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$\hat{g}$-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract. Characterizations and properties of $\mathcal{I}_{\hat{g}}$-closed sets and $\mathcal{I}_{\hat{g}}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\hat{g}}$-open sets. Also, it is established that an $\mathcal{I}_{\hat{g}}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

1. Introduction and Preliminaries

An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) A $\in \mathcal{I}$ and B $\subset$ A $\Rightarrow$ B $\in \mathcal{I}$ and (ii) A $\in \mathcal{I}$ and B $\in \mathcal{I} \Rightarrow$ A $\cup$ B $\in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [9] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A $\subseteq$ X, $A^*(\mathcal{I},\tau)=\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x)=\{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [8, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*(.) for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(A)=A \cup A^*(\mathcal{I},\tau)$ [18]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then (X,τ,$\mathcal{I}$) is called an ideal space. N is the ideal of all nowhere dense subsets in (X,τ). A subset A of an ideal space (X,τ,$\mathcal{I}$) is $\star$-closed [8] (resp. $\star$-dense in itself [6]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$). A subset A of an ideal space (X,τ,$\mathcal{I}$) is $\mathcal{I}_g$-closed [3] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open.

By a space, we always mean a topological space (X,τ) with no separation properties assumed. If A $\subseteq$ X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X,τ) and int*(A) will denote the interior of A in (X,τ^*). A subset A of a space (X,τ) is an α-open [15] (resp. semi-open [10], preopen [12]) set if A $\subseteq$ int(cl(int(A))) (resp. A $\subseteq$ cl(int(A)), A $\subseteq$ int(cl(A))). The family of all α-open sets in (X,τ), denoted by τ^α, is a topology on X finer than τ . The closure of A in (X,τ^α) is denoted by cl$_\alpha$(A). A subset A of a space (X,τ) is said to be g-closed [11] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open. A subset A of a space (X,τ) is said to be $\hat{g}$-closed [19] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. A subset A of a space (X,τ) is said to be $\hat{g}$-open [19] if its complement is $\hat{g}$-closed. The family of all $\hat{g}$-open sets in (X,τ) is a topology on X. The semi-closure [2] of a subset A of X, denoted by scl(A), is defined to be the intersection of all semi-closed sets containing A. An ideal $\mathcal{I}$ is said to be codense [4] or τ-boundary [14] if $\tau \cap \mathcal{I}=\{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [4] if PO(X) $\cap\, \mathcal{I}=\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X,τ). Every completely codense ideal is codense but not the converse [4]. The following Lemmas will be useful in the sequel.

Lemma 1.1. *Let (X,τ,$\mathcal{I}$) be an ideal space and A $\subseteq$ X. If A $\subseteq$ A^*, then A^* =cl(A^*) =cl(A)=cl*(A) [17, Theorem 5].*

Lemma 1.2. *Let (X,τ,$\mathcal{I}$) be an ideal space. Then $\mathcal{I}$ is codense if and only if G $\subseteq$ G^* for every semi-open set G in X [17, Theorem 3].*

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Lemma 1.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^* \subseteq \tau^\alpha$ [17, Theorem 6].*

Result 1.4. *If (X,τ) is a topological space, then every closed set is $\hat{g}$-closed but not conversely [1, Theorem 2.3].*

Lemma 1.5. *If $(X,\tau,\mathcal{I})$ is a $T_\mathcal{I}$ ideal space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [13, Corollary 2.2].*

Lemma 1.6. *Every g-closed set is $\mathcal{I}_g$-closed but not conversely [3, Theorem 2.1].*

2. $\mathcal{I}_{\hat{g}}$-CLOSED SETS

Definition 2.1. *A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}_{\hat{g}}$-closed if $A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open.*

Definition 2.2. *A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}_{\hat{g}}$-open if $X-A$ is $\mathcal{I}_{\hat{g}}$-closed.*

Theorem 2.3. *If $(X,\tau,\mathcal{I})$ is any ideal space, then every $\mathcal{I}_{\hat{g}}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Example 2.4. *Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{c\}\}$ and $\mathcal{I}=\{\emptyset\}$. Then $\mathcal{I}_{\hat{g}}$-closed sets are $\emptyset,X,\{a,b\}$ and $\mathcal{I}_g$-closed sets $\emptyset,X,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}$. It is clear that $\{a\}$ is $\mathcal{I}_g$-closed but it is not $\mathcal{I}_{\hat{g}}$-closed.*

The following theorem gives characterizations of $\mathcal{I}_{\hat{g}}$-closed sets.

Theorem 2.5. *If $(X,\tau,\mathcal{I})$ is any ideal space and $A \subseteq X$, then the following are equivalent.*
(a) A is $\mathcal{I}_{\hat{g}}$-closed.
(b) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
(c) For all $x \in cl^*(A)$, $scl(\{x\}) \cap A \neq \emptyset$.
(d) $cl^*(A)-A$ contains no nonempty semi-closed set.
(e) A^*-A contains no nonempty semi-closed set.

Proof. (a) $\Rightarrow$ (b) If A is $\mathcal{I}_{\hat{g}}$-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X and so $cl^*(A)=A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X. This proves (b).
(b) $\Rightarrow$ (c) Suppose $x \in cl^*(A)$. If $scl(\{x\}) \cap A=\emptyset$, then $A \subseteq X-scl(\{x\})$. By (b), $cl^*(A) \subseteq X-scl(\{x\})$, a contradiction, since $x \in cl^*(A)$.
(c) $\Rightarrow$ (d) Suppose $F \subseteq cl^*(A)-A$, F is semi-closed and $x \in F$. Since $F \subseteq X-A$ and F is semi-closed, then $A \subseteq X-F$ and F is semi-closed, $scl(\{x\}) \cap A = \emptyset$. Since $x \in cl^*(A)$ by (c), $scl(\{x\}) \cap A\neq\emptyset$. Therefore $cl^*(A)-A$ contains no nonempty semi-closed set.
(d) $\Rightarrow$ (e) Since $cl^*(A)-A=(A \cup A^*)-A= (A \cup A^*) \cap A^c=(A \cap A^c) \cup (A^* \cap A^c)=A^* \cap A^c= A^*-A$. Therefore A^*-A contains no nonempty semi-closed set.
(e) $\Rightarrow$ (a) Let $A \subseteq U$ where U is semi-open set. Therefore $X-U \subseteq X-A$ and so $A^* \cap (X-U) \subseteq A^* \cap (X-A)=A^*-A$. Therefore $A^* \cap (X-U) \subseteq A^*-A$. Since A^* is always closed set, so $A^* \cap (X-U)$ is a semi-closed set contained in A^*-A. Therefore $A^* \cap (X-U)=\emptyset$ and hence $A^* \subseteq U$. Therefore A is $\mathcal{I}_{\hat{g}}$-closed. □

Theorem 2.6. *Every $\star$-closed set is $\mathcal{I}_{\hat{g}}$-closed but not conversely.*

Proof. Let A be a $\star$-closed, then $A^* \subseteq A$. Let $A \subset U$ where U is semi-open. Hence $A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open. Therefore A is $\mathcal{I}_{\hat{g}}$-closed. □

Example 2.7. *Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{b,c\}\}$ and $\mathcal{I}=\{\emptyset,\{c\}\}$. Then $\mathcal{I}_{\hat{g}}$-closed sets are powerset of X and $\star$-closed sets are $\emptyset,X,\{a\},\{c\},\{a,c\},\{b,c\}$. It is clear that $\{b\}$ is $\mathcal{I}_{\hat{g}}$-closed set but it is not $\star$-closed.*

Theorem 2.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space. For every $A \in \mathcal{I}$, A is $\mathcal{I}_{\hat{g}}$-closed.*

Proof. Let A $\subseteq$ U where U is semi-open set. Since A*= $\emptyset$ for every A $\in \mathcal{I}$, then cl*(A)=A $\cup$ A*=A $\subseteq$ U. Therefore, by Theorem 2.5, A is $\mathcal{I}_{\hat{g}}$-closed. □

Theorem 2.9. *If $(X,\tau,\mathcal{I})$ is an ideal space, then A^* is always $\mathcal{I}_{\hat{g}}$-closed for every subset A of X.*

Proof. Let A* $\subseteq$ U where U is semi-open. Since (A*)* $\subseteq$ A* [8], we have (A*)* $\subseteq$ U whenever A* $\subseteq$ U and U is semi-open. Hence A* is $\mathcal{I}_{\hat{g}}$-closed. □

Theorem 2.10. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then every $\mathcal{I}_{\hat{g}}$-closed, semi-open set is $\star$-closed set.*

Proof. Since A is $\mathcal{I}_{\hat{g}}$-closed and semi-open. Then A* $\subseteq$ A whenever A $\subseteq$ A and A is semi-open. Hence A is $\star$-closed. □

Corollary 2.11. *If $(X,\tau,\mathcal{I})$ is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}_{\hat{g}}$-closed set, then A is $\star$-closed set.*

Corollary 2.12. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A be an $\mathcal{I}_{\hat{g}}$-closed set. Then the following are equivalent.*

a) A is a $\star$-closed set.

b) cl*(A)$-$A is a semi-closed set.

c) A^*-A is a semi-closed set.

Proof. (a) $\Rightarrow$ (b) If A is $\star$-closed, then A* $\subseteq$ A and so cl*(A)$-$A=(A $\cup$ A*)$-$A= $\emptyset$. Hence cl*(A)$-$A is semi-closed set.
(b) $\Rightarrow$ (c) Since cl*(A)$-$A=A$^*-$A and so A$^*-$A is semi-closed set.
(c) $\Rightarrow$ (a) If A$^*-$A is a semi-closed set, since A is $\mathcal{I}_{\hat{g}}$-closed set, by Theorem 2.5, A$^*-$A= $\emptyset$ and so A is $\star$-closed. □

Theorem 2.13. *Let $(X,\tau,\mathcal{I})$ be an ideal space . Then every $\hat{g}$-closed set is an $\mathcal{I}_{\hat{g}}$-closed set but not conversely.*

Proof. Let A be a $\hat{g}$-closed set. Then cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. We have cl*(A) $\subseteq$ cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. Hence A is $\mathcal{I}_{\hat{g}}$-closed. □

Example 2.14. *Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{a,c\}\}$ and $\mathcal{I}=\{\emptyset\ ,\{a\}\}$. Then $\mathcal{I}_{\hat{g}}$-closed sets are $\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\}$ and $\hat{g}$-closed sets are $\emptyset,X,\{b\},\{b,c\}$. It is clear that $\{a\}$ is $\mathcal{I}_{\hat{g}}$-closed set but it is not $\hat{g}$-closed.*

Theorem 2.15. *If $(X,\tau,\mathcal{I})$ is an ideal space and A is a $\star$-dense in itself, $\mathcal{I}_{\hat{g}}$-closed subset of X, then A is $\hat{g}$-closed.*

Proof. Suppose A is a $\star$-dense in itself, $\mathcal{I}_{\hat{g}}$-closed subset of X. Let A $\subseteq$ U where U is semi-open. Then by Theorem 2.5 (b), cl*(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. Since A is $\star$-dense in itself, by Lemma 1.1, cl(A)=cl*(A). Therefore cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. Hence A is $\hat{g}$-closed. □

Corollary 2.16. *If $(X,\tau,\mathcal{I})$ is any ideal space where $\mathcal{I}=\{\emptyset\}$, then A is $\mathcal{I}_{\hat{g}}$-closed if and only if A is $\hat{g}$-closed.*

Proof. From the fact that for $\mathcal{I}=\{\emptyset\}$, A*=cl(A) $\supseteq$ A. Therefore A is $\star$-dense in itself. Since A is $\mathcal{I}_{\hat{g}}$-closed, by Theorem 2.15, A is $\hat{g}$-closed. Conversely, by Theorem 2.13, every $\hat{g}$-closed set is $\mathcal{I}_{\hat{g}}$-closed set. □

Corollary 2.17. *If $(X,\tau,\mathcal{I})$ is any ideal space where $\mathcal{I}$ is codense and A is a semi-open, $\mathcal{I}_{\hat{g}}$-closed subset of X, then A is $\hat{g}$-closed.*

Proof. By Lemma 1.2, A is $\star$-dense in itself. By Theorem 2.15, A is $\hat{g}$-closed. □

Example 2.18. *Let* $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{a,c\}\}$ *and* $\mathcal{I}=\{\emptyset\}$. *Then g-closed sets are* $\emptyset,X,\{b\},\{a,b\},\{b,c\}$ *and* $\mathcal{I}_{\hat{g}}$*-closed sets are* $\emptyset,X,\{b\},\{b,c\}$. *It is clear that* $\{a,b\}$ *is g-closed set but it is not* $\mathcal{I}_{\hat{g}}$*-closed.*

Example 2.19. *Let* $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{a,c\}\}$ *and* $\mathcal{I}=\{\emptyset,\{a\}\}$. *Then g-closed sets are* $\emptyset,X,\{b\},\{a,b\},\{b,c\}$ *and* $\mathcal{I}_{\hat{g}}$*-closed sets are* $\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\}$. *It is clear that* $\{a\}$ *is* $\mathcal{I}_{\hat{g}}$*-closed set but it is not g-closed.*

Remark 2.20. *By Example 2.18 and Example 2.19, g-closed sets and* $\mathcal{I}_{\hat{g}}$*-closed sets are independent.*

Remark 2.21. *We have the following implications for the subsets stated above.*

$$\begin{array}{ccccc}
closed & \longrightarrow & \hat{g}-closed & \longrightarrow & g-closed \\
\downarrow & & \downarrow & & \downarrow \\
\star-closed & \longrightarrow & \mathcal{I}_{\hat{g}}-closed & \longrightarrow & \mathcal{I}_{g}-closed
\end{array}$$

Theorem 2.22. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space and* $A \subseteq X$. *Then A is* $\mathcal{I}_{\hat{g}}$*-closed if and only if* $A=F-N$ *where F is* $\star$*-closed and N contains no nonempty semi-closed set.*

Proof. If A is $\mathcal{I}_{\hat{g}}$-closed, then by Theorem 2.5 (e), $N=A^*-A$ contains no nonempty semi-closed set. If $F=cl^*(A)$, then F is $\star$-closed such that $F-N=(A \cup A^*)-(A^*-A)=(A \cup A^*) \cap (A^* \cap A^c)^c=(A \cup A^*) \cap ((A^*)^c \cup A)=(A \cup A^*) \cap (A \cup (A^*)^c)=A \cup (A^* \cap (A^*)^c)=A$.

Conversely, suppose $A=F-N$ where F is $\star$-closed and N contains no nonempty semi-closed set. Let U be a semi-open set such that $A \subseteq U$. Then $F-N \subseteq U \Rightarrow F \cap (X-U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$ then $A^* \subseteq F^*$ and so $A^* \cap (X-U) \subseteq F^* \cap (X-U) \subseteq F \cap (X-U) \subseteq N$. By hypothesis, since $A^* \cap (X-U)$ is semi-closed, $A^* \cap (X-U)=\emptyset$ and so $A^* \subseteq U$. Hence A is $\mathcal{I}_{\hat{g}}$-closed. □

Theorem 2.23. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space and* $A \subseteq X$. *If* $A \subseteq B \subseteq A^*$, *then* $A^*=B^*$ *and B is* $\star$*-dense in itself.*

Proof. Since $A \subseteq B$, then $A^* \subseteq B^*$ and since $B \subseteq A^*$, then $B^* \subseteq (A^*)^* \subseteq A^*$. Therefore $A^*=B^*$ and $B \subseteq A^* \subseteq B^*$. Hence proved. □

Theorem 2.24. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. If A and B are subsets of X such that* $A \subseteq B \subseteq cl^*(A)$ *and A is* $\mathcal{I}_{\hat{g}}$*-closed, then B is* $\mathcal{I}_{\hat{g}}$*-closed.*

Proof. Since A is $\mathcal{I}_{\hat{g}}$-closed, then by Theorem 2.5 (d), $cl^*(A)-A$ contains no nonempty semi-closed set. Since $cl^*(B)-B \subseteq cl^*(A)-A$ and so $cl^*(B)-B$ contains no nonempty semi-closed set. Hence B is $\mathcal{I}_{\hat{g}}$-closed. □

Corollary 2.25. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. If A and B are subsets of X such that* $A \subseteq B \subseteq A^*$ *and A is* $\mathcal{I}_{\hat{g}}$*-closed, then A and B are* $\hat{g}$*-closed sets.*

Proof. Let A and B be subsets of X such that $A \subseteq B \subseteq A^* \Rightarrow A \subseteq B \subseteq A^* \subseteq cl^*(A)$ and A is $\mathcal{I}_{\hat{g}}$-closed. By the above Theorem, B is $\mathcal{I}_{\hat{g}}$-closed. Since $A \subseteq B \subseteq A^*$, then $A^*=B^*$ and so A and B are $\star$-dense in itself. By Theorem 2.15, A and B are $\hat{g}$-closed. □

The following theorem gives a characterization of $\mathcal{I}_{\hat{g}}$-open sets.

Theorem 2.26. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space and* $A \subseteq X$. *Then A is* $\mathcal{I}_{\hat{g}}$*-open if and only if* $F \subseteq int^*(A)$ *whenever F is semi-closed and* $F \subseteq A$.

Proof. Suppose A is $\mathcal{I}_{\hat{g}}$-open. If F is semi-closed and F $\subseteq$ A, then X$-$A $\subseteq$ X$-$F and so cl*(X$-$A) $\subseteq$ X$-$F by Theorem 2.5 (b). Therefore F $\subseteq$ X$-$cl*(X$-$A)=int*(A). Hence F $\subseteq$ int*(A).

Conversely, suppose the condition holds. Let U be a semi-open set such that X$-$A $\subseteq$ U. Then X$-$U $\subseteq$ A and so X$-$U $\subseteq$ int*(A). Therefore cl*(X$-$A) $\subseteq$ U. By Theorem 2.5 (b), X$-$A is $\mathcal{I}_{\hat{g}}$-closed. Hence A is $\mathcal{I}_{\hat{g}}$-open. □

Corollary 2.27. *Let (X,τ,$\mathcal{I}$) be an ideal space and A $\subseteq$ X. If A is $\mathcal{I}_{\hat{g}}$-open, then F $\subseteq$ int*(A) whenever F is closed and F $\subseteq$ A.*

The following theorem gives a property of $\mathcal{I}_{\hat{g}}$-closed.

Theorem 2.28. *Let (X,τ,$\mathcal{I}$) be an ideal space and A $\subseteq$ X. If A is $\mathcal{I}_{\hat{g}}$-open and int*(A) $\subseteq$ B $\subseteq$ A, then B is $\mathcal{I}_{\hat{g}}$-open.*

Proof. Since A is $\mathcal{I}_{\hat{g}}$-open, then X$-$A is $\mathcal{I}_{\hat{g}}$-closed. By Theorem 2.5 (d), cl*(X$-$A)$-$(X$-$A) contains no nonempty semi-closed set. Since int*(A) $\subseteq$ int*(B) which implies that cl*(X$-$B) $\subseteq$ cl*(X$-$A) and so cl*(X$-$B)$-$(X$-$B) $\subseteq$ cl*(X$-$A)$-$(X$-$A). Hence B is $\mathcal{I}_{\hat{g}}$-open. □

The following theorem gives a characterization of $\mathcal{I}_{\hat{g}}$-closed sets in terms of $\mathcal{I}_{\hat{g}}$-open sets.

Theorem 2.29. *Let (X,τ,$\mathcal{I}$) be an ideal space and A $\subseteq$ X. Then the following are equivalent.*

(a) A is $\mathcal{I}_{\hat{g}}$-closed.

(b) A $\cup$ (X$-$A*) is $\mathcal{I}_{\hat{g}}$-closed.

(c) A$^*-$A is $\mathcal{I}_{\hat{g}}$-open.

Proof. (a) $\Rightarrow$ (b) Suppose A is $\mathcal{I}_{\hat{g}}$-closed. If U is any semi-open set such that A $\cup$ (X$-$A*) $\subseteq$ U, then X$-$U $\subseteq$ X$-$(A $\cup$ (X$-$A*))=X $\cap$ (A $\cup$ (A*)^c)^c=A* $\cap$ A^c=A$^*-$A. Since A is $\mathcal{I}_{\hat{g}}$-closed, by Theorem 2.5 (e), it follows that X$-$U=$\emptyset$ and so X=U. Therefore A $\cup$ (X$-$A*) $\subseteq$ U $\Rightarrow$ A $\cup$ (X$-$A*) $\subseteq$ X and so (A $\cup$ (X$-$A*))* $\subseteq$ X* $\subseteq$ X=U. Hence A $\cup$ (X$-$A*) is $\mathcal{I}_{\hat{g}}$-closed.
(b) $\Rightarrow$ (a) Suppose A $\cup$ (X$-$A*) is $\mathcal{I}_{\hat{g}}$-closed. If F is any semi-closed set such that F $\subseteq$ A$^*-$A, then F $\subseteq$ A* and F $\nsubseteq$ A $\Rightarrow$ X$-$A* $\subseteq$ X$-$F and A $\subseteq$ X$-$F. Therefore A $\cup$ (X$-$A*) $\subseteq$ A $\cup$ (X$-$F)= X$-$F and X$-$F is semi-open. Since (A $\cup$ (X$-$A*))* $\subseteq$ X$-$F $\Rightarrow$ A* $\cup$ (X$-$A*)* $\subseteq$ X$-$F and so A* $\subseteq$ X$-$F $\Rightarrow$ F $\subseteq$ X$-$A*. Since F $\subseteq$ A*, it follows that F=$\emptyset$. Hence A is $\mathcal{I}_{\hat{g}}$-closed.
(b) $\Leftrightarrow$ (c) Since X$-$(A$^*-$A)=X $\cap$ (A* $\cap$ A^c)^c=X $\cap$ ((A*)^c $\cup$ A)=(X $\cap$ (A*)^c) $\cup$ (X $\cap$ A)=A $\cup$ (X$-$A*). □

Theorem 2.30. *Let (X,τ,$\mathcal{I}$) be an ideal space. Then every subset of X is $\mathcal{I}_{\hat{g}}$-closed if and only if every semi-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{\hat{g}}$-closed. If U $\subseteq$ X is semi-open, then U is $\mathcal{I}_{\hat{g}}$-closed and so U* $\subseteq$ U. Hence U is $\star$-closed. Conversely, suppose that every semi-open set is $\star$-closed. If U is semi-open set such that A $\subseteq$ U $\subseteq$ X, then A* $\subseteq$ U* $\subseteq$ U and so A is $\mathcal{I}_{\hat{g}}$-closed. □

The following theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{\hat{g}}$-open sets.

Theorem 2.31. *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(a) X is normal.

(b) For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{\hat{g}}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(c) For any closed set A and open set V containing A, there exists an $\mathcal{I}_{\hat{g}}$-open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq V$.

Proof. (a) $\Rightarrow$ (b) The proof follows from the fact that every open set is $\mathcal{I}_{\hat{g}}$-open.
(b) $\Rightarrow$ (c) Suppose A is closed and V is an open set containing A. Since A and X$-$V are disjoint closed sets, there exist disjoint $\mathcal{I}_{\hat{g}}$-open sets U and W such that A $\subseteq$ U and X$-$V $\subseteq$ W. Since X$-$V is semi-closed and W is $\mathcal{I}_{\hat{g}}$-open, X$-$V $\subseteq$ int*(W) and so X$-$int*(W) $\subseteq$ V. Again U $\cap$ W=$\emptyset$ $\Rightarrow$ U $\cap$ int*(W)=$\emptyset$ and so U $\subseteq$ X$-$int*(W) $\Rightarrow$ cl*(U) $\subseteq$ X$-$int*(W) $\subseteq$ V. U is the required $\mathcal{I}_{\hat{g}}$-open sets with A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ V.
(c) $\Rightarrow$ (a) Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{\hat{g}}$-open set U such that A $\subseteq$ U $\subseteq$ cl*(U) $\subseteq$ X$-$B. Since U is $\mathcal{I}_{\hat{g}}$-open, A $\subseteq$ int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.3, $\tau^* \subseteq \tau^\alpha$ and so int*(U) and X$-$cl*(U)in τ^α. Hence A $\subseteq$ int*(U) $\subseteq$ int(cl(int(int*(U))))=G and B $\subseteq$ X$-$cl*(U) $\subseteq$ int(cl(int(X$-$cl*(U))))=H. G and H are the required disjoint open sets containing A and B respectively, which proves (a). □

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be an αgs-closed set [16] if cl$_\alpha$(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. The complement of αgs-closed is said to be an αgs-open set. If $\mathcal{I} = \mathcal{N}$, then $\mathcal{I}_{\hat{g}}$-closed sets coincide with αgs-closed sets and so we have the following Corollary.

Corollary 2.32. *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I} = \mathcal{N}$. Then the following are equivalent.*

(a) X is normal.

(b) For any disjoint closed sets A and B, there exist disjoint αgs-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(c) For any closed set A and open set V containing A, there exists an αgs-open set U such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.

A subset A of an ideal space is said to be $\mathcal{I}$-compact [5] or compact modulo $\mathcal{I}$ [14] if for every open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that A$-$ $\cup \{U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I}$. The space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.

Theorem 2.33. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [13, Theorem 2.17].*

Corollary 2.34. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is an $\mathcal{I}_{\hat{g}}$-closed subset of X, then A is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every $\mathcal{I}_{\hat{g}}$-closed set is $\mathcal{I}_g$-closed. □

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