the norm $\|f\|_{H_2^\omega(\Gamma)} := (f,f)^{1/2}_{H_2^\omega(\Gamma)}$.

There exists an alternative definition of the space $H_2^\omega(\Gamma)$ which shows that this space does not depend (up to equivalence of norms) on the choice of the local charts, the partition of unity, and that it is a Hilbert space.

Let a ΨDO A of order $m > 0$ be elliptic on Γ, and let it be a positive unbounded operator on the space $L^2(\Gamma)$. For instance, we can set $A := (1 - \triangle_\Gamma)^{1/2}$, where $\triangle_\Gamma$ is the Beltrami-Laplace operator on the Riemannian manifold Γ. Redefine the function $\omega \in \mathrm{OR}$ on the interval $0 < t < 1$ by $\omega(t) := \omega(1)$ and introduce the norm

$$f \mapsto \|\omega(A^{1/m})f\|_{L^2(\Gamma)}, \quad f \in C^\infty(\Gamma). \tag{5.1}$$

Theorem 5.1. *If $\omega \in \mathrm{OR}$, then the space $H_2^\omega(\Gamma)$ coincides, up to the equivalence of norms, with the completion of the linear space $C^\infty(\Gamma)$ with respect to the norm* (5.1).

Since the operator A has a discrete spectrum, the space $H_2^\omega(\Gamma)$ can be described by means of the Fourier series. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a monotonically non-decreasing, positive sequence of all eigenvalues of the operator A, enumerated according to their multiplicity. Let $\{h_k\}_{k\in\mathbb{N}}$ be an orthonormal basis in the space $L^2(\Gamma)$ formed by the corresponding eigenfunctions of the operator A, $Ah_k = \lambda_k h_k$. Then for any distribution, the following expansion into the Fourier series converging in the linear space $\mathfrak{D}'(\Gamma)$ holds:

$$f = \sum_{k=1}^{\infty} c_k(f)h_k, \quad f \in \mathfrak{D}'(\Gamma), \quad c_k(f) := (f, h_k). \tag{5.2}$$

Theorem 5.2. *The following formulae are fulfilled:*

$$H_2^\omega(\Gamma) = \Big\{ f = \sum_{k=1}^{\infty} c_k(f)h_k \in \mathfrak{D}'(\Gamma) \;\Big|\; \sum_{k=1}^{\infty} \omega^2(k^{1/n})|c_k(f)|^2 < \infty \Big\},$$

$$\|f\|^2_{H_2^\omega(\Gamma)} \asymp \sum_{k=1}^{\infty} \omega^2(k^{1/n})|c_k(f)|^2.$$

Note that for every distribution $f \in H_2^\omega(\Gamma)$, series (5.2) converges in the norm of the space $H_2^\omega(\Gamma)$. If values of the function ω are separated from zero, then $H_2^\omega(\Gamma) \subseteq L^2(\Gamma)$, and everywhere above we may replace the space $\mathfrak{D}'(\Gamma)$ by the space $L^2(\Gamma)$. For more details, see [21, 19, 20].

Example B. Let $\Gamma = \mathbb{T}$. Then $n = 1$, and we can choose $A = \left(1 - d^2/dx^2\right)^{1/2}$, where we denote by x the natural parametrization on $\mathbb{T}$. The eigenfunctions $h_k = e^{ik2\pi x}$, $k \in \mathbb{Z}$, of the operator A form an orthonormal basis in the space $L^2(\mathbb{T})$. For $\omega \in \mathrm{OR}$ we have

$$f \in H_2^\omega(\mathbb{T}) \Leftrightarrow f = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ik2\pi x}, \qquad \sum_{k\in\mathbb{Z}\setminus\{0\}} |\widehat{f}(k)|^2\omega^2(|k|) < \infty.$$

In this case, the function f is real-valued if and only if $\widehat{f}(k) = \overline{\widehat{f}(-k)}$, $k \in \mathbb{Z}$. Therefore *the class H^ω coincides with the Hörmander space $H_2^\omega(\mathbb{T},\mathbb{R})$* with the weight function $\omega(\sqrt{1+\xi^2})$ if $\omega \in \mathrm{OR}$.

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