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A q-DIFFERENCE OPERATOR WITH DISCRETE AND SIMPLE SPECTRUM

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Abstract. We continue our study of a q-difference version of a second-order differential operator which depends on a real parameter. This version was introduced in our previous article. For values of the parameter for which the difference operator is self adjoint, we show that the spectrum of the operator is discrete and simple. When q approaches 1, the spectrum fills the whole positive or negative semiaxis.

1. Introduction

During the last decade, the investigation of pairs of operators $\{A, B\}$ that satisfy the formal algebraic relation

$$AB = qBA, \quad \text{where} \quad q > 1,$$

has received a considerable interest. These studies were motivated by the development of the theory of quantum groups and quantum algebras (see, e.g., [12, 13, 22]) and, of course, by the development of operator theory [18, 19].

In [20], the case $B = A^*$ was considered. The corresponding operators A satisfying

$$AA^* = qA^*A, \quad \text{where} \quad q > 1,$$

are called q-normal. Such operators as well as some other classes of q-deformed operators (q-quasinormal, q-hyponormal) were also investigated in [20]. In that article, it was pointed out, in particular, that the case of pairs $\{A, B\}$ with unitary A and self-adjoint B can be reduced to that of q-normal operators.

In [9], the authors considered a one-parameter family $\{U_s\}$ ($s \in S \subset \mathbb{R}$) of unitary operators acting in some Hilbert space $\mathcal{H}$ and a linear operator A ($\neq 0$) acting in $\mathcal{H}$ such that

$$U_s A = p(s) A U_s, \quad s \in S,$$

where p is a real-valued function. In [9], such operators were called $p(s)$-homogeneous. In the particular case $S = \mathbb{Z}$, $\{U_s\}$ is a group, $p(s) = r^s$ for $s \in \mathbb{Z}$, where $r \neq 1$ is a constant, while in the case $S = \mathbb{R}$, $\{U_s\}$ is a group, and $p(s) = e^s$, and hence one obtains a scale-invariant operator A.

Positive symmetric scale-invariant operators were considered in [15], and using 'real' Caley transforms in [2, 3]. In those articles, it was proved that a positive symmetric scale-invariant operator always admits a positive scale-invariant self-adjoint extension. In particular, the extreme extensions, the so-called Friedrichs and Kreĭn extensions, are scale-invariant. Moreover, in [2, 3], it was proved that if the index of defect of a positive scale-invariant symmetric operator is $(1, 1)$, then only the extreme self-adjoint extensions are scale-invariant and if $S = \mathbb{R}$, then any positive scale-invariant symmetric operator

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with index of defect $(1, 1)$ is unitarily equivalent to the operator on $L^2(\mathbb{R}_+, \mathrm{d}x)$ defined by means of the differential expression

$$\mathcal{H}_0 f = -\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\alpha}{t^2} x$$

for some α that satisfies the condition $-1/4 \le \alpha < 3/4$ (see [2]).

In our previous article [4], we constructed a discrete version of the aforementioned differential operator (see also [6, 7, 10]) and showed that our model closely resembles some properties of that operator. The operator constructed in that article is scale invariant with $S = \mathbb{Z}$. Below we briefly repeat our arguments from [4]. Let $\mathcal{M}$ be the linear space of all sequences $x = \{x_n\}_{n\in\mathbb{Z}}$ with complex entries. Select a number $q > 1$ and consider the points $t_n = q^n$, $n \in \mathbb{Z}$, as points of discretization. The first and second derivatives of a function x, defined on $(0, \infty)$, such that $x = \{x_n\}_{n\in\mathbb{Z}} = \{x(q^n)\}_{n\in\mathbb{Z}}$ are replaced by

$$(D_q x)_n = \frac{x_{n+1} - x_n}{q^{n+1} - q^n} = \frac{x_{n+1} - x_n}{q^n(q-1)}$$

and

$$(D_q^2 x)_{n-1} = \frac{(D_q x)_n - (D_q x)_{n-1}}{q^n - q^{n-1}} = \frac{x_{n+1} - (1+q)x_n + q x_{n-1}}{q^{2n-1}(q-1)^2},$$

respectively. For $\alpha \in \mathbb{R}$ and $\beta = 1 + q + (q-1)^2\alpha$, we consider a linear mapping $\mathcal{L} : \mathcal{M} \to \mathcal{M}$ defined as

$$(\mathcal{L}x)_n = -(D_q^2 x)_{n-1} + \frac{\alpha}{q^{n-1}q^n} x_n = -\frac{x_{n+1} - \beta x_n + q x_{n-1}}{q^{2n-1}(q-1)^2}. \tag{1}$$

Instead of $L^2(\mathbb{R}_+)$, we consider the linear subset of $\mathcal{M}$ consisting of sequences $\{x_n\}$ that satisfy condition

$$\|x\|^2 = \frac{q-1}{q} \sum_{n=-\infty}^{\infty} q^n |x_n|^2 < \infty. \tag{2}$$

The set $\mathcal{H}$ of all such sequences is a Hilbert space with inner product $\langle x, y\rangle$ defined as

$$\langle x, y\rangle = \frac{q-1}{q} \sum_{n=-\infty}^{\infty} q^n x_n \bar{y_n}. \tag{3}$$

We denote this Hilbert space by $l^2(\mathbb{Z}; q)$. The vectors $\{e^{(k)}\}_{k=-\infty}^{\infty}$ with

$$e^{(k)} = \left\{e_n^{(k)}\right\}_{n=-\infty}^{\infty} = \left\{\frac{q^{-(k-1)/2}}{\sqrt{q-1}}\delta_{kn}\right\}_{n=-\infty}^{\infty}$$

form an orthonormal basis in the space $\mathcal{H} = l^2(\mathbb{Z}; q)$.

The difference expression (1) defines some unbounded linear operator L in $l^2(\mathbb{Z}; q)$. Some of the properties of this operator that depend on β (whether the operator symmetric or self adjoint, semibounded or not semibounded) were investigated in [4]. In [4], it was proved that the operator L generated by the expression (1) for

$$\beta \ge \beta_+ = 2\sqrt{q} \quad \text{and} \quad \beta \le \beta_- = -2\sqrt{q},$$

i.e.,

$$\alpha \ge \alpha_+ = -\frac{1}{\left(\sqrt{q}+1\right)^2} \quad \text{and} \quad \alpha \le \alpha_- = -\frac{1}{\left(\sqrt{q}-1\right)^2},$$

is positive semibounded and negative semibounded, respectively, and is self adjoint for

$$\beta \ge \beta_{++} = \frac{q^2+1}{\sqrt{q}} \quad \text{and} \quad \beta \le \beta_{--} = -\frac{q^2+1}{\sqrt{q}},$$

i.e.,

$$\alpha \geq \alpha_{++} = \frac{\sqrt{q}+1+\frac{1}{\sqrt{q}}}{\left(\sqrt{q}+1\right)^2} \quad \text{and} \quad \alpha \leq \alpha_{--} = -\frac{\sqrt{q}-1+\frac{1}{\sqrt{q}}}{\left(\sqrt{q}-1\right)^2}.$$

It was proved that

$$\mathcal{D}(L) = \{x \in \mathcal{H} : \mathcal{L}x \in \mathcal{H}\} \tag{4}$$

and

$$Lx = \mathcal{L}x, \quad x \in \mathcal{D}(L), \tag{5}$$

where $\mathcal{D}(L)$ denotes the domain of the operator L. It was also proved that for $\beta \geq \beta_{++}$, one has $\langle Lx, x\rangle \geq 0$ ($x \in \mathcal{D}(L)$), while for $\beta \leq \beta_{--}$, the operator L is negative, that is, $\langle Lx, x\rangle \leq 0$.

Denote by U the operator on $\mathcal{H}$ defined by

$$(Ux)_n = \frac{1}{\sqrt{q}} x_{n-1}. \tag{6}$$

The operator U^* is then given by

$$(U^*x)_n = \sqrt{q} x_{n+1}. \tag{7}$$

The operator U is unitary ($U^*U = UU^* = I$), $U\mathcal{D}(L) = \mathcal{D}(L)$ and satisfies the relation

$$ULx = q^2 LUx, \quad x \in \mathcal{D}(L). \tag{8}$$

In [4], operators L that satisfy (8) were called (q^2, U)-invariant (see also [2, 3]).

Remark 1. Let us show that the commutative relation (8) and the unitary operator U define the three-term difference expression (1) (bi-infinite Jacobi matrix) in an essentially unique way. For the sake of simplification we start from the space $l^2(\mathbb{Z})$ of all sequences $\{x_n\}_{n\in\mathbb{Z}}$ with complex entries such that

$$\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty.$$

Let $\hat{U}$ be the bilateral shift operator on $l^2(\mathbb{Z})$, that is,

$$(\hat{U}x)_n = x_{n-1}.$$

Let $\hat{J}$ be the bi-infinite symmetric Jacobi matrix given by

$$\hat{J} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\ddots\ddots & & & \\ \dots & 0 & b_{-2} & a_{-1} & b_{-1} & 0 & \dots & \dots & \\ \dots & \dots & 0 & b_{-1} & a_0 & b_0 & 0 & \dots & \\ \ddots & \ddots & \ddots & 0 & b_0 & a_1 & b_1 & 0\dots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where all a_k and b_k, $k \in \mathbb{Z}$, are real. This matrix $\hat{J}$ defines some linear (symmetric or self adjoint) operator $\hat{L}$ on the Hilbert space $l^2(\mathbb{Z})$. The domain of this operator includes all sequences with finite support. On such sequences, we have

$$(\hat{L}x)_n = b_{n-1}x_{n-1} + a_n x_n + b_n x_{n+1}.$$

We want to select the entries of $\hat{J}$ in such a way that

$$\hat{U}\hat{L}x = q^2 \hat{L}\hat{U}x, \quad \text{where} \quad q > 1.$$

Thus, for all sequences x with finite support, the equality

$$b_{n-2}x_{n-2} + a_{n-1}x_{n-1} + b_{n-1}x_n = q^2(b_{n-1}x_{n-2} + a_n x_{n-1} + b_n x_n)$$

holds, from which one deduces

$$b_{n+1} = q^{-2} b_n, \quad a_{n+1} = q^{-2} a_n, \quad n \in \mathbb{Z},$$

i.e.,

$$a_n = a_0 q^{-2n}, \quad b_n = b_0 q^{-2n}, \quad n \in \mathbb{Z}.$$

Taking $a_0 = \beta$ and $b_0 = -q^{1/2}$, we obtain the Jacobi matrix considered, in fact, in Section 2 below. It only remains to make a unitary transform from $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z}; q)$. The additional coefficients in the definition of the norm in $l^2(\mathbb{Z}; q)$ are such that for $q \to 1$, the values of α_+ and α_{++} tend to the known values of α_+ and α_{++} for the differential operator, while the values of α_- and α_{--} disappear.

In this article, we investigate the spectrum of the operator L in the self-adjoint case, that is, for $\beta \geq \beta_{++}$ or for $\beta \leq \beta_{--}$. Since the arguments for both cases are similar, we provide them only for the case $\beta \geq \beta_{++}$. For the case $\beta \leq \beta_{--}$, we only state the corresponding results.

The setup of this paper is as follows. In Section 2, we consider polynomials associated with the q-difference operator under consideration, together with convergence or divergence of the corresponding series. In Section 3, we discuss the general structure of (q^2, U)-invariant operators, while we show in Section 4 that the operator under consideration has simple spectrum. Finally, in Section 5, we prove that the operator under consideration has discrete spectrum.

2. Associated polynomials

Denote by $P(z) = \{P_n(z)\}_{n\in\mathbb{Z}}$ the set of polynomials of z that satisfy the difference equation $\mathcal{L}P = zP$, that is,

$$-\frac{q}{(q-1)^2}\frac{P_{n+1}(z) - \beta P_n(z) + qP_{n-1}(z)}{q^{2n}} = zP_n(z), \tag{9}$$

together with the "initial" condition

$$P_{-1}(z) = 0, \quad P_0(z) = 1. \tag{10}$$

For $n \in \mathbb{N}$, each P_n is a polynomial of degree n. In particular,

$$P_1(z) = \beta - \frac{(q-1)^2}{q} z,$$
$$P_2(z) = \left(\beta - q^2 \frac{(q-1)^2}{q} z\right)\left(\beta - \frac{(q-1)^2}{q} z\right) - q,$$

and so on. For $n \in \mathbb{Z}$ such that $n \leq -2$, the polynomials P_n have degree $|n| - 2$. In particular,

$$P_{-2} = -\frac{1}{q},$$
$$P_{-3} = -\frac{1}{q^2}\left(\beta - q^{-4}\frac{(q-1)^2}{q} z\right),$$

and so on. Each polynomial P_n ($n \notin \{-2, -1, 0\}$) has real and simple roots. For $n \in \mathbb{N}$, the roots of two consecutive polynomials P_n and P_{n+1} alternate, that is, between any two consecutive roots of the polynomial P_{n+1}, there is exactly one root of the polynomial P_n. The same property is fulfilled for the polynomials P_n with negative $n \in \mathbb{Z}$. Indeed, denote by $Y_n(z)$, $n \in \mathbb{Z}$, the polynomials defined by $Y_n(z) = q^{n/2} P_n(z)$. Then $Y_{-1} = 0$,

$Y_0 = 1$, and it is easily seen that the polynomials Y_n, $n \in \mathbb{N}_0$, are associated with the infinite Jacobi matrix

$$\begin{pmatrix} \beta & -1/\sqrt{q} & 0 & 0 & \dots & \dots \\ -1/\sqrt{q} & \beta/q^2 & -1/\sqrt{q^5} & 0 & \dots & \dots \\ 0 & -1/\sqrt{q^5} & \beta/q^4 & -1/\sqrt{q^9} & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

while Y_n, $n \leq -2$, are associated with the infinite Jacobi matrix

$$\begin{pmatrix} \beta q^4 & -q^{11/2} & 0 & 0 & \dots\dots & \\ -q^{11/2} & \beta q^6 & -q^{15/2} & 0 & \dots & \dots \\ 0 & -q^{15/2} & \beta q^8 & -q^{19/2} & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Therefore, polynomials $\{Y_n\}$ for $n \in \mathbb{N}_0$ and $\{Y_n\}$ for $n \leq -2$ are orthogonal with respect to some measures from which the declared properties about their roots follow (see, for example, [1, Chap. 1, Sec. 2]). Note also that since the operator L is positive, all roots of polynomials P_n are located on positive semi-axis.

Denote by $Q(z) = \{Q_n(z)\}_{n\in\mathbb{Z}}$ the set of polynomials that satisfies the equation $\mathcal{L}Q = zQ$ and the "initial condition"

$$Q_{-1}(z) = 1, \quad Q_0(z) = 0. \tag{11}$$

There are some algebraic relations between the polynomials P_n and Q_n. Denote by W_n the expression

$$W_n(z) = P_n(z)Q_{n-1}(z) - P_{n-1}(z)Q_n(z). \tag{12}$$

Multiplying

$$-\frac{P_{n+1}(z) - \beta P_n(z) + qP_{n-1}(z)}{q^{2n}} = \frac{(q-1)^2}{q} zP_n(z)$$

by $Q_n(z)$, multiplying the corresponding equation for $Q_n(z)$ by $P_n(z)$, and subtracting, one obtains $W_{n+1}(z) = qW_n(z)$. The formulas (10) and (11) show that $W_0(z) = 1$. Therefore $W_n = W_n(z)$ does not depend on z and

$$W_n = q^n, \quad n \in \mathbb{Z}. \tag{13}$$

The following formulas (14) and (15) algebraically relate the values of the polynomials P_n and Q_n outside the annulus $\{z \in \mathbb{C} : \ 0 < a \leq |z| < aq^2\}$ with their values on this annulus. These relations may be viewed as the algebraic expression of the fact that the operator generated by the difference expression (1) is scale invariant.

Theorem 1. *For $n, k \in \mathbb{Z}$, the polynomials P_n and Q_n are related by the equations*

$$P_{n-k}(z) = \frac{Q_{k-1}(z/q^{2k})P_n(z/q^{2k}) - P_{k-1}(z/q^{2k})Q_n(z/q^{2k})}{q^k} \tag{14}$$

and

$$Q_{n-k}(z) = \frac{P_k(z/q^{2k})Q_n(z/q^{2k}) - Q_k(z/q^{2k})P_n(z/q^{2k})}{q^k}. \tag{15}$$

Proof. Fix $k \in \mathbb{Z}$ and denote by $\hat{P}(z) = \{\hat{P}_n(z)\}_{n\in\mathbb{Z}}$ the sequence of polynomials defined by $\hat{P}_n(z) = P_{n-k}(z)$. Then

$$-\frac{\hat{P}_{n+1}(z) - \beta\hat{P}_n(z) + q\hat{P}_{n-1}(z)}{q^{2n}} = \frac{(q-1)^2}{q}\frac{z}{q^{2k}}\hat{P}_n(z),$$

which means that there exist $a(z)$ and $b(z)$ such that

$$\hat{P}(z) = a(z)P(z/q^{2k}) + b(z)Q(z/q^{2k}),$$

that is,

$$P_{n-k}(z) = a(z)P_n(z/q^{2k}) + b(z)Q_n(z/q^{2k}). \tag{16}$$

Put now $n = k$ and $n = k-1$ and obtain

$$\begin{aligned} 1 &= a(z)P_k(z/q^{2k}) + b(z)Q_k(z/q^{2k}),\\ 0 &= a(z)P_{k-1}(z/q^{2k}) + b(z)Q_{k-1}(z/q^{2k}). \end{aligned}$$

Solving this system (observe (13)) for $a(z)$ and $b(z)$ and substituting into (16), one obtains (14). The proof of (15) is similar. □

From relations (14) and (15), it follows, for example, that

$$P_{k-1}(z) = -\frac{Q_k(z/q^2)}{q}, \tag{17}$$

$$P_{-k}(z) = \frac{Q_{k-1}(z/q^{2k})}{q^k}, \tag{18}$$

$$Q_{-k}(z) = -\frac{Q_k(z/q^{2k})}{q^k}. \tag{19}$$

Now the properties of the polynomials Q_n can be obtained from the corresponding properties of the polynomials P_n. In particular, all roots of the polynomial Q_n ($n \notin \{-1,0,1\}$) are real and simple. Any two consecutive polynomials Q_n and Q_{n+1} have alternating roots. The roots of Q_n are located on the positive semi-axis.

Observe that $P_+(z) = \{P_n(z)\}_{n\in\mathbb{N}_0}$ is a solution of the equation

$$-\frac{x_{n+1} - \beta x_n + qx_{n-1}}{q^{2n-1}(q-1)^2} = zx_n, \quad n \in \mathbb{N}$$

with initial conditions

$$x_0 = P_0(z) = 1 \quad \text{and} \quad x_1 = P_1(z) = \beta - \frac{(q-1)^2}{q}z.$$

The operator A on $l^2(\mathbb{N}_0; q)$ defined by that difference expression is compact, self adjoint, and positive (see [4, Formula (4.1)]). In the proof of [4, Lemma 4.1], it was shown that

$$\left\|Ae^{(k)}\right\| = \frac{C}{q^{2k}}, \quad \text{where} \quad C = \frac{\sqrt{q(1+\beta^2 q + q^4)}}{(q-1)^2}.$$

Thus

$$\sum_{k=0}^{\infty}\left\|Ae^{(k)}\right\|^p < \infty$$

for any $p > 0$, that is, the operator A belongs to the von Neumann–Schatten class for any $p > 0$ (see, for example, [8, Chap. 3]). Therefore

$$\sum_{n=0}^{\infty} q^n |P_n(z)|^2 = \infty \tag{20}$$

for all but countably many values of $z = \lambda_s > 0$. Those values of λ_s satisfy the condition

$$\sum \lambda_s^p < \infty$$

for any $p > 0$.

Now the relation (17) gives

$$\sum_{n=0}^{\infty} q^n |Q_n(\lambda_s/q^2)|^2 < \infty.$$

Since the operator L is self adjoint and positive, for $z \notin [0, \infty)$ one has

$$\sum_{n=-\infty}^{\infty} q^n |P_n(z)|^2 = \infty.$$

Using [4, Theorem 3.4 and Remark 3.6], one can prove that

$$\sum_{n=-1}^{-\infty} q^n |P_n(z)|^2 = \infty, \quad z < 0.$$

As a summary, we state the following theorem.

Theorem 2. *Let P_k and Q_k be the polynomials discussed in this section. Then the following results hold:*

(i) *For all $z \in \mathbb{C}$, the series*

$$\sum_{k=-\infty}^{\infty} q^k |P_k(z)|^2 \quad \textit{and} \quad \sum_{k=-\infty}^{\infty} q^k |Q_k(z)|^2 \quad \textit{diverge.}$$

(ii) *For all but countable values of $z = \lambda_s > 0$, $s \in \mathbb{N}_0$, the series*

$$\sum_{k=0}^{\infty} q^k |P_k(z)|^2 \quad \textit{and} \quad \sum_{k=0}^{\infty} q^k \left| Q_k\left(\frac{z}{q^2}\right)\right|^2 \quad \textit{diverge.}$$

Moreover, we have

$$\sum_{s=0}^{\infty} \lambda_s^p < \infty.$$

(iii) *For all $z \in \mathbb{C} \setminus [0, \infty)$, the series*

$$\sum_{k=-\infty}^{0} q^k |P_k(z)|^2 \quad \textit{and} \quad \sum_{k=-\infty}^{0} q^k |Q_k(z)|^2 \quad \textit{diverge.}$$

3. Scale-invariant self-adjoint operators

From (8) it follows that the spectrum $S(L)$ of the operator L is invariant with respect to multiplication by q^2, $q^2 S(L) = S(L)$. In particular, if a point $\lambda_0 > 0$ is an eigenvalue of L, then all points $\lambda_k = q^{2k}\lambda_0$ are eigenvalues of L of the same multiplicity.

We will use the following generalized Stieltjes inversion formula which was proved by M. Livsic (see [14, Lemma 2.1]).

Lemma 1. *Let $\sigma(\lambda) = (\sigma(\lambda + 0) + \sigma(\lambda - 0))/2$ $(-\infty < \lambda < \infty)$ be some function which is of bounded variation on each finite interval, such that the integral*

$$\Phi(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z}$$

converges absolutely. Let $\varphi(\lambda)$ be some function which is analytic on the closed interval $\Delta = [\alpha, \beta]$. Denote by Δ_ε the broken path of integration consisting of the directed segment $[\alpha - i\varepsilon, \beta - i\varepsilon]$ and the antiparallel segment $[\beta + i\varepsilon, \alpha + i\varepsilon]$. Then

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Delta_\varepsilon} \varphi(z)\Phi(z)\, dz = -\int_\alpha^\beta \varphi(\lambda)\, d\sigma(\lambda). \tag{21}$$

The Stieltjes inversion formula is obtained when $\varphi(\lambda) \equiv 1$ in Lemma 1.

Denote by $E(\lambda)$, $\lambda \in \mathbb{R}$, the resolution of identity of the operator L. Since L is a positive operator, $E(\lambda) = 0$ for $\lambda \le 0$. We normalize $E(\lambda)$ in such a way that $E(\lambda) = [E(\lambda + 0) + E(\lambda - 0)]/2$.

Theorem 3. *Suppose U is a unitary operator, L is a self-adjoint operator, $q > 0$, and $\mathcal{B}(\mathbb{R})$ is the Borel field of $\mathbb{R}$. Then the following statements are equivalent:*

(i) *The operator L is $(q^2, U)-$invariant.*
(ii) *For any $\Delta \in \mathcal{B}(\mathbb{R})$, we have*

$$UE(\Delta) = E(\Delta/q^2)U. \tag{22}$$

Proof. We first show that (i) implies (ii). It is easy to check that for a (q^2, U)-invariant self-adjoint operator L, we have

$$U(L - zI)^{-1} = (q^2L - zI)^{-1}U, \quad \operatorname{Im} z \ne 0,$$

which means that for any $\phi, \psi \in \mathcal{H}$

$$\int_0^\infty \frac{\langle dE(\lambda)U^*\phi, U^*\psi\rangle}{\lambda - z} = \int_0^\infty \frac{\langle dE(\lambda)\phi, \psi\rangle}{q^2\lambda - z} = \int_0^\infty \frac{\left\langle dE(\lambda/q^2)\phi, \psi\right\rangle}{\lambda - z}.$$

Now using Lemma 1, one obtains that (22) is fulfilled for any compact interval $\Delta \subset \mathbb{R}$. To obtain (22) for an arbitrary Borel set $\Delta \in \mathcal{B}(\mathbb{R})$, we use the standard procedure of measure extension.

Now we show that (ii) implies (i). From (22), it follows that

$$U \int_0^\infty \lambda\, dE(\lambda)U^* = q^2 \int_0^\infty \lambda\, dE(\lambda),$$

and a change of variables gives $UL = q^2LU$, which is (8). □

Recall that for any Borel set Δ, the value of $E(\Delta)$ is an orthogonal projection on $\mathcal{H}$. From (22), it follows that the subspaces

$$\mathcal{H}_\lambda = [E(\lambda) - E(0)]\mathcal{H} = E(\lambda)\mathcal{H} \quad \text{and} \quad \mathcal{H}_{\lambda/q^2} = E(\lambda/q^2)\mathcal{H}$$

have the same dimensions, i.e., $\dim \mathcal{H}_\lambda = \dim \mathcal{H}_{\lambda/q^2}$. This means that each $\mathcal{H}_\lambda = E(\lambda)\mathcal{H}$ $(\lambda > 0)$ is infinite dimensional. Indeed, since $q > 1$, the subspace $\mathcal{H}_{\lambda/q^{2m}}$, $m \in \mathbb{N}$, is contained in $\mathcal{H}_\lambda$. At the same time, they have the same dimensions since orthogonal projections onto these subspaces are unitarily equivalent.

Theorem 4. *Let L be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ with dense range. Suppose that the operator L is (q^2, U)-invariant. Then there are Hilbert spaces $\mathcal{H}_k$, $k \in \mathbb{Z}$, such that the following statements hold:*

(i) *Each $\mathcal{H}_k$ reduces L.*
(ii) *$\mathcal{H}_k = U^k\mathcal{H}_0$.*
(iii) *We have*

$$\mathcal{H} = \sum_{k=-\infty}^{\infty} \oplus \mathcal{H}_k. \tag{23}$$

(iv) *With respect to the decomposition* (23), *the operator L is representable as*

$$L = \sum_{k=-\infty}^{\infty} \oplus q^{-2k} U^k L_0 U^{*k}, \tag{24}$$

where L_0 is a positive bounded and invertible self-adjoint operator.

Conversely, if (i)–(iv) are fulfilled, then L is a (q^2, U)-invariant positive self-adjoint operator with dense range.

Remark 2. In the representations (23) and (24), the domain $\mathcal{D}(L)$ of the operator L consists of all $f = \{f_k\}_{k\in\mathbb{Z}} \subset \mathcal{H}_k$ such that

$$\sum_{k=-\infty}^{\infty} \|f_k\|^2 < \infty \quad \text{and} \quad \sum_{k=-\infty}^{\infty} q^{-2k} \|L_0 U^{*k} f_k\|^2 < \infty.$$

Convergence of the series (24) is understood in the sense that

$$Lf = \lim_{m\to-\infty} \lim_{n\to\infty} \sum_{k=m}^{n} q^{-2k} U^k L_0 U^{*k} f_k,$$

where the limits are taken in the norm of the space $\mathcal{H}$.

Proof of Theorem 4. If (i)–(iv) are fulfilled, then

$$UL = \sum_{k=-\infty}^{\infty} q^{-2k} U^{k+1} L_0 U^k = q^2 \sum_{k=-\infty}^{\infty} q^{-2k-2} U^{k+1} L_0 U^{*(k+1)} U = q^2 LU,$$

that is, L is (q^2, U)-invariant. Positivity of L is obvious. Assume now that L is a positive (q^2, U)-self-adjoint operator. Pick an arbitrary $a > 0$ and denote $\Delta_k = [aq^{2k}, aq^{2k+2})$, $k \in \mathbb{Z}$. One has $\Delta_k \cap \Delta_j = \emptyset$ for $k \neq j$ and $\bigcup_{k=-\infty}^{\infty} \Delta_k = (0, \infty)$. For the resolution of identity of the operator L, one has that $E(\Delta_k)$, $k \in \mathbb{Z}$, are mutually orthogonal projections. Because of the hypothesis that L has dense range, hence does not annihilate any vector, so

$$I = \sum_{k=-\infty}^{\infty} E(\Delta_k).$$

Put $\mathcal{H}_k = E(\Delta_k)\mathcal{H}$. Then each $\mathcal{H}_k$ reduces L. Denote by L_k the part of L in $\mathcal{H}_k$. Then each L_k is a positive invertible operator, $L = \sum_{k=-\infty}^{\infty} \oplus L_k$. From Theorem 3, it immediately follows that $L_k = q^{-2k} U^k L_0 U^{*k}$. This completes the proof. $\square$

Remark 3. (i) If we remove the assumption that the range of L is dense, then it will be necessary to add one more term to the right-hand side of (23), namely the null space of the operator L, and the projection on that space to the right-hand side of (24).

(ii) If a (q^2, U)-invariant operator L is not semibounded, then it is representable as a direct sum of two semibounded (q^2, U)-invariant operators, one of which is positive and the other is negative.

Let $R(z) = (L - zI)^{-1}$, $z \notin [0, \infty)$, be the resolvent operator for the operator L. From (8), it follows that

$$q^2 U R(zq^2) = R(z) U. \tag{25}$$

As a consequence of (25), one obtains that in order to calculate action of the resolvent operator on an arbitrary vector f, it is sufficient to calculate $R(z)e^{(0)}$. Indeed from (6), it follows that $e^{(k)} = U^k e^{(0)}$. Therefore,

$$R(z)e^{(k)} = R(z)U^k e^{(0)} = q^{2k} U^k R(zq^{2k}) e^{(0)}$$

and

$$R(z)f = \sum_{k=-\infty}^{\infty} c_k q^{2k} U^k R(zq^{2k}) e^{(0)}, \quad f = \sum_{k=-\infty}^{\infty} c_k e^{(k)},$$

where the series converge in the norm of the space $\mathcal{H}$.

A general approach to the spectral analysis of difference operators generated by bi-infinite Jacobi matrices is pointed out in [5, Chap. VII, Sec. 3]. Following [5], we introduce

$$\Phi(g;\lambda) = \sum_{n=-\infty}^{\infty} q^n g_n \begin{pmatrix} P_n(\lambda) \\ Q_n(\lambda) \end{pmatrix}, \quad g = \{g_n\}_{n\in\mathbb{Z}} \quad \text{has finite support.} \tag{26}$$

For each $\lambda \in \mathbb{R}$, $\Phi(g;\lambda)$ takes values in $\mathbb{C}^2$. Then there exists a 2×2 nondecreasing matrix-valued function $\sigma(\lambda)$, $\sigma(\lambda) = (\sigma(\lambda_0) + \sigma(\lambda + 0))/2$, such that for any $f, g \in \mathcal{H}$ with finite support, Parseval's identity

$$\langle f, g\rangle = \int_0^\infty \langle d\sigma(\lambda)\Phi(f;\lambda), \Phi(g;\lambda)\rangle \tag{27}$$

holds, where the inner product inside the integral is taken in the sense of $\mathbb{C}^2$. Since the operator L is self adjoint, the matrix-valued function $\sigma(\lambda)$ is uniquely defined. Positivity of the operator L implies that $\sigma(\lambda) = 0$ for $\lambda \le 0$. The mapping Φ is extended by continuity onto $\mathcal{H}$. Denote by $d\rho(\lambda) = d\sigma_{11}(\lambda) + d\sigma_{22}(\lambda)$ and by $\tau(\lambda)$ a 2×2 matrix-valued function whose entries are Radon–Nikodým derivatives of the corresponding entries of σ with respect to ρ. Note that the matrix-valued function $\tau(\lambda)$ is nonnegative, i.e., $\tau(\lambda) \ge 0$, and $\tau_{11}(\lambda) + \tau_{22}(\lambda) = 1$. Then (27) can be written as

$$\langle f, g\rangle = \int_0^\infty \langle \tau(\lambda)\Phi(f;\lambda), \Phi(g;\lambda)\rangle \, d\rho(\lambda). \tag{28}$$

For the detailed construction of a Hilbert space generated by a nondecreasing matrix-valued function see, for example, [11, 16]. The following two important formulas are valid (see [14, Theorem 1 and Formula (3.6), respectively]):

$$\Phi(Lf;\lambda) = \lambda\Phi(f;\lambda), \quad f \in \mathcal{D}(L) \tag{29}$$

and

$$\begin{aligned} \langle R(z)f, g\rangle &= \int_0^\infty \frac{1}{\lambda - z} \langle d\sigma(\lambda)\Phi(f;\lambda), \Phi(g;\lambda)\rangle \\ &= \int_0^\infty \frac{1}{\lambda - z} \langle \tau(\lambda)\Phi(f;\lambda), \Phi(g;\lambda)\rangle \, d\rho(\lambda). \end{aligned} \tag{30}$$

4. Multiplicity of the spectrum

We will show that the matrix-valued function $\tau(\lambda)$ is singular ρ-almost everywhere. According to [16, Definition 4.5], this means that the following theorem is valid.

Theorem 5. *The operator L has simple spectrum, that is, the multiplicity of the spectrum is one.*

Proof. From (27), it follows that

$$\begin{aligned} \left\langle e^{(k)}, e^{(l)} \right\rangle &= \int_0^\infty \left\langle d\sigma(\lambda)\Phi^{(k)}, \Phi^{(l)} \right\rangle \\ &= \int_0^\infty \left\langle \tau(\lambda)\Phi^{(k)}(\lambda), \Phi^{(l)}(\lambda) \right\rangle d\rho(\lambda) = \delta_{kl}, \end{aligned}$$

where

$$\Phi^{(k)}(\lambda) = \Phi(e^{(k)};\lambda) = \sqrt{\frac{q^{k+1}}{q-1}} \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix}. \tag{31}$$

In particular,

$$1 = \left\langle e^{(0)}, e^{(0)} \right\rangle = \frac{q}{q-1} \int_0^\infty d\sigma_{11}(\lambda),$$

hence $\int_0^\infty d\sigma_{11}(\lambda) < \infty$. Also,

$$1 = \left\langle e^{(-1)}, e^{(-1)} \right\rangle = \frac{1}{q-1} \int_0^\infty d\sigma_{22}(\lambda),$$

that is, $\int_0^\infty d\sigma_{22}(\lambda) < \infty$. Therefore

$$\int_0^\infty d\rho(\lambda) < \infty. \tag{32}$$

From (25), it follows that

$$\begin{aligned} \left\langle R(z)e^{(0)}, e^{(0)} \right\rangle &= \frac{q}{q-1} \int_0^\infty \frac{d\sigma_{11}(\lambda)}{\lambda - z} = \left\langle U^k R(z)e^{(0)}, U^k e^{(0)} \right\rangle \\ &= \frac{1}{q^{2k}} \left\langle R(z/q^{2k})e^{(k)}, e^{(k)} \right\rangle \\ &= \frac{1}{q^{2k}} \frac{q^{k+1}}{q-1} \int_0^\infty \frac{1}{\lambda - z/q^{2k}} \left\langle d\sigma(\lambda) \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix}, \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix} \right\rangle, \end{aligned}$$

that is,

$$\int_0^\infty \frac{d\sigma_{11}(\lambda)}{\lambda - z} = \int_0^\infty \frac{1}{\lambda - z} q^k \left\langle d\sigma(\lambda/q^{2k}) \begin{pmatrix} P_k(\lambda/q^{2k}) \\ Q_k(\lambda/q^{2k}) \end{pmatrix}, \begin{pmatrix} P_k(\lambda/q^{2k}) \\ Q_k(\lambda/q^{2k}) \end{pmatrix} \right\rangle.$$

From the last expression, using Lemma 1, one deduces that for any interval $\Delta = [a, b)$

$$\int_\Delta d\sigma_{11}(\lambda) = \int_{\Delta/q^{2k}} q^k \left\langle d\sigma(\lambda) \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix}, \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix} \right\rangle. \tag{33}$$

Pick now an arbitrary $a > 0$ and put $\Delta_k = [aq^{2k}, aq^{2k+2})$ for $k \in \mathbb{Z}$. Then $\Delta_k \cap \Delta_j = \emptyset$ for $k \neq j$ and $\bigcup_{k=-\infty}^\infty \Delta_k = \mathbb{R}_+ \setminus \{0\}$. Therefore

$$\begin{aligned} \infty &> \int_{0+}^\infty d\sigma_{11}(\lambda) = \sum_{k=-\infty}^\infty \int_{\Delta_k} d\sigma_{11}(\lambda) \\ &= \sum_{k=-\infty}^\infty \int_{\Delta_0} q^k \left\langle d\sigma(\lambda) \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix}, \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix} \right\rangle. \end{aligned}$$

All terms on the right-hand side of the last equality are nonnegative. According to Beppo Levi's theorem (see, e.g., [21, Chap. 2]), the series

$$\sum_{k=-\infty}^\infty q^k \left\langle \tau(\lambda) \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix}, \begin{pmatrix} P_k(\lambda) \\ Q_k(\lambda) \end{pmatrix} \right\rangle$$

converges ρ-almost everywhere on Δ_0. Since $a > 0$ was selected arbitrarily, we conclude that the series converges for ρ-almost all $\lambda > 0$. Now, if the matrix-valued function $\tau(\lambda)$ is nonsingular on a set of positive ρ-measure, then for any λ_0 from this set, there exists a number $c(\lambda_0) > 0$ (the smallest eigenvalue of $\tau(\lambda_0)$) such that

$$\left\langle \tau(\lambda_0) \begin{pmatrix} P_k(\lambda_0) \\ Q_k(\lambda_0) \end{pmatrix}, \begin{pmatrix} P_k(\lambda_0) \\ Q_k(\lambda_0) \end{pmatrix} \right\rangle \geq c(\lambda_0) \left[|P_k(\lambda_0)|^2 + |Q_k(\lambda_0)|^2 \right].$$

Hence,

$$\sum_{k=-\infty}^\infty q^k \left[|P_k(\lambda_0)|^2 + |Q_k(\lambda_0)|^2 \right] < \infty.$$

Since by Theorem 2 the last series diverges, we obtain a contradiction. The matrix $\tau(\lambda_0)$ cannot be invertible. The proof is complete. $\square$

5. Structure of the spectrum

According to Theorem 5, for ρ-almost all λ, there is a 2×2 unitary matrix $\Gamma(\lambda)$ such that

$$\Gamma(\lambda)^*\tau(\lambda)\Gamma(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, Parseval's identity (27) takes the form

$$\langle f, g\rangle = \int_0^\infty \Psi(f;\lambda)\overline{\Psi(g;\lambda)}\, d\rho(\lambda), \tag{34}$$

where

$$\Psi(f;\lambda) = \sum_{k=-\infty}^{\infty} q^k J_k(\lambda) f_k \quad \text{with} \quad J_k = \Gamma_{11}P_k + \Gamma_{12}Q_k \tag{35}$$

and $\int_0^\infty d\rho(\lambda) < \infty$. The mapping $f \mapsto \Psi(f;\lambda)$ is a generalized Fourier transform, generated by the operator L, and can be considered as a discrete version of the Hankel transform (see [17, Chap. 6, Sec. 21]). Recall that the measure $d\rho(\lambda)$ is defined uniquely and the spectrum of the operator L coincides with the set of points of growth of the function ρ. The functions

$$\Psi^{(k)}(\lambda) = \Psi(e^{(k)};\lambda) = \sqrt{\frac{q^{k+1}}{q-1}}J_k(\lambda)$$

form an orthonormal basis of the space $L^2(d\sigma, \mathbb{R}^+)$. In particular,

$$1 = \left\langle e^{(k)}, e^{(k)}\right\rangle = \frac{q^{k+1}}{q-1}\int_0^\infty |J_k(\lambda)|^2 d\rho(\lambda).$$

Hence, the functions $J_n(\lambda)$ are summable with respect to the measure ρ.

Recall that $\Gamma_{11}(\lambda)$ and $\Gamma_{12}(\lambda)$ form the first row of the unitary matrix $\Gamma(\lambda)$. Therefore, $|\Gamma_{11}(\lambda)|^2 + |\Gamma_{12}(\lambda)|^2 = 1$ for all $\lambda \in \mathbb{R}_+$. Another relation between these functions follows from the fact that the operator L is (q^2, U)-invariant. Indeed,

$$\begin{aligned} \left\langle R(z)e^{(-1)}, e^{(-1)}\right\rangle &= \int_0^\infty \frac{1}{\lambda - z}|\Psi^{(-1)}(\lambda)|^2 d\rho(\lambda) \\ &= \frac{1}{q-1}\int_0^\infty \frac{|\Gamma_{12}(\lambda)|^2}{\lambda - z} d\rho(\lambda). \end{aligned}$$

The same arguments as in Section 4 give

$$\left\langle R(z)e^{(-1)}, e^{(-1)}\right\rangle = \frac{1}{q^2}\left\langle R(z/q^2)e^{(0)}, e^{(0)}\right\rangle = \frac{1}{q(q-1)}\int_0^\infty \frac{|\Gamma_{11}(\lambda)|^2}{\lambda - z/q^2} d\rho(\lambda).$$

From the last expression, the following statement follows.

Theorem 6. *For all $\lambda \in \mathbb{R}_+$, we have*

$$q\int_0^{\lambda/q^2} |\Gamma_{11}(s)|^2 d\rho(s) = \int_0^\lambda |\Gamma_{12}(s)|^2 d\rho(s). \tag{36}$$

Using the same arguments as in Section 4, one obtains that the series

$$\sum_{n=-\infty}^{\infty} q^n |J_n(\lambda)|^2$$

converges for ρ-almost all $\lambda \in \mathbb{R}^+$. Now we invoke [5, Theorem 1.17 from Chap. VII, Sec. 1]. According to that theorem, we have

$$\rho(\lambda+0)-\rho(\lambda)=\frac{q-1}{q}\frac{1}{\sum_{n=-\infty}^{\infty} q^n|J_n(\lambda)|^2}. \tag{37}$$

Aforementioned theorem was proved in [5] for operators generated by infinite Jacobi matrices, not by bi-infinite ones like in the case under consideration. Nevertheless, the analysis of the proof shows that the only requirements are summability of $J_n(\lambda)$ and the fact that they form an orthogonal basis for $L^2(\mathbb{R}^+, d\rho)$.

Combining (37), the fact that $\rho(\lambda)$ is monotonically nondecreasing and bounded, and convergence of $\sum\limits_{n=-\infty}^{\infty} q^n|J_n(\lambda)|^2$ ρ-almost everywhere, one obtains the following theorem.

Theorem 7. *The spectrum of the operator L is discrete. Every interval of the form $[a, q^2a)$, $a>0$, contains at least one point of the spectrum of L.*

A similar statement is true for $\beta \le \beta_{--}$.

Theorem 8. *For $\beta \le \beta_{--} = -\dfrac{q^2+1}{\sqrt{q}}$, the spectrum of the operator L is discrete and simple. Every interval of the form $[aq^2, a)$, $a<0$, contains at least one point of the spectrum.*

Remark 4. In Theorems 7 and 8, the cardinalities of the sets

$$S(L)\cap[aq^{2k}, aq^{2(k+1)}) \quad \text{and} \quad S(L)\cap[aq^{2(k+1)}, aq^{2k}),$$

respectively, do not depend on $k \in \mathbb{Z}$. This is a consequence of Theorem 4.

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References

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publishing Co., New York, 1965. (Russian edition: Fizmatlit, Moscow, 1961)
2. Miron B. Bekker, *On a class of nondensely defined Hermitian contractions*, Adv. Dyn. Syst. Appl. **2** (2007), no. 2, 141–165.
3. Miron B. Bekker, *On non-densely defined invariant Hermitian contractions*, Methods Funct. Anal. Topology **13** (2007), no. 3, 223–235.
4. Miron B. Bekker, Martin J. Bohner, Alexander N. Herega, and Hristo Voulov, *Spectral analysis of a q-difference operator*, J. Phys. A: Math. Theor. **43** (2010), no. 14, 145–207.
5. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
6. Martin Bohner and Allan Peterson, *Dynamic Equations on Time Scales*, An introduction with applications, Birkhäuser Boston Inc., Boston, 2001.
7. Martin Bohner and Mehmet Ünal, *Kneser's theorem in q-calculus*, J. Phys. A: Math. Theor. **38** (2005), no. 30, 6729–6739.
8. I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc., Providence, RI, 1969. (Russian edition: Nauka, Moscow, 1965)
9. Seppo Hassi and Sergii Kuzhel, *On symmetries in the theory of finite rank singular perturbations*, J. Funct. Anal. **256** (2009), no. 3, 777–809.
10. Adil Huseynov and Elgiz Bairamov, *An eigenvalue problem for quadratic pencils of q-difference equations and its applications*, Appl. Math. Lett. **22** (2009), no. 4, 521–527.
11. I. S. Kac, *On Hilbert spaces generated by monotone Hermitian matrix functions*, Kharkov, Zap. Nauchno-issl. Inst. Mat. i Mekh. i Khark. Mat. Obshch. **22** (1950), 95–113. (Russian)
12. Christian Kassel, *Quantum Groups*, Graduate Texts in Mathematics, Vol. 155, Springer-Verlag, New York, 1995.
13. Anatoli Klimyk and Konrad Schmüdgen, *Quantum Groups and their Representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.

14. M. G. Kreĭn, *The fundamental propositions of the theory of representations of Hermitian operators with deficiency index* (m, m), Ukrain. Mat. Zh. **1** (1949), no. 2, 3–66. (Russian)
15. K. A. Makarov and E. Tsekanovskii, *On* μ*-scale invariant operators*, Methods Funct. Anal. Topology **13** (2007), no. 2, 181–186.
16. M. M. Malamud and S. M. Malamud, *Spectral theory of operator measures in a Hilbert space*, St. Petersburg Math. J. **15** (2004), no. 3, 323–373.
17. M. A. Naĭmark, *Linear Differential Operators*, Second edition, revised and augmented, With an appendix by V. È. Ljance, Nauka, Moscow, 1969. (Russian)
18. Vasyl L. Ostrovskyĭ and Yuriĭ S. Samoĭlenko, *Unbounded operators satisfying non-Lie commutation relations*, Rep. Math. Phys. **28** (1989), no. 1, 91–104.
19. Vasyl L. Ostrovskyĭ and Yuriĭ S. Samoĭlenko, *Representations of quadratic* $*$*-algebras by bounded and unbounded operators*, Mathematics as language and art (Bialowieża, 1993), Rep. Math. Phys. **35** (1995), no. 2-3, 283–301.
20. Schôichi Ôta, *Some classes of q-deformed operators*, J. Operator Theory **48** (2002), no. 1, 151–186.
21. Frigyes Riesz and Béla Sz.-Nagy, *Functional Analysis*, Dover Books on Advanced Mathematics, Dover Publications Inc., New York, 1990. (Translated from 2nd French edition. 1955 edition)
22. Konrad Schmüdgen, *Operator representations of a q-deformed Heisenberg algebra*, J. Math. Phys. **40** (1999), no. 9, 4596–4605.

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