

\hat{g} -CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT. Characterizations and properties of $\mathcal{I}_{\hat{g}}$ -closed sets and $\mathcal{I}_{\hat{g}}$ -open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{\hat{g}}$ -open sets. Also, it is established that an $\mathcal{I}_{\hat{g}}$ -closed subset of an \mathcal{I} -compact space is \mathcal{I} -compact.

1. INTRODUCTION AND PRELIMINARIES

An ideal \mathcal{I} on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal \mathcal{I} on X and if $\wp(X)$ is the set of all subsets of X , a set operator $(\cdot)^* : \wp(X) \rightarrow \wp(X)$, called a local function [9] of A with respect to τ and \mathcal{I} is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [8, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the \star -topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [18]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. If \mathcal{I} is an ideal on X , then (X, τ, \mathcal{I}) is called an ideal space. N is the ideal of all nowhere dense subsets in (X, τ) . A subset A of an ideal space (X, τ, \mathcal{I}) is \star -closed [8] (resp. \star -dense in itself [6]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$). A subset A of an ideal space (X, τ, \mathcal{I}) is $\mathcal{I}_{\hat{g}}$ -closed [3] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X, τ) and $int^*(A)$ will denote the interior of A in (X, τ^*) . A subset A of a space (X, τ) is an α -open [15] (resp. semi-open [10], preopen [12]) set if $A \subseteq int(cl(int(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(A))$). The family of all α -open sets in (X, τ) , denoted by τ^α , is a topology on X finer than τ . The closure of A in (X, τ^α) is denoted by $cl_\alpha(A)$. A subset A of a space (X, τ) is said to be g -closed [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open. A subset A of a space (X, τ) is said to be \hat{g} -closed [19] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. A subset A of a space (X, τ) is said to be \hat{g} -open [19] if its complement is \hat{g} -closed. The family of all \hat{g} -open sets in (X, τ) is a topology on X . The semi-closure [2] of a subset A of X , denoted by $scl(A)$, is defined to be the intersection of all semi-closed sets containing A . An ideal \mathcal{I} is said to be codense [4] or τ -boundary [14] if $\tau \cap \mathcal{I} = \{\emptyset\}$. \mathcal{I} is said to be completely codense [4] if $PO(X) \cap \mathcal{I} = \{\emptyset\}$, where $PO(X)$ is the family of all preopen sets in (X, τ) . Every completely codense ideal is codense but not the converse [4]. The following Lemmas will be useful in the sequel.

Lemma 1.1. *Let (X, τ, \mathcal{I}) be an ideal space and $A \subseteq X$. If $A \subseteq A^*$, then $A^* = cl(A^*) = cl(A) = cl^*(A)$ [17, Theorem 5].*

Lemma 1.2. *Let (X, τ, \mathcal{I}) be an ideal space. Then \mathcal{I} is codense if and only if $G \subseteq G^*$ for every semi-open set G in X [17, Theorem 3].*

2000 *Mathematics Subject Classification.* Primary 54A05; Secondary 54D15, 54D30.

Key words and phrases. \hat{g} -closed set, $\mathcal{I}_{\hat{g}}$ -closed set and \mathcal{I} -compact space.

Lemma 1.3. *Let (X, τ, \mathcal{I}) be an ideal space. If \mathcal{I} is completely codense, then $\tau^* \subseteq \tau^\alpha$ [17, Theorem 6].*

Result 1.4. *If (X, τ) is a topological space, then every closed set is \hat{g} -closed but not conversely [1, Theorem 2.3].*

Lemma 1.5. *If (X, τ, \mathcal{I}) is a $T_{\mathcal{I}}$ ideal space and A is an \mathcal{I}_g -closed set, then A is a \star -closed set [13, Corollary 2.2].*

Lemma 1.6. *Every g -closed set is \mathcal{I}_g -closed but not conversely [3, Theorem 2.1].*

2. $\mathcal{I}_{\hat{g}}$ -CLOSED SETS

Definition 2.1. *A subset A of an ideal space (X, τ, \mathcal{I}) is said to be $\mathcal{I}_{\hat{g}}$ -closed if $A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open.*

Definition 2.2. *A subset A of an ideal space (X, τ, \mathcal{I}) is said to be $\mathcal{I}_{\hat{g}}$ -open if $X - A$ is $\mathcal{I}_{\hat{g}}$ -closed.*

Theorem 2.3. *If (X, τ, \mathcal{I}) is any ideal space, then every $\mathcal{I}_{\hat{g}}$ -closed set is \mathcal{I}_g -closed but not conversely.*

Example 2.4. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\mathcal{I}_{\hat{g}}$ -closed sets are $\emptyset, X, \{a, b\}$ and \mathcal{I}_g -closed sets $\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$. It is clear that $\{a\}$ is \mathcal{I}_g -closed but it is not $\mathcal{I}_{\hat{g}}$ -closed.*

The following theorem gives characterizations of $\mathcal{I}_{\hat{g}}$ -closed sets.

Theorem 2.5. *If (X, τ, \mathcal{I}) is any ideal space and $A \subseteq X$, then the following are equivalent.*

- (a) A is $\mathcal{I}_{\hat{g}}$ -closed.
- (b) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X .
- (c) For all $x \in cl^*(A)$, $scl(\{x\}) \cap A \neq \emptyset$.
- (d) $cl^*(A) - A$ contains no nonempty semi-closed set.
- (e) $A^* - A$ contains no nonempty semi-closed set.

Proof. (a) \Rightarrow (b) If A is $\mathcal{I}_{\hat{g}}$ -closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X and so $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X . This proves (b).

(b) \Rightarrow (c) Suppose $x \in cl^*(A)$. If $scl(\{x\}) \cap A = \emptyset$, then $A \subseteq X - scl(\{x\})$. By (b), $cl^*(A) \subseteq X - scl(\{x\})$, a contradiction, since $x \in cl^*(A)$.

(c) \Rightarrow (d) Suppose $F \subseteq cl^*(A) - A$, F is semi-closed and $x \in F$. Since $F \subseteq X - A$ and F is semi-closed, then $A \subseteq X - F$ and F is semi-closed, $scl(\{x\}) \cap A = \emptyset$. Since $x \in cl^*(A)$ by (c), $scl(\{x\}) \cap A \neq \emptyset$. Therefore $cl^*(A) - A$ contains no nonempty semi-closed set.

(d) \Rightarrow (e) Since $cl^*(A) - A = (A \cup A^*) - A = (A \cup A^*) \cap A^c = (A \cap A^c) \cup (A^* \cap A^c) = A^* \cap A^c = A^* - A$. Therefore $A^* - A$ contains no nonempty semi-closed set.

(e) \Rightarrow (a) Let $A \subseteq U$ where U is semi-open set. Therefore $X - U \subseteq X - A$ and so $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Therefore $A^* \cap (X - U) \subseteq A^* - A$. Since A^* is always closed set, so $A^* \cap (X - U)$ is a semi-closed set contained in $A^* - A$. Therefore $A^* \cap (X - U) = \emptyset$ and hence $A^* \subseteq U$. Therefore A is $\mathcal{I}_{\hat{g}}$ -closed. □

Theorem 2.6. *Every \star -closed set is $\mathcal{I}_{\hat{g}}$ -closed but not conversely.*

Proof. Let A be a \star -closed, then $A^* \subseteq A$. Let $A \subseteq U$ where U is semi-open. Hence $A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open. Therefore A is $\mathcal{I}_{\hat{g}}$ -closed. □

Example 2.7. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then $\mathcal{I}_{\hat{g}}$ -closed sets are powerset of X and \star -closed sets are $\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}$. It is clear that $\{b\}$ is $\mathcal{I}_{\hat{g}}$ -closed set but it is not \star -closed.*

Theorem 2.8. *Let (X, τ, \mathcal{I}) be an ideal space. For every $A \in \mathcal{I}$, A is $\mathcal{I}_{\hat{g}}$ -closed.*

Proof. Let $A \subseteq U$ where U is semi-open set. Since $A^* = \emptyset$ for every $A \in \mathcal{I}$, then $\text{cl}^*(A) = A \cup A^* = A \subseteq U$. Therefore, by Theorem 2.5, A is $\mathcal{I}_{\hat{g}}$ -closed. \square

Theorem 2.9. *If (X, τ, \mathcal{I}) is an ideal space, then A^* is always $\mathcal{I}_{\hat{g}}$ -closed for every subset A of X .*

Proof. Let $A^* \subseteq U$ where U is semi-open. Since $(A^*)^* \subseteq A^*$ [8], we have $(A^*)^* \subseteq U$ whenever $A^* \subseteq U$ and U is semi-open. Hence A^* is $\mathcal{I}_{\hat{g}}$ -closed. \square

Theorem 2.10. *Let (X, τ, \mathcal{I}) be an ideal space. Then every $\mathcal{I}_{\hat{g}}$ -closed, semi-open set is \star -closed set.*

Proof. Since A is $\mathcal{I}_{\hat{g}}$ -closed and semi-open. Then $A^* \subseteq A$ whenever $A \subseteq U$ and U is semi-open. Hence A is \star -closed. \square

Corollary 2.11. *If (X, τ, \mathcal{I}) is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}_{\hat{g}}$ -closed set, then A is \star -closed set.*

Corollary 2.12. *Let (X, τ, \mathcal{I}) be an ideal space and A be an $\mathcal{I}_{\hat{g}}$ -closed set. Then the following are equivalent.*

- a) A is a \star -closed set.
- b) $\text{cl}^*(A) - A$ is a semi-closed set.
- c) $A^* - A$ is a semi-closed set.

Proof. (a) \Rightarrow (b) If A is \star -closed, then $A^* \subseteq A$ and so $\text{cl}^*(A) - A = (A \cup A^*) - A = \emptyset$. Hence $\text{cl}^*(A) - A$ is semi-closed set.

(b) \Rightarrow (c) Since $\text{cl}^*(A) - A = A^* - A$ and so $A^* - A$ is semi-closed set.

(c) \Rightarrow (a) If $A^* - A$ is a semi-closed set, since A is $\mathcal{I}_{\hat{g}}$ -closed set, by Theorem 2.5, $A^* - A = \emptyset$ and so A is \star -closed. \square

Theorem 2.13. *Let (X, τ, \mathcal{I}) be an ideal space. Then every \hat{g} -closed set is an $\mathcal{I}_{\hat{g}}$ -closed set but not conversely.*

Proof. Let A be a \hat{g} -closed set. Then $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. We have $\text{cl}^*(A) \subseteq \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. Hence A is $\mathcal{I}_{\hat{g}}$ -closed. \square

Example 2.14. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\mathcal{I}_{\hat{g}}$ -closed sets are $\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}$ and \hat{g} -closed sets are $\emptyset, X, \{b\}, \{b, c\}$. It is clear that $\{a\}$ is $\mathcal{I}_{\hat{g}}$ -closed set but it is not \hat{g} -closed.*

Theorem 2.15. *If (X, τ, \mathcal{I}) is an ideal space and A is a \star -dense in itself, $\mathcal{I}_{\hat{g}}$ -closed subset of X , then A is \hat{g} -closed.*

Proof. Suppose A is a \star -dense in itself, $\mathcal{I}_{\hat{g}}$ -closed subset of X . Let $A \subseteq U$ where U is semi-open. Then by Theorem 2.5 (b), $\text{cl}^*(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. Since A is \star -dense in itself, by Lemma 1.1, $\text{cl}(A) = \text{cl}^*(A)$. Therefore $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. Hence A is \hat{g} -closed. \square

Corollary 2.16. *If (X, τ, \mathcal{I}) is any ideal space where $\mathcal{I} = \{\emptyset\}$, then A is $\mathcal{I}_{\hat{g}}$ -closed if and only if A is \hat{g} -closed.*

Proof. From the fact that for $\mathcal{I} = \{\emptyset\}$, $A^* = \text{cl}(A) \supseteq A$. Therefore A is \star -dense in itself. Since A is $\mathcal{I}_{\hat{g}}$ -closed, by Theorem 2.15, A is \hat{g} -closed. Conversely, by Theorem 2.13, every \hat{g} -closed set is $\mathcal{I}_{\hat{g}}$ -closed set. \square

Corollary 2.17. *If (X, τ, \mathcal{I}) is any ideal space where \mathcal{I} is codense and A is a semi-open, $\mathcal{I}_{\hat{g}}$ -closed subset of X , then A is \hat{g} -closed.*

Proof. By Lemma 1.2, A is \star -dense in itself. By Theorem 2.15, A is \hat{g} -closed. \square

Example 2.18. Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{a,c\}\}$ and $\mathcal{I}=\{\emptyset\}$. Then g -closed sets are $\emptyset,X,\{b\},\{a,b\},\{b,c\}$ and $\mathcal{I}_{\hat{g}}$ -closed sets are $\emptyset,X,\{b\},\{b,c\}$. It is clear that $\{a,b\}$ is g -closed set but it is not $\mathcal{I}_{\hat{g}}$ -closed.

Example 2.19. Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{a,c\}\}$ and $\mathcal{I}=\{\emptyset,\{a\}\}$. Then g -closed sets are $\emptyset,X,\{b\},\{a,b\},\{b,c\}$ and $\mathcal{I}_{\hat{g}}$ -closed sets are $\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\}$. It is clear that $\{a\}$ is $\mathcal{I}_{\hat{g}}$ -closed set but it is not g -closed.

Remark 2.20. By Example 2.18 and Example 2.19, g -closed sets and $\mathcal{I}_{\hat{g}}$ -closed sets are independent.

Remark 2.21. We have the following implications for the subsets stated above.

$$\begin{array}{ccccc} \text{closed} & \longrightarrow & \hat{g} - \text{closed} & \longrightarrow & g - \text{closed} \\ \downarrow & & \downarrow & & \downarrow \\ \star - \text{closed} & \longrightarrow & \mathcal{I}_{\hat{g}} - \text{closed} & \longrightarrow & \mathcal{I}_g - \text{closed} \end{array}$$

Theorem 2.22. Let (X,τ,\mathcal{I}) be an ideal space and $A \subseteq X$. Then A is $\mathcal{I}_{\hat{g}}$ -closed if and only if $A=F-N$ where F is \star -closed and N contains no nonempty semi-closed set.

Proof. If A is $\mathcal{I}_{\hat{g}}$ -closed, then by Theorem 2.5 (e), $N=A^*-A$ contains no nonempty semi-closed set. If $F=\text{cl}^*(A)$, then F is \star -closed such that $F-N=(A \cup A^*)-(A^*-A)=(A \cup A^*) \cap (A^* \cap A^c)^c=(A \cup A^*) \cap ((A^*)^c \cup A)=(A \cup A^*) \cap (A \cup (A^*)^c)=A \cup (A^* \cap (A^*)^c)=A$.

Conversely, suppose $A=F-N$ where F is \star -closed and N contains no nonempty semi-closed set. Let U be a semi-open set such that $A \subseteq U$. Then $F-N \subseteq U \Rightarrow F \cap (X-U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$ then $A^* \subseteq F^*$ and so $A^* \cap (X-U) \subseteq F^* \cap (X-U) \subseteq F \cap (X-U) \subseteq N$. By hypothesis, since $A^* \cap (X-U)$ is semi-closed, $A^* \cap (X-U)=\emptyset$ and so $A^* \subseteq U$. Hence A is $\mathcal{I}_{\hat{g}}$ -closed. \square

Theorem 2.23. Let (X,τ,\mathcal{I}) be an ideal space and $A \subseteq X$. If $A \subseteq B \subseteq A^*$, then $A^*=B^*$ and B is \star -dense in itself.

Proof. Since $A \subseteq B$, then $A^* \subseteq B^*$ and since $B \subseteq A^*$, then $B^* \subseteq (A^*)^* \subseteq A^*$. Therefore $A^*=B^*$ and $B \subseteq A^* \subseteq B^*$. Hence proved. \square

Theorem 2.24. Let (X,τ,\mathcal{I}) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq \text{cl}^*(A)$ and A is $\mathcal{I}_{\hat{g}}$ -closed, then B is $\mathcal{I}_{\hat{g}}$ -closed.

Proof. Since A is $\mathcal{I}_{\hat{g}}$ -closed, then by Theorem 2.5 (d), $\text{cl}^*(A)-A$ contains no nonempty semi-closed set. Since $\text{cl}^*(B)-B \subseteq \text{cl}^*(A)-A$ and so $\text{cl}^*(B)-B$ contains no nonempty semi-closed set. Hence B is $\mathcal{I}_{\hat{g}}$ -closed. \square

Corollary 2.25. Let (X,τ,\mathcal{I}) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq A^*$ and A is $\mathcal{I}_{\hat{g}}$ -closed, then A and B are \hat{g} -closed sets.

Proof. Let A and B be subsets of X such that $A \subseteq B \subseteq A^* \Rightarrow A \subseteq B \subseteq A^* \subseteq \text{cl}^*(A)$ and A is $\mathcal{I}_{\hat{g}}$ -closed. By the above Theorem, B is $\mathcal{I}_{\hat{g}}$ -closed. Since $A \subseteq B \subseteq A^*$, then $A^*=B^*$ and so A and B are \star -dense in itself. By Theorem 2.15, A and B are \hat{g} -closed. \square

The following theorem gives a characterization of $\mathcal{I}_{\hat{g}}$ -open sets.

Theorem 2.26. Let (X,τ,\mathcal{I}) be an ideal space and $A \subseteq X$. Then A is $\mathcal{I}_{\hat{g}}$ -open if and only if $F \subseteq \text{int}^*(A)$ whenever F is semi-closed and $F \subseteq A$.

Proof. Suppose A is $\mathcal{I}_{\tilde{g}}$ -open. If F is semi-closed and $F \subseteq A$, then $X-A \subseteq X-F$ and so $\text{cl}^*(X-A) \subseteq X-F$ by Theorem 2.5 (b). Therefore $F \subseteq X-\text{cl}^*(X-A)=\text{int}^*(A)$. Hence $F \subseteq \text{int}^*(A)$.

Conversely, suppose the condition holds. Let U be a semi-open set such that $X-A \subseteq U$. Then $X-U \subseteq A$ and so $X-U \subseteq \text{int}^*(A)$. Therefore $\text{cl}^*(X-A) \subseteq U$. By Theorem 2.5 (b), $X-A$ is $\mathcal{I}_{\tilde{g}}$ -closed. Hence A is $\mathcal{I}_{\tilde{g}}$ -open. \square

Corollary 2.27. *Let (X, τ, \mathcal{I}) be an ideal space and $A \subseteq X$. If A is $\mathcal{I}_{\tilde{g}}$ -open, then $F \subseteq \text{int}^*(A)$ whenever F is closed and $F \subseteq A$.*

The following theorem gives a property of $\mathcal{I}_{\tilde{g}}$ -closed.

Theorem 2.28. *Let (X, τ, \mathcal{I}) be an ideal space and $A \subseteq X$. If A is $\mathcal{I}_{\tilde{g}}$ -open and $\text{int}^*(A) \subseteq B \subseteq A$, then B is $\mathcal{I}_{\tilde{g}}$ -open.*

Proof. Since A is $\mathcal{I}_{\tilde{g}}$ -open, then $X-A$ is $\mathcal{I}_{\tilde{g}}$ -closed. By Theorem 2.5 (d), $\text{cl}^*(X-A)-(X-A)$ contains no nonempty semi-closed set. Since $\text{int}^*(A) \subseteq \text{int}^*(B)$ which implies that $\text{cl}^*(X-B) \subseteq \text{cl}^*(X-A)$ and so $\text{cl}^*(X-B)-(X-B) \subseteq \text{cl}^*(X-A)-(X-A)$. Hence B is $\mathcal{I}_{\tilde{g}}$ -open. \square

The following theorem gives a characterization of $\mathcal{I}_{\tilde{g}}$ -closed sets in terms of $\mathcal{I}_{\tilde{g}}$ -open sets.

Theorem 2.29. *Let (X, τ, \mathcal{I}) be an ideal space and $A \subseteq X$. Then the following are equivalent.*

- (a) A is $\mathcal{I}_{\tilde{g}}$ -closed.
- (b) $A \cup (X-A^*)$ is $\mathcal{I}_{\tilde{g}}$ -closed.
- (c) A^*-A is $\mathcal{I}_{\tilde{g}}$ -open.

Proof. (a) \Rightarrow (b) Suppose A is $\mathcal{I}_{\tilde{g}}$ -closed. If U is any semi-open set such that $A \cup (X-A^*) \subseteq U$, then $X-U \subseteq X-(A \cup (X-A^*))=X \cap (A \cup (A^*)^c)^c=A^* \cap A^c=A^*-A$. Since A is $\mathcal{I}_{\tilde{g}}$ -closed, by Theorem 2.5 (e), it follows that $X-U=\emptyset$ and so $X=U$. Therefore $A \cup (X-A^*) \subseteq U \Rightarrow A \cup (X-A^*) \subseteq X$ and so $(A \cup (X-A^*))^* \subseteq X^* \subseteq X=U$. Hence $A \cup (X-A^*)$ is $\mathcal{I}_{\tilde{g}}$ -closed.

(b) \Rightarrow (a) Suppose $A \cup (X-A^*)$ is $\mathcal{I}_{\tilde{g}}$ -closed. If F is any semi-closed set such that $F \subseteq A^*-A$, then $F \subseteq A^*$ and $F \not\subseteq A \Rightarrow X-A^* \subseteq X-F$ and $A \subseteq X-F$. Therefore $A \cup (X-A^*) \subseteq A \cup (X-F)=X-F$ and $X-F$ is semi-open. Since $(A \cup (X-A^*))^* \subseteq X-F \Rightarrow A^* \cup (X-A^*)^* \subseteq X-F$ and so $A^* \subseteq X-F \Rightarrow F \subseteq X-A^*$. Since $F \subseteq A^*$, it follows that $F=\emptyset$. Hence A is $\mathcal{I}_{\tilde{g}}$ -closed.

(b) \Leftrightarrow (c) Since $X-(A^*-A)=X \cap (A^* \cap A^c)^c=X \cap ((A^*)^c \cup A)=(X \cap (A^*)^c) \cup (X \cap A)=A \cup (X-A^*)$. \square

Theorem 2.30. *Let (X, τ, \mathcal{I}) be an ideal space. Then every subset of X is $\mathcal{I}_{\tilde{g}}$ -closed if and only if every semi-open set is \star -closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{\tilde{g}}$ -closed. If $U \subseteq X$ is semi-open, then U is $\mathcal{I}_{\tilde{g}}$ -closed and so $U^* \subseteq U$. Hence U is \star -closed. Conversely, suppose that every semi-open set is \star -closed. If U is semi-open set such that $A \subseteq U \subseteq X$, then $A^* \subseteq U^* \subseteq U$ and so A is $\mathcal{I}_{\tilde{g}}$ -closed. \square

The following theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{\tilde{g}}$ -open sets.

Theorem 2.31. *Let (X, τ, \mathcal{I}) be an ideal space where \mathcal{I} is completely codense. Then the following are equivalent.*

(a) X is normal.

(b) For any disjoint closed sets A and B , there exist disjoint $\mathcal{I}_{\hat{g}}$ -open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(c) For any closed set A and open set V containing A , there exists an $\mathcal{I}_{\hat{g}}$ -open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq V$.

Proof. (a) \Rightarrow (b) The proof follows from the fact that every open set is $\mathcal{I}_{\hat{g}}$ -open.

(b) \Rightarrow (c) Suppose A is closed and V is an open set containing A . Since A and $X - V$ are disjoint closed sets, there exist disjoint $\mathcal{I}_{\hat{g}}$ -open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Since $X - V$ is semi-closed and W is $\mathcal{I}_{\hat{g}}$ -open, $X - V \subseteq int^*(W)$ and so $X - int^*(W) \subseteq V$. Again $U \cap W = \emptyset \Rightarrow U \cap int^*(W) = \emptyset$ and so $U \subseteq X - int^*(W) \Rightarrow cl^*(U) \subseteq X - int^*(W) \subseteq V$. U is the required $\mathcal{I}_{\hat{g}}$ -open sets with $A \subseteq U \subseteq cl^*(U) \subseteq V$.

(c) \Rightarrow (a) Let A and B be two disjoint closed subsets of X . By hypothesis, there exists an $\mathcal{I}_{\hat{g}}$ -open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq X - B$. Since U is $\mathcal{I}_{\hat{g}}$ -open, $A \subseteq int^*(U)$. Since \mathcal{I} is completely codense, by Lemma 1.3, $\tau^* \subseteq \tau^\alpha$ and so $int^*(U)$ and $X - cl^*(U)$ in τ^α . Hence $A \subseteq int^*(U) \subseteq int(cl(int(int^*(U)))) = G$ and $B \subseteq X - cl^*(U) \subseteq int(cl(int(X - cl^*(U)))) = H$. G and H are the required disjoint open sets containing A and B respectively, which proves (a). \square

A subset A of an ideal space (X, τ, \mathcal{I}) is said to be an α gs-closed set [16] if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open. The complement of α gs-closed is said to be an α gs-open set. If $\mathcal{I} = \mathcal{N}$, then $\mathcal{I}_{\hat{g}}$ -closed sets coincide with α gs-closed sets and so we have the following Corollary.

Corollary 2.32. *Let (X, τ, \mathcal{I}) be an ideal space where $\mathcal{I} = \mathcal{N}$. Then the following are equivalent.*

(a) X is normal.

(b) For any disjoint closed sets A and B , there exist disjoint α gs-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(c) For any closed set A and open set V containing A , there exists an α gs-open set U such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.

A subset A of an ideal space is said to be \mathcal{I} -compact [5] or compact modulo \mathcal{I} [14] if for every open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of A , there exists a finite subset Δ_0 of Δ such that $A - \cup \{U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I}$. The space (X, τ, \mathcal{I}) is \mathcal{I} -compact if X is \mathcal{I} -compact as a subset.

Theorem 2.33. *Let (X, τ, \mathcal{I}) be an ideal space. If A is an \mathcal{I}_g -closed subset of X , then A is \mathcal{I} -compact [13, Theorem 2.17].*

Corollary 2.34. *Let (X, τ, \mathcal{I}) be an ideal space. If A is an $\mathcal{I}_{\hat{g}}$ -closed subset of X , then A is \mathcal{I} -compact.*

Proof. The proof follows from the fact that every $\mathcal{I}_{\hat{g}}$ -closed set is \mathcal{I}_g -closed. \square

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Received 24/01/2011