

DUNFORD-PETTIS PROPERTY OF THE PRODUCT OF SOME OPERATORS

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ABSTRACT. We establish a sufficient condition under which the product of an order bounded almost Dunford-Pettis operator and an order weakly compact operator is Dunford-Pettis. And we derive some consequences.

1. DEFINITIONS AND NOTATION

Recall that a *vector lattice* E is an ordered vector space in which $x \vee y := \sup(x, y)$ and $x \wedge y := \inf(x, y)$ exists for every $x, y \in E$. For any vector x in a vector lattice, the element $x^+ := x \vee 0$ is called the *positive part*, $x^- := (-x) \vee 0$ is called the *negative part*, and $|x| := x \vee (-x)$ called the *absolute value* of x . Note that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. A sequence (x_n) in a vector space is said to be *disjoint* whenever $|x_n| \wedge |x_m| = 0$ holds for $n \neq m$. A subset A of a vector lattice E is said to be *solid* if it follows from $|y| \leq |x|$ with $x \in A$ and $y \in E$ that $y \in A$. The *solid hull* of a subset W of E is the smallest solid set including W and is exactly the set $\text{Sol}(W) := \{x \in E : \exists y \in W \text{ with } |x| \leq |y|\}$. An *order ideal* of a vector lattice E is a solid subspace. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] := \{z \in E : x \leq z \leq y\}$ is called an *order interval*. A subset of E is said to be *order bounded* if it is included in some order interval. A *Banach lattice* is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E' , endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is *order continuous* if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ for E , the sequence (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice E is said to have *weakly sequentially continuous lattice operations* whenever $x_n \rightarrow 0$ for $\sigma(E, E')$ implies $|x_n| \rightarrow 0$ for $\sigma(E, E')$.

We will use the term operator $T : E \rightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is *positive* if $T(x) \geq 0$ for F whenever $x \geq 0$ for E . The operator T is *regular* if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F . Note that each positive linear mapping on a Banach lattice is continuous. An operator $T : E \rightarrow F$ is said to be *order bounded* if it maps order bounded subsets of E to order bounded subsets of F .

We refer reader to [1] for unexplained terminology on Banach lattice theory and positive operators.

2. MAIN RESULTS

Recall that an operator T from a Banach lattice E into a Banach space X is called *almost Dunford-Pettis* if $\|T(x_n)\| \rightarrow 0$ for every disjoint sequence $(x_n) \subset E$ satisfying

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$x_n \rightarrow 0$ for the topology $\sigma(E, E')$. It follows from a Remark of Wnuk ([5], Remark 1, p. 228) that T is almost Dunford-Pettis if and only if $\|T(x_n)\| \rightarrow 0$ for every weakly null disjoint sequence (x_n) in E^+ .

An operator between two Banach spaces is called *Dunford-Pettis*, whenever it maps weakly null sequences into norm null sequences. It is evident that every Dunford-Pettis operator from a Banach lattice E into a Banach space F is almost Dunford-Pettis, but the converse is false in general. In fact, the identity operator of $L^1([0, 1])$ is almost Dunford-Pettis, but it is not Dunford-Pettis.

Also, an operator T from a Banach lattice E into a Banach space X is said to be *order weakly compact* if for each $x \in E^+$, the subset $T([0, x])$ is relatively weakly compact in X . Note that each almost Dunford-Pettis operator T from a Banach lattice E into a Banach space X is order weakly compact (In fact, if (x_n) is an order bounded disjoint sequence of E then $x_n \rightarrow 0$ for the topology $\sigma(E, E')$ ([1], p. 192) and so $\|T(x_n)\| \rightarrow 0$. Hence, Dodds's Theorem ([1], Theorem 5.57) implies that T is order weakly compact). But an order weakly compact operator is not necessary almost Dunford-Pettis. In fact, the identity operator $Id_{c_0} : c_0 \rightarrow c_0$ is order weakly compact (because the norm of c_0 is order continuous and hence each order interval of c_0 is weakly compact (see [1], Theorem 12.9)) but it fails to be almost Dunford-Pettis.

Let ρ be a lattice seminorm on a Banach lattice E . A subset A of E is said to be ρ -almost order bounded (see Zaanen [6], p. 525) (or *approximately order bounded* with respect to ρ (see [3], Remark, p. 73)) if for every $\varepsilon > 0$ there exists $u \in E^+$ such that $A \subset [-u, u] + \varepsilon B_\rho$ where $B_\rho = \{x \in E : \rho(x) \leq 1\}$ is the closed unit ball associated to ρ . Since B_ρ is a solid subset of E , then it follows from [3, p. 73] that

$$A \subset [-u, u] + \varepsilon B_\rho \quad \text{if and only if} \quad \rho(|y| - u)^+ \leq \varepsilon \quad \text{for all } y \in A.$$

Let T be an operator from a Banach lattice E into a Banach space X . We will need the lattice seminorm q_T on E , which is defined in ([3], p. 192), by the following formula:

$$q_T(x) := \sup \{\|T(y)\| : |y| \leq |x|\}, \quad x \in E.$$

It is easy to see that $\|T(x)\| \leq q_T(x) \leq \|T\| \cdot \|x\|$ holds for each $x \in E$. So, the lattice seminorm q_T is continuous for the norm of E .

Proposition 2.1. *Let T be an almost Dunford-Pettis operator from a Banach lattice E into a Banach space X . Then each weakly relatively compact subset W of E is approximately order bounded with respect to the lattice seminorm q_T .*

If, in addition, T is order bounded, then $T(W)$ is an almost order bounded subset of F , i.e., for every $\varepsilon > 0$ there exists some $v \in F^+$ such that $\|(|T(x)| - v)^+\| \leq \varepsilon$ holds for all $x \in W$.

Proof. Let W be a weakly relatively compact subset of E and let $\varepsilon > 0$. It follows from Theorem 4.34 of [1] that every disjoint sequence, in the solid hull A of W , converges weakly to zero. Let (x_n) be a disjoint sequence (x_n) in A . We claim that $q_T(x_n) \rightarrow 0$. From $q_T(x_n) = \sup \{\|T(y)\| : |y| \leq |x_n|\}$, it follows that for each n there exists an element y_n in A such that $|y_n| \leq |x_n|$ and

$$q_T(x_n) \leq 2\|T(y_n)\|.$$

We note that the sequence $(y_n) \subset A$ is disjoint and hence it converges weakly to zero. Next, as the operator $T : E \rightarrow X$ is almost Dunford-Pettis, we obtain $\|T(y_n)\| \rightarrow 0$, and hence $q_T(x_n) \rightarrow 0$ holds, as desired.

After that, if $Id_E : E \rightarrow E$ is the identity operator of E , then $q_T(Id_E(x_n)) \rightarrow 0$ holds for each disjoint sequence (x_n) in A . Thus, by Theorem 4.36 of [1], there exists some $u \in E^+$, lying in the order ideal generated by A , such that

$$q_T(Id_E(|x| - u)^+) \leq \varepsilon$$

for all $x \in A$, i.e., $q_T(|x| - u)^+ \leq \varepsilon$ for all $x \in A$ (and hence for all $x \in W$). This implies that W is approximately order bounded with respect to the lattice seminorm q_T .

If, in addition, T is order bounded, then there exists some $v \in F^+$ such that $T([0, u]) \subseteq [-v, v]$. Let first $0 \leq x \in A$. It follows from $x = x \wedge u + (x - u)^+$ and $|T(x \wedge u)| \leq v$ that

$$|T(x)| - v \leq |T(x)| - |T(x \wedge u)| \leq |T(x) - T(x \wedge u)| = |T((x - u)^+)|.$$

Then $(|T(x)| - v)^+ \leq |T((x - u)^+)|$ and hence

$$\|(|T(x)| - v)^+\| \leq \| |T((x - u)^+)| \| \leq q_T((x - u)^+) \leq \varepsilon.$$

If $x \in A$ is arbitrary, then

$$|T(x)| - 2v \leq (|T(x^+)| - v) + (|T(x^-)| - v),$$

and so $\|(|T(x)| - 2v)^+\| \leq 2\varepsilon$. This shows that $T(W)$ is an almost order bounded subset of F . \square

Our result gives a generalization of Example 4 of ([5], p. 230).

Theorem 2.2. *Let E and F be two Banach lattices and let X be a Banach space. Suppose that F has weakly sequentially continuous lattice operations. If $T : E \rightarrow F$ is an order bounded almost Dunford-Pettis operator and $S : F \rightarrow X$ is an order weakly compact operator, then $S \circ T$ is Dunford-Pettis.*

Proof. Let (x_n) be a weakly null sequence of E . To this end, we have to show that $\|S[T(x_n)]\| \rightarrow 0$. Let $y_n = [T(x_n)]^+$ and $z_n = [T(x_n)]^-$. It suffices to show that $\|S(y_n)\| \rightarrow 0$ and $\|S(z_n)\| \rightarrow 0$. To this end, let $\varepsilon > 0$ be given. Since $W := \{x_n : n \in \mathbb{N}\}$ is a weakly relatively compact subset of E , it follows from Proposition 2.1 that there exists some $v \in F^+$ such that $\|(|T(x_n)| - v)^+\| \leq \varepsilon$ holds for all n . From the inequalities $(y_n - v)^+ \leq (|T(x_n)| - v)^+$, we see that $\|(y_n - v)^+\| \leq \|(|T(x_n)| - v)^+\| \leq \varepsilon$ holds for all n .

On the other hand, it is clear that $T(x_n) \rightarrow 0$ for the topology $\sigma(F, F')$. Since F has weakly sequentially continuous lattice operations, then $|T(x_n)| \rightarrow 0$ for the topology $\sigma(F, F')$. Next, from the inequalities $0 \leq y_n \wedge v \leq y_n \leq |T(x_n)|$ for each n , we see that $y_n \wedge v \rightarrow 0$ for the topology $\sigma(F, F')$. Hence $(y_n \wedge v)$ is an order bounded weakly null sequence of F^+ . Now, since $S : F \rightarrow X$ is order weakly compact, it follows from Corollary 3.4.9 of [3] that $\|S(y_n \wedge v)\| \rightarrow 0$. So, there exists some n_0 such that $\|S(y_n \wedge v)\| < \varepsilon$ holds for all $n \geq n_0$.

Finally, from the identity $y_n = (y_n - v)^+ + (y_n \wedge v)$, we see that

$$\begin{aligned} \|S(y_n)\| &\leq \|S(y_n - v)^+\| + \|S(y_n \wedge v)\| \\ &\leq \|S\| \cdot \|(y_n - v)^+\| + \varepsilon \leq (\|S\| + 1)\varepsilon \end{aligned}$$

holds for all $n \geq n_0$. And this implies that $\|S(y_n)\| \rightarrow 0$.

Similarly, it can be shown that $\|S(z_n)\| \rightarrow 0$. Since $S[T(x_n)] = S(y_n) - S(z_n)$ holds for all n , we see that $\|S[T(x_n)]\| \rightarrow 0$ and this completes the proof of the Theorem. \square

A Banach lattice E has the *positive Schur property* if weakly null sequences with positive terms are norm null. For example, the Banach lattice $L^1([0, 1])$ has the positive Schur property.

Note that if E has the positive Schur property, then every operator T from E into an arbitrary Banach space is almost Dunford-Pettis.

Remark 2.3. *The order boundedness of the operator T in the Theorem 2.2 is essential. In fact, if we take $E = L^1[0, 1]$ and $F = c_0$, and if we consider the operator $T : L^1[0, 1] \rightarrow c_0$ defined by the equality*

$$T(f) = \left(\int_0^1 f(t) \sin(nt) dt \right)_{n=1}^{\infty}.$$

Note that T is not order bounded (see Exercise 10, p. 289 of [1]). Since $L^1[0, 1]$ has the positive Schur property, it follows from Proposition 3.7.24 of [3] that T is not Dunford-Pettis. However, T is almost Dunford-Pettis.

Also, note that the identity operator $Id_{c_0} : c_0 \rightarrow c_0$ is order weakly compact but the composed $Id_{c_0} \circ T = T$ is not Dunford-Pettis.

Recall from [1] that an operator $T : X \rightarrow Y$ between two Banach spaces is said to be *weak Dunford-Pettis* whenever $x_n \rightarrow 0$ weakly in X and $f_n \rightarrow 0$ weakly in Y' imply $f_n(T(x_n)) \rightarrow 0$. On the other hand, we say that a Banach space X has the *Dunford-Pettis property* whenever $x_n \rightarrow 0$ weakly in X and $f_n \rightarrow 0$ weakly in X' imply $f_n(x_n) \rightarrow 0$.

A weak Dunford-Pettis operator $T : E \rightarrow Y$ from a Banach lattice E into a Banach space X is not necessary almost Dunford-Pettis operator. In fact, the identity operator $Id_{c_0} : c_0 \rightarrow c_0$ is weak Dunford-Pettis but it fails to be almost Dunford-Pettis. Conversely, there exists an almost Dunford-Pettis operator which is not weak Dunford-Pettis. In fact, it follows from the Remark 3 of Wnuk ([4], p. 19) that the Lorenz space $E = \Lambda(w, 1)$ has the positive Schur property but it fails to have the Dunford-Pettis property. Hence, the identity operator $Id_{\Lambda(w, 1)} : \Lambda(w, 1) \rightarrow \Lambda(w, 1)$ is almost Dunford-Pettis but it fails to be weak Dunford-Pettis.

As a consequence of Theorem 2.2 and Theorem 5.99 of [1], we obtain

Corollary 2.4. *Let E and F be two Banach lattices. If F has weakly sequentially continuous lattice operations, then each order bounded almost Dunford-Pettis operator $T : E \rightarrow F$ is weak Dunford-Pettis.*

Proof. Assume that F has weakly sequentially continuous lattice operations and let $T : E \rightarrow F$ be an order bounded almost Dunford-Pettis operator. It follows from Theorem 2.2 that for every weakly compact operator S from F into an arbitrary Banach space X , the composed operator $S \circ T$ is Dunford-Pettis. Thus, we deduce that $T : E \rightarrow F$ is a weak Dunford-Pettis operator (by using the equivalence (1) \iff (3) of Theorem 5.99 of [1]). \square

Recall that a nonzero element x of a vector lattice E is *discrete* if the order ideal generated by x equals the subspace generated by x . The vector lattice E is *discrete*, if it admits a complete disjoint system of discrete elements.

Recall from Wnuk ([5], Example 4, p. 230) that if F is a discrete Banach lattice with an order continuous norm, then a positive operator $T : E \rightarrow F$ is almost Dunford-Pettis iff T is Dunford-Pettis.

Another a consequence of Theorem 2.2, we give a generalization of this result of Wnuk [5].

Corollary 2.5. *Let E and F be two Banach lattices. If F is discrete with an order continuous norm, then an order bounded operator $T : E \rightarrow F$ is almost Dunford-Pettis if and only if T is Dunford-Pettis.*

Proof. Assume that $T : E \rightarrow F$ is an order bounded almost Dunford-Pettis operator. It follows from Proposition 2.5.23 of [3] that F has weakly sequentially continuous lattice operations.

On the other hand, since the norm of F is order continuous, then each order interval of F is weakly compact (see [1], Theorem 4.9), and hence its identity operator $Id_F : F \rightarrow F$ is order weakly compact. So, by Theorem 2.2, the composed $T = Id_F \circ T$ is Dunford-Pettis. This completes the proof. \square

Remark 2.6. *We can changed the assumption “ F is discrete with an order continuous norm” of Corollary 2.5 by the assumption “ F has weakly sequentially continuous lattice operations and the norm of F is order continuous”. Because these two assumptions are equivalent. In fact, it follows from Corollary 2.3 of Chen-Wickstead [2] that if the norm of F is order continuous, then F is discrete if and only if F has weakly sequentially continuous lattice operations.*

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