Methods of Functional Analysis and Topology
Vol. 17 (2011), no. 4, pp. 300–318

FUNCTIONAL EVOLUTIONS FOR HOMOGENEOUS STATIONARY DEATH-IMMIGRATION SPATIAL DYNAMICS

D. FINKELSHTEIN

Abstract. We discover death-immigration non-equilibrium stochastic dynamics in the continuum also known as the Surgailis process. Explicit expression for the correlation functions is presented. Dynamics of states and their generating functionals are studied. Ergodic properties for the evolutions are considered.

1. Introduction

Complex systems theory is a growing interdisciplinary area with a very broad spectrum of motivations and applications. One may characterize complex systems by properties as diversity and individuality of components, localization of interactions among components, and the outcomes of interactions used for replication or enhancement of components. In the study of these systems, proper language and techniques are delivered by the interacting particle models which form a rich and powerful direction in modern stochastic and infinite dimensional analysis. Interacting particle systems have a wide use as models in condensed matter physics, chemical kinetics, population biology, ecology, sociology and economics.

Mathematical realizations of such models may be considered as a dynamics of collections of points in proper spaces. The possible positions of points may be fixed due to the structure of space, e.g., dynamics on graphs, or, in particular, on lattices. Another area of models connects with free positions of points in the continuum, say, in the Euclidean space $\mathbb{R}^d$. However, as was shown in statistical physics, many empirical effects, such as phase transitions, are impossible in systems with a finite number of points. Due to this, one can consider infinite point systems as a mathematical approximation for realistic systems with a high number of elements. The connection with the reality, where infinite systems are absent, is given by the restriction of the study to locally finite systems (configurations) which have only finite number of elements in any finite volume.

Depending on applications, the points of such a system may be interpreted as molecules in physics, plants in ecology, animals in biology, infected people in medicine, companies in economics, market agents in finance, and so on. For study stochastic dynamics of such systems we may consider different mechanisms of (random) evolutions of their points. Existing points may disappear from the configuration that is naturally called 'death'. Each existing point may change own position due to some moving or hop; this mechanism traditionally is called 'emigration'. Each existing point may produce a new one, that is called 'birth'. There exists also another possibility for appearing a new element in the configuration coming from outside; this is called 'immigration'. Mathematically, the

2000 *Mathematics Subject Classification.* Primary 82C22, 82C21; Secondary 60K35.
Key words and phrases. Continuous systems, Markov evolution, spatial birth-death-migration processes, correlation functions, evolution equations, Surgailis process.
Author would like to thank Prof. Dr. Yuri Kondratiev for useful discussions. This work was supported by DFG through SFB-701, Bielefeld University, Germany.

random evolution of the system is described by a heuristic Markov generator which includes parts corresponding to different mechanisms above.

Rigorous mathematical results concerning stochastic dynamics of configurations in the continuum have not very reach history. One of the pioneering work in this area was [10]. Special class of models introduced therein have been recently studied in [8, 9]. We mentioned also [20]– [22], and references therein. During the last decade a functional approach for studying of the stochastic dynamics above was discovered. It was considered the evolutional equations connected with considered stochastic dynamics, namely, equations on states of systems and their correlation functions, equations on generating functionals and so on. Studying this evolutional equations yields not only existence (in different senses) of dynamics but their qualitative and quantitative properties also. For general description of this approach see, e.g., [7, 14], and for particular models see, e.g., [3]– [6], [15]. In the present paper we consider one of the simplest model, where only independent (constant) death and immigration appear. The corresponding stochastic process is the well-known Surgailis process [16, 25, 26]. For this model we find explicit expression for correlation functions that gives us a way to improve general results as well as to obtain new ones. The structure of the paper is the following. We describe the model and present necessary knowledge on configuration space techniques in Section 2. Section 3 is devoted to the evolutions of correlation functions and measures (states) of the system. The ergodic properties of the dynamics as well as evolution of the generating functionals are presented in Section 4. Finally, Section 5 deals with the so-called dynamics of quasi-observables.

We also note that the main results obtained in this work may be generalized to death and immigration rates whose are independent of other points of a configuration, however, they may depend on the position of each point and time. We will consider this case of non-homogeneous non-stationary death-immigration process in a forthcoming publication.

2. Description of model

The simplest economic model in the description of spatial dynamics is the model of free development when particles (which we may interpret, for instance, as companies on the market) appears independently without any influence of existing ones. On the other language, they migrate from the outside without any motivation due to situation inside the system. Of course, companies on real market never have infinite life time. We consider model with global regulation. This means that any points of configuration has exponentially distributed (with some positive parameter m) random life time. Hence, again a death (bankruptcy) appears due to "request" from the outside.

The state space of this model is the space $\Gamma = \Gamma_{\mathbb{R}^d}$ of all locally finite subsets (configurations) in $\mathbb{R}^d$

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \;\middle|\; |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\}.$$

Here $\gamma_\Lambda = \gamma \cap \Lambda$, $|\cdot|$ means cardinality of a set, $\mathcal{B}_c(\mathbb{R}^d)$ denote the system of all bounded Borel sets in $\mathbb{R}^d$. We consider the σ-algebra $\mathcal{B}(\Gamma)$ as the smallest σ-algebra for which all the mappings $N_\Lambda : \Gamma \to \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $N_\Lambda(\gamma) := |\gamma_\Lambda|$ are measurable for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a projection $p_\Lambda : \Gamma \to \Gamma_\Lambda := \{\gamma \in \Gamma \mid \gamma \subset \Lambda\}$; $p_\Lambda(\gamma) := \gamma_\Lambda$ and w.r.t. this projections Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$. One can consider also the σ-algebra $\mathcal{B}_\Lambda(\Gamma)$ as the smallest σ-algebra for which all the mappings $N_{\Lambda'} : \Gamma \to \mathbb{N}_0$ are measurable for all $\Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$, $\Lambda' \subset \Lambda$.

On Γ we consider the set of a cylinder functions $\mathcal{F}L^0(\Gamma)$, i.e. the set of all measurable function F on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. These functions are characterized by the following relation: $F(\gamma) = F\restriction_{\Gamma_\Lambda}(\gamma_\Lambda)$.

Let $\mathcal{M}^1_{\rm fm}(\Gamma)$ be the set of all probability measures μ on $\bigl(\Gamma,\mathcal{B}(\Gamma)\bigr)$ which have finite local moments of all orders, i.e. $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$ for all $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ and $n\in\mathbb{N}_0$.

To describe a (pre-)generator of a dynamics above we consider for fixed $m>0$, $\sigma\geq 0$ and for any $F\in\mathcal{F}L^0(\Gamma)$ the following expression

$$(LF)(\gamma) = m\sum_{x\in\gamma}\bigl[F(\gamma\setminus x)-F(\gamma)\bigr] + \sigma\int_{\mathbb{R}^d}\left[F(\gamma\cup x)-F(\gamma)\right]dx, \tag{2.1}$$

which is well-defined since, by the definition of $\mathcal{F}L^0(\Gamma)$, there exists $\Lambda\in\mathcal{B}_{\rm b}(\mathbb{R}^d)$ such that $F(\gamma\setminus x)=F(\gamma)$ for any $x\in\gamma_{\Lambda^c}$ and $F(\gamma\cup x)=F(\gamma)$ for any $x\in\Lambda^c$; therefore, sum and integral in (2.1) are finite. Stress that L is the generator of the (non-equilibrium) Surgailis process, see [16, 25, 26].

We consider now the space of finite configurations on $\mathbb{R}^d$. The space of n-point configuration is

$$\Gamma^{(n)} := \left\{\,\eta\subset\mathbb{R}^d \bigm| |\eta| = n\right\},\quad n\in\mathbb{N}_0.$$

As a set, $\Gamma^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{(\mathbb{R}^d)^n} = \left\{\,(x_1,\ldots,x_n)\in(\mathbb{R}^d)^n \bigm| x_k\neq x_l \text{ if } k\neq l\right\}.$$

Hence, $\Gamma^{(n)}_0$ inherits the structure of an $n\cdot d$-dimensional manifold. Applying this we can define Borel σ-algebra $\mathcal{B}(\Gamma^{(n)}_0)$. Also one can consider a measure $\mathrm{m}^{(n)}$ as image of product $\mathrm{m}^{\otimes n}$ of Lebesgue measures $d\mathrm{m}(x)=dx$ on $\bigl(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)\bigr)$.

The space of finite configurations

$$\Gamma_0 := \bigsqcup_{n\in\mathbb{N}_0}\Gamma^{(n)}_0$$

has structure of disjoint union, therefore, one can define the Borel σ-algebra $\mathcal{B}(\Gamma_0)$. A set $B\in\mathcal{B}(\Gamma_0)$ is called bounded if there exists a $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ and an $N\in\mathbb{N}$ such that $B\subset\bigsqcup_{n=0}^N\Gamma^{(n)}_\Lambda$, where $\Gamma^{(n)}_\Lambda := \left\{\eta\subset\Lambda \bigm| |\eta|=n\right\}$.

We will use also the following two classes of functions on Γ_0: $L^0_{\rm ls}(\Gamma_0)$ is the set of all measurable functions on Γ_0 which have a local support, i.e. $G\in L^0_{\rm ls}(\Gamma_0)$ if there exists $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ such that $G\restriction_{\Gamma_0\setminus\Gamma_\Lambda}=0$; $B_{\rm bs}(\Gamma_0)$ is the set of bounded measurable functions with bounded support: $G\restriction_{\Gamma_0\setminus B}=0$ for some bounded $B\in\mathcal{B}(\Gamma_0)$.

The Lebesgue—Poisson measure λ_z on $\bigl(\Gamma_0,\mathcal{B}(\Gamma_0)\bigr)$ is defined as

$$\lambda_z := \sum_{n=0}^\infty \frac{z^n}{n!}\mathrm{m}^{(n)}. \tag{2.2}$$

Here $z>0$ is the so called activity parameter. The restriction of λ_z to Γ_Λ will be also denoted by λ_z. Let λ be the Lebesgue-Poisson measure on Γ_0 (and Γ_Λ) with activity parameter equal to 1.

The Poisson measure π_z on $\bigl(\Gamma,\mathcal{B}(\Gamma)\bigr)$ is given as the projective limit of the family of measures $\{\pi^\Lambda_z\}_{\Lambda\in\mathcal{B}_c(\mathbb{R}^d)}$, where π^Λ_z is the measure on Γ_Λ defined by $\pi^\Lambda_z := e^{-z\mathrm{m}(\Lambda)}\lambda_z$. Again, we will omit index in the case $z=1$.

The following mapping between functions on Γ_0, e.g. $L^0_{\rm ls}(\Gamma_0)$, and functions on Γ, e.g. $\mathcal{F}L^0(\Gamma)$, plays an important role in our further considerations:

$$KG(\gamma) := \sum_{\eta\Subset\gamma} G(\eta),\quad \gamma\in\Gamma, \tag{2.3}$$

where $G\in L^0_{\rm ls}(\Gamma_0)$, see, e.g., [12, 18, 19]. The summation in the latter expression is extend over all finite subconfigurations of γ, in symbols $\eta\Subset\gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}F(\xi), \quad \eta\in\Gamma_0. \tag{2.4}$$

We consider now a mapping $\hat{L}G := K^{-1}LKG$ which is well-defined on functions $G \in L^0_{\mathrm{ls}}(\Gamma_0)$. By, e.g., [7], we have

$$\big(\hat{L}G\big)(\eta) = -m|\eta|G(\eta) + \sigma\int_{\mathbb{R}^d} G(\eta\cup x)\,dx. \tag{2.5}$$

Let now $C > 1$ be fixed. Applying results from [6] to the zero-potential case, we obtain that (2.5) provides a linear operator on the Banach space of $\mathcal{B}(\Gamma_0)$-measurable functions

$$\mathcal{L}_C := \Big\{G : \Gamma_0 \to \mathbb{R}\;\Big|\; \|G\|_C := \int_{\Gamma_0}|G(\eta)|C^{|\eta|}d\lambda(\eta) < \infty\Big\} \tag{2.6}$$

with dense domain $\mathcal{L}_{2C} \subset \mathcal{L}_C$. If additionally,

$$C \ge \frac{\sigma}{m} \tag{2.7}$$

then $\big(\hat{L}, \mathcal{L}_{2C}\big)$ is closable linear operator in $\mathcal{L}_C$ and its closure $\big(\hat{L}, D(\hat{L})\big)$ generates a strongly continuous contraction semigroup $\hat{T}(t)$ on $\mathcal{L}_C$.

3. Correlation functions evolution

3.1. **Notion of correlation functions.** A measure ρ on $\big(\Gamma_0, \mathcal{B}(\Gamma_0)\big)$ is called locally finite iff $\rho(A) < \infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$, the set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$. One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the K-transform, i.e., for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

$\rho_\mu := K^*\mu$ we call the correlation measure corresponding to μ.

As shown in [12] for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G \in L^1(\Gamma_0, \rho_\mu)$ the series (2.3) is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$\int_{\Gamma_0} G(\eta)\,\rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\,\mu(d\gamma). \tag{3.1}$$

Among the elements in the domain of the K-transform are also the so-called coherent states $e_\lambda(f)$. By definition, for any $\mathcal{B}(\mathbb{R}^d)$-measurable function f,

$$e_\lambda(f,\eta) := \prod_{x\in\eta} f(x),\ \ \eta\in\Gamma_0\setminus\{\emptyset\}, \quad e_\lambda(f,\emptyset) := 1.$$

Then, by (2.2), for $f \in L^1(\mathbb{R}^d, dx)$ we obtain $e_\lambda(f) \in L^1(\Gamma_0, d\lambda)$ and

$$\int_{\Gamma_0} e_\lambda(f,\eta)d\lambda(\eta) = \exp\{\langle f\rangle\}, \tag{3.2}$$

here and below $\langle f\rangle = \int_{\mathbb{R}^d} f(x)dx$.

Note that

$$\big(Ke_\lambda(f)\big)(\gamma) = \prod_{x\in\gamma}\big(1 + f(x)\big), \quad \mu\text{–a.a. } \gamma\in\Gamma, \tag{3.3}$$

for all $\mathcal{B}(\mathbb{R}^d)$-measurable functions f such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$, see, e.g., [12].

Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. If for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the projection $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to (w.r.t.) π^Λ on Γ_Λ then $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t. λ on Γ_0. The corresponding Radon–Nikodym derivative

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda}(\eta), \quad \eta \in \Gamma_0$$

is called a correlation functional of a measure μ. The functions

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+, \tag{3.4}$$

given by

$$k_\mu^{(n)}(x_1, \dots, x_n) := \begin{cases} k_\mu(\{x_1, \dots, x_n\}), & \text{if } (x_1, \dots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases},$$

are well known correlation functions of statistical physics, see e.g [23, 24].

Obviously, not any positive function on Γ_0 is a correlation functional of a some measure on Γ. To describe sufficient condition on this we will do in the following manner. Given G_1 and G_2 two $\mathcal{B}(\Gamma_0)$-measurable functions, let us consider the $\star$-convolution between G_1 and G_2,

$$(G_1 \star G_2)(\eta) := \sum_{\eta_1 \sqcup \eta_2 \sqcup \eta_3 = \eta} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3), \tag{3.5}$$

where sign $\sqcup$ denotes disjoint union (parts may be empty), see [12] for a details. It is straightforward to verify that the space of all $\mathcal{B}(\Gamma_0)$-measurable functions endowed with this product has the structure of a commutative algebra with unit element $e_\lambda(0)$. Furthermore, for every $G_1, G_2 \in B_{bs}(\Gamma_0)$ we have $G_1 \star G_2 \in B_{bs}(\Gamma_0)$, and

$$K\left(G_1 \star G_2\right) = (KG_1) \cdot (KG_2) \tag{3.6}$$

cf. [12]. Note that

$$e_\lambda(f) \star e_\lambda(g) = e_\lambda(f + g + fg) \tag{3.7}$$

for all $\mathcal{B}(\mathbb{R}^d)$-measurable functions f and g.

The following theorem shows when we can reconstruct a measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ by the system of symmetric functions (3.4).

Theorem 3.1. (*[12]). Let $k : \Gamma_0 \to \mathbb{R}_+$ be measurable function such that $k d\lambda \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, $k(\emptyset) = 1$, there exists $C > 0, \varepsilon > 0$ such that $k(\eta) \le C^{|\eta|}\big(|\eta|!\big)^{1-\varepsilon}$, $\eta \in \Gamma_0$ and the function k is positive definite in the sense that for any $G \in B_{\mathrm{bs}}(\Gamma_0)$*

$$\int_{\Gamma_0} (G \star \bar{G})(\eta) k(\eta) d\lambda(\eta) \ge 0. \tag{3.8}$$

Then there exists a unique measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that $k = k_\mu$.

3.2. **Evolution of correlation functions.** The space $(\mathcal{L}_C)' = \big(L^1(\Gamma_0, d\lambda_C)\big)' = L^\infty(\Gamma_0, d\lambda_C)$ is the topologically dual space to the space $\mathcal{L}_C$. The space $L^\infty(\Gamma_0, d\lambda_C)$ is isometrically isomorphic to the Banach space

$$\mathcal{K}_C := \left\{ k : \Gamma_0 \to \mathbb{R} \;\middle|\; k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda) \right\}$$

with the norm $\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|} k(\cdot)\|_{L^\infty(\Gamma_0,\lambda)}$, where the isomorphism is provided by the isometry R_C

$$(\mathcal{L}_C)' \ni k \longmapsto R_C k := k \cdot C^{|\cdot|} \in \mathcal{K}_C. \tag{3.9}$$

In fact, we may say about a duality between Banach spaces $\mathcal{L}_C$ and $\mathcal{K}_C$, which is given by the following expression

$$\langle\!\langle G,\, k\rangle\!\rangle := \int_{\Gamma_0} G\cdot k\, d\lambda, \quad G\in\mathcal{L}_C, \quad k\in\mathcal{K}_C \tag{3.10}$$

with

$$|\langle\!\langle G, k\rangle\!\rangle| \le \|G\|_C\cdot\|k\|_{\mathcal{K}_C}. \tag{3.11}$$

It is clear that for any $k\in\mathcal{K}_C$

$$|k(\eta)|\le \|k\|_{\mathcal{K}_C}\, C^{|\eta|} \quad \text{for } \lambda\text{-a.a. } \eta\in\Gamma_0. \tag{3.12}$$

Let $\big(\hat{L}', D(\hat{L}')\big)$ be an operator in $(\mathcal{L}_C)'$ which is dual to the closed operator $\big(\hat{L}, D(\hat{L})\big)$. We consider also its image in $\mathcal{K}_C$ under isometry R_C, namely, let $\hat{L}^* = R_C\hat{L}'R_{C^{-1}}$ with a domain $D(\hat{L}^*) = R_C D(\hat{L}')$. Then, for any $G\in\mathcal{L}_C$, $k\in D(\hat{L}^*)$

$$\begin{aligned}\int_{\Gamma_0} G\cdot \hat{L}^* k d\lambda &= \int_{\Gamma_0} G\cdot R_C\hat{L}'R_{C^{-1}} k d\lambda = \int_{\Gamma_0} G\cdot \hat{L}'R_{C^{-1}} k d\lambda_C\\ &= \int_{\Gamma_0} \hat{L}G\cdot R_{C^{-1}} k d\lambda_C = \int_{\Gamma_0}\hat{L}G\cdot k d\lambda,\end{aligned}$$

therefore, $\hat{L}^*$ is the dual operator to $\hat{L}$ w.r.t. duality (3.10).

By, e.g., [7], we have the precise form of $\hat{L}^*$ on $D(\hat{L}^*)$

$$\big(\hat{L}^* k\big)(\eta) = -m|\eta|k(\eta) + \sigma\sum_{x\in\eta} k\,(\eta\setminus x)\,. \tag{3.13}$$

In the same way one can consider the adjoint contraction semigroup $\hat{T}'(t)$ in $(\mathcal{L}_C)'$ and its image $\hat{T}^*(t)$ in $\mathcal{K}_C$. Now, we may apply general results about adjoint semigroups (see, e.g., [1]) onto the contraction semigroup $\hat{T}^*(t)$. The last semigroup will be weak*-continuous, moreover, weak*-differentiable at 0 and $\hat{L}^*$ will be weak*-generator of $\hat{T}^*(t)$. Here and below we mean "weak*-properties" w.r.t. duality (3.10). Let $\mathring{\mathcal{K}}_C = \big\{k\in \mathcal{K}_C \;\big|\; \exists \lim_{t\downarrow 0}\big\|\hat{T}^*(t)k - k\big\|_{\mathcal{K}_C} = 0\big\}$. Then $\mathring{\mathcal{K}}_C$ is closed, weak*-dense, $\hat{T}^*(t)$-invariant linear subspace of $\mathcal{K}_C$. Moreover, $\mathring{\mathcal{K}}_C = \overline{D(\hat{L}^*)}$ (the closure is in the norm of $\mathcal{K}_C$). Let $\hat{T}^{\odot}(t)$ denote the restriction of $\hat{T}^*(t)$ onto Banach space $\mathring{\mathcal{K}}_C$. Then $\hat{T}^{\odot}(t)$ is a contraction C_0-semigroup on $\mathring{\mathcal{K}}_C$ and its generator $\hat{L}^{\odot}$ will be part of $\hat{L}^*$, namely, $D(\hat{L}^{\odot}) = \big\{k\in D(\hat{L}^*) \;\big|\; \hat{L}^*k\in\overline{D(\hat{L}^*)}\big\}$ and $\hat{L}^*k = \hat{L}^{\odot}k$ for any $k\in D(\hat{L}^{\odot})$.

Using simple reccurent structure of the operator (3.13) we may find explicit expression for the action of the contraction semigroup $\hat{T}^*(t)$ from the solution of the Cauchy problem

$$\frac{\partial}{\partial t}k_t = \hat{L}^* k_t, \qquad k_t\,\big|_{t=0} = k_0. \tag{3.14}$$

To do this let us define the following associative and commutative convolution on measurable functions on Γ_0

$$(G_1 * G_2)(\eta) = \sum_{\xi\subset\eta} G_1(\xi)G_2(\eta\setminus\xi), \quad \eta\in\Gamma_0. \tag{3.15}$$

One can consider an algebra of measurable functions on Γ_0 with such a product and the unit element $1^*(\eta) := 0^{|\eta|}$. Note that,

$$e_\lambda(f) * e_\lambda(g) = e_\lambda(f+g), \tag{3.16}$$

$$e_\lambda(f)\big(G_1 * G_2\big) = \big(e_\lambda(f)G_1\big) * \big(e_\lambda(f)G_2\big). \tag{3.17}$$

Theorem 3.2. *The function*

$$k_t(\eta) = e^{-tm|\eta|}\left(e_\lambda\left(\frac{\sigma}{m}(e^{tm}-1)\right) * k_0\right)(\eta) \tag{3.18}$$

$$= \left(e_\lambda\left(\frac{\sigma}{m}(1-e^{-tm})\right) * \left(e_\lambda(e^{-tm})k_0\right)\right)(\eta) \tag{3.19}$$

is a well-defined point-wise differentiable function which satisfied (3.14).

Proof. By (3.13), (3.14) implies

$$\frac{\partial}{\partial t}k_t^{(1)}(x_1) = -mk_t^{(1)}(x_1) + \sigma,$$

that yields

$$k_t^{(1)}(x_1) = e^{-mt}k_0^{(1)}(x_1) + \sigma\int_0^t e^{-m(t-s)}ds = e^{-mt}\left(k_0^{(1)}(x_1) + \frac{\sigma}{m}(e^{mt}-1)\right).$$

Suppose that (3.18) holds for $|\eta| = n-1$, namely,

$$k_t^{(n-1)}(x_1,\dots,x_{n-1}) = e^{-m(n-1)t}\sum_{\xi\subset\{x_1,\dots,x_{n-1}\}} k_0^{(|\xi|)}(\xi)\left(\frac{\sigma}{m}(e^{mt}-1)\right)^{n-1-|\xi|}.$$

Then, by (3.13) and (3.14) we obtain

$$\begin{aligned}
&k_t^{(n)}(x_1,\dots,x_n)\\
&= e^{-mnt}k_0^{(n)}(x_1,\dots,x_n) + \sigma\int_0^t e^{-mn(t-s)}\sum_{i=1}^n k_s^{(n-1)}(x_1,\dots,\check{x}_i,\dots,x_n)\,ds\\
&= e^{-mnt}k_0^{(n)}(x_1,\dots,x_n)\\
&\quad + \sigma e^{-mnt}\int_0^t e^{mns}\sum_{i=1}^n e^{-m(n-1)s}\sum_{\xi\subset\{x_1,\dots,\check{x}_i,\dots,x_n\}} k_0^{(|\xi|)}(\xi)\left(\frac{\sigma}{m}(e^{mt}-1)\right)^{n-1-|\xi|}ds\\
&= e^{-mnt}k_0^{(n)}(x_1,\dots,x_n)\\
&\quad + e^{-mnt}\sum_{\xi\subsetneq\{x_1,\dots,x_n\}}(n-|\xi|)\,k_0^{(|\xi|)}(\xi)\left(\frac{\sigma}{m}\right)^{n-|\xi|} m\int_0^t (e^{ms}-1)^{n-1-|\xi|}e^{ms}ds\\
&= e^{-mnt}k_0^{(n)}(x_1,\dots,x_n) + e^{-mnt}\sum_{\xi\subsetneq\{x_1,\dots,x_n\}} k_0^{(|\xi|)}(\xi)\left(\frac{\sigma}{m}(e^{mt}-1)\right)^{n-|\xi|}\\
&= e^{-mnt}\sum_{\xi\subset\{x_1,\dots,x_n\}} k_0^{(|\xi|)}(\xi)\left(\frac{\sigma}{m}(e^{mt}-1)\right)^{n-|\xi|}\\
&= \sum_{\xi\subset\{x_1,\dots,x_n\}} e^{-m|\xi|t}k_0^{(|\xi|)}(\xi)\left(\frac{\sigma}{m}(1-e^{-mt})\right)^{n-|\xi|}.
\end{aligned}$$

By a mathematical induction principle, the statement is proved. □

Remark 3.3. *Note that, by* (3.18), $k_0(\emptyset) = 1$ *implies* $k_t(\emptyset) = 1$ *as well as* $k_0 > 0$ *implies* $k_t > 0$.

Proposition 3.4. *Let* $k_0 \in \mathcal{K}_C$ *and* k_t *is the solution of* (3.14). *Then* $k_t \in \mathcal{K}_{C'}$, *where* $C' = \max\{C; \frac{\sigma}{m}\}$. *More precisely,*

$$\left|k_t(\eta)\right| \le \|k_0\|_{\mathcal{K}_C}\left(\max\left\{C; \frac{\sigma}{m}\right\}\right)^{|\eta|}, \quad \eta\in\Gamma_0.$$

Proof. By (3.18), (3.12), and (3.16), one get

$$
\begin{aligned}
|k_t(\eta)| &\le e^{-tm|\eta|}\Big(e_\lambda\Big(\frac{\sigma}{m}(e^{mt}-1)\Big) * \big(\|k_0\|_{\mathcal{K}_C} e_\lambda(C)\big)\Big)(\eta)\\
&= \|k_0\|_{\mathcal{K}_C} e^{-tm|\eta|} e_\lambda\Big(C+\sigma\frac{e^{tm}-1}{m},\eta\Big)\\
&= \|k_0\|_{\mathcal{K}_C} e_\lambda\Big(Ce^{-tm}+\sigma\frac{1-e^{-tm}}{m},\eta\Big) \le \|k_0\|_{\mathcal{K}_C} e_\lambda\Big(\max\Big\{C;\frac{\sigma}{m}\Big\},\eta\Big),
\end{aligned}
$$

since

$$
Ce^{-tm}+\sigma\frac{1-e^{-tm}}{m} = \Big(C-\frac{\sigma}{m}\Big)e^{-tm}+\frac{\sigma}{m} \le \max\Big\{C;\frac{\sigma}{m}\Big\}. \tag{3.20}
$$

Hence, this dynamics stays so-called *sub-Poissonian* (cf. Remark 3.5 below). □

Remark 3.5. *Let us stress that if we start in* (3.14) *from the Poisson distribution* $\mu_0 = \pi_A$ *with* $k_{\mu_0}(\eta) = k_0(\eta) = A^{|\eta|}$, $A > 0$ *then the distribution stays Poissonian during dynamics*

$$
k_t(\eta) = \Big(\Big(A-\frac{\sigma}{m}\Big)e^{-tm}+\frac{\sigma}{m}\Big)^{|\eta|}. \tag{3.21}
$$

Corollary 3.6. *Let* $C \ge \dfrac{\sigma}{m}$. *Then for any* $k \in \mathcal{K}_C$

$$
\big(\hat{T}^*(t)k\big)(\eta) := e^{-tm|\eta|}\Big(e_\lambda\Big(\frac{\sigma}{m}(e^{tm}-1)\Big) * k\Big)(\eta), \quad \eta\in\Gamma_0, \quad t>0. \tag{3.22}
$$

As was noted in [4], $\mathcal{K}_{\alpha C} \subset D(\hat{L}^*)$ for any $\alpha \in (0;1)$. Moreover, by Proposition 3.4, if $k \in \mathcal{K}_{\alpha C}$ then $k_t = \hat{T}^*(t)k = \hat{T}^{\odot}(t)k \in \mathcal{K}_{C'}$, where $C' = \max\{\alpha C; \frac{\sigma}{m}\}$. Therefore, the following improvement of the result from [4] holds.

Proposition 3.7. *Let* $C > \dfrac{\sigma}{m}$. *Then for any* $\alpha \in \Big(\dfrac{\sigma}{mC};1\Big)$ *the Banach subspace* $\overline{\mathcal{K}_{\alpha C}}$ *of the Banach space* $\mathcal{K}_C$ *is* $\hat{T}^{\odot}(t)$*-invariant. Here closure is taken in the norm of* $\mathcal{K}_C$. *The restriction* $\hat{T}^{\odot\alpha}(t)$ *of* $\hat{T}^{\odot}(t)$ *onto* $\overline{\mathcal{K}_{\alpha C}}$ *is a contraction* C_0*-semigroup.*

As a result, we have that for any $C \ge \dfrac{\sigma}{m}$ the Cauchy problem (3.14) is solvable on $\mathcal{K}_C$. Moreover, for $C > \dfrac{\sigma}{m}$ and $\alpha \in \Big(\dfrac{\sigma}{mC};1\Big)$ this problem is solvable on $\overline{\mathcal{K}_{\alpha C}}$.

At the end let us us find an expression for the resolvent $R_z^{\odot}$ of the generator $\hat{L}^{\odot}$ of the semigroup $\hat{T}^{\odot}(t)$.

Proposition 3.8. *For any* z *with* $\operatorname{Re} z > 0$ *there exists a bounded operator* $R_z^{\odot} = (z - \hat{L}^{\odot})^{-1}$ *on the space* $\mathring{\mathcal{K}}_C$ *such that for any* $k \in \mathring{\mathcal{K}}_C$

$$
\big(R_z^{\odot}k\big)(\eta) = \frac{1}{m}\sum_{\xi\subset\eta}\Big(\frac{\sigma}{m}\Big)^{|\xi|} B\Big(\frac{z}{m}+|\eta|-|\xi|,|\xi|+1\Big)k(\eta\setminus\xi),
$$

where $B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\,ds$ *is the Euler beta function.*

Proof. We have

$$\begin{aligned}\left(R_z^{\odot} k\right)(\eta) &= \int_0^\infty e^{-zt}\, e^{-tm|\eta|} \left(e_\lambda\left(\frac{\sigma}{m}(e^{tm}-1)\right) * k\right)(\eta)\, dt \\ &= \sum_{\xi\subset\eta} k\left(\eta\setminus\xi\right) \int_0^\infty e^{-(z+m|\eta|)t} \left(\frac{\sigma}{m}(e^{tm}-1)\right)^{|\xi|} dt \\ &= \sum_{\xi\subset\eta} k\left(\eta\setminus\xi\right) \left(\frac{\sigma}{m}\right)^{|\xi|} \int_0^\infty e^{-(z+m|\eta\setminus\xi|)t} \left(1-e^{-tm}\right)^{|\xi|} dt.\end{aligned}$$

Using substitution $s = e^{-tm}$ we obtain for $\operatorname{Re} z > 0$

$$\begin{aligned}\int_0^\infty e^{-(z+m|\eta\setminus\xi|)t}\left(1-e^{-tm}\right)^{|\xi|} dt &= \frac{1}{m}\int_0^1 s^{\frac{z}{m}+|\eta\setminus\xi|-1}\left(1-s\right)^{|\xi|} ds \\ &= \frac{1}{m} B\left(\frac{z}{m}+|\eta|-|\xi|, |\xi|+1\right),\end{aligned} \tag{3.23}$$

that proves the assertion. □

3.3. **Evolution of measures.** In [4], it was shown that dynamics $\hat{T}^{\odot}(t)$ preserves so-called Lenard-positivity property on the subspace $\overline{D(\hat{L}^*)}$. We recall that a measurable function $k : \Gamma_0 \to \mathbb{R}$ is to be called a positive defined function in the sense of Lenard if for any $G \in B_{bs}\left(\Gamma_0\right)$ such that $KG \geq 0$ the following inequality holds $\int_{\Gamma_0} G\left(\eta\right) k\left(\eta\right) d\lambda\left(\eta\right) \geq 0$. By (3.6), any such a function will be positive defined in the sense of (3.8) too.

We extend now this preservation of positive-definiteness (in the sense of (3.8)) on the whole space $\mathcal{K}_C$.

We start from the following lemma which is seems to be important itself.

Lemma 3.9. *Let $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and suppose that their correlation function $k_0 = k_{\mu_0}$ exists and belongs to $\mathcal{K}_C$. Let $f \in L^1(\mathbb{R}^d, dx)$ and $0 \leq f(x) \leq 1$, $x \in \mathbb{R}^d$. Then function $k(\eta) = e_\lambda(f,\eta)k_0(\eta)$ is a positive definite in the sense of* (3.8).

Proof. Using classical measure theory arguments it is enough to proof (3.8) for function $G : \Gamma_0 \to \mathbb{C}$ of the form

$$G(\eta) = \sum_{i=1}^N b_i e_\lambda(g_i,\eta), \quad N\in\mathbb{N}, \quad b_i\in\mathbb{C}, \quad g_i \in C_0(\mathbb{R}^d\to\mathbb{C}), \tag{3.24}$$

where $C_0(\mathbb{R}^d \to \mathbb{C})$ is the space of all complex-valued continuous functions on $\mathbb{R}^d$ with compact supports.

Note that, by (3.12) and (3.2), for any $g \in C_0(\mathbb{R}^d \to \mathbb{C}) \subset L^1(\mathbb{R}^d \to \mathbb{C}, dx)$

$$\int_{\Gamma_0} |e_\lambda(g,\eta)| k_0(\eta) d\lambda(\eta) \leq \|k_0\|_{\mathcal{K}_C} \int_{\Gamma_0} e_\lambda(C|g|,\eta) d\lambda(\eta) < \infty. \tag{3.25}$$

By (3.3) and (3.1), inequality (3.25) implies $\prod_{x\in\gamma}(1+|g(x)|) \in L^1(\Gamma, d\mu_0)$ for any $g \in C_0(\mathbb{R}^d)$. Moreover, $\prod_{x\in\gamma}(1+|g(x)|) \in \mathcal{F}L^0(\Gamma)$, hence,

$$\int_\Gamma \prod_{x\in\gamma}(1+g(x)) d\mu_0(\gamma) = \int_{\Gamma_\Lambda} \prod_{x\in\gamma_\Lambda}(1+g(x)) d\mu_0^\Lambda(\gamma_\Lambda),$$

where Λ is the support of g and the measure μ_0^Λ is the projection of the measure μ_0 onto Γ_Λ.

Let G has the form (3.24). Then, taking Λ equal to union of the supports of functions g_i, $i = 1, \ldots, n$, we obtain

$$
\begin{aligned}
&\int_{\Gamma_0} \left(G \star \bar{G}\right)(\eta)\, e_\lambda(f,\eta)\, k_0(\eta)\, d\lambda(\eta) \\
&= \sum_{i,j=1}^N b_i \bar{b}_j \int_{\Gamma_0} e_\lambda\left(g_i + \bar{g}_j + g_i \bar{g}_j, \eta\right) e_\lambda(f,\eta)\, k_0(\eta)\, d\lambda(\eta) \\
&= \sum_{i,j=1}^N b_i \bar{b}_j \int_{\Gamma} K\left(e_\lambda\left(f g_i + f \bar{g}_j + f g_i \bar{g}_j\right)\right)(\gamma)\, d\mu_0(\gamma) \\
&= \sum_{i,j=1}^N b_i \bar{b}_j \int_{\Gamma} \prod_{x\in\gamma} \Big(1 - f(x) + f(x)\big(1 + g_i(x)\big)\big(1 + \bar{g}_j(x)\big)\Big) d\mu_0(\gamma) \\
&= \sum_{i,j=1}^N b_i \bar{b}_j \int_{\Gamma_\Lambda} \prod_{x\in\gamma_\Lambda} \Big(1 - f(x) + f(x)\big(1 + g_i(x)\big)\big(1 + \bar{g}_j(x)\big)\Big) d\mu_0^\Lambda(\gamma_\Lambda) \\
&= \sum_{i,j=1}^N b_i \bar{b}_j \int_{\Gamma_\Lambda} \sum_{\eta\subset\gamma_\Lambda} e_\lambda(1 - f, \eta)\, e_\lambda\left(f(1 + g_i)(1 + \bar{g}_j), \gamma_\Lambda \setminus \eta\right) d\mu_0^\Lambda(\gamma_\Lambda) \\
&= \int_{\Gamma_\Lambda} \sum_{\eta\subset\gamma_\Lambda} e_\lambda(1 - f, \eta) \sum_{i,j=1}^N b_i \bar{b}_j e_\lambda(1 + g_i, \gamma_\Lambda \setminus \eta) \\
&\quad \times e_\lambda(1 + \bar{g}_j, \gamma_\Lambda \setminus \eta)\, e_\lambda(f, \gamma_\Lambda \setminus \eta)\, d\mu_0^\Lambda(\gamma_\Lambda) \\
&= \int_{\Gamma_\Lambda} \sum_{\eta\subset\gamma_\Lambda} e_\lambda(1 - f, \eta) \left| \sum_{i=1}^N b_i e_\lambda(1 + g_i, \gamma_\Lambda \setminus \eta) \right|^2 e_\lambda(f, \gamma_\Lambda \setminus \eta)\, d\mu_0^\Lambda(\gamma_\Lambda) \geq 0,
\end{aligned}
$$

since $0 \leq f(x) \leq 1$, $x \in \mathbb{R}^d$. □

As we noted before not all elements of $\mathcal{K}_C$ are correlation functions of some measures. Next theorem shows that we really have correlation functions evolutions and, as a result, evolution of states (measures) on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$.

Theorem 3.10. *Let $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and $k_0 = k_{\mu_0} \in \mathcal{K}_C$, $C > 0$ be the corresponding correlation function on Γ_0. Then for any $t > 0$ the solution k_t of (3.14) is a correlation function of a unique measure $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$.*

Proof. By (3.18), k_t is positive measurable function and $k_t(\emptyset) = 1$. Proposition 3.4 implies sub-Poissonian bounds for k_t. Hence, for apply Theorem 3.1 we should check (3.8) only.

By Lemma 3.9, $e^{-tm|\cdot|}k_0 = e_\lambda(e^{-tm})k_0$ is a positive defined function in the sense of (3.8) (cf. [11, Corollary 3]). Clearly, this function belongs to $\mathcal{K}_C$. Therefore, by Theorem 3.1, there exists a unique measure from $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ whose correlation function is $e_\lambda(e^{-tm})k_0$.

Next, $e_\lambda\left(\frac{\sigma}{m}(1 - e^{-tm})\right)$ is the correlation function of the Poisson measure with intensity $\frac{\sigma}{m}(1 - e^{-tm})$.

By [2], Ruelle convolution of correlation functions $e_\lambda\left(\frac{\sigma}{m}(1 - e^{-tm})\right)$ and $e_\lambda(e^{-tm})k_0$ will be positive defined in the sense of (3.8) too. Hence, the assertion is followed by Theorem 3.2. □

As it was shown in [2], the $*$-convolution of correlation functions k_{μ_1} and k_{μ_2} is the correlation function of the convolution of measures μ_1 and μ_2, where by definition $\mu =$

$\mu_1 * \mu_2$ is the probability measure on $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ such that for any measurable F with $\widetilde{F} \in L^1(\Gamma \times \Gamma, d\mu_1 \times d\mu_2)$, where

$$\widetilde{F}(\gamma_1, \gamma_2) = F(\gamma_1 \cup \gamma_2), \quad \gamma_{1,2} \in \Gamma,$$

the following equality holds:

$$\int_\Gamma F(\gamma) d\mu(\gamma) = \int_\Gamma \int_\Gamma F(\gamma_1 \cup \gamma_2)\, d\mu_1(\gamma_1)\, d\mu_2(\gamma_2).$$

Let now $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and consider weak evolution equation for measures

$$\frac{\partial}{\partial t} \int_\Gamma F(\gamma) d\mu_t(\gamma) = \int_\Gamma (LF)(\gamma) d\mu_t(\gamma)$$

for any $F \in \mathcal{F}L^0(\Gamma)$ provided both parts exist and, of course, $\mu_t\big|_{t=0} = \mu_0$. Let $\nu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be solution of a corresponding pure death evolution equation

$$\frac{\partial}{\partial t} \int_\Gamma F(\gamma) d\nu_t(\gamma) = m \int_\Gamma \sum_{x \in \gamma} \bigl(F(\gamma \setminus x) - F(\gamma)\bigr) d\nu_t(\gamma)$$

with the same initial condition $\nu_t\big|_{t=0} = \mu_0$. Then, by Theorem 3.2 for the case $\sigma = 0$, we obtain $k_{\nu_t}(\eta) = e^{-tm|\eta|} k_0(\eta)$. As a result,

$$\mu_t = \pi_{z_t} * \nu_t, \tag{3.26}$$

where

$$z_t = \frac{\sigma}{m}(1 - e^{-tm}).$$

4. Ergodicity

4.1. **Ergodic properties of correlation functions.** We recall that a measure $\mu_{\mathrm{inv}} \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called invariant for the operator L if for any $F \in \mathcal{F}L^0(\Gamma)$

$$\int_\Gamma (LF)(\gamma) d\mu_{\mathrm{inv}}(\gamma) = 0.$$

If k_{inv} is the corresponding correlation function then for any $G \in L^0_{\mathrm{ls}}(\Gamma_0)$

$$\int_{\Gamma_0} (\hat{L}^* k_{\mathrm{inv}})(\eta) G(\eta) d\lambda(\eta) = \int_{\Gamma_0} (\hat{L} G)(\eta) k_{\mathrm{inv}}(\eta) d\lambda(\eta) = 0,$$

and, therefore, $(\hat{L}^* k_{\mathrm{inv}})(\eta) = 0$, $\eta \in \Gamma_0$. As a result, by (3.13),

$$m|\eta| k_{\mathrm{inv}}(\eta) = \sigma \sum_{x \in \eta} k_{\mathrm{inv}}\left(\eta \setminus x\right).$$

Iterating the last equation, we easily can see that it implies

$$k_{\mathrm{inv}}(\eta) = \left(\frac{\sigma}{m}\right)^{|\eta|} = e_\lambda\left(\frac{\sigma}{m}, \eta\right). \tag{4.1}$$

As result, Poisson measure $\pi_{\frac{\sigma}{m}}$ is a unique invariant measure of our evolution.

Note also that the condition $k_0(\emptyset) = 1$ implies that point-wisely we obtain

$$k_t(\eta) = \left(\frac{\sigma}{m}\bigl(1 - e^{-mt}\bigr)\right)^{|\eta|} + \sum_{\xi \subsetneq \eta} k_0(\eta \setminus \xi) e^{-mt|\eta \setminus \xi|} \left(\frac{\sigma}{m}\bigl(1 - e^{-mt}\bigr)\right)^{|\xi|} \to \left(\frac{\sigma}{m}\right)^{|\eta|}$$

as $t \to \infty$. Taking into account (4.1) and Proposition 3.4, we may expect that our non-equilibrium dynamics are ergodic in the space $\mathcal{K}_C$ for big enough C. In the next theorem we explain more exact conditions for this ergodicity.

Theorem 4.1. *Let* $C > \dfrac{\sigma}{m}$, $k_0 \in \mathcal{K}_C$ *and* $k_0(\emptyset) = 1$. *Then*

$$\left\|k_t - k_{\mathrm{inv}}\right\|_{\mathcal{K}_C} < \|k_0 - k_{\mathrm{inv}}\|_{\mathcal{K}_C} \frac{e^{-mt}}{1 - \dfrac{\sigma}{Cm}}, \quad t > 0. \tag{4.2}$$

Proof. First of all note that, by (4.1) and Corollary 3.4, for any $C > \dfrac{\sigma}{m}$

$$\{k_t, t > 0; k_{\mathrm{inv}}\} \subset \mathcal{K}_C.$$

Next, by (3.16),

$$k_{\mathrm{inv}} = e_\lambda\Big(\frac{\sigma}{m}\Big) = e_\lambda\Big(\frac{\sigma}{m}(1 - e^{-mt})\Big) * e_\lambda\Big(\frac{\sigma}{m}e^{-mt}\Big).$$

Therefore,

$$k_t(\eta) - k_{\mathrm{inv}}(\eta) = \sum_{\xi \subsetneq \eta} \left(k_0(\eta \setminus \xi) - \Big(\frac{\sigma}{m}\Big)^{|\eta\setminus\xi|} \right) e^{-mt|\eta\setminus\xi|} \left(\frac{\sigma}{m}\big(1 - e^{-mt}\big)\right)^{|\xi|}$$

and one can estimate

$$\begin{aligned} & C^{-|\eta|}\Big|k_t(\eta) - k_{\mathrm{inv}}(\eta)\Big| \\ & \leq C^{-|\eta|} \sum_{\xi \subsetneq \eta} \left| k_0(\eta \setminus \xi) - \Big(\frac{\sigma}{m}\Big)^{|\eta\setminus\xi|} \right| e^{-mt|\eta\setminus\xi|} \left(\frac{\sigma}{m}\big(1 - e^{-mt}\big)\right)^{|\xi|} \\ & \leq \|k_0 - k_{\mathrm{inv}}\|_{\mathcal{K}_C} C^{-|\eta|} \sum_{\xi \subsetneq \eta} C^{|\eta\setminus\xi|} e^{-mt|\eta\setminus\xi|} \left(\frac{\sigma}{m}\big(1 - e^{-mt}\big)\right)^{|\xi|} \\ & = \|k_0 - k_{\mathrm{inv}}\|_{\mathcal{K}_C} \left[\left(e^{-mt} + \frac{\sigma}{Cm}\big(1 - e^{-mt}\big)\right)^{|\eta|} - \left(\frac{\sigma}{Cm}\big(1 - e^{-mt}\big)\right)^{|\eta|} \right]. \end{aligned} \tag{4.3}$$

Let us recall, that $e^{-mt} + \frac{\sigma}{Cm}\big(1 - e^{-mt}\big) < 1$, $t > 0$.

To find uniform, by $|\eta|$, estimate for the r.h.s. of (4.3) let us consider for any fixed $0 < a < b < 1$, $n \in \mathbb{N}$ the difference

$$\begin{aligned} b^n - a^n =& (b - a) \sum_{j=0}^{n-1} a^j b^{n-1-j} < (b - a) \sum_{j=0}^{n-1} a^j \\ =& (b - a) \frac{1 - a^n}{1 - a} < \frac{b - a}{1 - a}. \end{aligned}$$

As result, using (4.3), obvious estimate $\dfrac{\sigma}{Cm}\big(1 - e^{-mt}\big) < \dfrac{\sigma}{Cm}$, $t > 0$, and the fact that $\dfrac{1}{1 - a}$ is a strictly increasing function of $a \in (0; 1)$ we obtain (4.2). $\square$

Remark 4.2. *Note that if we consider corresponding general result from [4] in the zero-potential case and for* $m = 1$ *we obtain more weaker inequality*

$$\|k_t - k_{\mathrm{inv}}\|_{\mathcal{K}_C} \leq e^{-(1-\nu)t} \|k_0 - k_{\mathrm{inv}}\|_{\mathcal{K}_C}, \quad 1 > \nu > \frac{\sigma}{C}.$$

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and denote the projection of the measure μ_t, $t \geq 0$ on Γ_Λ by μ_t^Λ. Then, in the same notations, $\mu_{inv}^\Lambda = \pi_{\sigma/m}^\Lambda$.

Corollary 4.3. *Let* $C > \dfrac{\sigma}{m}$ *and* $A = \Big(1 - \dfrac{\sigma}{mC}\Big)^{-1}$. *Then for any* $t > 0$

$$\left\| \frac{d\mu_t^\Lambda}{d\lambda} - \frac{d\mu_{\mathrm{inv}}^\Lambda}{d\lambda} \right\|_{\mathcal{K}_C} \leq A e^{-tm} \exp\{C|\Lambda|\}. \tag{4.4}$$

Proof. Since, clearly, $\int_{\Gamma_\Lambda} 2^{|\eta|} k_t(\eta) d\lambda(\eta) < +\infty$, $t \geq 0$ then (see, e.g., [12])

$$\frac{d\mu_t^\Lambda}{d\lambda}(\eta) = \int_{\Gamma_\Lambda} (-1)^{|\xi|} k_t(\eta \cup \xi) \, d\lambda(\xi), \quad \eta \in \Gamma_\Lambda. \tag{4.5}$$

Hence, by Theorem 4.1, for $C > \frac{\sigma}{m}, t > 0$, we have

$$\begin{aligned} &\left| \frac{d\mu_t^\Lambda}{d\lambda}(\eta) - \frac{d\mu_{\mathrm{inv}}^\Lambda}{d\lambda}(\eta) \right| \\ &\leq \int_{\Gamma_\Lambda} |k_t(\eta \cup \xi) - k_{\mathrm{inv}}(\eta \cup \xi)| \, d\lambda(\xi) \\ &= \int_{\Gamma_\Lambda} \frac{|k_t(\eta \cup \xi) - k_{\mathrm{inv}}(\eta \cup \xi)|}{C^{|\eta \cup \xi|}} C^{|\eta \cup \xi|} d\lambda(\xi) \\ &\leq \|k_t - k_{\mathrm{inv}}\|_{\mathcal{K}_C} C^{|\eta|} \exp\{C|\Lambda|\} \\ &\leq A C^{|\eta|} e^{-tm} \exp\{C|\Lambda|\}, \end{aligned}$$

that proves the assertion. □

For any $\eta \in \Gamma_0$, $y \in \mathbb{R}^d$, $t \geq 0$ we define

$$v_t(\eta, y) := k_t(\eta \cup y) - k_t(\eta) k_t(y). \tag{4.6}$$

Clearly, Remark 3.5 implies that if $k_0(\eta) = A^{|\eta|}$, $A > 0$ then $v_t(\eta, y) \equiv 0$.

Our dynamics at moment t is said to be satisfied *the decay of correlation principle* if

$$\lim_{|y| \to \infty} v_t(\eta, y) = 0, \quad \eta \in \Gamma_0. \tag{4.7}$$

Next theorem shows preserving the decay of correlation principle during our dynamics.

Theorem 4.4. *Let $C > \dfrac{\sigma}{m}$, $k_0(\emptyset) = 1$ and let*

$$a(y) := \left\| v_0(\cdot, y) \right\|_{\mathcal{K}_C} \in [0; \infty), \quad y \in \mathbb{R}^d.$$

Then

$$\left\| v_t(\cdot, y) \right\|_{\mathcal{K}_C} \leq a(y) e^{-tm}, \quad y \in \mathbb{R}^d.$$

Proof. Let $y \in \mathbb{R}^d$ be fixed. Consider the mapping

$$(D_y G)(\eta) := G(\eta \cup \{y\}). \tag{4.8}$$

By direct computations, we obtain from (3.15) that D_y is satisfied chain rule

$$D_y(G_1 * G_2) = (D_y G_1) * G_2 + G_1 * (D_y G_2). \tag{4.9}$$

Therefore,

$$\begin{aligned} D_y \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0 \right) = &\frac{\sigma}{m} \left(e^{tm} - 1 \right) \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0 \right) \\ &+ \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0(\cdot \cup y) \right). \end{aligned}$$

Hence, using equality

$$k_t(y) = e^{-tm} \left(k_0(y) + \frac{\sigma}{m} \left(e^{tm} - 1 \right) \right), \tag{4.10}$$

we obtain

$$\begin{aligned} v_t(\eta, y) =& e^{-tm(|\eta|+1)} D_y \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0 \right)(\eta) \\ &- e^{-tm|\eta|} \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0 \right)(\eta)\, k_t(y) \\ =& e^{-tm(|\eta|+1)} \frac{\sigma}{m} \left(e^{tm} - 1 \right) \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0 \right)(\eta) \\ &+ e^{-tm(|\eta|+1)} \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0(\cdot \cup y) \right)(\eta) \\ &- e^{-tm(|\eta|+1)} \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0 \right)(\eta) \left[k_0(y) + \frac{\sigma}{m} \left(e^{tm} - 1 \right) \right] \\ =& e^{-tm(|\eta|+1)} \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0(\cdot \cup y) \right)(\eta) \\ &- e^{-tm(|\eta|+1)} k_0(y) \left(e_\lambda \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right) * k_0 \right)(\eta) \\ =& e^{-tm(|\eta|+1)} \sum_{\xi \subset \eta} \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right)^{|\eta \setminus \xi|} v_0(\xi, y). \end{aligned}$$

Therefore, for any $\eta \in \Gamma_0$ one has

$$\begin{aligned} C^{-|\eta|} \left| v_t(\eta, y) \right| \le& C^{-|\eta|} e^{-tm(|\eta|+1)} \sum_{\xi \subset \eta} \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right)^{|\eta \setminus \xi|} C^{|\xi|} C^{-|\xi|} \left| v_0(\xi, y) \right| \\ \le& a(y) C^{-|\eta|} e^{-tm(|\eta|+1)} \sum_{\xi \subset \eta} \left(\frac{\sigma}{m} \left(e^{tm} - 1 \right) \right)^{|\eta \setminus \xi|} C^{|\xi|} \\ =& a(y) C^{-|\eta|} e^{-tm(|\eta|+1)} \left(C + \frac{\sigma}{m} \left(e^{tm} - 1 \right) \right)^{|\eta|} \\ =& a(y) e^{-tm} \left(e^{-tm} + \frac{\sigma}{Cm} \left(1 - e^{-tm} \right) \right)^{|\eta|} \le a(y) e^{-tm}. \end{aligned}$$

The statement is proved. □

Remark 4.5. *From the proof of the Theorem 4.4 one can see that if* (4.7) *holds for* $t = 0$ *then it holds for any* $t > 0$ *as well.*

Remark 4.6. *More traditional object for studying decay of correlation principle is the so-called Ursell functions (or truncated correlation functions). We recall (see [2] and references therein) that the function* $u_t : \Gamma_0 \to \mathbb{R}$ *is called Ursell function for* k_t *if*

$$k_t = \exp^* u_t := \sum_{n=0}^{\infty} \frac{1}{n!} u_t^{*n}, \quad u^{*0} := 1^*.$$

The condition $k_t(\emptyset) = 1$ *guarantees existence of* u_t *with* $u_t(\emptyset) = 0$ *(see e.g. [2] for details). Then, by* (3.19) *and [2], we obtain that* u_t *is equal to sum of Ursell functions, corresponding to correlation functions of measures* π_{z_t} *and* ν_t *from* (3.26)*. It's easy to see that the Ursell function corresponding to the Poisson measure* π_{z_t} *is equal to* $\chi_{\{|\eta|=1\}} \dfrac{\sigma}{m}(1 - e^{-tm})$*. Next,*

$$\begin{aligned} e^{-tm|\eta|} k_0(\eta) =& e^{-tm|\eta|} \sum_{n=1}^{\infty} \frac{1}{n!} u_0^*(\eta) \\ =& e^{-tm|\eta|} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\eta_1 \sqcup \ldots \sqcup \eta_n = \eta \\ \eta_i \neq \emptyset,\ 1 \le i \le n}} u(\eta_1) \ldots u(\eta_n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\eta_1 \sqcup \ldots \sqcup \eta_n = \eta \\ \eta_i \neq \emptyset,\ 1 \le i \le n}} e^{-tm|\eta_1|} u(\eta_1) \ldots e^{-tm|\eta_n|} u(\eta_n) \\
&= \exp^*\bigl(e^{-tm|\cdot|} u_0\bigr)(\eta).
\end{aligned}
$$

Therefore,

$$
u_t(\eta) = e^{-tm|\eta|} u_0(\eta) + \chi_{\{|\eta|=1\}} \frac{\sigma}{m}(1 - e^{-tm}).
$$

In particular, if for any $n \ge 2$ the symmetric function $u_0^{(n)}$ is integrable by j variables ($1 \le j \le n-1$) then $u_t^{(n)}$ has this property too.

4.2. **Evolution of Bogolyubov functional.** Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that for any $\theta \in L^1(\mathbb{R}^d, dx)$ the following so-called Bogolyubov functional there exists:

$$
B_\mu(\theta) := \int_\Gamma \prod_{x \in \gamma} \bigl(1 + \theta(x)\bigr) d\mu(\gamma). \tag{4.11}
$$

By (3.1) and (3.3), we have an another representation

$$
B_\mu(\theta) = \int_{\Gamma_0} e_\lambda(\theta, \eta) k_\mu(\eta) d\lambda(\eta). \tag{4.12}
$$

In particular, if there exists $C > 0$ such that $k_\mu(\eta) \le \mathrm{const} \cdot C^{|\eta|}$, $\eta \in \Gamma_0$ then, by (3.2) and (3.3), the r.h.s. of (4.12) as well as (4.11) are finite.

Proposition 4.7. *Let $C > \frac{\sigma}{m}$, $k_0 \in \mathcal{K}_C$ and $k_0(\emptyset) = 1$. Let $B_t(\theta) := B_{\mu_t}(\theta)$, $B_{\mathrm{inv}}(\theta) := B_{\mu_{\mathrm{inv}}}(\theta)$. Then*

$$
\bigl|B_t(\theta) - B_{\mathrm{inv}}(\theta)\bigr| \le e^{-mt} \|k_0 - k_{\mathrm{inv}}\|_{\mathcal{K}_C} \frac{\exp\bigl\{C\|\theta\|_{L^1}\bigr\}}{1 - \dfrac{\sigma}{Cm}}.
$$

Proof. First of all let us note that, by Proposition 3.4, B_t exists. Then, by Theorem 4.1, we have

$$
\begin{aligned}
\bigl|B_t(\theta) - B_{\mathrm{inv}}(\theta)\bigr| &= \left| \int_{\Gamma_0} e_\lambda(\theta, \eta) k_t(\eta) d\lambda(\eta) - \int_{\Gamma_0} e_\lambda(\theta, \eta) k_{\mathrm{inv}}(\eta) d\lambda(\eta) \right| \\
&\le \int_{\Gamma_0} e_\lambda(|\theta|, \eta) \bigl|k_t(\eta) - k_{\mathrm{inv}}(\eta)\bigr| d\lambda(\eta) \\
&\le \|k_t - k_{\mathrm{inv}}\|_{\mathcal{K}_C} \int_{\Gamma_0} e_\lambda(|\theta|, \eta) C^{|\eta|} d\lambda(\eta) \\
&\le \|k_0 - k_{\mathrm{inv}}\|_{\mathcal{K}_C} \frac{e^{-mt}}{1 - \dfrac{\sigma}{Cm}} \exp\bigl\{C\|\theta\|_{L^1}\bigr\}.
\end{aligned}
$$

□

Remark 4.8. *Note that, by (3.18), we have*

$$
\begin{aligned}
B_t(\theta) &= \int_{\Gamma_0} e_\lambda(\theta, \eta)\, k_t(\eta)\, d\lambda(\eta) \\
&= \int_{\Gamma_0} e_\lambda(\theta, \eta) e^{-tm|\eta|} \int_{\Gamma_0} e_\lambda(\theta, \xi)\, e_\lambda\Bigl(\frac{\sigma}{m}\bigl(1 - e^{-tm}\bigr), \xi\Bigr) d\lambda(\xi)\, k_0(\eta)\, d\lambda(\eta) \\
&= \int_{\Gamma_0} e_\lambda(\theta, \xi) e_\lambda\Bigl(\frac{\sigma}{m}\bigl(1 - e^{-tm}\bigr), \xi\Bigr) d\lambda(\xi) \int_{\Gamma_0} e^{-tm|\eta|} e_\lambda(\theta, \eta)\, k_0(\eta)\, d\lambda(\eta) \\
&= \exp\Bigl\{\frac{\sigma}{m}\bigl(1 - e^{-tm}\bigr)\langle\theta\rangle\Bigr\} B_0\bigl(e^{-tm}\theta\bigr),
\end{aligned} \tag{4.13}
$$

that corresponds to (3.26).

Since, by (4.12) *and* (3.2), $B_{\mathrm{inv}}(\theta) = \exp\Big\{\dfrac{\sigma}{m}\langle\theta\rangle\Big\}$, *we obtain from* (4.13)

$$
\begin{aligned}
&B_t(\theta) - B_{\mathrm{inv}}(\theta)\\
&= \exp\left\{\frac{\sigma}{m}\left(1-e^{-tm}\right)\langle\theta\rangle\right\}\Big(B_0\left(e^{-tm}\theta\right) - \exp\left\{\frac{\sigma}{m}e^{-tm}\langle\theta\rangle\right\}\Big)\\
&= \exp\left\{\frac{\sigma}{m}\left(1-e^{-tm}\right)\langle\theta\rangle\right\}\int_{\Gamma_0} e^{-tm|\eta|}e_\lambda(\theta,\eta)\big(k_0(\eta)-k_{\mathrm{inv}}(\eta)\big)d\lambda(\eta).
\end{aligned} \tag{4.14}
$$

As a result, if, e.g., $k_0(\eta) \le \left(\dfrac{\sigma}{m}\right)^{|\eta|} = k_{\mathrm{inv}}(\eta)$, $\eta\in\Gamma_0$ *and* $k_0(\emptyset)=1$, *then for any* $0\le\theta\in L^1(\mathbb{R}^d,dx)$ *one has*

$$0 \le B_{\mathrm{inv}}(\theta) - B_t(\theta) \le e^{-tm}\exp\left\{\frac{\sigma}{m}\left(1-e^{-tm}\right)\langle\theta\rangle\right\}\big(B_{\mathrm{inv}}(\theta)-B_0(\theta)\big).$$

One can consider now the state space where the evolution $B_0(\theta)\mapsto B_t(\theta)$ lives. Let $E = L^1(\mathbb{R}^d,dx)$. We recall (see, e.g., [13] and references therein) that a functional $A: E\to\mathbb{C}$ is called entire on E whenever A is locally bounded and for all $\theta_0,\theta\in E$ the mapping $\mathbb{C}\ni z\mapsto A(\theta_0+z\theta)\in\mathbb{C}$ is entire. For any $\alpha>0$ we consider a Banach space $E^{(\alpha)}$ of entire functionals on E with norm

$$\|A\|_\alpha := \sup_{\theta\in E}\big(|A(\theta)|e^{-\alpha\|\theta\|_E}\big) < \infty.$$

Then for any $\alpha \ge \dfrac{\sigma}{m}$ we have

$$
\begin{aligned}
\|B_t(\theta)\|_\alpha &= \sup_{\theta\in E}\Big(\exp\left\{\frac{\sigma}{m}(1-e^{-tm})\langle\theta\rangle\right\}|B_0(e^{-tm}\theta)|\exp\{-\alpha\|\theta\|_E\}\Big)\\
&\le \|B_0(\theta)\|_\alpha\sup_{\theta\in E}\Big(\exp\left\{\frac{\sigma}{m}(1-e^{-tm})\|\theta\|_E\right\}\exp\left\{\alpha(e^{-tm}-1)\|\theta\|_E\right\}\Big)\\
&= \|B_0(\theta)\|_\alpha\sup_{\theta\in E}\Big(\exp\Big\{\Big(\frac{\sigma}{m}-\alpha\Big)(1-e^{-tm})\|\theta\|_E\Big\}\Big) \le \|B_0(\theta)\|_\alpha.
\end{aligned}
$$

Therefore, the evolution $B_0(\theta)\mapsto B_t(\theta)$ preserves balls in $E^{(\alpha)}$.

5. Evolution on $\mathcal{L}_C$

We recall now without a proof the partial case of the well-known lemma (cf., [17]).

Lemma 5.1. *For any measurable function* $H:\Gamma_0\times\Gamma_0\times\Gamma_0\to\mathbb{R}$

$$\int_{\Gamma_0}\sum_{\xi\subset\eta}H(\xi,\eta\setminus\xi,\eta)\,d\lambda(\eta) = \int_{\Gamma_0}\int_{\Gamma_0}H(\xi,\eta,\eta\cup\xi)\,d\lambda(\xi)\,d\lambda(\eta) \tag{5.1}$$

if only both sides of the equality make sense.

Next statements present explicit form for the semigroup on $\mathcal{L}_C$ and resolvent of its generator and show mean-ergodic properties of this semigroup (see, e.g., [1] for a terminology).

Proposition 5.2. *Let* $C \ge \dfrac{\sigma}{m}$. *Then for any* $G\in\mathcal{L}_C$

$$\big(\hat{T}(t)G\big)(\eta) = e^{-tm|\eta|}\int_{\Gamma_0}G(\eta\cup\xi)\,e_\lambda\Big(\frac{\sigma}{m}\left(1-e^{-tm}\right),\xi\Big)\,d\lambda(\xi). \tag{5.2}$$

Moreover, for any $z\in\mathbb{C}$ *with* $\operatorname{Re}z>0$ *there exist bounded resolvent operator* $R_z = (\hat{L}-z)^{-1}$ *and for any* $G\in\mathcal{L}_C$

$$(R_zG)(\eta) = \frac{1}{m}\int_{\Gamma_0}G(\eta\cup\xi)\left(\frac{\sigma}{m}\right)^{|\xi|}B\Big(\frac{z}{m}+|\eta|,|\xi|+1\Big)\,d\lambda(\xi), \tag{5.3}$$

where $B(x,y)=\int_0^1 s^{x-1}(1-s)^{y-1}\,ds$ *is the Euler beta function.*

Proof. Let $C\geq\dfrac{\sigma}{m}$ and $G\in\mathcal{L}_C$. Then, $\hat{T}(t)G\in\mathcal{L}_C$ and for any $k\in\mathcal{K}_C$, by Corollary 3.6 and Lemma 5.1, one has

$$\begin{aligned}\int_{\Gamma_0}\big(\hat{T}(t)G\big)(\eta)\,k(\eta)\,d\lambda(\eta)&=\int_{\Gamma_0}G(\eta)\big(\hat{T}^*(t)k\big)(\eta)\,d\lambda(\eta)\\ =\int_{\Gamma_0}G(\eta)\,e^{-tm|\eta|}\Big(e_\lambda\Big(\frac{\sigma}{m}\left(e^{tm}-1\right)\Big)*k\Big)(\eta)\,d\lambda(\eta)\\ =\int_{\Gamma_0}\int_{\Gamma_0}G(\eta\cup\xi)\,e^{-tm|\eta|}e^{-tm|\xi|}e_\lambda\Big(\frac{\sigma}{m}\left(e^{tm}-1\right),\xi\Big)\,k(\eta)\,d\lambda(\xi)\,d\lambda(\eta),\end{aligned}$$

that implies (5.2).

Since $\hat{T}(t)$ is a C_0-semigroup with generator $\big(\hat{L},D(\hat{L})\big)$ then for any $z\in\mathbb{C}$ with $\operatorname{Re}z>0$

$$R_z=\int_0^\infty e^{-zt}U(t)\,dt.$$

Then, by direct computation,

$$\begin{aligned}(R_zG)(\eta)&=\int_0^\infty e^{-zt}e^{-tm|\eta|}\int_{\Gamma_0}G(\eta\cup\xi)\,e_\lambda\Big(\frac{\sigma}{m}\left(1-e^{-tm}\right),\xi\Big)\,d\lambda(\xi)\,dt\\ &=\int_{\Gamma_0}G(\eta\cup\xi)\left(\frac{\sigma}{m}\right)^{|\xi|}\int_0^\infty e^{-(z+m|\eta|)t}\left(1-e^{-tm}\right)^{|\xi|}dt\,d\lambda(\xi),\end{aligned}$$

and the assertion follows from (3.23). □

Theorem 5.3. *Let* $C\geq\max\Big(\dfrac{\sigma}{m};1\Big)$ *and* $G\in\mathcal{L}_{2C}$ *then*

$$\frac{1}{t}\int_0^t\hat{T}(s)\,G\,ds\to\chi_{\Gamma^{(0)}}\cdot\int_{\Gamma_0}G(\xi)\,k_{\mathrm{inv}}(\xi)\,d\lambda(\xi)$$

as $t\to\infty$ *in* $\mathcal{L}_C$.

Proof. Using equality

$$1_{\Gamma^{(0)}}(\eta)\cdot\int_{\Gamma_0}G(\xi)\,k_{\mathrm{inv}}(\xi)\,d\lambda(\xi)=\int_{\Gamma_0}G(\xi\cup\eta)\left(\frac{\sigma}{m}\right)^{|\xi|}0^{|\eta|}d\lambda(\xi)$$

we have

$$\begin{aligned}&\left\|\frac{1}{t}\int_0^t\hat{T}(s)\,G\,ds-\chi_{\Gamma^{(0)}}\cdot\int_{\Gamma_0}G(\xi)\left(\frac{\sigma}{m}\right)^{|\xi|}d\lambda(\xi)\right\|_{\mathcal{L}_C}\\ &\leq\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta\cup\xi)|\left(\frac{\sigma}{m}\right)^{|\xi|}\left|\frac{1}{t}\int_0^t e^{-sm|\eta|}\left(1-e^{-sm}\right)^{|\xi|}ds-0^{|\eta|}\right|C^{|\eta|}d\lambda(\xi)\,d\lambda(\eta).\end{aligned}$$

We have

$$\begin{aligned}&\frac{1}{t}\int_0^t e^{-sm|\eta|}\left(1-e^{-sm}\right)^{|\xi|}ds\\ &=\sum_{j=0}^{|\xi|}\binom{|\xi|}{j}(-1)^j\frac{1}{t}\int_0^t e^{-sm(|\eta|+j)}ds\\ &=\begin{cases}\displaystyle\sum_{j=0}^{|\xi|}\binom{|\xi|}{j}(-1)^j\frac{1}{t}\frac{1-e^{-tm(|\eta|+j)}}{m\left(|\eta|+j\right)}, & |\eta|\neq0,\\ \displaystyle 1+\sum_{j=1}^{|\xi|}\binom{|\xi|}{j}(-1)^j\frac{1}{t}\frac{1-e^{-tmj}}{mj}, & |\eta|=0.\end{cases}\end{aligned}$$

Therefore, for any $\xi, \eta \in \Gamma_0$

$$\frac{1}{t}\int_0^t e^{-sm|\eta|}\left(1-e^{-sm}\right)^{|\xi|} ds \to 0^{|\eta|}, \quad t\to\infty.$$

Using trivial estimate $\left|\frac{1}{t}\int_0^t e^{-sm|\eta|}\left(1-e^{-sm}\right)^{|\xi|} ds - 0^{|\eta|}\right| \le 1$ we obtain the assertion by the dominated convergence theorem since

$$\begin{aligned}&\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta\cup\xi)|\left(\frac{\sigma}{m}\right)^{|\xi|}C^{|\eta|}d\lambda(\xi)\,d\lambda(\eta)\\ &=\int_{\Gamma_0}|G(\eta)|\left(1+\frac{\sigma}{Cm}\right)^{|\eta|}C^{|\eta|}d\lambda(\eta)\\ &\le\int_{\Gamma_0}|G(\eta)|\,2^{|\eta|}C^{|\eta|}d\lambda(\eta)=\|G\|_{2C}<+\infty.\end{aligned}$$

The statement is proved. □

References

1. K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, Vol. 194, Springer-Verlag, New York, 2000.
2. D. Finkelshtein, *Towards convolutions on configuration spaces*, in preparation.
3. D. Finkelshtein and Y. Kondratiev, *Regulation mechanisms in spatial stochastic development models*, J. Stat. Phys. **136** (2009), 103–115.
4. D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, *Correlation functions evolution for the Glauber dynamics in continuum*. http://arxiv.org/abs/1107.3487.
5. D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, *Individual based model with competition in spatial ecology*, SIAM J. Math. Anal. **41** (2009), 297–317.
6. D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and E. Zhizhina, *An approximative approach for construction of the Glauber dynamics in continuum*. http://arxiv.org/abs/0910.4241 (to appear in Math. Nachr.).
7. D. Finkelshtein, Y. Kondratiev, and M. J. Oliveira, *Markov evolutions and hierarchical equations in the continuum.* I. *One-component systems*, J. Evol. Equ. **9** (2009), 197–233.
8. N. L. Garcia and T. G. Kurtz, *Spatial birth and death processes as solutions of stochastic equations*, ALEA Lat. Am. J. Probab. Math. Stat. **1** (2006), 281–303 (electronic).
9. N. L. Garcia and T. G. Kurtz, *Spatial point processes and the projection method*, Progress in Probability **60** (2008), 271–298.
10. R. A. Holley and D. W. Stroock, *Nearest neighbor birth and death processes on the real line*, Acta Math. **140** (1978), 103–154.
11. Y. Kondratiev and A. M. Chebotarev, *Bernstein theorems and transformations of correlation measures in statistical physics*, Mat. Zametki **79** (2006), 700–716.
12. Y. Kondratiev and T. Kuna, *Harmonic analysis on configuration space.* I. *General theory*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2002), 201–233.
13. Y. Kondratiev, T. Kuna, and M. J. Oliveira, *Holomorphic Bogoliubov functionals for interacting particle systems in continuum*, J. Funct. Anal. **238** (2006), 375–404.
14. Y. Kondratiev, O. Kutoviy, and R. Minlos, *On non-equilibrium stochastic dynamics for interacting particle systems in continuum*, J. Funct. Anal. **255** (2008), 200–227.
15. Y. Kondratiev, O. Kutoviy, and E. Zhizhina, *Nonequilibrium Glauber-type dynamics in continuum*, J. Math. Phys. **47** (2006), no. 11, 113501, 17.
16. Y. Kondratiev, E. Lytvynov, and M. Röckner, *Non-equilibrium stochastic dynamics in continuum: The free case*, Cond. Matter Phys. **11** (2008), no. 4, 701–721.
17. Y. Kondratiev, R. Minlos, and E. Zhizhina, *One-particle subspace of the Glauber dynamics generator for continuous particle systems*, Rev. Math. Phys. **16** (2004), 1073–1114.
18. A. Lenard, *States of classical statistical mechanical systems of infinitely many particles.* I, Arch. Rational Mech. Anal. **59** (1975), 219–239.
19. A. Lenard, *States of classical statistical mechanical systems of infinitely many particles.* II. *Characterization of correlation measures*, Arch. Rational Mech. Anal. **59** (1975), 241–256.
20. M. D. Penrose, *Existence and spatial limit theorems for lattice and continuum particle systems*, Probability Surveys **5** (2008), 1–36.

21. C. Preston, *Spatial birth-and-death processes*, Bull. Inst. Internat. Statist. **46** (1975), no. 2, 371–391, 405–408.
22. X. Qi, *A functional central limit theorem for spatial birth and death processes*, Adv. Appl. Probab. **40** (2008), no. 3, 759–797.
23. D. Ruelle, *Statistical mechanics: Rigorous results*, W. A. Benjamin, Inc., New York—Amsterdam, 1969.
24. D. Ruelle, *Superstable interactions in classical statistical mechanics*, Comm. Math. Phys. **18** (1970), 127–159.
25. D. Surgailis, *On Poisson multiple stochastic integrals and associated equilibrium Markov processes*, Theory and Application of Random Fields, Proc. IFIP-WG 7/1 Working Conf. (Bangalore, India, 1982), Lecture Notes Control Inf. Sci., Vol. 49, Springer, Berlin, 1983, pp. 233–248.
26. D. Surgailis, *On multiple Poisson stochastic integrals and associated Markov semigroups*, Probab. Math. Statist. **3** (1984), 217–239.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: fdl@imath.kiev.ua

Received 19/07/2011