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ON ASYMPTOTIC BEHAVIOR OF THE CONSTANTS IN GENERALIZED KHINTCHINE'S INEQUALITY

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Abstract. We establish an asymptotic behavior of the constants in Khintchine's inequality for independent random variables of mean zero.

1. Introduction

Let $(r_n)_{n=1}^{\infty}$ be the sequence of Rademacher functions

$$r_n(t) = \text{sign}\,\sin(2^n \pi t), \quad t \in [0,1].$$

According to Khintchine's inequality [1] for each $n \in \mathbb{N}$, $p \in [1,+\infty)$, and a sequence $(a_k)_{k=1}^{\infty}$ of reals $a_k \in \mathbb{R}$ we have

$$A_p^{(0)}\Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}} \le \Big\|\sum_{k=1}^{n} a_k r_k\Big\|_p \le B_p^{(0)}\Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}}, \tag{1.1}$$

where $||\cdot||_p$ is the norm of $L_p = L_p[0,1]$,

$$A_p^{(0)} = \begin{cases} \frac{1}{\sqrt{2}}, & 1 \le p < 2, \\ 1, & 2 \le p < +\infty, \end{cases} \qquad B_p^{(0)} = \begin{cases} 1, & 1 \le p \le 2, \\ O(\sqrt{p}), & 2 < p < +\infty. \end{cases}$$

Recall that a sequence $(f_n)_{n=1}^{\infty}$ of measurable functions $f_n : [0,1] \to \mathbb{R}$ is called *a sequence of independent random variables (i.r.v.)* if for each $n \in \mathbb{N}$ and any intervals $[a_1,b_1],\dots,[a_n,b_n] \subseteq \mathbb{R}$, the following equality holds:

$$\mu\Big(\bigcap_{k=1}^{n} f_k^{-1}([a_k,b_k])\Big) = \prod_{k=1}^{n} \mu\{f_k^{-1}([a_k,b_k])\}.$$

On the other hand, Khintchine's inequality was generalized in [2] for k-tuple products of mean zero i.r.v. In particular, for each $n \in \mathbb{N}$, each collection $(y_k)_{k=1}^{n}$ of i.r.v. $y_k \in L_p$, $p > 1$, such that $\int_0^1 y_k d\mu = 0$ and for each collection of reals, $(a_k)_{k=1}^{n}$ we have

$$\frac{\alpha_q A_p^{(0)}}{p^*}\Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}} \le \Big\|\sum_{k=1}^{n} a_k y_k\Big\|_p \le \beta_r B_p^{(0)} p^*\Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}}, \tag{1.2}$$

where $r = \max\{p,2\}, q = \min\{p,2\}, p^* = \max\{p, \frac{p}{p-1}\} - 1, a_k \in \mathbb{R}, 1 \le k \le n$, $\alpha_q = \inf\limits_{1\le k\le n} ||y_n||_q$ and $\beta_r = \sup\limits_{1\le k\le n} ||y_n||_r$.

We remark that inequality (1.2) can be obtained for the case of $p = 1$ ([2]).

Moreover, inequality (1.2) can be written in the following way:

$$\alpha_q A_p\Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}} \le \Big\|\sum_{k=1}^{n} a_k y_k\Big\|_p \le \beta_r B_p\Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}}, \tag{1.3}$$

where A_p and B_p are the largest and the smallest constants respectively such that this inequality holds for each $n \in \mathbb{N}$, each collection $(a_k)_{k=1}^{n}$ of reals a_k and each collection $(y_k)_{k=1}^{n}$ of mean zero independent random variables y_k.

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In particular, we have

$$B_p = \sup\left\{ \frac{\|\sum_{k=1}^{n} a_k y_k\|_p}{\beta_r} : n \in \mathbb{N}, \sum_{k=1}^{n} a_k^2 = 1, y_1, \ldots, y_n \in L_p\text{– i.r.v.} \right\}$$

for each $p \geq 1$.

Note that (1.2) and $B_p^{(0)} \leq \sqrt{p}$ imply $B_p \leq p\sqrt{p}$. So, the question on the order of magnitude for the function $f(p) = B_p$ naturally arises.

In [3], using standard methods and some combinatorial arguments we proved that $B_{2p} \leq p$ for all $p \in \mathbb{N}$.

So, the question of a similar estimation for B_p for each $p \geq 1$ naturally arises as well.

In this paper we clarify arguments of [3], and with the help of a generalization of the Riesz-Thorin theorem we show that $B_p \leq 2\sqrt{e(p+2)}$ for all $p \geq 1$.

2. The upper estimate of B_{2p} for $p \in \mathbb{N}$

In this section we develop a method from [3] and obtain an upper estimate for B_{2p} with $p \in \mathbb{N}$.

We recall that $(a_1 + \cdots + a_n)^m = \sum_{k_1+\cdots+k_n=m} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k^n}$ for each reals $a_1, \ldots, a_n \in \mathbb{R}$ and $m \in \mathbb{N}$, where $\gamma(k_1, \ldots, k_n) = \frac{m!}{k_1! \ldots k_n!}$.

We need the following combinatorial statements.

Proposition 2.1. *Let $k_1, \ldots, k_n$, $m_1, \ldots, m_n$ be natural positive reals such that $k_1 + \cdots + k_n = m_1 + \cdots + m_n = 2p$, $k_1, \ldots, k_n, m_1, \ldots, m_n \neq 1$ and $|k_i - m_i| \leq 1$ for $1 \leq i \leq n$. Then $\frac{\gamma(k_1,\ldots,k_n)}{\gamma(m_1,\ldots,m_n)} \leq \left(\frac{2e}{3}\right)^p$.*

Proof. Since $k_1 + \cdots + k_n = m_1 + \cdots + m_n$, the sets $\{i : k_i = m_i + 1\}$ and $\{i : k_i = m_i - 1\}$ have the same quantity of elements which we denote by s. Let $m_1 = k_1 - 1, m_2 = k_2 + 1, \ldots, m_{2s-1} = k_{2s-1} - 1, m_{2s} = k_{2s} + 1$. Then

$$A = \frac{\gamma(k_1, \ldots, k_n)}{\gamma(m_1, \ldots, m_n)} = \frac{k_2 + 1}{k_1} \cdot \frac{k_4 + 1}{k_3} \cdots \frac{k_{2s} + 1}{k_{2s-1}}.$$

The expression A attains it's most value when $k_1, k_3, \ldots, k_{2s-1}$ attain their least values. Since $m_{2i-1} \neq 1$ and $k_{2i-1} = m_{2i-1} + 1 \neq 1$, $m_{2i-1} \geq 2$ and $k_{2i-1} \geq 3$ for $1 \leq i \leq s$. Then using the Cauchy inequality we obtain

$$A \leq \frac{(k_2+1)\ldots(k_{2s}+1)}{3^s} \leq \frac{\left(\frac{k_2+\cdots+k_{2s}+s}{s}\right)^s}{3^s} \leq \left(\frac{k_1+\cdots+k_{2s}-2s}{3s}\right)^s \leq \left(\frac{2p-2s}{3s}\right)^s.$$

We consider the function $f(x) = \left(\frac{p}{x} - 1\right)^x$ where $x \in (0, \frac{p}{2}]$. Since $f(\frac{p}{2}) = \lim_{x\to 0} f(x) = 1$, by Rolle's theorem, the function f reaches it's maximum at some point $x_0 \in (0, \frac{p}{2})$, where $f'(x_0) = 0$, i.e. $\ln f(x_0) = \frac{px_0}{p-x_0}$. Hence,

$$f(x_0) = e^{\frac{px_0}{p-x_0}} \leq e^p.$$

Therefore, taking into account that $s \leq p$ and $f(x) \leq 1$ for each $x \in [\frac{p}{2}, p]$ we obtain

$$A \leq \left(\frac{2}{3}\right)^s \cdot \left(\frac{p}{s} - 1\right)^s \leq \left(\frac{2}{3}\right)^s f(x_0) \leq \left(\frac{2e}{3}\right)^p.$$

$\square$

Proposition 2.2. *Let $p \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$.*

Then $\sum_{\substack{k_1+\cdots+k_n=2p \\ k_1,\ldots,k_n\neq 1}} \gamma(k_1, \ldots, k_n) a_1^{k_1} \ldots a_n^{k_n} \leq \left(2ep \cdot \sum_{k=1}^{n} a_k^2\right)^p.$

Proof. Without loss of generality we can assume that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Let $A = \{(k_1, k_2, \ldots, k_n) : k_i \neq 1,\ k_1 + k_2 + \cdots + k_n = 2p\}$ and $B = \{(m_1, m_2, \ldots, m_n) : m_i \text{ is even}, \ m_1 + m_2 + \cdots + m_n = 2p\}$. Now we construct a mapping $\varphi : A \to B$. Fix

$(k_1, \dots, k_n) \in A$. Denote by I the set of all numbers $1 \le i \le n$ such that k_i is even, and by J the set of all numbers $1 \le i \le n$ such that k_i is odd. Since $k_1 + \dots + k_n = 2p$, the set J has even quantity of elements. Let $J = \{j_1, \dots, j_{2s}\}$ where $j_1 \le j_2 \le \dots \le j_{2s}$. Then we set

$$m_i = \begin{cases} k_i, & i \in I, \\ k_i + 1, & i = j_{2l-1}, \quad 1 \le l \le s, \\ k_i - 1, & i = j_{2l}, \quad 1 \le l \le s, \end{cases}$$

for each $1 \le i \le n$ and $(m_1, \dots, m_n) = \varphi(k_1, \dots, k_n)$.

Since the reals a_i decrease and the indices j_l increase,

$$a_1^{k_1} \dots a_n^{k_n} \le a_1^{m_1} \dots a_n^{m_n}. \tag{2.1}$$

Now since $|k_i - m_i| \le 1$ for $1 \le i \le n$, by Proposition 2.1 we have $\gamma(k_1, \dots, k_n) \le \left(\frac{2e}{3}\right)^p \gamma(m_1, \dots, m_n)$.

Thus, if $(m_1, \dots, m_n) = \varphi(k_1, \dots, k_n)$ then

$$\gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} \le \left(\frac{2e}{3}\right)^p \gamma(m_1, \dots, m_n) a_1^{m_1} \dots a_n^{m_n}. \tag{2.2}$$

For each $(m_1, \dots, m_n) \in B$ we put $C = C_{m_1, \dots, m_n} = \{(k_1, \dots, k_n) : \varphi(k_1, \dots, k_n) = (m_1, \dots, m_n)\}$ and $N = N_{m_1, \dots, m_n} = \{i \le n : m_i \ne 0\}$.

Note that for each $(k_1, \dots, k_n) \in C$ we have $k_i = m_i$ for $i \notin N$ and $k_i \in \{m_i - 1, m_i, m_i + 1\}$ for $i \in N$. Furthermore, $m_i \ge 2$ for $i \in N$. So $|N| \le p$. Thus, by the multiplication principle of ([4]) we have $|C| \le 3^p$.

Taking into account (2.2) we obtain

$$\begin{aligned} \sum_{(k_1, \dots, k_n) \in A} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} &\le \left(\frac{2e}{3}\right)^p \sum_{(k_1, \dots, k_n) \in A} \gamma(\varphi(k_1, \dots, k_n)) a_1^{m_1} \dots a_n^{m_n} \\ &\le \left(\frac{2e}{3}\right)^p \sum_{(m_1, \dots, m_n) \in B} |C_{m_1, \dots, m_n}| \, \gamma(m_1, \dots, m_n) a_1^{m_1} \dots a_n^{m_n} \\ &\le (2e)^p \sum_{l_1 + \dots + l_n = p} \gamma(2l_1, \dots, 2l_n) a_1^{2l_1} \dots a_n^{2l_n}. \end{aligned}$$

Now since

$$\begin{aligned} \gamma(2l_1, \dots, 2l_s) &= \frac{2p!}{(2l_1)! \dots (2l_s)!} \\ &= \frac{(p+1) \dots 2p}{(l_1 + 1) \dots 2l_1 \cdot (2l_s + 1) \dots 2l_s} \cdot \gamma(l_1, ..., l_s) \\ &\le \frac{(2p)^p}{2^p} \gamma(l_1, \dots, l_s) = p^p \gamma(l_1, \dots, l_s) \end{aligned}$$

for each collection of reals $l_1, \dots, l_s \ge 1, l_1 + \dots + l_s = p$, one has that

$$\begin{aligned} &\sum_{(k_1, \dots, k_n) \in A} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} \\ &\le (2ep)^p \sum_{l_1 + \dots + l_n = p} \gamma(l_1, \dots, l_n) a_1^{2l_1} \dots a_n^{2l_n} = \Big(2ep \cdot \sum_{k=1}^n a_k^2\Big)^p. \end{aligned}$$

□

The following theorem is the main result of this section.

Theorem 2.3. $B_{2p} \le \sqrt{2ep}$ *for each* $p \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ and a collection $(y_k)_{k=1}^n$ of i.r.v. $y_k \in L_p$ of mean zero. Note that since the random variables $y_1, \dots, y_n$ are independent,

$$\int_0^1 y_1^{k_1} \dots y_n^{k_n} d\mu = \int_0^1 y_1^{k_1} d\mu \cdots \int_0^1 y_n^{k_n} d\mu = 0,$$

if $k_i = 1$ for at least one of the integers. On the other hand,

$$\int_0^1 |y_1^{k_1} \dots y_n^{k_n}|\, d\mu = \int_0^1 |y_1^{k_1}|\, d\mu \dots \int_0^1 |y_n^{k_n}|\, d\mu$$
$$\le ||y_1||_{k_1}^{k_1} \dots ||y_n||_{k_n}^{k_n} \le ||y_1||_{2p}^{k_1} \dots ||y_n||_{2p}^{k_n} \le \beta_{2p}^{k_1+\dots+k_n} = \beta_{2p}^{2p}.$$

Observe that it is sufficient to consider the case of $a_1, \dots, a_n \ge 0$.

Now we show that

$$\Big\| \sum_{k=1}^n a_k y_k \Big\|_{2p} \le \beta_{2p} \sqrt{2ep} \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}}.$$

Using Proposition 2.2, we obtain

$$\int_0^1 \Big(\sum_{k=1}^n a_k y_k \Big)^{2p} d\mu = \sum_{k_1+\dots+k_n=2p} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} \cdot \int_0^1 y_1^{k_1} \dots y_n^{k_n} d\mu$$
$$= \sum_{\substack{k_1+\dots+k_n=2p \\ k_i \ne 1}} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} \cdot \int_0^1 y_1^{k_1} \dots y_n^{k_n} d\mu$$
$$\le \sum_{\substack{k_1+\dots+k_n=2p \\ k_i \ne 1}} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} \cdot \int_0^1 |y_1^{k_1} \dots y_n^{k_n}| d\mu$$
$$\le \beta_{2p}^{2p} \sum_{\substack{k_1+\dots+k_n=2p \\ k_1, \dots, k_n \ne 1}} \gamma(k_1, \dots, k_n) a_1^{k_1} \dots a_n^{k_n} \le \beta_{2p}^{2p} \cdot \Big(2ep \cdot \sum_{k=1}^n a_k^2 \Big)^p.$$

Then

$$\Big\| \sum_{k=1}^n a_k y_k \Big\|_{2p} \le \beta_{2p} \sqrt{2ep} \Big(\sum_{k=1}^n a_k^2 \Big)^{\frac{1}{2}},$$

and thus,

$$B_{2p} \le \sqrt{2ep}.$$

□

3. A generalization of the Riesz-Thorin theorem

In this section we prove a version of the Riesz-Thorin interpolation theorem [5] which we will use to obtain an upper estimate of B_p for each $p \ge 1$. The proof of this version is similar to the proof of the classic theorem (see [5]).

We need Hadamard's theorem about three lines [5].

Theorem 3.1. (Hadamard). *Let $\Pi = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$, $f : \Pi \to \mathbb{C}$ be a bounded and analytic in Π function. Then for each $\theta \in [0, 1]$ we have*

$$M_\theta \le M_0^{1-\theta} \cdot M_1^\theta,$$

where $M_\theta = \sup\{|f(\theta + iy)| : y \in \mathbb{R}\}$.

Now assume $n \in \mathbb{N}$, reals satisfy $1 \le p_0^{(1)} \le p_1^{(1)} < +\infty, \dots, 1 \le p_0^{(n)} \le p_1^{(n)} < +\infty$, $1 \le q_0 \le q_1 < +\infty$, $L_{p_j^{(i)}}, 1 \le i \le n, j \in \{0, 1\}$ are the spaces of $p_j^{(i)}$-integrable functions $z : [0, 1] \to \mathbb{K}$, where $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, $X_1 \subseteq L_{p_1^{(1)}}, \dots, X_n \subseteq L_{p_1^{(n)}}$ are linear subspaces which contain all finite valued simple functions on $[0, 1]$, $Y \subseteq L_{q_1}$ is a linear subspace, $T_1 : X_1 \to Y, \dots, T_n : X_n \to Y$ are operators, $X = \bigoplus_{i=1}^n X_i$ and $T : X \to Y$, $T(x_1 + \dots + x_n) = T_1(x_1) + \dots + T_n(x_n)$, where $x_1 \in X_1, \dots, x_n \in X_n$.

For each $\alpha \in (0, 1)$ and $1 \le k \le n$ we denote by $p_\alpha^{(k)}$ the real $p_\alpha^{(k)} \in [p_0^{(k)}, p_1^{(k)}]$ such that $\frac{1}{p_\alpha^{(k)}} = \frac{1-\alpha}{p_0^{(k)}} + \frac{\alpha}{p_1^{(k)}}$ and by q_α the real $q_\alpha \in [q_0, q_1]$ such that $\frac{1}{q_\alpha} = \frac{1-\alpha}{q_0} + \frac{\alpha}{q_1}$.

Let $\tau : \mathbb{R}^n \to \mathbb{R}$ be any norm on $\mathbb{R}^n$. For each $\alpha \in [0,1]$ by τ_α we denote the norm on X defined by

$$\tau_\alpha(x) = \tau_\alpha(x_1 + \cdots + x_n) = \tau(\|x_1\|_{p_\alpha^{(1)}}, \ldots, \|x_n\|_{p_\alpha^{(n)}}).$$

Then we set $S_\alpha = T : (X, \tau_\alpha) \to (Y, \|\cdot\|_{q_\alpha})$.

Theorem 3.2. *Let $\mathbb{K} = \mathbb{C}$. Then $\|S_\alpha\| \le \|S_0\|^{1-\alpha} \cdot \|S_1\|^\alpha$ for each $\alpha \in [0,1]$.*

Proof. Fix $\alpha \in [0,1]$. Denote $q = q_\alpha$, $p_1 = p_\alpha^{(1)}$, …, $p_n = p_\alpha^{(n)}$ and $K = \|S_0\|^{1-\alpha} \cdot \|S_1\|^\alpha$. It is enough to prove that

$$\|S_\alpha x\|_q \le K \cdot \tau_\alpha(x) \tag{3.1}$$

for each $x \in X$.

Since all norms on $\mathbb{R}^n$ are equivalent, the τ-convergence is equivalent to the coordinatewise convergence in $\mathbb{R}^n$. Moreover, the set of all simple functions is dense in every space L_p. So it is sufficient to prove (3.1) for any simple functions $x_1, \ldots, x_n$.

Let $x_k = \sum\limits_{i=1}^{m_k} a_i^{(k)} \chi_{A_{i,k}}$, $x = x_1 + \cdots + x_n$ and $\tau_\alpha(x) = 1$. Then (3.1) takes the following form:

$$\|S_\alpha x\|_q \le K. \tag{3.2}$$

Hahn-Banach's theorem implies that (3.2) is equivalent to the inequality $|f(S_\alpha x)| \le K$ for each $f \in (L_q)^*$ with $\|f\| = 1$. Thus, it is sufficient to check that for each simple function $v = \sum\limits_{i=1}^{l} b_i \chi_{B_1}$ with $\|v\|_{q'} = 1$ where $\frac{1}{q} + \frac{1}{q'} = 1$ the following inequality holds

$$\Big| \int_0^1 v(t) S_\alpha x(t) d\mu \Big| \le K. \tag{3.3}$$

We consider the functions $\varphi_1, \ldots, \varphi_n, \psi : \mathbb{C} \to \mathbb{C}$, $\varphi_k(z) = \frac{1-z}{p_0^{(k)}} + \frac{z}{p_1^{(k)}}$, $\psi(z) = \frac{1-z}{q_0'} + \frac{z}{q_1'}$, where $\frac{1}{q_0} + \frac{1}{q_0'} = \frac{1}{q_1} + \frac{1}{q_1'} = 1$. For each $z \in \mathbb{C}$ we set

$$x_{k,z} = \sum_{i=1}^{m_k} |a_i^{(k)}|^{p_k \cdot \varphi_k(z)} \cdot \text{sign } a_i^{(k)} \cdot (\|x_k\|_{p_k})^{1 - p_k \varphi_k(z)} \cdot \chi_{A_{i,k}}$$

for $1 \le k \le n$ if $x_k \ne 0$ and $x_{k,z} = 0$ if $x_k = 0$,

$$x_z = (x_{1,z}, \ldots, x_{n,z})$$

and

$$v_z = \sum_{i=1}^{l} |b_i|^{q' \psi(z)} \text{sign } b_i \chi_{B_i},$$

where $\text{sign } z = \begin{cases} 0, & z = 0, \\ \frac{z}{|z|}, & z \ne 0 \end{cases}$ for any $z \in \mathbb{C}$.

Observe that $x_\alpha = x$, $v_\alpha = v$,

$$p_k \cdot \varphi_k(iy) = p_k \bigg(\frac{1 - iy}{p_0^{(k)}} + \frac{iy}{p_1^{(k)}} \bigg) = \frac{p_k}{p_0^{(k)}} + iy \bigg(\frac{p_k}{p_1^{(k)}} - \frac{p_k}{p_0^{(k)}} \bigg).$$

Analogously

$$p_k \cdot \varphi_k(1 + iy) = \frac{p_k}{p_1^{(k)}} + iy \bigg(\frac{p_k}{p_1^{(k)}} - \frac{p_k}{p_0^{(k)}} \bigg),$$

$$q' \cdot \psi(iy) = \frac{q'}{q_0'} + iy \left(\frac{q'}{q_1'} - \frac{q'}{q_0'} \right), \quad q' \cdot \psi(1 + iy) = \frac{q'}{q_1'} + iy \left(\frac{q'}{q_1'} - \frac{q'}{q_0'} \right).$$

For each k, $1 \le k \le n$, we have

$$\begin{aligned}
\|x_{k,iy}\|_{p_0^{(k)}} &= \left(\int_0^1 \left| \sum_{j=1}^{m_k} |a_j^{(k)}|^{p_k\varphi_k(iy)} \cdot \operatorname{sign} a_j^{(k)} \cdot (\|x_k\|_{p_k})^{1-p_k\varphi_k(iy)} \cdot \chi_{A_{jk}} \right|^{p_0^{(k)}} d\mu \right)^{\frac{1}{p_0^{(k)}}} \\
&= \|x_k\|_{p_k}^{1-\frac{p_k}{p_0^{(k)}}} \left(\sum_{j=1}^{m_k} \int_{A_{jk}} \left| |a_j^{(k)}|^{p_k\varphi_k(iy)} \right|^{p_0^{(k)}} d\mu \right)^{\frac{1}{p_0^{(k)}}} \\
&= \|x_k\|_{p_k}^{1-\frac{p_k}{p_0^{(k)}}} \left(\sum_{j=1}^{m_k} \int_{A_{jk}} \left(|a_j^{(k)}|^{\frac{p_k}{p_0^{(k)}}} \right)^{p_0^{(k)}} d\mu \right)^{\frac{1}{p_0^{(k)}}} \\
&= \|x_k\|_{p_k}^{1-\frac{p_k}{p_0^{(k)}}} \cdot \|x_k\|_{p_k}^{\frac{p_k}{p_0^{(k)}}} = \|x_k\|_{p_k}.
\end{aligned}$$

Analogously $\|x_{k,1+iy}\|_{p_1^{(k)}} = \|x_k\|_{p_k}$, $\|v_{iy}\|_{q_0'} = \|v_{1+iy}\|_{q_1'} = \|v\|_{q'} = 1$. Then $(\|x_{1,iy}\|_{p_0^{(1)}}, \dots, \|x_{n,iy}\|_{p_0^{(n)}}) = (\|x_1\|_{p_1}, \dots, \|x_n\|_{p_n})$, $(\|x_{1,1+iy}\|_{p_1^{(1)}}, \dots, \|x_{n,1+iy}\|_{p_1^{(n)}}) = (\|x_1\|_{p_1}, \dots, \|x_n\|_{p_n})$ and

$$\tau_0(x_{iy}) = 1, \quad \tau_1(x_{1+iy}) = 1. \tag{3.4}$$

We define a function $f : \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned}
f(z) &= \int_0^1 v_z T x_z d\mu = \int_0^1 \left(\sum_{i=1}^{l} |b_i|^{q'\psi(z)} \operatorname{sign} b_i \chi_{B_i} \right) \cdot T(x_{1,z} + \dots + x_{n,z})\, d\mu \\
&= \sum_{i=1}^{l} \sum_{k=1}^{n} \sum_{j=1}^{m_k} (\|x_k\|_{p_k})^{1-p_k\varphi_k(z)} |b_i|^{q'\psi(z)} |a_j^{(k)}|^{p_k\varphi_k(z)} \operatorname{sign} b_i \operatorname{sign} a_j^{(k)} \int_{B_i} T\chi_{A_{j,k}} d\mu.
\end{aligned}$$

Since all the functions $a^z, a > 0$, are analytic and bounded in Π, the function f is the same in Π, as a linear combination of such exponential functions.

Now according to Holder's inequality and (3.4), we obtain

$$\begin{aligned}
|f(iy)| &= \left| \int_0^1 v_{iy} T x_{iy} d\mu \right| \le \|v_{iy}\|_{q_0'} \cdot \|T x_{iy}\|_{q_0} = \|v_{iy}\|_{q_0'} \cdot \|S_0 x_{iy}\|_{q_0} \\
&\le \|v_{iy}\|_{q_0'} \cdot \|S_0\| \cdot \tau_0(x_{iy}) = \|S_0\|.
\end{aligned}$$

Analogously $|f(1+iy)| \le \|S_1\|$. Now using Theorem 3.1 and the fact that $x_\alpha = x$ and $v_\alpha = v$, we obtain

$$\begin{aligned}
\left| \int_0^1 v_\alpha S_\alpha x d\mu \right| &= \left| \int_0^1 v_\alpha T x_\alpha d\mu \right| = |f(\alpha)| \le \sup_{y \in \mathbb{R}} |f(\alpha + iy)| \\
&\le (\sup_{y \in \mathbb{R}} |f(iy)|)^{1-\alpha} \cdot (\sup_{y \in \mathbb{R}} |f(1+iy)|)^{\alpha} \le \|S_0\|^{1-\alpha} \cdot \|S_1\|^{\alpha} = K.
\end{aligned}$$

□

The following result gives an estimate for the norm of operator S_α in the case when $\mathbb{K} = \mathbb{R}$.

Corollary 3.3. *Let $\mathbb{K} = \mathbb{R}$. Then $\|S_\alpha\| \le 2\|S_0\|^{1-\alpha} \cdot \|S_1\|^{\alpha}$ for each $\alpha \in [0,1]$.*

Proof. Let for each $\alpha \in [0,1]$ $S_\alpha^{\mathbb{C}}$ be the extension of the operator S_α to the space $L_{p_\alpha}^{\mathbb{C}}$ of functions $z : [0,1] \to \mathbb{C}$.

Note that $\|S_\alpha^{\mathbb{C}}\| \ge \|S_\alpha\|$ and $\|S_\alpha^{\mathbb{C}}\| \le \sup_{\|z\| \le 1} (\|S_\alpha x\| + \|S_\alpha y\|) \le 2\|S_\alpha\|$. By Theorem 2.2 we have

$$\|S_\alpha\| \le \|S_\alpha^{\mathbb{C}}\| \le \|S_0^{\mathbb{C}}\|^{1-\alpha} \cdot \|S_1^{\mathbb{C}}\|^{\alpha} \le 2 \cdot \|S_0\|^{1-\alpha} \cdot \|S_1\|^{\alpha}.$$

□

4. The main result

The following theorem gives an upper estimate for B_p.

Theorem 4.1. *Let* $2 \le p_0 < p_1$, $q \ge 2$, $\beta \in [0,1]$ *and* $\frac{1}{q} = \frac{1-\beta}{p_0} + \frac{\beta}{p_1}$. *Then* $B_q \le 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}$.

Proof. Fix $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ such that $\sum\limits_{k=1}^{n} a_k^2 = 1$ and i.r.v. $x_1, \dots, x_n \in L_q$. It is sufficient to prove that

$$\frac{\|\sum\limits_{k=1}^{n} a_k x_k\|_q}{\sup\limits_{1\le k\le n} \|x_k\|_q} \le 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}.$$

At first we consider the case $x_1, \dots, x_n \in L_{p_1}$.

For each $1 \le k \le n$ we set $\psi_k : L_{p_1}[0,1] \to L_{p_1}([0,1]^n)$, $\psi_k(x)(t_1, \dots, t_n) = x(t_k)$. Observe that all the mappings ψ_k are isometric embeddings.

Since the spaces $L_{p_1}[0,1]$ and $L_{p_1}([0,1]^n)$ are isometrically isomorphic [6], there exists a linear isometry $\psi : L_{p_1}([0,1]^n) \to L_{p_1}[0,1]$. Now let $\varphi_k = \psi \circ \psi_k$ for each $1 \le k \le n$.

Note that for any $z_1, \dots, z_n \in L_{p_1}$ the functions $\psi_1(z_1), \dots, \psi_n(z_n)$ are i.r.v. Moreover, for each i.r.v. $z_1, \dots, z_n \in L_{p_1}$ and reals $b_1, \dots, b_n \in \mathbb{R}$ we have

$$\|\sum_{k=1}^{n} b_k z_k\|_s = \|\sum_{k=1}^{n} b_k \varphi_k(z_k)\|_s = \|\sum_{k=1}^{n} b_k \psi_k(z_k)\|_s \tag{4.1}$$

for any $s \in [p_0, p_1]$.

Let $q_0 = p_0^{(1)} = p_0^{(2)} = \dots = p_0^{(n)} = p_0$, $q_1 = p_1^{(1)} = p_1^{(2)} = \dots = p_1^{(n)} = p_1$, $X_1 = X_2 = \dots = X_n = L_{p_1}$, $Y = L_{p_1}$, $T_1 = a_1\varphi_1, \dots, T_n = a_n\varphi_n$, $X = \bigoplus\limits_{k=1}^{n} X_k$ and operator $T : X \to Y$ acts by the following rule $T(z_1 + \dots + z_n) = T_1(z_1) + \dots + T_n(z_n) = \sum\limits_{k=1}^{n} a_k \varphi_k(z_k)$, where $z_1 \in X_1, \dots, z_n \in X_n$. Furthermore, by τ we denote the maximum-norm on $\mathbb{R}^n$. Let for each $\alpha \in [0,1]$ and $1 \le k \le n$ reals $p_\alpha^{(k)}$ and q_α, norm τ_α and operator S_α are defined analogously as in Section 3.

Taking into account that $\psi_1(z_1), \dots, \psi_n(z_n)$ are i.r.v. and (4.1) we have

$$\|S_0\| = \sup_{z\ne 0} \frac{\|S_0(z)\|_{p_0}}{\tau_0(z)} = \sup_{z\ne 0} \frac{\|\sum\limits_{k=1}^{n} a_k\varphi_k(z_k)\|_{p_0}}{\sup\limits_{1\le k\le n} \|z_k\|_{p_0}} = \sup_{z\ne 0} \frac{\|\sum\limits_{k=1}^{n} a_k\psi_k(z_k)\|_{p_0}}{\sup\limits_{1\le k\le n} \|\psi_k(z_k)\|_{p_0}} \le B_{p_0},$$

where $z = z_1 + \dots + z_n$. Similarly $\|S_1\| \le B_{p_1}$.

Now by Corollary 3.3, equality (4.1) and equality $q = q_\beta$ we have

$$\frac{\|\sum\limits_{k=1}^{n} a_k x_k\|_q}{\sup\limits_{1\le k\le n} \|x_k\|_q} \overset{(4.1)}{=} \frac{\|\sum\limits_{k=1}^{n} a_k \varphi_k(x_k)\|_q}{\sup\limits_{1\le k\le n} \|x_k\|_q} = \frac{\|S_\beta(x_1 + \dots + x_n)\|_q}{\tau_\beta(x_1 + \dots + x_n)}$$
$$\le \|S_\beta\| \le 2\|S_0\| \cdot \|S_1\| \le 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}.$$

Now we prove the general case $x_1, \dots, x_n \in L_p$. We consider a sequence $\left((x_k^{(i)})_{k=1}^n\right)_{i=1}^{\infty}$ of collections $x_1^{(i)}, \dots, x_n^{(i)}$ of i.r.v. $x_k^{(i)} \in L_\infty$ such that $\|x_k^{(i)} - x_k\|_q \le \frac{1}{i}$ for each $i \in \mathbb{N}$ and $k = 1, \dots, n$. Accordingly to the proved above we have

$$\frac{\|\sum\limits_{k=1}^{n} a_k x_k^{(i)}\|_q}{\sup\limits_{1\le k\le n} \|x_k^{(i)}\|_q} \le 2B_{p_0}^{1-\beta} \cdot B_{p_1}^{\beta}.$$

It remains to pass i to infinity. □

The following result gives an upper and a lower estimates for B_p if $p \ge 1$.

Theorem 4.2. $\sqrt{\frac{[\frac{p}{2}]}{2}} \le B_p \le 2\sqrt{e(p+2)}$ *for each* $p \ge 1$.

Proof. We remark that $B_p \ge B_p^{(0)}$ for each $p \ge 1$. We will show that $B_p^{(0)} \ge \sqrt{\frac{[\frac{p}{2}]}{2}}$.

Fix $p, n \in \mathbb{N}$, $n \ge 2p$ and reals $a_1 = \cdots = a_n = 1$. Observe that

$$\begin{aligned}\|a_1 r_1 + \cdots + a_n r_n\|_{2p}^{2p} = \|r_1 + \cdots + r_n\|_{2p}^{2p} = \sum_{k_1+\cdots+k_n=p} \gamma(2k_1, \ldots, 2k_n) \\ \ge C_n^p \gamma(2, \ldots, 2) = \frac{n!}{p!(n-p)!} \cdot \frac{(2p)!}{(2!)^p} \\ = \frac{n(n-1)\ldots(n-p+1)(p+1)\ldots 2p}{2^p} \ge \Big(\frac{np}{2}\Big)^p,\end{aligned}$$

that is,

$$\|r_1 + \cdots + r_n\|_{2p} \ge \sqrt{\frac{np}{2}} = \Big(\sum_{k=1}^n a_k^2\Big)^{\frac{1}{2}} \cdot \sqrt{\frac{p}{2}}.$$

Then for every $p \ge 1$ we have

$$\|r_1 + \cdots + r_n\|_{2p} \ge \|r_1 + \cdots + r_n\|_{2[\frac{p}{2}]} \ge \Big(\sum_{k=1}^n a_k^2\Big)^{\frac{1}{2}} \cdot \sqrt{\frac{[\frac{p}{2}]}{2}}$$

and thus, $B_p^{(0)} \ge \sqrt{\frac{[\frac{p}{2}]}{2}}$.

Now we will show that $B_p \le 2\sqrt{e(p+2)}$. Let $p \ge 2$. Then there exists a number $q \in \mathbb{N}$ such that $2q \le p \le 2(q+1)$. Now choose $\alpha \in [0,1]$ such that $\frac{1}{p} = \frac{1-\alpha}{2q} + \frac{\alpha}{2q+2}$. Using theorems 2.3 and 4.1 we obtain

$$\begin{aligned}B_p \le 2B_{2q}^{1-\alpha} \cdot B_{2q+2}^{\alpha} \le 2\left(\sqrt{2eq}\right)^{1-\alpha} \cdot \left(\sqrt{2e(q+1)}\right)^{\alpha} \\ \le 2\sqrt{e(2q+2)} \le 2\sqrt{e(p+2)}.\end{aligned}$$

Now let $1 \le p < 2$. Then

$$B_p \le B_2 \le \sqrt{2e} < 2\sqrt{e(p+2)}.$$

□

The following question naturally arises in a connection with Theorem 4.2.

Question 4.3. *Does there exist* $\lim\limits_{p\to\infty} \frac{B_p}{\sqrt{p}}$?

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