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ON INFINITESIMAL STRUCTURE OF A HYPERGROUP THAT ORIGINATES FROM A LIE GROUP

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Abstract. We describe an infinitesimal algebra to a hypergroup constructed from a Lie group and a conditional expectation. We also prove a theorem on a decomposition of the conditional expectation into the product of a counital conditional expectation and the one that arises in the double coset construction.

1. Introduction

The notion of a locally compact hypergroup is a generalization of that of a group and allows to carry main results of the harmonic analysis on a group to this more general object [1, 2, 3]. However an attempt to construct, for a hypergroup, a theory similar to the Lie theory, as in the case of a Lie group, meets a number of difficulties. It turns out that a hypergroup obtained from a Lie group may not be a manifold, the relations that the generators of the hypergroup satisfy may not be of Lie type, and the generators themselves may not be differential operators.

A construction of a Lie theory for generalized translation operators was initiated by Delsart [4] and Levitan [5]. The latter author studied the case where the generators satisfied Lie type relations. Some classes of hypergroups where the generators satisfied relations more general than Lie type relations were studied in [6, 7, 8].

In this paper, we describe an infinitesimal algebra for a hypergroup, constructed from a Lie group and a conditional expectation [9]. Such a hypergroup generalizes the one constructed in [10] from an orbital morphism. The main instrument we use is that of a conditional expectation on a C^*-algebra of continuous functions on a Lie group. There are two essentially different cases arise. The one corresponds to a so-called counital conditional expectation, see Definition 3.3, and the other one to a conditional expectation connected with a double coset construction, Remark 2.2. In both cases, we describe the infinitesimal algebra in terms of the universal enveloping algebra of the Lie group.

The paper is organized as follows. In Section 2 we recall some definitions and set forth the notation used in the paper.

Section 3 discusses constructing a new locally compact hypegroup with a use of a conditional expectation, and contains a theorem on the structure of this hypergroup. We also recall the connection between an orbital morphism and the corresponding conditional expectation.

Section 4 describes the differential structure on a locally compact hypergroup obtained from a Lie group and a conditional expectation, where two cases of the expectation being counital and not are treated.

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2. Initial definitions and notations

Let Q be a locally compact second countable Hausdorff topological space.

The spaces of complex-valued functions that are continuous, continuous and bounded, continuous with compact supports, continuous and equal to zero at infinity are denoted by $\mathcal{C}(Q)$, $\mathcal{C}_b(Q)$, $\mathcal{C}_c(Q)$, $\mathcal{C}_0(Q)$, respectively. For $f \in \mathcal{C}_c(Q)$, $\operatorname{supp} f$ denotes support of the function f. The linear spaces $\mathcal{C}_0(Q)$ and $\mathcal{C}_b(Q)$ have the structure of a C^*-algebra with respect to the pointwise multiplication and complex conjugation, endowed with the norm $\|f\| = \sup_{r\in Q}|f(r)|$ for $f \in \mathcal{C}_b(Q)$ or $f \in \mathcal{C}_0(Q)$.

Everywhere in the sequel a *measure* will mean a Radon measure on Q [11]. The integral of f, $f \in \mathcal{C}_c(Q)$, with respect to a measure μ is denoted by $\mu(f) = \int_Q f(p)\, d\mu(p)$. The Dirac measure at a point $q \in Q$ is denoted by ε_q, i.e., $\varepsilon_q(f) = f(q)$, $f \in \mathcal{C}(Q)$.

Let A be a C^*-algebra and $B \subset A$ a C^*-subalgebra of A. A bounded linear map $P\colon A \to B$ is called a conditional expectation if it satisfies the following properties [12]:

(i) P is a projection onto and has norm 1, that is, $P^2 = P$ and $\|P\| = 1$;
(ii) P is positive, that is $P(a^*a) \geq 0$ for any $a \in A$;
(iii) $P(b_1 a b_2) = b_1 P(a) b_2$ for any $a \in A$ and $b_1, b_2 \in B$;
(iv) $P(a^*)P(a) \leq P(a^*a)$ for all $a \in A$.

It follows from (ii) and the polarization identity that

(v) $P(a^*) = P(a)^*$, $a \in A$.

It also follows from [13] that (i) implies (ii), (iii), and (iv).

Now we recall the definition of a locally compact hypergroup, see [1, Definition 2.1].

Definition 2.1. Let Q be a locally compact space with an involutive homeomorphism $*\colon Q \to Q$ and a point $e \in Q$, $e^* = e$, and let the following conditions be satisfied.

(H_1) There is a $\mathbb{C}$-linear mapping $\Delta\colon \mathcal{C}_b(Q) \to \mathcal{C}_b(Q \times Q)$ such that

(a) Δ is coassociative, that is,

$$(\Delta \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \Delta) \circ \Delta; \tag{1}$$

(b) Δ is positive, that is, $\Delta f \geq 0$ for all $f \in \mathcal{C}_b(Q)$ such that $f \geq 0$;
(c) Δ preserves the identity, that is, $(\Delta 1)(p, q) = 1$, for all $p, q \in Q$;
(d) for all $f, g \in \mathcal{C}_c(Q)$, we have $(1 \otimes f) \cdot (\Delta g) \in \mathcal{C}_c(Q \times Q)$ and $(f \otimes 1) \cdot (\Delta g) \in \mathcal{C}_c(Q \times Q)$.

(H_2) The homomorphism $\epsilon\colon \mathcal{C}_b(Q) \to \mathbb{C}$ defined on the C^*-algebra $\mathcal{C}_b(Q)$ by $\epsilon(f) = f(e)$ satisfies the counit property, that is,

$$(\epsilon \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \epsilon) \circ \Delta = \mathrm{id}, \tag{2}$$

in other words, $(\Delta f)(e, p) = (\Delta f)(p, e) = f(p)$ for all $p \in Q$.

(H_3) The function $\check{f}$ defined by $\check{f}(q) = f(q^*)$ for $f \in \mathcal{C}_b(Q)$ satisfies

$$(\Delta \check{f})(p, q) = (\Delta f)(q^*, p^*). \tag{3}$$

(H_4) There exists a positive measure μ on Q, $\operatorname{supp} \mu = Q$, such that

$$\int_Q (\Delta f)(p, q)\, g(q)\, d\mu\,(q) = \int_Q f(q)\, (\Delta g)(p^*, q)\, d\mu\,(q) \tag{4}$$

for all $f \in \mathcal{C}_b(Q)$ and $g \in \mathcal{C}_c(Q)$, or $f \in \mathcal{C}_c(Q)$ and $g \in \mathcal{C}_b(Q)$, $p \in Q$.

Then $(Q, *, e, \Delta, \mu)$, or simply Q, is called a *locally compact hypergroup.*

The measure μ will be called a *left Haar measure* on Q.

Remark 2.2. If G is a locally compact group, then it is a locally compact hypergroup with respect to the following operations:

$$\Delta(f)(p,q) = f(p\cdot q), \quad \epsilon(f) = f(e), \quad q^* = q^{-1} \tag{5}$$

for all $p, q \in G$, with $\cdot$, e, and $^{-1}$ being the multiplication, the unit, and the operation of taking the inverse in the group G, correspondingly. The measure μ is a left invariant measure on G.

3. A hypergroup constructed from a conditional expectation

Let $(Q, *, e, \Delta, \mu)$ be a locally compact hypergroup, A denote the C^*-algebra $\mathcal{C}_b(Q)$, A_0 its C^*-subalgebra $\mathcal{C}_0(Q)$, and let I be the ideal of A consisting of functions with compact support.

Theorem 3.1. *Let A be as above, $P\colon A \to A$ be a conditional expectation such that $B = P(A_0)$ is a C^*-algebra, $P(I) \subset I$, and let the following hold:*

$$\begin{gathered}\big((P\times \mathrm{id})\circ\Delta\circ P\big)(f) = \big((\mathrm{id}\times P)\circ\Delta\circ P\big)(f) = \big((P\times P)\circ\Delta\big)(f),\\ P(\check f) = \big(P(f)\big)\check{}\,,\end{gathered} \tag{6}$$

for all $f \in A$.

Denote by $\tilde Q$ the spectrum of the commutative algebra B, which is a Hausdorff locally compact space. For each $g \in B \subset A$, let

$$\tilde\Delta(g) = \big((P\times P)\circ\Delta\big)(g). \tag{7}$$

If $\tilde q \in \tilde Q$ and $g \in B$, then we set

$$\tilde q^*(g) = \check g(q), \quad \tilde e = \epsilon, \tag{8}$$

and let $\tilde\mu$ be defined by

$$\tilde\mu = \mu\circ P. \tag{9}$$

Then $(\tilde Q, *, \tilde e, \tilde\Delta, \tilde\mu)$ is a locally compact hypergroup.

Proof. (H_1) (a). Using (7) and property (6) we have

$$(\tilde\Delta\times\mathrm{id})\circ\tilde\Delta = (P\times P\times\mathrm{id})\circ(\Delta\times\mathrm{id})\circ(P\times P)\circ\Delta = (P\times P\times P)\circ(\Delta\times\mathrm{id})\circ\Delta.$$

On the other hand,

$$(\mathrm{id}\times\tilde\Delta)\circ\tilde\Delta = (\mathrm{id}\times P\times P)\circ(\mathrm{id}\times\Delta)\circ(P\times P)\circ\Delta = (P\times P\times P)\circ(\mathrm{id}\times\Delta)\circ\Delta.$$

Now, the result follows from coassociativity of Δ.

(H_1) (b) is trivial, since Δ and P are positive.

(H_1) (c). The identity in B is $\tilde 1 = P(1)$. Indeed, $P(1)\cdot P(f) = P(1\cdot f) = P(f)$ for any $P(f) \in B$. By writing property (H_1) (c) as $\Delta(1) = 1\otimes 1$, we get

$$\tilde\Delta(\tilde 1) = \big((P\otimes P)\circ\Delta\big)(\tilde 1) = \big((P\otimes P)\circ\Delta\circ P\big)(1) = \big((P\otimes P)\circ\Delta\big)(1) = (P\otimes P)(1\otimes 1) = \tilde 1\otimes\tilde 1.$$

(H_1) (d). For $f, g \in \mathcal{C}_c(\tilde Q)) \subset \mathcal{C}_c(Q)$, we have

$$(1\otimes f)\cdot(\tilde\Delta(g)) = (1\otimes f)\cdot\big((P\otimes P)\circ\Delta(g)\big) = (\mathrm{id}\otimes P)\big((1\otimes f)\cdot\Delta(g)\big).$$

Since $(1\otimes f)\cdot\Delta(g) \in \mathcal{C}_c(Q\times Q)$, the result follows.

(H_2). For $f \in B$, we have

$$(\epsilon\times\mathrm{id})\circ\tilde\Delta(f) = (\epsilon\times\mathrm{id})\circ(P\times P)\circ\Delta(P(f)) = (\epsilon\times P)\circ\Delta(P(f)) = P(P(f)) = f.$$

(H_3). First note that $\check{g} \in B$ for $g \in B$, since $P(\check{g}) = P(g)\check{} = \check{g}$ by (6). Now,

$$(\tilde{\Delta}\check{f})(p,q) = \big((P \times P) \circ \Delta\check{f}\big)(p,q) = \big((P \times P) \circ ({}^\vee \times {}^\vee) \circ \Delta\big)(f)(q,p)$$
$$= \big((P \times P) \circ \Delta\big)(f)(q^*, p^*) = (\tilde{\Delta}f)(q^*, p^*)$$

for $p, q \in \tilde{Q}$, $f \in B$.

(H_4). Let us rewrite identity (4) as

$$(\mathrm{id} \times \tilde{\mu})\big(\tilde{\Delta}(f) \cdot (1 \otimes g)\big) = (\mathrm{id} \times \tilde{\mu})\big((1 \otimes f) \cdot (({}^\vee \times \mathrm{id}) \circ \tilde{\Delta}(g))\big).$$

Consider the left-hand side and substitute the definitions of $\tilde{\mu}$ and $\tilde{\Delta}$,

$$(\mathrm{id} \times \mu \circ P)\big(((P \times P) \circ \Delta(Pf)) \cdot (1 \otimes Pg)\big) = (\mathrm{id} \times \mu)\big((\mathrm{id} \times P) \circ \Delta(Pf) \cdot (1 \otimes Pg)\big)$$
$$= (\mathrm{id} \times \mu)\big((P \times \mathrm{id}) \circ \Delta(Pf) \cdot (1 \otimes Pg)\big) = P\big((\mathrm{id} \times \mu)(\Delta(Pf) \cdot (1 \otimes Pg))\big).$$

Now using (4) we get

$$P\big((\mathrm{id} \times \mu)(\Delta(Pf) \cdot (1 \otimes Pg))\big) = P\big((\mathrm{id} \times \mu)((1 \otimes Pf) \cdot ({}^\vee \times \mathrm{id}) \circ \Delta(Pg))\big)$$
$$= (\mathrm{id} \times \mu)\big((1 \otimes Pf) \cdot ({}^\vee \circ P \times \mathrm{id}) \circ \Delta(Pg)\big)$$
$$= (\mathrm{id} \times \mu)\big((1 \otimes Pf) \cdot ({}^\vee \circ P \times P) \circ \Delta(Pg)\big)$$
$$= (\mathrm{id} \times \mu \circ P)\big((1 \otimes Pf) \cdot ({}^\vee \times P) \circ \Delta(pg)\big)$$
$$= (\mathrm{id} \times \tilde{\mu})\big((1 \otimes Pf) \cdot ({}^\vee \times \mathrm{id}) \circ \tilde{\Delta}(Pg)\big).$$

□

Remark 3.2. The statement of Theorem 3.1 remains true [9] if we replace the first condition in (6) with the following two more general conditions: 1) $(P \times P) \circ \Delta = (P \times P) \circ \Delta \circ P$, and 2) ϵ is a counit with respect to $\tilde{\Delta} = (P \times P) \circ \Delta$.

Definition 3.3. Let Q be a locally compact hypergroup. A conditional expectation P is called *counital* if $\epsilon \circ P = \epsilon$ on A.

Remark 3.4. Let G be a locally compact group and H its compact subgroup. Let A denote the C^*-algebra $\mathcal{C}_b(Q)$. A function $f \in A$ is called H-*biinvariant* if $f(h_1 g h_2) = f(g)$ for all $g \in G$, $h_1, h_2 \in H$. Then $P\colon A \to A$ defined by

$$(Pf)(g) = \int_{H^2} f(h_1 g h_2)\, d\mu_H(h_1)\, d\mu_H(h_2), \quad g \in G, \tag{10}$$

where μ_H is the normalized Haar measure on H, is a conditional expectation on A satisfying all conditions of Theorem 3.1. The corresponding comultiplication $\tilde{\Delta}$ is given by

$$(\tilde{\Delta}f)(g_1, g_2) = \int_H f(g_1, h g_2)\, d\mu_H(h), \quad g_1, g_2 \in G,$$

and $\tilde{Q} = H \backslash G / H$ is the double coset hypergroup [14]. Since

$$(\epsilon \circ P)(f) = \int_H f(h)\, d\mu(h),$$

the conditional expectation P is not counital.

Lemma 3.5. *Let G be a locally compact group with a locally compact hypergroup structure as in Remark 2.2, $A = \mathcal{C}_b(G)$, and $A_0 = \mathcal{C}_0(G)$. Let $P\colon A \to A$ be a conditional expectation satisfying (6). Denote $\tilde{\epsilon} = \epsilon \circ P$. Assume that $\tilde{\epsilon}(A_0) \neq \{0\}$ and $P(A_0)$ is a C^*-subalgebra of A_0. Define mappings $P^l, P^r\colon A \to A$ by*

$$P^l = (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta, \quad P^r = (\mathrm{id} \times \tilde{\epsilon}) \circ \Delta. \tag{11}$$

Then $P^l(\mathcal{C}_c(G)) \subset \mathcal{C}_c(G)$, $P^r(\mathcal{C}_c(G)) \subset \mathcal{C}_c(G)$ *and*

$$P^l \circ P^l = P^l, \quad P^r \circ P^r = P^r, \tag{12}$$

$$P^l \circ P^r = P^r \circ P^l, \tag{13}$$

$$P \circ P^l = P^l \circ P = P \circ P^r = P^r \circ P = P. \tag{14}$$

In particular, P^l *and* P^r *are conditional expectations on* A_0*.*

Proof. First, let us show that $P^l(A_0) \subset A_0$. Indeed, since $\tilde{\epsilon}(A_0) \neq \{0\}$, we have that $\tilde{\epsilon}(\mathcal{C}_c(G)) \neq \{0\}$. Let $g \in \mathcal{C}_c(G)$ be such that $P(g) = g$ and $\epsilon(g) = 1$. Then, for any $f \in \mathcal{C}_c(G)$, we have

$$P^l(f) = (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta(f) = (\tilde{\epsilon}(g) \otimes 1) \cdot (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta(f) = (\tilde{\epsilon} \times \mathrm{id})\big((g \otimes 1) \cdot \Delta(f)\big).$$

Since $(g \otimes 1) \cdot \Delta(f) \in \mathcal{C}_c(G \times G)$ for $f, g \in \mathcal{C}_c(G)$, the claim follows. In the same way, we prove that $P^r(A_0) \subset A_0$.

Let us prove the first identity in (12). We have

$$\begin{aligned} P^l \circ P^l &= (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta \circ (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta \\ &= (\epsilon \circ P \times \epsilon \circ P \times \mathrm{id}) \circ (\Delta \times \mathrm{id}) \circ \Delta = (\epsilon \circ P \times \epsilon \times \mathrm{id}) \circ (\Delta \circ P \times \mathrm{id}) \circ \Delta \\ &= (\epsilon \circ P \circ P \times \mathrm{id}) \circ \Delta = (\epsilon \circ P \times \mathrm{id}) \circ \Delta = P^l. \end{aligned}$$

The second identity in (12) is proved similarly.

Consider now identity (13). We have

$$P^l \circ P^r = (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta \circ (\mathrm{id} \times \tilde{\epsilon}) \circ \Delta = (\tilde{\epsilon} \times \mathrm{id} \times \tilde{\epsilon}) \circ (\Delta \times \mathrm{id}) \circ \Delta.$$

On the other hand,

$$P^r \circ P^l = (\mathrm{id} \times \tilde{\epsilon}) \circ \Delta \circ (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta = (\tilde{\epsilon} \times \mathrm{id} \times \tilde{\epsilon}) \circ (\mathrm{id} \times \Delta) \circ \Delta,$$

hence, identity (13) follows from coassociativity of Δ, see condition (1).

To prove (14), consider

$$P \circ P^l = (\epsilon \circ P \times P) \circ \Delta = (\epsilon \times P) \circ \Delta \circ P = P \circ P = P.$$

The other identities in (14) are proved similarly.

Finally, since Δ and ϵ are C^*-homomorphisms, P is a conditional expectation, their composition has C^*-norm 1 and thus, by [12], P^l and P^r are conditional expectations. □

Corollary 3.6. *Let* P *be as in Lemma 3.5. Then*

$$(\tilde{\epsilon} \times \tilde{\epsilon})\big((1 \otimes a) \cdot \Delta(f)\big) = (\tilde{\epsilon} \times \tilde{\epsilon})\big(\Delta(f) \cdot (1 \otimes a)\big) = \tilde{\epsilon}(a) \cdot \tilde{\epsilon}(f), \tag{15}$$

$$(\tilde{\epsilon} \times \tilde{\epsilon})\big((a \otimes 1) \cdot \Delta(f)\big) = (\tilde{\epsilon} \times \tilde{\epsilon})\big(\Delta(f) \cdot (a \otimes 1)\big) = \tilde{\epsilon}(a) \cdot \tilde{\epsilon}(f) \tag{16}$$

for all $a, f \in A_0$*.*

Proof. Note that

$$\epsilon \circ P^l = \epsilon(\tilde{\epsilon} \times \mathrm{id}) \circ \Delta = (\tilde{\epsilon} \times \epsilon) \circ \Delta = \tilde{\epsilon}.$$

Since P^l is a conditional expectation, $P^l(aP^l(f)) = P^l(a)P^l(f)$. Hence, applying ϵ to both sides, we get $\tilde{\epsilon}(aP^l(f)) = \tilde{\epsilon}(a) \cdot \tilde{\epsilon}(f)$, or

$$(\tilde{\epsilon} \times \tilde{\epsilon})\big((1 \otimes a) \cdot \Delta(f)\big) = \tilde{\epsilon}(a) \cdot \tilde{\epsilon}(b).$$

This proves the first part of (15).

Considering the identity $P^l(P^l(a)f) = P^l(a)P^l(b)$ and applying ϵ to both sides we get the second identity in (15).

Identity (16) is proved similarly. □

Proposition 3.7. *Let G be a locally compact group, Δ, ϵ be given by (5), $A = \mathcal{C}_b(Q)$, and $P\colon A \to A$ be a conditional expectation satisfying the conditions of Theorem 3.1 and such that $\epsilon \circ P(A_0) \neq \{0\}$. Then there exists a compact subgroup H of G such that*

$$P^l(A_0) = H\backslash A_0, \quad P^r(A_0) = A_0/H, \tag{17}$$

where $H\backslash A_0$ (resp. A_0/H) is a C^*-subalgebra of A_0 of H-left invariant (resp. H-right invariant) functions on G tending to zero at infinity.

Moreover, the measure μ_H given, for any $f \in \mathcal{C}_0(G)$, by $\mu_H(f\restriction_H) = \tilde{\epsilon}(f)$ is a normalized Haar measure on H and the expectations P^l and P^r are given by

$$P^l(f) = \int_H f(hg)\,d\mu_H(h), \quad P^r(f) = \int_H f(gh)\,d\mu_H(h). \tag{18}$$

Proof. Let

$$J = \{f \in A_0 : (\epsilon \circ P)(|f|^2) = 0\}.$$

The linear functional $\epsilon \circ P$ on A_0 is positive, hence J is an ideal [15]. Let $B_0 = A_0/J$ and $\pi\colon A_0 \to B_0$ be the corresponding map onto the quotient C^*-algebra B. Denote $H = \operatorname{spec} B_0$, the spectrum of B_0 [15]. Since π is a C^*-algebra homomorphism,

$$H = \operatorname{spec} B_0 \subseteq \operatorname{spec} A_0 = G.$$

Also, by the definition, $h \in H$ if and only if $f(h) = 0$ for all $f \in J$.

Let us show that H is a subgroup of G. Indeed, let $f \in J$ and $h_1, h_2 \in H$. Then $f(h_1h_2) = \Delta f(h_1, h_2)$. Hence, $h_1h_2 \in H$ if and only if $(\epsilon \circ P \times \epsilon \circ P)\Delta f = 0$ if $f \in J$. But using the first identity in (6) we get

$$\big((\epsilon \circ P \times \epsilon \circ P) \circ \Delta\big)(f) = \big((\epsilon \circ P \times \epsilon) \circ \Delta \circ P\big)(f) = (\epsilon \circ P)(Pg) = (\epsilon \circ P)(f) = 0.$$

In the same way, the second identity in (6) yields get that $h^{-1} \in H$ if $h \in H$. Hence, H is a subgroup of G.

To show that $P^l(A_0) = H\backslash A_0$, let us first prove that $\operatorname{Im} P^l \subset H\backslash A_0$. Since $H = \operatorname{supp} \tilde{\epsilon}$, for an arbitrary $h \in H$, choose a sequence $e_n \in A_0$ such that $\tilde{\epsilon}(e_n) = 1$ and $\tilde{\epsilon}(e_n f) \to f(h)$ for any $f \in A_0$. Then $f \in H\backslash A_0$ if, for all $h \in H$, $f(hg) = (\Delta f)(h, g) = f(g)$. If $f \in \operatorname{Im} P^l$, then $f = P^l(f)$ and

$$\begin{aligned}(\tilde{\epsilon} \times \mathrm{id})\big((e_n \otimes 1) \cdot \Delta(P^l(f))\big) &= (\tilde{\epsilon} \times \mathrm{id})\big((e_n \otimes 1) \cdot (\tilde{\epsilon} \times \mathrm{id} \times \mathrm{id}) \circ (\mathrm{id} \times \Delta) \circ \Delta(f)\big)\\ &= (\tilde{\epsilon} \times \tilde{\epsilon} \times \mathrm{id})\big((1 \otimes e_n \otimes 1) \cdot ((\Delta \times \mathrm{id}) \circ \Delta)(f)\big)\\ &= \tilde{\epsilon}(e_n) \cdot (\tilde{\epsilon} \times \mathrm{id}) \circ \Delta(f) = P^l(f) = f,\end{aligned}$$

where we used identity (15).

To show that $\operatorname{Im} P^l \supset H\backslash A_0$, let $f \in H\backslash A_0$, that is $f(hg) = f(g)$ for all $h \in H$ and $g \in G$ or, which is the same thing, $(\pi \times \mathrm{id}) \circ \Delta f = 1 \otimes f$. Since $\operatorname{supp} \tilde{\epsilon} = H$, we have that $\tilde{\epsilon} \circ \pi = \tilde{\epsilon}$ and, hence,

$$P^l(f) = \big((\tilde{\epsilon} \times \mathrm{id}) \circ \Delta\big)(f) = \big((\tilde{\epsilon} \circ \pi \times \mathrm{id}) \circ \Delta\big)(f) = (\tilde{\epsilon} \times \mathrm{id})(1 \otimes f) = f.$$

The second equality in (17) is proved similarly.

If ν is any Radon measure on H, then

$$(\mu_H * \nu)(f) = (\tilde{\epsilon} \times \nu) \circ \Delta(f) = \nu(P^l(f)) = \nu(1)\tilde{\epsilon}(f),$$

hence $\tilde{\epsilon}$ is a left-invariant measure. Since $\tilde{\epsilon}(1) = 1$, the group H is compact and μ_H is the normalized Haar measure.

Formulas (18) are immediate. □

Theorem 3.8. *Let G be a group, $P\colon A \to A$ a conditional expectation satisfying the conditions of Proposition 3.7. Then there is a conditional expectation $P_2\colon H\backslash A/H \to H\backslash A/H$ such that $P = P_2 \circ P_1$, where $P_1 = P^l \circ P^r$ defined by (18) and P_2 is counital.*

Proof. By Lemma 3.5, $P_1 \circ P = P$, hence P factors through P_1, that is, $P = P_2 \circ P_1$, where P_2 is defined by $P_2(f) = P(f)$ for any $f \in H \backslash A/H$. Thus we only need to show that P_2 is counital. But this is immediate, since the counit on $H \backslash A/H$ is $\tilde{\epsilon} = \epsilon \circ P$ and $\tilde{\epsilon} \circ P_2 = \epsilon \circ P \circ P = \epsilon \circ P = \tilde{\epsilon}$. □

3.1. **The connection between conditional expectations and orbital morphisms.** In what follows we discuss conditional expectations on usual hypergroups and a construction of orbital morphisms [10]. For simplicity, we consider only the case of compact hypergroups. Let Q be a compact DJS-hypergroup with involution $Q \ni p \mapsto p^* \in Q$, comultiplication $\Delta : C(Q) \to C(Q \times Q)$, a neutral element (counit) e, and a Haar measure μ, and let Y be a compact Hausdorff space. Let ϕ be an open continuous mapping from Q onto Y (the orbital mapping). The closed sets $\phi^{-1}(y)$, $y \in Y$, are called *ϕ-orbits*. Let B be a C^*-subalgebra of $C(Q)$ consisting of functions constant on ϕ-orbits. Obviously, a mapping $\phi : C(Y) \to B$ defined by $(\phi f)(x) = f(\phi(x))$ is an isomorphism of the C^*-algebras. Denote by $\phi_* : M(Q) \to M(Y)$ the corresponding mapping of Radon measures, $\langle \phi_*(\mu), f\rangle = \langle \mu, f \circ \phi\rangle$, $\mu \in M(Q)$, $f \in C(Y)$. The measure $\mu \in M(Q)$ is called *ϕ-consistent*, if $\phi_*(\mu * \nu) = 0 = \phi_*(\nu * \mu)$ whenever $\phi_*(\nu) = 0$.

The following proposition clarifies the concept of ϕ-consistency.

Proposition 3.9. *Let $\phi : Q \to Y$ be an orbital mapping and denote $\tilde{e} = \phi(e)$. Suppose that the following conditions are satisfied:*

- (a) $\phi^{-1}(\tilde{e}) = \{e\}$,
- (b) *if A is an ϕ-orbit then so is A^*,*
- (c) *for any $y \in Y$ there exists a probability measure $q_y \in M(Q)$ such that supp $q_y \subset \phi^{-1}(y)$,*
- (d) *each measure q_y is ϕ-consistent.*

Define the linear mapping $P : C(Q) \to B$ as follows:

$$(Pf)(x) = \left\langle q_{\phi(x)}, f \right\rangle. \tag{19}$$

Then P is a conditional expectation and satisfies all hypothesis of Remark 3.2. Conversely, if conditions (a)–(c) are satisfied and the linear mapping P defined by (19) is a conditional expectation from $C(Q)$ to B satisfying all hypothesis of Remark 3.2, then each q_y is ϕ-consistent.

Proof. In virtue of (b) we can define an involutive homeomorphism $* : Y \to Y$ as follows: if $y = \phi(x)$, then $y^* = \phi(x^*)$. Theorem 13.5A in [10] states that there exists a unique convolution $*$ in $M(Y)$ such that Y is a DJS-hypergroup and ϕ is an orbital morphism, i.e.,

- (i) $\delta_y * \delta_z = \phi_*(q_y * q_z)$ for any $y, z \in Y$,
- (ii) $q_{y^*} = (q_y)^*$,
- (iii) $\operatorname{supp} q_y = \phi^{-1}(y)$ and

$$m = \int_Q q_{\phi(x)} m(dx), \tag{20}$$

where m is a Haar measure on the hypergroup Q.

Thus we can define a mapping $\phi^* : M(Y) \to M(Q)$ by setting $\phi^*(\delta_y) = q_y$ and

$$\phi^*(\nu) = \int_Y q_z \; \nu(dz),$$

for $y \in Y$ and $\nu \in M(Y)$. In virtue of Lemma 13.6A in [10] the orbital morphism ϕ is consistent, i.e., the mapping ϕ^* is a $*$-homomorphism. It also follows from the proof of Theorem 13.5A cited above that the mapping $Y \ni y \mapsto q_y \in M(q)$ is continuous in the weak topology. Thus P is well-defined and is, indeed, a conditional expectation. Let $f \in \ker P$. Then $\langle q_z, f\rangle = 0$ for all $z \in Y$, and for $x_1, x_2 \in Q$ we have

$$\begin{aligned}((P\otimes P)\Delta f)(x_1,x_2) &= \left\langle q_{\phi(x_1)}\otimes q_{\phi(x_2)}, \Delta f\right\rangle \\ &= \left\langle q_{\phi(x_1)} * q_{\phi(x_2)}, f\right\rangle = \left\langle \phi^*(\delta_{\phi(x_1)}) * \phi^*(\delta_{\phi(x_2)}), f\right\rangle \\ &= \left\langle \phi^*(\delta_{\phi(x_1)} * \delta_{\phi(x_2)}), f\right\rangle = \int_Y \langle q_z, f\rangle \ (\delta_{\phi(x_1)} * \delta_{\phi(x_2)})(dz) \\ &= \int_Y (Pf)(\phi(x)) \ (\delta_{\phi(x_1)} * \delta_{\phi(x_2)})(dz) = 0,\end{aligned}$$

where $x \in \phi^{-1}(z)$. Hence, $\ker P$ is a coideal. In virtue of (iii), P is m-invariant and it follows from (a) that P is counital. At last, the equality $P \circ * = * \circ P$ follows from (b).

Let us prove the converse statement. Define the convolution in $M(Y)$ according to (i). Since B is isomorfic to $C(Y)$, it follows from Theorem 2.1 that Y is a DJS-hypergroup. To prove the result, we need to show that ϕ is a consistent orbital morphism. Then the result follows from Theorem 13.6B in [10]. Indeed, (ii) follows from (b) and equality (20) follows from the fact that P is m-invariant. Let us show that $\operatorname{supp} q_y = \phi^{-1}(y)$. Suppose that $x_0 \in \phi^{-1}(y)$ and $x_0 \notin \operatorname{supp} q_y$. Then there exists an open neighborhood O_{x_0} of x_0 such that $O_{x_0} \cap \operatorname{supp} q_y = \varnothing$. Let $f \in C(Q)$ be a positive function such that $f(x_0) = 0$ and $f(x) = 1$ for $x \in Q\backslash O_{x_0}$. Since $Pf \in C(Q)$ and $(Pf)(x_0) = \langle q_y, f\rangle = 1$, one can find, for an arbitrary $0 < \varepsilon < 1/2$, an open neighborhood $V \subset O_{x_0}$ of x_0 such that $f(x) < \epsilon$ and $(Pf)(x) > 1 - \varepsilon$ for $x \in V$. Denote by i_V the indicator of V. Since

$$\left\langle q_{\phi(x)}, i_V f\right\rangle = \int_V f(t) q_{\phi(x)}(dt) \geq (1-\varepsilon) q_{\phi(x)}(V),$$

we have, by using (20),

$$\begin{aligned}\varepsilon m(V) &\geq \int_V f(x) m(dx) = \int_Q \left\langle q_{\phi(x)}, i_V f\right\rangle m(dx) \\ &\geq (1-\varepsilon)\int_Q q_{\phi(x)}(V) m(dx) = (1-\varepsilon) m(V).\end{aligned}$$

Since m is positive on open sets, we have that $\operatorname{supp} q_y = \phi^{-1}(y)$. The fact that the orbital morphism ϕ is consistent follows from the following fact: Let P be a conditional expectation satisfying all hypotesis of the Remark 3.2. Denote $P' : A' \to A'$ as follows $< P\xi, f > = < \xi, Pf >$. Then P' is a $*$-homomorphism. □

4. Differential structure on the hypergroup constructed from a Lie group and a conditional expectation

Let G be a Lie group. We regard it with the structure of a locally compact hypergroup as in Remark 2.2. By $A^\infty = C^\infty(G)$, we denote the algebra of infinitely differentiable functions on G, $\mathfrak{g}$ denotes the Lie algebra of G, and $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. We regard $U(\mathfrak{g})$ as a linear space spanned by the left invariant differential operators X^α on A^∞, where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ and

$$(X^\alpha f)(g) = \partial^{\alpha_1}_{t_1^{\alpha_1}} \dots \partial^{\alpha_m}_{t_m^{\alpha_m}} f(g e^{t_1 X_1} \dots e^{t_m X_m})\big|_{t_1 = \dots = t_m = 0}, \quad X_1, \dots, X_m \in \mathfrak{g}.$$

The universal enveloping algebra is considered with a Hopf algebra structure, $(U(\mathfrak{g}), \Delta', \epsilon', S)$ [16], where $\Delta' \colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $S \colon U(\mathfrak{g}) \to U(\mathfrak{g})$, and $\epsilon' \colon U(\mathfrak{g}) \to \mathbb{C}$ are given on $\mathfrak{g}$ by

(21) $$\Delta' X = 1 \otimes X + X \otimes 1, \quad S(X) = -X, \quad \epsilon'(X) = 0, \quad (X \in \mathfrak{g}),$$

and then extended, correspondingly, to $U(\mathfrak{g})$. For $X^\alpha \in U(\mathfrak{g})$ and $f \in A^\infty$, $\langle X^\alpha, f\rangle$ denotes the evaluation $(X^\alpha f)(e)$. It is well known [16] that

$$\langle X^\alpha Y^\beta, f\rangle = \langle X^\alpha \otimes Y^\beta, \Delta(f)\rangle, \tag{22}$$

where $X^\alpha, Y^\beta \in U(\mathfrak{g})$, $f \in C^\infty$.

We can also identify $U(\mathfrak{g})$ with the algebra of distributions having support in $\{e\}$ with convolution as the product [17].

Let P be a conditional expectation on the C^*-algebra A_0 satisfying the conditions of Theorem 3.1. Assuming that P gives rise to a continuous mapping $P\colon A^\infty \to A^\infty$ and following [18], one can define right generators X_Q^α, $\alpha \in \mathbb{N}^m$, of the hypergroup Q acting on functions in $P(A^\infty)$ as follows:

$$(X_Q^\alpha f)(g) = D_h^\alpha(\tilde{\Delta} f)(g,h)\big|_{h=e} = \big((\mathrm{id} \times \epsilon) \circ (\mathrm{id} \times D^\alpha) \circ \tilde{\Delta}\big)(f)(g), \quad f \in P(A^\infty). \tag{23}$$

The algebra $\mathfrak{B}$ generated by all right generators is called an *infinitesimal algebra* of the hypergroup Q.

The main purpose of this section is to give a description of the algebra $\mathfrak{B}$. As follows from Theorem 3.8, it is sufficient to consider the case where the conditional expectation P is counital and the case of a double coset hypergroup, that is, where $P = P^l \circ P^r$ as in Theorem 3.8.

4.1. **The case of a counital conditional expectation.** Let P be counital. Everywhere in this subsection, we also assume that P satisfies the following condition:

(**B**) if $e \notin \operatorname{supp} f$ for $f \in \mathcal{C}_0^\infty(G)$, then $e \notin \operatorname{supp} P(f)$.

Condition (**B**) is satisfied for counital conditional expectations constructed from counital orbital morphisms and, in particular, for expectations related to the Delsart construction, see Proposition 4.2.

Lemma 4.1. *Let $P'\colon U(\mathfrak{g}) \to U(\mathfrak{g})$ be defined by*

$$\langle P'(X^\alpha), f\rangle = \langle X^\alpha, P(f)\rangle, \quad X^\alpha \in U(\mathfrak{g}), \quad f \in A^\infty. \tag{24}$$

Then P' satisfies the following:

$$P' \circ P' = P', \tag{25}$$

$$P'\big(P'(X^\alpha) X^\beta P'(X^\gamma)\big) = P'(X^\alpha) P'(X^\beta) P'(X^\gamma), \quad X^\alpha, X^\beta, X^\gamma \in U(\mathfrak{g}), \tag{26}$$

$$(P' \otimes \mathrm{id}) \circ \Delta' \circ P' = (P' \otimes P') \circ \Delta' = (\mathrm{id} \otimes P') \circ \Delta' \circ P'. \tag{27}$$

Proof. First of all, let us show that P' is well defined. It is clear that $P'(X^\alpha)$ is a distributions for $X^\alpha \in U(\mathfrak{g})$. Let us show that $\operatorname{supp} P'(X^\alpha) = \{e\}$. Let $g \neq e$. Choose a neighborhood U_g of the point g such that $e \notin U_g$. Then, for all $f \in A^\infty$ such that $\operatorname{supp} f \subset U_g$, we have that $e \notin \operatorname{supp} P(f)$ by condition (**B**). Then $\langle P'(X^\alpha), f\rangle = \langle X^\alpha, P(f)\rangle = 0$, hence $e \notin \operatorname{supp} P'(X^\alpha)$, and $P'(X^\alpha) \in U(\mathfrak{g})$.

Property (25) directly follows from $P^2 = P$.

Let us prove (26). Let $X^\alpha, X^\beta, X^\gamma \in U(\mathfrak{g})$ and $f \in A^\infty$. Using (6) we get

$$\begin{aligned}\big\langle P'\big(P'(X^\alpha) X^\beta P'(X^\gamma)\big), f\big\rangle &= \langle P'(X^\alpha) X^\beta P'(X^\gamma), P(f)\rangle \\ &= \big\langle P'(X^\alpha) \otimes X^\beta \otimes P'(X^\gamma), \big((\Delta \times \mathrm{id}) \circ \Delta\big)\big(P(f)\big)\big\rangle \\ &= \big\langle X^\alpha \otimes X^\beta \otimes X^\gamma, \big((P \times P \times P) \circ (\Delta \times \mathrm{id}) \circ \Delta\big)(f)\big\rangle \\ &= \langle P'(X^\alpha) \otimes P'(X^\beta) \otimes P'(X^\gamma), f\rangle,\end{aligned}$$

which proves (26).

Finally, consider (27). For $X^\alpha \in U(\mathfrak{g})$ and $f_1, f_2 \in A^\infty$, we have

$$\begin{aligned}\big\langle\big((P'\otimes \mathrm{id})\circ\Delta'\big)(P'(X^\alpha)), f_1\otimes f_2\big\rangle &= \langle P'(X^\alpha), P(f_1)f_2\rangle \\ &= \langle X^\alpha, P(P(f_1)f_2)\rangle = \langle X^\alpha, P(f_1)P(f_2)\rangle \\ &= \big\langle\big((P'\otimes P')\circ\Delta'\big)(X^\alpha), f_1\otimes f_2\big\rangle.\end{aligned}$$

the second identity in (27) is proved similarly. □

Proposition 4.2. *Let $\phi\colon G\to Q$ be a counital orbital morphism from a Lie group G to hypergroup a Q and $(Pf)(g) =< q_{\phi(g)}, f >$ be the corresponding conditional expectation. Then the condition (**B**) is satisfied.*

Proof. It is sufficient to prove that $\operatorname{supp}(P'X^\alpha) = \{e\}$. Let $g \neq e$. Then there exists an open neighborhood U_g of g such that $e \notin U_g$ and for all functions $f \in C_0^\infty(U_g)$ we have that $e \notin \operatorname{supp}(Pf)$. Indeed, since ϕ is an open continuous mapping, $\phi(U_g)$ is open set. Therefore $\operatorname{supp}(Pf) \subset \phi(U_g)$. But $\phi(e) \notin \phi(U_g)$ since $\phi^{-1}(\phi(e)) = \{e\}$. Therefore, $e \notin \operatorname{supp}(Pf)$. □

Denote $\mathfrak{A} = P'(U(\mathfrak{g}))$. It follows from (26), (27) that $\mathfrak{A}$ is a subalgebra of $U(\mathfrak{g})$. Moreover, there is a comultiplication $\tilde{\Delta}'$ defined by

$$\tilde{\Delta}' = (P'\otimes P')\circ\Delta'. \tag{28}$$

Theorem 4.3. *The algebra $\mathfrak{A}$ is isomorphic to the infinitesimal algebra $\mathfrak{B}$ of the hypergroup Q.*

Proof. Let $f \in A^\infty$, $X^\alpha \in U(\mathfrak{g})$, and $g \in G$. Then

$$\begin{aligned}(X_Q^\alpha f)(g) &= \big((\mathrm{id}\otimes\epsilon)\circ(\mathrm{id}\otimes D^\alpha)\circ(P\otimes P)\circ\Delta\big)(f)(g) \\ &= \big((\mathrm{id}\otimes\epsilon\circ P)\circ(\mathrm{id}\otimes D^\alpha)\circ(P\otimes \mathrm{id})\circ\Delta\circ P(f)\big)(g) \\ &= \big((P\otimes\epsilon)\circ(1\otimes D^\alpha)\circ\Delta\big)(Pf)(g) = \big(P(X^\alpha(Pf))\big)(g).\end{aligned}$$

Hence, the generators of the hypergroup Q and generators of the group G are related via

$$X_Q^\alpha = P\circ X^\alpha\circ P. \tag{29}$$

Identity (29) permits to define a linear map $\lambda\colon \mathfrak{B}\to\mathfrak{A}$ by

$$\lambda(X_Q^\alpha) = P'(X^\alpha). \tag{30}$$

It is clear that the map λ is invertible. Let us show that it is a homomorphism. Note that

$$\begin{aligned}X_Q^\alpha X_Q^\beta &= (\mathrm{id}\otimes\varepsilon)(\mathrm{id}\otimes D^\alpha)(P\otimes P)\Delta(\mathrm{id}\otimes\varepsilon)(\mathrm{id}\otimes D^\beta)(P\otimes P)\Delta \\ &= (\mathrm{id}\otimes\varepsilon\otimes\varepsilon)(\mathrm{id}\otimes D^\alpha\otimes\mathrm{id})(P\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\circ P\otimes\mathrm{id})(P\otimes D^\beta)\Delta\circ P \\ &= (P\otimes\varepsilon\otimes\varepsilon)(\mathrm{id}\otimes D^\alpha\otimes D^\beta)(\Delta\circ P\otimes\mathrm{id})\Delta\circ P \\ &= (P\otimes\varepsilon)(\mathrm{id}\otimes D^\alpha)\Delta\circ P(\mathrm{id}\otimes\varepsilon)(\mathrm{id}\otimes D^\beta)\Delta\circ P.\end{aligned}$$

Hence, $X_Q^\alpha X_Q^\beta = PX^\alpha PX^\beta P$. Therefore

$$\lambda(X_Q^\alpha X_Q^\beta) = P'(X^\alpha(P'(X^\beta))) = P'(X^\alpha)P'(X^\beta) = \lambda(X_Q^\alpha)\lambda(X_Q^\beta),$$

i.e. λ is a homomorphism. □

4.2. **The case where the conditional expectation is defined by the double coset construction.** Here $\mathfrak{B}$ still denotes the infinitesimal algebra of the hypergroup and we keep the notations of Theorem 3.1. As before $\mathfrak{g}$ denotes the Lie algebra of the Lie group G and we denote by $\mathfrak{h}$ the Lie algebra of H. The corresponding universal enveloping algebras are denoted by $U(\mathfrak{g})$ and $U(\mathfrak{h})$. For a set $S \subset U(\mathfrak{g})$, by S^H we denote a subset of S of all H-invariant elements, that is,

$$S^H = \{s \in S : \mathrm{Ad}_h(s) = s,\ h \in H\},$$

where Ad is the restriction to H of the adjoint action of G on $U(\mathfrak{g})$ [16].

The following theorem has been proved in [7].

Theorem 4.4. *Let $I = \mathfrak{h}U(\mathfrak{g})$. Then I^H is an ideal in $U(\mathfrak{g})^H$ and $\mathfrak{B} \approx U^H/I^H$.*

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