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ELIMINATION OF JACOBI EQUATION IN EXTREMAL VARIATIONAL PROBLEMS

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Abstract. It is shown that the extremal problem for the one–dimensional Euler–Lagrange variational functional in $C^1[a;b]$ under a strengthened Legendre condition can be solved without using the Jacobi equation. In this case, exactly one of the two possible cases requires a restriction to the length of $[a;b]$, defined only by the form of the integrand. The result is extended to the case of compact extremum in $H^1[a;b]$.

0. Introduction

The classical scheme of a study of a local extremum for the one–dimensional Euler–Lagrange variational functional

$$\Phi(y) = \int_a^b f(x, y, y')dx \mapsto \text{extr} \quad (y \in C^1[a;b])$$

at an extremal point y assumes [1], [2] checking the strengthened Legendre condition $f_{y'y'}(x, y, y') \neq 0$ and the Jacobi condition $U(x) \neq 0$ $(a < x \leq b)$ for the Jacobi equation

$$-\frac{d}{dx}\Big[f_{y'y'}(x, y, y')U'\Big] + \Big[-\frac{d}{dx}\big(f_{yy'}(x, y, y')\big) + f_{y^2}(x, y, y')\Big]U = 0$$
$$(U(a) = 0,\ U'(a) = 1).$$

The second step is the most laborious, it requires to solve a fairly complicated equation for obtaining, actually, very little information about the behavior of the solution $U(x)$.

Moreover, the initial conditions $U(a) = 0$, $U'(a) = 1$, as a consequence of the Jacobi condition, is automatically satisfied near a. The question is only what is the length of a suitable interval ?

Note that all the classical sufficiency theorems in the theory of calculus of variations assume the Legendre–Jacobi conditions. Because of this fact, G. M. Ewing mentioned in [3] that there is a gap between the necessary and sufficient conditions for optimality. He showed that it is possible to partially close this gape by adding a *penalty term*. However, this technique may not hold in general.

In more recent years, a study of the second order conditions for optimality in the theory of calculus of variations and optimal control has provided an extensive literature (see e.g. [4], [5], [6], [7], [8] and references therein). A known approach by M. R. Hestenes [9] treats explicitly with the positivity of the second variation and it is implicitly based on the concept of a directionally convergent sequence of trajectories. A generalization of this method which covers optimal control problems can be found in the works by J. F. Rosenblueth and G. S. Licea [10]–[14]. Here a proof of sufficiency modifies the classical necessary Weierstrass condition under certain supplementary assumptions on

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the functions delimiting the problem and it makes no use of Hamilton–Jacobi theory and conjugate points.

Recently, in [15], by applying a similar technique G. S. Licea obtained a new sufficiency theorem which does not include the strengthened Legendre–Clebsch condition and Jacobi condition. This result assumes, with respect to a given extremal, the usual Legendre–Clebsch condition, the positivity of the second variation and the special conditions related to the Weierstrass excess function.

In contrast to the above mentioned approach, the aim of the present work is to show that the interval satisfying the Jacobi condition can be chosen depending only on the form of the integrand f and not depending on a concrete extremal. It enables us to exclude Jacobi condition under preserving the strengthened Legendre condition and without any supplementary condition to second variation, Weierstrass function, etc. More precisely, the main result (Theorem 1.1) distinguishes two cases depending on the range of the coefficients in the Jacobi equation. For the first case, an extremum is guaranteed without any restriction to the length of $[a;b]$ being imposed, for the second one, such a restriction is given. The result above remains valid when passing to the case of finding a compact extremum in the Sobolev space $H^1[a;b]$.

The first part of the work deals with elimination of the Jacobi equation in case of zero extremal in $C^1[a;b]$. The second part contains a quadratic estimate of tending Φ to a minimal value via the norm of y in $H^1[a;b]$. The third part contains a passage to the case of an arbitrary C^2–smooth extremal in $C^1[a;b]$ and the last, the fourth part contains a passage to the case of a compact extremum in $H^1[a;b]$.

1. Elimination of the Jacobi condition: the case of zero extremal

Let us consider the classical Euler–Lagrange variational functional

$$\Phi(y) = \int_a^b f(x,y,y')dx \quad (y \in C^1[a;b],\, y(a)=y(b)=0,\, f \in C^2,\, f_{yz} \in C^1). \tag{1.1}$$

We are going to show that, if the Euler–Lagrange variational equation and the strengthened Legendre condition at zero are satisfied, then the functional (1.1) *always* attains a strong local extremum at zero. However, in addition, two different possible cases defined by the form of the integrand f, arise. One of the cases assumes a restriction on the length of $[a;b]$, in the second case, any restriction is absent.

So, let us split the integrand $f(x,y,z)$ into two terms:

$$\begin{aligned} f_1(x,y,z) &= f(x,y,z) - f(x,0,0) - [f_y(x,0,0)\cdot y + f_z(x,0,0)\cdot z] \\ &\quad - \frac{1}{2}\left[f_{y^2}(x,0,0)\cdot y^2 + 2f_{yz}(x,0,0)\cdot yz + \lambda\cdot f_{z^2}(x,0,0)\cdot z^2\right] \quad (0<\lambda<1), \\ f_2(x,y,z) &= f(x,y,z) - f_1(x,y,z) = f(x,0,0) + [f_y(x,0,0)\cdot y + f_z(x,0,0)\cdot z] \\ &\quad + \frac{1}{2}\left[f_{y^2}(x,0,0)\cdot y^2 + 2f_{yz}(x,0,0)\cdot yz + \lambda\cdot f_{z^2}(x,0,0)\cdot z^2\right]. \end{aligned}$$

Set, respectively,

$$\Phi_i(y) = \int_a^b f_i(x,y,y')dx \quad (i=1,2); \quad \Phi(y) = \Phi_1(y) + \Phi_2(y).$$

1) Let us investigate Φ_1 for a local extremum (minimum, for definiteness) at zero with the help of the Euler–Lagrange, Legendre and Jacobi conditions.

(i) *The Euler–Lagrange equation.* Because

$$\begin{aligned}\big(f_{1,y}(x,y,z) &= f_y(x,y,z) - f_y(x,0,0) - f_{y^2}(x,0,0)\cdot y - f_{yz}(x,0,0)\cdot z\big)\\ &\Rightarrow \big(f_{1,y}(x,0,0)=0\big),\\ \big(f_{1,z}(x,y,z) &= f_z(x,y,z) - f_z(x,0,0) - f_{yz}(x,0,0)\cdot y - \lambda\cdot f_{z^2}(x,0,0)\cdot z\big)\\ &\Rightarrow \big(f_{1,z}(x,0,0)=0\big),\end{aligned}$$

the Euler–Lagrange equation for Φ_1 at zero,

$$f_{1,y}(x,0,0) - \frac{d}{dx}\big[f_{1,z}(x,0,0)\big] = 0,$$

holds automatically, i.e., $y_0(x)\equiv 0$ is an extremal of the functional Φ_1.

(ii) *The strengthened Legendre condition.* Because

$$\big(f_{1,z^2}(x,y,z) = f_{z^2}(x,y,z) - \lambda\cdot f_{z^2}(x,0,0)\big) \Rightarrow \big(f_{1,z^2}(x,0,0) = (1-\lambda)\cdot f_{z^2}(x,0,0)\big),$$

under the additional requirement

$$p(x) := f_{z^2}(x,0,0) > 0 \quad (a\le x\le b), \tag{1.2}$$

the strengthened Legendre condition for a strong minimum at zero holds.

(iii) *The Jacobi equation and the Jacobi condition.* Because

$$\big(f_{1,yz}(x,y,z) = f_{yz}(x,y,z) - f_{yz}(x,0,0)\big) \Rightarrow \big(f_{1,yz}(x,0,0)=0\big),$$
$$\big(f_{1,y^2}(x,y,z) = f_{y^2}(x,y,z) - f_{y^2}(x,0,0)\big) \Rightarrow \big(f_{1,y^2}(x,0,0)=0\big),$$

the Jacobi equation for Φ_1 at zero takes the form

$$\begin{aligned}-\frac{d}{dx}\Big[(1-\lambda)\cdot f_{z^2}(x,0,0)U'\Big] &+ \Big[-\frac{d}{dx}\big(f_{1,yz}(x,0,0)\big) + f_{1,y^2}(x,0,0)\Big]U\\ &= -\frac{d}{dx}\Big[(1-\lambda)p(x)U'\Big] = 0 \quad (U(a)=0,\ U'(a)=1).\end{aligned}$$

Hence, in view of condition (1.2), the required result

$$\left(U(x) = p(a)\cdot\int_a^x \frac{dt}{p(t)}\right) \Rightarrow \Big(U(x)\neq 0 \text{ for } a<x\le b\Big)$$

holds, i.e., the strengthened Jacobi condition at zero for a strong minimum of Φ_1 takes place. Thus, under the condition (1.2), Φ_1 attains a strong local minimum at zero.

2) Let us study now Φ_2 for a local extremum at zero immediately. Note at first that $\Phi_2(0)=\Phi(0)$.

(i) Suppose that the Euler–Lagrange equation for Φ at zero,

$$f_y(x,0,0) - f_{xz}(x,0,0) = 0 \quad (a\le x\le b) \tag{1.3}$$

is satisfied. Then, by integrating by parts, we get

$$\begin{aligned}\Phi_2(y) &= \int_a^b f(x,0,0)dx + \int_a^b \left[f_y(x,0,0)\cdot y + f_z(x,0,0)\cdot y'\right]dx\\ &+ \int_a^b \left[\frac{1}{2}f_{y^2}(x,0,0)\cdot y^2 + f_{yz}(x,0,0)\cdot yy'\right]dx + \frac{\lambda}{2}\cdot\int_a^b \cdot f_{z^2}(x,0,0)\cdot y'^2dx\\ &= \Phi_2(0) + \left[\int_a^b \left(f_y - f_{xz}\right)(x,0,0)dx + f_z(x,0,0)\cdot y\Big|_a^b\right]\\ &+ \left[\frac{1}{2}\int_a^b \left(f_{y^2} - f_{xyz}\right)(x,0,0)\cdot y^2dx + \frac{1}{2}f_{yz}(x,0,0)\cdot y^2\Big|_a^b\right] + \frac{\lambda}{2}\int_a^b p(x)\cdot y'^2dx.\end{aligned}$$

From here, denoting by

$$q(x) := \left(f_{y^2} - f_{xyz}\right)(x,0,0),$$

it follows that

$$\Phi_2(y) = \Phi_2(0) + \frac{1}{2}\int_a^b \left[\lambda \cdot p(x) \cdot y'^2 + q(x) \cdot y^2\right] dx. \tag{1.4}$$

(ii) Denote

$$p := \min_{a \le x \le b} p(x) > 0, \quad q := \min_{a \le x \le b} q(x) \tag{1.5}$$

and consider, at first, the case $q \ge 0$. Then

$$\lambda p(x)y'^2 + q(x)y^2 \ge \lambda p \cdot y'^2 + q \cdot y^2 > 0 \quad \text{as} \quad y' \ne 0,$$

whence, in view of (1.4), the inequality

$$\Phi_2(y) > \Phi_2(0) \quad \text{as} \quad y(x) \ne 0$$

follows. Thus, in this case, Φ_2 attains a strong absolute minimum at zero. Hence, in view of what has been proved in i.1), Φ attains a strong local minimum at zero (without any restriction on the length of $[a;b]$).

(iii) Let us consider now the case $q < 0$. Then, using Friederichs inequality (see, e.g., [16], Ch. 18), it follows that

$$\begin{aligned} \Phi_2(y) - \Phi_2(0) &= \frac{1}{2}\int_a^b \left[\lambda \cdot p(x) \cdot y'^2 + q(x) \cdot y^2\right] dx \\ &\ge \frac{1}{2}\int_a^b \left[\lambda \cdot p \cdot y'^2 - |q| \cdot y^2\right] dx \ge \frac{1}{2}\int_a^b \left[\lambda \cdot p \cdot y'^2 - \frac{16(b-a)^2}{\pi^2}|q| \cdot y'^2\right] dx \\ &= \frac{1}{2}\left(\lambda \cdot p - \frac{16(b-a)^2}{\pi^2}|q|\right) \cdot \int_a^b y'^2 dx. \end{aligned} \tag{1.6}$$

Let us require that the coefficient in front of the last integral in (1.6) be strictly positive,

$$\left(\lambda \cdot p - \frac{16(b-a)^2}{\pi^2}|q| > 0\right) \Leftrightarrow \left(b - a < \frac{\pi}{4}\sqrt{\frac{\lambda p}{|q|}}\right). \tag{1.7}$$

It follows from (1.6) and (1.7) that $\Phi_2(y) > \Phi_2(0)$ as $y \ne 0$, i.e., Φ_2 attains a strong absolute minimum at zero and, hence, in virtue of i.1), Φ attains a strong local minimum at zero under the restriction (1.7) on the length of $[a;b]$.

Finally, passing to the limits in (1.7) as $\lambda \to 1 - 0$, the last statement can be extended to the case of the estimate of the length of $[a;b]$ not depending on λ

$$b - a < \frac{\pi}{4}\sqrt{\frac{p}{|q|}}.$$

So, the following was proved.

Theorem 1.1. *Let the variational functional* (1.1) *satisfy, at zero, the Euler–Lagrange equation* (1.3) *under the conditions $y(a) = y(b) = 0$. Then, with the notation* (1.5),

1) *for $p > 0$, $q \ge 0$, $\Phi(y)$ attains a strong local minimum at zero (without any restriction on the length of $[a;b]$);*
2) *for $p > 0$, $q < 0$, with the restriction on the length of $[a;b]$,*

$$b - a < \frac{\pi}{4}\sqrt{\frac{p}{|q|}}, \tag{1.8}$$

$\Phi(y)$ attains a strong local minimum at zero as well.

2. Quadratic estimation from below of tending Φ to minimum at zero

It is easy to see that the estimate (1.8) in Theorem 1.1 is not optimal. For example, the generalized harmonic oscillator

$$\Phi(y) = \int_0^T (py'^2 - qy^2)dx \qquad (p > 0,\ q > 0)$$

on the zero extremal reduces to the Jacobi equation

$$pU'' + qU = 0 \qquad (U(0) = 0,\ U'(0) = 1)$$

having the solution

$$U(x) = \sqrt{\frac{p}{q}} \sin \sqrt{\frac{q}{p}} x\,,$$

satisfying Jacobi condition $U(x) \neq 0$ as $0 < x < T$ for $T < \pi\sqrt{\frac{p}{q}}$.

At the same time, the estimate (1.8) for this case leads to the inequality $T < \frac{\pi}{4}\sqrt{\frac{p}{q}}$. However, as it is easily seen, an advantage of estimate (1.8) consists of the possibility to get a useful quadratic estimate from below for $\Phi(y)$ tending to the minimal value by means of the norm of y in the Sobolev space $H^1[a;b]$.

1) First, let us consider the case $p > 0$, $q > 0$. The equality (1.4) implies

$$\Phi_2(y) - \Phi_2(0) \geq \frac{1}{2}\min(p,q) \cdot \int_a^b (y'^2 + y^2)dx = \frac{1}{2}\min(p,q) \cdot \|y\|^2_{H^1[a;b]}.$$

Since $\Phi(y) - \Phi(0) \geq \Phi_2(y) - \Phi_2(0)$ in a sufficiently small neighborhood of zero, then, given a neighborhood of zero, the inequality

$$\Phi(y) - \Phi(0) \geq \frac{1}{2}\min(p,q) \cdot \|y\|^2_{H^1[a;b]}$$

is true.

2) Let us pass to the case $p > 0$, $q < 0$. The inequality (1.6) leads to the estimate

$$\Phi_2(y) - \Phi_2(0) \geq \frac{1}{2}\left[p - \frac{16(b-a)^2}{\pi^2}|q|\right] \cdot \int_a^b y'^2 dx\ .$$

Since the Friederichs inequality implies

$$\int_a^b y'^2 dx \geq \frac{\pi^2}{\pi^2 + 16(b-a)^2} \cdot \|y\|^2_{H^1[a;b]}\ , \tag{2.1}$$

by combining the last two inequalities for a sufficiently small neighborhood of zero, under the conditions of inequality (1.6), we get

$$\Phi(y) - \Phi(0) \geq \frac{\pi^2 p - 16(b-a)^2|q|}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|^2_{H^1[a;b]}\,.$$

3) Note that the estimate (2.1) can be applied as well in the case $p > 0$, $q \geq 0$, whence the inequality

$$\Phi(y) - \Phi(0) \geq \frac{\pi^2 p}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|^2_{H^1[a;b]}$$

follows. This gives the following.

Theorem 2.1. *Under the conditions and notation of Theorem 1.1, the following statements are valid:*

1) *in the case* $p > 0$, $q > 0$, *in a sufficiently enough neighborhood of zero in* $C^1[a;b]$, *the estimate*

$$\Phi(y) - \Phi(0) \geq \frac{1}{2}\min(p,q) \cdot \|y\|^2_{H^1[a;b]}$$

holds;

2) *in the case* $p > 0$, $q \geq 0$, *in a sufficiently small neighborhood of zero in* $C^1[a;b]$, *the estimate*

$$\Phi(y) - \Phi(0) \geq \frac{\pi^2 p}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|^2_{H^1[a;b]}$$

holds;

3) *in the case* $p > 0$, $q < 0$, *in a sufficiently small neighborhood of zero in* $C^1[a;b]$, *under the condition of estimate* (1.8), *the estimate*

$$\Phi(y) - \Phi(0) \geq \frac{\pi^2 p - 16(b-a)^2|q|}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|^2_{H^1[a;b]}$$

holds.

3. Case of arbitrary C^2–smooth extremal in $C^1[a;b]$

Let us fix an arbitrary C^2–smooth function $y_0(x)$, $a \leq x \leq b$, and consider the question of eliminating the Jacobi condition for the local minimum of the variational functional (1.1) at the point $y_0(\cdot)$ under the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$.

To pass to the case (i.1) considered above of the zero extremal, it suffices to consider an auxiliary variational functional,

$$\widetilde{\Phi}(y) = \Phi(y + y_0) = \int_a^b f(x, y + y_0(x), y' + y_0'(x))dx =: \int_a^b \widetilde{f}(x, y, y')dx$$

$$(y(a) = y(b) = 0).$$

In this connection, the condition $y_0(\cdot) \in C^2$ guarantees fulfillment of the condition from (1.1) for the auxiliary integrand $\widetilde{f}$ and permits to apply Theorem 1.1 to $\widetilde{\Phi}$. A not complicated calculation shows the following.

Theorem 3.1. *Let the variational functional* (1.1) *satisfy, at a point* $y_0(\cdot) \in C^2[a;b]$, *the Euler–Lagrange equation*

$$f_y(x, y_0, y_0') - \frac{d}{dx}\big[f_z(x, y_0, y_0')\big] = 0\,. \tag{3.1}$$

Denote

$$p := \min_{a \leq x \leq b} f_{z^2}(x, y_0(x), y_0'(x));$$

$$q := \min_{a \leq x \leq b}\left[f_{y^2}(x, y_0(x), y_0'(x)) - \frac{d}{dx}\big(f_{yz}(x, y_0(x), y_0'(x))\big)\right].$$

Then, under the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$ *we have the following:*

1) *for* $p > 0$, $q \geq 0$, $\Phi(y)$ *attains a strong local minimum at* $y_0(\cdot)$ *(without any restriction on the length of* $[a;b]$*);*

2) *for* $p > 0$, $q < 0$, *and under the restriction*

$$b - a < \frac{\pi}{4}\sqrt{\frac{p}{|q|}} \tag{3.2}$$

on the length of $[a;b]$, $\Phi(y)$ *attains a strong local minimum at* $y_0(\cdot)$ *as well.*

Analogously, applying Theorem 2.1 to $\widetilde{\Phi}$ leads to a general quadratic estimate for tending Φ to a local minimum at y_0.

Theorem 3.2. *Under the conditions and notation of Theorem 3.1 we have the following:*

1) *for $p > 0$, $q > 0$, in some neighborhood of $y_0(\cdot)$ in $C^1[a;b]$, the estimate*

$$\Phi(y) - \Phi(y_0) \geq \frac{1}{2}\min(p,q) \cdot \|y\|^2_{H^1[a;b]} \tag{3.3}$$

holds;

2) *for $p > 0$, $q \geq 0$, in some neighborhood of $y_0(\cdot)$ in $C^1[a;b]$, the estimate*

$$\Phi(y) - \Phi(y_0) \geq \frac{\pi^2 p}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|^2_{H^1[a;b]} \tag{3.4}$$

holds;

3) *for $p > 0$, $q < 0$, under the restriction* (3.2) *on the length of* $[a;b]$*, in some neighborhood of $y_0(\cdot)$ in $C^1[a;b]$, the estimate*

$$\Phi(y) - \Phi(y_0) \geq \frac{\pi^2 p - 16(b-a)^2|q|}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|^2_{H^1[a;b]} \tag{3.5}$$

holds.

4. The case of compact extremum in $H^1[a;b]$

In the Hilbert–Sobolev space $W^{1,2}[a;b] = H^1[a;b]$ equipped with the norm

$$\|y\|^2_{H^1[a;b]} = \int_a^b (y^2 + y'^2)dx\ , \tag{4.1}$$

as it is well known, by virtue of I. V. Skrypnik theorem ([17], Ch. 11), the non-absolute local extrema of the variational functionals are practically absent. Note that in the present work the norm (4.1) has appeared above (Theorem 2.1, 3.2) in a natural way even for extremal problems in $C^1[a;b]$.

In our works [18]–[20] and in the works by E. V. Bozhonok [21]–[23] a general concept of *compact extremum* (or *K–extremum*) of a functional was studied (see, also, [24]). It has been shown there that the both classical necessary and sufficient conditions for a local extremum of the variational functional in $C^1[a;b]$ can be extended to the case of the K–extremum in $H^1[a;b]$. In this case, K–extrema inherit important properties of the local extrema and can be considered as an analog in the case of variational functionals in $H^1[a;b]$.

Definition 4.1. Let a real functional $\Phi : H \to \mathbb{R}$ be defined on a Hilbert space H. We say that Φ has a *compact minimum* (or *K–minimum*) at a point $y_0 \in H$ if, for each absolutely convex (a.c.) compact set $C \subset H$, the restriction of f to the subspace $(y_0 + span\, C)$ has a local minimum at y_0 with respect to the Banach norm $\|\cdot\|_C$ in $span\, C$ generated by C. In other words, for each a.c. compactum $C \subset H$ there exists $\varepsilon = \varepsilon(C) > 0$ such that $\varphi(y) \geq \varphi(y_0)$ as $y - y_0 \in \varepsilon \cdot C$.

The well posedness and the validity for the case of K–extremum of the variational functional (1.1) of the classical extreme conditions in C^1 (Euler–Lagrange equation, Legendre condition, Jacobi condition) require, as it was shown in [20], belonging coefficient $R(x,y,z)$ in the pseudoquadratic representation of the integrand f:

$$f(x,y,z) = P(x,y) + Q(x,y)\cdot z + \frac{1}{2}R(x,y,z)\cdot z^2$$

to an appropriate *dominated mixed smoothness space* C^2_{xy} (see [25], [26]). Namely, for the arbitrary compacta $C_x,\ C_y \subset \mathbb{R}$ the following property holds:

$$(x \in C_x\,,\ y \in C_y\,,\ -\infty < z < +\infty) \Rightarrow (R(x,y,z)\ \textit{is uniformly continuous and bounded, together with its first and second partial derivatives}).$$

Under the conditions above, the Euler–Lagrange equation, the Legendre condition, the strengthened Legendre condition, and the Jacobi condition for the Jacobi equation are extended to the case of the K–extremum in an arbitrary $W^{2,2}$–smooth point $y_0(\cdot) \in H^1[a;b]$. It enables us to extend the results of i.4 to the case of a K–minimum in $H^1[a;b]$. Let us give the corresponding formulations.

Theorem 4.1. *Let the variational functional* (1.1)*, at a* $W^{2,2}$*–smooth point*

$$y_0(\cdot) \in H^1[a;b],$$

satisfy the Euler–Lagrange equation (3.1)*, and* $R(x,y,z) \in C^2_{xy}$*. Then, under the conditions and the notation of Theorem 3.1, we have the following:*

1) *for* $p > 0$*,* $q \geq 0$*,* $\Phi(y)$ *attains a strong* K*–minimum at* $y_0(\cdot)$ *(without any restriction on the length of* $[a;b]$*);*
2) *for* $p > 0$*,* $q < 0$*, and under the restriction* (3.2) *on the length of* $[a;b]$*,* $\Phi(y)$ *attains a strong* K*–minimum at* $y_0(\cdot)$ *as well.*

Theorem 4.2. *Under the conditions and the notation of Theorem 4.1, we have the following:*

1) *for* $p > 0$*,* $q > 0$*, for each a.c. compactum* $C \subset H^1[a;b]$ *there exists* $\varepsilon = \varepsilon(C) > 0$ *such that the inclusion* $y - y_0 \in \varepsilon \cdot C$ *implies estimate* (3.3)*;*
2) *for* $p > 0$*,* $q \geq 0$*, for each a.c. compactum* $C \subset H^1[a;b]$ *there exists* $\varepsilon = \varepsilon(C) > 0$ *such that the inclusion* $y - y_0 \in \varepsilon \cdot C$ *implies estimate* (3.4)*;*
3) *for* $p > 0$*,* $q < 0$*, under the restriction* (3.2) *on the length of* $[a;b]$*, for each a.c. compactum* $C \subset H^1[a;b]$ *there exists* $\varepsilon = \varepsilon(C) > 0$ *such that the inclusion* $y - y_0 \in \varepsilon \cdot C$ *implies estimate* (3.5)*.*

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