Methods of Functional Analysis and Topology
Vol. 17 (2011), no. 4, pp. 295–299

DUNFORD-PETTIS PROPERTY OF THE PRODUCT OF SOME OPERATORS

BELMESNAOUI AQZZOUZ, OTHMAN ABOUTAFAIL, AND AZIZ ELBOUR

Abstract. We establish a sufficient condition under which the product of an order bounded almost Dunford-Pettis operator and an order weakly compact operator is Dunford-Pettis. And we derive some consequences.

1. Definitions and Notation

Recall that a *vector lattice* E is an ordered vector space in which $x \vee y := \sup(x, y)$ and $x \wedge y := \inf(x, y)$ exists for every $x, y \in E$. For any vector x in a vector lattice, the element $x^+ := x \vee 0$ is called the *positive part*, $x^- := (-x) \vee 0$ is called the *negative part*, and $|x| := x \vee (-x)$ called the *absolute value* of x. Note that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. A sequence (x_n) in a vector space is said to be *disjoint* whenever $|x_n| \wedge |x_m| = 0$ holds for $n \neq m$. A subset A of a vector lattice E is said to be *solid* if it follows from $|y| \leq |x|$ with $x \in A$ and $y \in E$ that $y \in A$. The *solid hull* of a subset W of E is the smallest solid set including W and is exactly the set $\mathrm{Sol}(W) := \{x \in E : \exists y \in A \text{ with } |x| \leq |y|\}$. An *order ideal* of a vector lattice E is a solid subspace. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] := \{z \in E \ : \ x \leq z \leq y\}$ is called an *order interval*. A subset of E is said to be *order bounded* if it is included in some order interval. A *Banach lattice* is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is *order continuous* if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ for E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice E is said to have *weakly sequentially continuous lattice operations* whenever $x_n \to 0$ for $\sigma(E, E')$ implies $|x_n| \to 0$ for $\sigma(E, E')$.

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is *positive* if $T(x) \geq 0$ for F whenever $x \geq 0$ for E. The operator T is *regular* if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous. An operator $T : E \longrightarrow F$ is said to be *order bounded* if it maps order bounded subsets of E to order bounded subsets of E.

We refer reader to [1] for unexplained terminology on Banach lattice theory and positive operators.

2. Main results

Recall that an operator T from a Banach lattice E into a Banach space X is called *almost Dunford-Pettis* if $\|T(x_n)\| \to 0$ for every disjoint sequence $(x_n) \subset E$ satisfying

2000 *Mathematics Subject Classification.* 46A40, 46B40, 46B42.
Key words and phrases. Almost Dunford-Pettis operator, Dunford-Pettis operator, order weakly compact operator.

$x_n \to 0$ for the topology $\sigma(E, E')$. It follows from a Remark of Wnuk ([5], Remark 1, p. 228) that T is almost Dunford-Pettis if and only if $\|T(x_n)\| \to 0$ for every weakly null disjoint sequence (x_n) in E^+.

An operator between two Banach spaces is called *Dunford-Pettis*, whenever it maps weakly null sequences into norm null sequences. It is evident that every Dunford-Pettis operator from a Banach lattice E into a Banach space F is almost Dunford-Pettis, but the converse is false in general. In fact, In fact, the identity operator of $L^1([0,1])$ is almost Dunford-Pettis, but it is not Dunford-Pettis.

Also, an operator T from a Banach lattice E into a Banach space X is said to be *order weakly compact* if for each $x \in E^+$, the subset $T([0,x])$ is relatively weakly compact in X. Note that each almost Dunford-Pettis operator T from a Banach lattice E into a Banach space X is order weakly compact (In fact, if (x_n) is an order bounded disjoint sequence of E then $x_n \to 0$ for the topology $\sigma(E, E')$ ([1], p. 192) and so $\|T(x_n)\| \to 0$. Hence, Dodds's Theorem ([1], Theorem 5.57) implies that T is order weakly compact). But an order weakly compact operator is not necessary almost Dunford-Pettis. In fact, the identity operator $Id_{c_0} : c_0 \to c_0$ is order weakly compact (because the norm of c_0 is order continuous and hence each order interval of c_0 is weakly compact (see [1], Theorem 12.9)) but it fails to be almost Dunford-Pettis.

Let ρ be a lattice seminorm on a Banach lattice E. A subset A of E is said to be *ρ-almost order bounded* (see Zaanen [6], p. 525) (or *approximately order bounded* with respect to ρ (see [3], Remark, p. 73)) if for every $\varepsilon > 0$ there exists $u \in E^+$ such that $A \subset [-u,u] + \varepsilon B_\rho$ where $B_\rho = \{x \in E : \rho(x) \leq 1\}$ is the closed unit ball associated to ρ. Since B_ρ is a solid subset of E, then it follows from [3, p. 73] that

$$A \subset [-u,u] + \varepsilon B_\rho \quad \text{if and only if} \quad \rho\big((|y| - u)^+ \big) \leq \varepsilon \quad \text{for all} \quad y \in A.$$

Let T be an operator from a Banach lattice E into a Banach space X. We will need the lattice seminorm q_T on E, which is defined in ([3], p. 192), by the following formula:

$$q_T(x) := \sup\{\|T(y)\| : |y| \leq |x|\}, \quad x \in E.$$

It is easy to see that $\|T(x)\| \leq q_T(x) \leq \|T\| \cdot \|x\|$ holds for each $x \in E$. So, the lattice seminorm q_T is continuous for the norm of E.

Proposition 2.1. *Let T be an almost Dunford-Pettis operator from a Banach lattice E into a Banach space X. Then each weakly relatively compact subset W of E is approximately order bounded with respect to the lattice seminorm q_T.*

If, in addition, T is order bounded, then $T(W)$ is an almost order bounded subset of F, i.e., for every $\varepsilon > 0$ there exists some $v \in F^+$ such that $\big\| (|T(x)| - v)^+ \big\| \leq \varepsilon$ holds for all $x \in W$.

Proof. Let W be a weakly relatively compact subset of E and let $\varepsilon > 0$. It follows from Theorem 4.34 of [1] that every disjoint sequence, in the solid hull A of W, converges weakly to zero. Let (x_n) be a disjoint sequence (x_n) in A. We claim that $q_T(x_n) \to 0$. From $q_T(x_n) = \sup\{\|T(y)\| : |y| \leq |x_n|\}$, it follows that for each n there exists an element y_n in A such that $|y_n| \leq |x_n|$ and

$$q_T(x_n) \leq 2\|T(y_n)\|.$$

We note that the sequence $(y_n) \subset A$ is disjoint and hence it converges weakly to zero. Next, as the operator $T : E \to X$ is almost Dunford-Pettis, we obtain $\|T(y_n)\| \to 0$, and hence $q_T(x_n) \to 0$ holds, as desired.

After that, if $Id_E : E \to E$ is the identity operator of E, then $q_T(Id_E(x_n)) \to 0$ holds for each disjoint sequence (x_n) in A. Thus, by Theorem 4.36 of [1], there exists some $u \in E^+$, lying in the order ideal generated by A, such that

$$q_T\big(Id_E(|x| - u)^+\big) \leq \varepsilon$$

for all $x \in A$, i.e., $q_T\big((|x| - u)^+ \big) \leq \varepsilon$ for all $x \in A$ (and hence for all $x \in W$). This implies that W is approximately order bounded with respect to the lattice seminorm q_T.

If, in addition, T is order bounded, then there exists some $v \in F^+$ such that $T([0,u]) \subseteq [-v, v]$. Let first $0 \leq x \in A$. It follows from $x = x \wedge u + (x-u)^+$ and $|T\,(x \wedge u)| \leq v$ that

$$|T\,(x)| - v \leq |T\,(x)| - |T\,(x \wedge u)| \leq |T\,(x) - T\,(x \wedge u)| = \big|T\big((x-u)^+ \big)\big|.$$

Then $(|T\,(x)| - v)^+ \leq \big|T\big((x-u)^+ \big)\big|$ and hence

$$\big\| \,(|T\,(x)| - v)^+ \,\big\| \leq \big\|T\big((x-u)^+ \big)\big\| \leq q_T\big((x-u)^+ \big) \leq \varepsilon.$$

If $x \in A$ is arbitrary, then

$$|T\,(x)| - 2v \leq \big(\big|T\,\big(x^+\big)\big| - v\big) + \big(\big|T\,\big(x^-\big)\big| - v\big),$$

and so $\big\| \,(|T\,(x)| - 2v)^+ \,\big\| \leq 2\varepsilon$. This shows that $T\,(W)$ is an almost order bounded subset of F. □

Our result gives a generalization of Example 4 of ([5], p. 230).

Theorem 2.2. *Let E and F be two Banach lattices and let X be a Banach space. Suppose that F has weakly sequentially continuous lattice operations. If $T : E \to F$ is an order bounded almost Dunford-Pettis operator and $S : F \to X$ is an order weakly compact operator, then $S \circ T$ is Dunford-Pettis.*

Proof. Let (x_n) be a weakly null sequence of E. To this end, we have to show that $\|S\,[T\,(x_n)]\| \to 0$. Let $y_n = [T\,(x_n)]^+$ and $z_n = [T\,(x_n)]^-$. It suffices to show that $\|S\,(y_n)\| \to 0$ and $\|S\,(z_n)\| \to 0$. To this end, let $\varepsilon > 0$ be given. Since $W := \{x_n : n \in \mathbb{N}\}$ is a weakly relatively compact subset of E, it follows from Proposition 2.1 that there exists some $v \in F^+$ such that $\big\| \,(|T\,(x_n)| - v)^+ \,\big\| \leq \varepsilon$ holds for all n. From the inequalities $(y_n - v)^+ \leq (|T\,(x_n)| - v)^+$, we see that $\big\| \,(y_n - v)^+ \,\big\| \leq \big\| \,(|T\,(x_n)| - v)^+ \,\big\| \leq \varepsilon$ holds for all n.

On the other hand, it is clear that $T\,(x_n) \to 0$ for the topology $\sigma\,(F, F')$. Since F has weakly sequentially continuous lattice operations, then $|T\,(x_n)| \to 0$ for the topology $\sigma\,(F, F')$. Next, from the inequalities $0 \leq y_n \wedge v \leq y_n \leq |T\,(x_n)|$ for each n, we see that $y_n \wedge v \to 0$ for the topology $\sigma\,(F, F')$. Hence $(y_n \wedge v)$ is an order bounded weakly null sequence of F^+. Now, since $S : F \to X$ is order weakly compact, it follows from Corollary 3.4.9 of [3] that $\|S\,(y_n \wedge v)\| \to 0$. So, there exists some n_0 such that $\|S\,(y_n \wedge v)\| < \varepsilon$ holds for all $n \geq n_0$.

Finally, from the identity $y_n = (y_n - v)^+ + (y_n \wedge v)$, we see that

$$\begin{array}{rcl} \|S\,(y_n)\| & \leq & \big\|S\,(y_n - v)^+ \,\big\| + \|S\,(y_n \wedge v)\| \\ & \leq & \|S\| \cdot \big\| \,(y_n - v)^+ \,\big\| + \varepsilon \leq (\|S\| + 1)\,\varepsilon \end{array}$$

holds for all $n \geq n_0$. And this implies that $\|S\,(y_n)\| \to 0$.

Similarly, it can be shown that $\|S\,(z_n)\| \to 0$. Since $S\,[T\,(x_n)] = S\,(y_n) - S\,(z_n)$ holds for all n, we see that $\|S\,[T\,(x_n)]\| \to 0$ and this completes the proof of the Theorem. □

A Banach lattice E has the *positive Schur property* if weakly null sequences with positive terms are norm null. For example, the Banach lattice $L^1\,([0,1])$ has the positive Schur property.

Note that if E has the positive Schur property, then every operator T from E into an arbitrary Banach space is almost Dunford-Pettis.

Remark 2.3. *The order boundedness of the operator T in the Theorem 2.2 is essential. In fact, if we take $E = L^1[0,1]$ and $F = c_0$, and if we consider the operator $T : L^1[0,1] \to c_0$ defined by the equality*

$$T(f) = \left(\int_0^1 f(t) \sin(nt)\, dt \right)_{n=1}^{\infty}.$$

Note that T is not order bounded (see Exercise 10, p. 289 of [1]). Since $L^1[0,1]$ has the positive Schur property, it follows from Proposition 3.7.24 of [3] that T is not Dunford-Pettis. However, T is almost Dunford-Pettis.

Also, note that the identity operator $Id_{c_0} : c_0 \to c_0$ is order weakly compact but the composed $Id_{c_0} \circ T = T$ is not Dunford-Pettis.

Recall from [1] that an operator $T : X \longrightarrow Y$ between two Banach spaces is said to be *weak Dunford-Pettis* whenever $x_n \longrightarrow 0$ weakly in X and $f_n \longrightarrow 0$ weakly in Y' imply $f_n(T(x_n)) \longrightarrow 0$. On the other hand, we say that a Banach space X has the *Dunford-Pettis property* whenever $x_n \longrightarrow 0$ weakly in X and $f_n \longrightarrow 0$ weakly in X' imply $f_n(x_n) \longrightarrow 0$.

A weak Dunford-Pettis operator $T : E \to Y$ from a Banach lattice E into a Banach space X is not necessary almost Dunford-Pettis operator. In fact, the identity operator $Id_{c_0} : c_0 \to c_0$ is weak Dunford-Pettis but it fails to be almost Dunford-Pettis. Conversely, there exists an almost Dunford-Pettis operator which is not weak Dunford-Pettis. In fact, it follows from the Remark 3 of Wnuk ([4], p. 19) that the Lorenz space $E = \Lambda(w,1)$ has the positive Schur property but it fails to have the Dunford-Pettis property. Hence, the identity operator $Id_{\Lambda(w,1)} : \Lambda(w,1) \to \Lambda(w,1)$ is almost Dunford-Pettis but it fails to be weak Dunford-Pettis.

As a consequence of Theorem 2.2 and Theorem 5.99 of [1], we obtain

Corollary 2.4. *Let E and F be two Banach lattices. If F has weakly sequentially continuous lattice operations, then each order bounded almost Dunford-Pettis operator $T : E \to F$ is weak Dunford-Pettis.*

Proof. Assume that F has weakly sequentially continuous lattice operations and let $T : E \to F$ be an order bounded almost Dunford-Pettis operator. It follows from Theorem 2.2 that for every weakly compact operator S from F into an arbitrary Banach space X, the composed operator $S \circ T$ is Dunford-Pettis. Thus, we deduce that $T : E \to F$ is a weak Dunford-Pettis operator (by using the equivalence (1)$\Longleftrightarrow$(3) of Theorem 5.99 of [1]). $\square$

Recall that a nonzero element x of a vector lattice E is *discrete* if the order ideal generated by x equals the subspace generated by x. The vector lattice E is *discrete*, if it admits a complete disjoint system of discrete elements.

Recall from Wnuk ([5], Example 4, p. 230) that if F is a discrete Banach lattice with an order continuous norm, then a positive operator $T : E \to F$ is almost Dunford-Pettis iff T is Dunford-Pettis.

Another a consequence of Theorem 2.2, we give a generalization of this result of Wnuk [5].

Corollary 2.5. *Let E and F be two Banach lattices. If F is discrete with an order continuous norm, then an order bounded operator $T : E \to F$ is almost Dunford-Pettis if and only if T is Dunford-Pettis.*

Proof. Assume that $T : E \to F$ is an order bounded almost Dunford-Pettis operator. It follows from Proposition 2.5.23 of [3] that F has weakly sequentially continuous lattice operations.

On the other hand, since the norm of F is order continuous, then each order interval of F is weakly compact (see [1], Theorem 4.9), and hence its identity operator $Id_F : F \to F$ is order weakly compact. So, by Theorem 2.2, the composed $T = Id_F \circ T$ is Dunford-Pettis. This completes the proof. □

Remark 2.6. *We can changed the assumption "F is discrete with an order continuous norm" of Corollary 2.5 by the assumption "F has weakly sequentially continuous lattice operations and the norm of F is order continuous". Because these two assumptions are equivalent. In fact, it follows from Corollary 2.3 of Chen-Wickstead* [2] *that if the norm of F is order continuous, then F is discrete if and only if F has weakly sequentially continuous lattice operations.*

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Université Mohammed V-Souissi, Faculté des Sciences Economiques, Juridiques et Sociales, Département d'Economie, B.P. 5295, SalaAljadida, Morocco

E-mail address: baqzzouz@hotmail.com

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra, Morocco

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra, Morocco

Received 31/03/2011; Revised 25/07/2011