Methods of Functional Analysis and Topology
Vol. 18 (2012), no. 1, pp. 4–18

ON THE COMPLETENESS OF GENERAL BOUNDARY VALUE PROBLEMS FOR 2×2 FIRST-ORDER SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

A. V. AGIBALOVA, M. M. MALAMUD, AND L. L. ORIDOROGA

Dedicated to the blessed memory of A. G. Kostyuchenko.

Abstract. Let $B = \operatorname{diag}(b_1^{-1}, b_2^{-1}) \neq B^*$ be a 2×2 diagonal matrix with $b_1^{-1} b_2 \notin \mathbb{R}$ and let Q be a smooth 2×2 matrix function. Consider the system

$$-iBy' + Q(x)y = \lambda y, \quad y = \operatorname{col}(y_1, y_2), \quad x \in [0,1],$$

of ordinary differential equations subject to general linear boundary conditions $U_1(y) = U_2(y) = 0$. We find sufficient conditions on Q and U_j that guaranty completeness of root vector system of the boundary value problem.

Moreover, we indicate a condition on Q that leads to a completeness criterion in terms of the linear boundary forms $U_j,\ j \in \{1,2\}$.

1. Introduction

Spectral theory of non-selfadjoint boundary value problems (BVP) for nth order ordinary differential equations (ODE)

$$y^{(n)} + q_1 y^{(n-2)} + \cdots + q_{n-1} y = \lambda^n y \tag{1.1}$$

on a finite interval takes its origin in the classical papers by Birkhoff [2] and Tamarkin [21]. They have introduced a concept of regular boundary conditions (BC) and investigated asymptotic behavior of the eigenvalues and the eigenfunctions of such problems for ODE. Moreover, they have proved that the system of root functions, i.e. eigenfunctions and associated functions (EAF) of the regular BVP is complete. Their results are also treated in classical monographs (see, for instance, [18, Section 2]).

Note however, that some natural and important boundary conditions are not regular. For instance, boundary value problem with separated boundary conditions is regular if and only if $n = 2l$, l is the number of BC at the left (right) endpoint of the interval. Later on, completeness of the system of EAF of such boundary value problems had been announced by M. V. Keldysh in his famous communication [7] and was proved by A. A. Shkalikov [20]. The completeness property of other non-regular BVP for nth order ordinary differential equations on $[0,1]$ has been investigated by A. G. Kostyuchenko, G. V. Radzievsky and A. A. Shkalikov ([9], [10]), A. P. Khromov [8], V. S. Rykhlov [19] and others.

Consider first-order system of ODE of the form

$$Ly := L(Q)y := \frac{1}{i} B \frac{dy}{dx} + Q(x)y = \lambda y, \quad y = \operatorname{col}(y_1, \ldots, y_n), \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 34L10; Secondary 47E05.

Key words and phrases. Boundary value problem, Dirac-type system, root vector system, completeness property, Phragmén-Lindelöf theorem.

We are grateful to Anton Lunyov for careful reading of the manuscript and making numerous useful remarks.

where B is a non-singular diagonal $n \times n$ matrix

$$B = \operatorname{diag}(b_1^{-1} I_{n_1}, \dots, b_r^{-1} I_{n_r}) \in \mathbb{C}^{n\times n}, \quad n = n_1 + \dots + n_r, \tag{1.3}$$

with complex entries satisfying $b_j \neq b_k$ for $j \neq k$, and $Q = (Q_{jk})_{j,k=1}^r$ is a potential matrix with respect to the orthogonal decomposition $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_r}$, $Q(\cdot) \in L^2([0,1];\mathbb{C}^{n\times n})$.

Systems (1.2) form more general object than ordinary differential equations. They are of significant interest in some theoretical and practical questions. More precisely, for $n = 2m$, $B = \operatorname{diag}(I_m, -I_m)$ and $Q_{11} = Q_{22} = 0$, the system (1.2) is equivalent to the Dirac system [17]. For $r = n$ and $b_j = \exp(2\pi i j/n)$, an nth-order differential equation is reduced to the system (1.2) (see [13]).

To obtain a BVP, we adjoin to equation (1.2) boundary conditions

$$Cy(0) + Dy(1) = 0, \quad C = (c_{jk}), \quad D = (d_{jk}) \in \mathbb{C}^{n\times n}. \tag{1.4}$$

Denote by $L_{C,D} := L_{C,D}(Q)$ the operator in $L^2([0,1];\mathbb{C}^n)$ associated with the BVP (1.2)–(1.4). Moreover, in what follows we impose the maximality condition

$$\operatorname{rank}(C\ D) = n, \tag{1.5}$$

which is equivalent to $\ker(CC^* + DD^*) = \{0\}$.

Recall some results from [16]. For this we need the following construction. Let $A = \operatorname{diag}(a_1, \dots, a_n)$ be a diagonal matrix with entries a_k (not necessarily distinct) that are not lying on the imaginary axis, $\operatorname{Re} a_k \neq 0$. Starting from arbitrary matrices $C, D \in \mathbb{C}^{n\times n}$, we define the auxiliary matrix $T_A(C,D)$ as follows:

- if $\operatorname{Re} a_k > 0$, then the kth column in the matrix $T_A(C,D)$ coincides with the kth column of the matrix C,
- if $\operatorname{Re} a_k < 0$, then the kth column in the matrix $T_A(C,D)$ coincides with the kth column of the matrix D.

Definition 1.1. *The boundary conditions* (1.4) *are called* weakly B-regular *(or, simply, weakly regular) if there exist three complex numbers z_1, z_2, z_3 satisfying the following conditions:*

(a) the origin is an interior point of the triangle $\triangle_{z_1 z_2 z_3}$;
(b) $\det T_{z_j B}(C,D) \neq 0$ *for* $j \in \{1,2,3\}$.

Theorem 1.2. [14, 16, Theorem 1.2]. *Let $Q \in L^1[0,1] \otimes \mathbb{C}^{n\times n}$ and let boundary conditions* (1.4) *be weakly B-regular. Then the system of root functions of the BVP* (1.2)–(1.4) *(of the operator $L_{C,D}(Q)$) is complete and minimal in $L^2[0,1] \otimes \mathbb{C}^n$. Moreover, the root vector system of the operator $L_{C,D}(Q)^*$ is complete and minimal too.*

In the case of $B = \operatorname{diag}(b_1^{-1}, b_2^{-1}) = B^*$, more general BVP that include irregular and even degenerate boundary conditions were investigated in papers [15] and [16]. The later publications have been inspired by the results of [12] on the completeness property of boundary value problems for Sturm–Liouville operators with degenerate BC.

In this connection we also mention the recent papers by F. Gesztesy and V. Tkachenko [6] and P. Djakov and B. Mityagin [4], [5], devoted to Riesz basis property for boundary value problems for Sturm–Liouville and Dirac operators.

In what follows we consider only 2×2-systems

$$-iBy' + Q(x)y = \lambda y, \quad y = \operatorname{col}(y_1, y_2), \quad x \in [0,1], \tag{1.6}$$

with the matrix $B = \operatorname{diag}(b_1^{-1}, b_2^{-1}) \neq B^*$, assuming that $b_1^{-1} b_2 \notin \mathbb{R}$,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad Q(\cdot) \in L^2([0,1];\mathbb{C}^{2\times 2}). \tag{1.7}$$

Without loss of generality one can assume that $Q_{11} = Q_{22} = 0$ (see Lemma 2.6 below). To the system (1.6) we join the boundary conditions

$$U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1,2\}. \tag{1.8}$$

We put $A_{jk} := \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}$ and $J_{jk} := \det A_{jk},\ j,k \in \{1,2,3,4\}$.

It follows from Definition 1.1 that boundary conditions (1.8) are weakly B-regular if and only if one of the following conditions is satisfied:

$$(i) \quad J_{14}J_{23} \neq 0 \quad \text{or} \quad (ii) \quad J_{12}J_{34} \neq 0. \tag{1.9}$$

If conditions (1.9) are violated then boundary conditions (1.8) are equivalent (see Lemma 2.7 below) either to the boundary conditions

$$\begin{cases} U_1(y) := y_1(0) = 0, \\ U_2(y) := a_{22}y_2(0) + a_{23}y_1(1) + a_{24}y_2(1) = 0 \end{cases} \tag{1.10}$$

or to the boundary conditions

$$\begin{cases} y_1(0) + \widetilde{a}_{12}y_2(0) = 0, \\ y_1(1) + \widetilde{a}_{22}y_2(0) = 0, \end{cases} \tag{1.11}$$

where $\widetilde{a}_{12}\widetilde{a}_{22} \neq 0$.

Despite of the fact that conditions (1.11) are not weakly regular, the following result on completeness property holds.

Theorem 1.3. [15, 16, Theorem 6.1]. *Let $B = \operatorname{diag}(b_1^{-1}, b_2^{-1})$, $b_1^{-1}b_2 \notin \mathbb{R}$, $Q \in L^1[0,1] \otimes \mathbb{C}^{2\times 2}$ and $\widetilde{a}_{12}\widetilde{a}_{22} \neq 0$. Then the root vector system of the problem* (1.6), (1.11) *(of the operator $L_{C,D}(Q)$) is complete and minimal in $L^2\big([0,1];\mathbb{C}^2\big)$.*

It is also shown in [16] that in the case $B = \operatorname{diag}(b_1^{-1}, b_2^{-1}) \neq B^*$ *weak B-regularity of boundary conditions* (1.4) *is equivalent to the completeness of both operators $L_{C,D}(0)$ and $L_{C,D}(0)^*$ with $Q = 0$.*

Since boundary conditions (1.11) are not weakly B-regular, *the root vector system of the corresponding adjoint operator $L_{C,D}(Q)^*$ is not complete in general.* For instance, the operator $L_{C,D}(0)^*$ with $Q = 0$ is not complete.

In this paper we study completeness of the root vectors of system (1.6) subject to boundary conditions that are not covered by Theorems 1.2 and 1.3. In particular, we complete Theorem 1.3 by investigating completeness property of the adjoint operator $L_{C,D}(Q)^*$. More precisely, we investigate the BVP (1.6), (1.10), assuming that

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix}, \quad b \in \mathbb{C}\backslash\mathbb{R}, \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \tag{1.12}$$

and $Q_{12}(\cdot)$ and $Q_{21}(\cdot)$ admit an analytic continuation to the disk $\mathbb{D}_R$ (in short $Q(\cdot) \in A(\mathbb{D}_R) \otimes \mathbb{C}^{2\times 2}$) for some sufficiently large R.

Now we can state the main results on the completeness property of BVP (1.6), (1.10), (1.12). As it is already mentioned, completeness depends on a potential matrix $Q(\cdot)$.

Theorem 1.4. *Let $a_{23}a_{24} \neq 0$ and $Q_{21}(1) \neq 0$. Then the root vector system of the boundary value problem* (1.6), (1.10), (1.12) *(the operator $L_{C,D}(Q)$) is complete and minimal in $L_2\left([0,1];\mathbb{C}^2\right)$. Moreover, the adjoint operator $L_{C,D}(Q)^* := (L_{C,D}(Q))^*$ is complete and minimal too.*

Theorem 1.5. *Let $a_{23} = 0$, $a_{24} \neq 0$ and $Q_{12}(0)Q_{21}(1) \neq 0$. Then both the operator $L_{C,D}(Q)$ and its adjoint $L_{C,D}(Q)^*$ are complete and minimal in $L_2\left([0,1];\mathbb{C}^2\right)$.*

Theorem 1.6. *Let $a_{23} \neq 0$, $a_{24} = 0$ and $Q_{21}(0)Q_{21}(1) \neq 0$. Then both the operator $L_{C,D}(Q)$ and its adjoint $L_{C,D}(Q)^*$ are complete and minimal in $L_2\left([0,1];\mathbb{C}^2\right)$.*

Finally, we indicate a condition on a potential matrix $Q(\cdot)$ that leads to completeness criterion for $L_{C,D}(Q)$ in terms of boundary conditions.

Corollary 1.7. *Let* $Q \in A(\mathbb{D}_R) \otimes \mathbb{C}^{2\times 2}$ *and* $Q_{12}(0)Q_{12}(1)Q_{21}(0)Q_{21}(1) \neq 0$. *Then the boundary value problem* (1.6)–(1.8) *is incomplete if and only if BC* (1.8) *are equivalent to one of the "Volterra" boundary conditions:* $y_1(0) = y_2(0) = 0$ *or* $y_1(1) = y_2(1) = 0$.

The proof is immediate from Theorems 1.2, 1.3, 1.4, 1.5, 1.6. We mention also that BC (1.8) are not equivalent to "Volterra" conditions if and only if $A_{12} \neq 0$ and $A_{34} \neq 0$.

2. Preliminary and auxiliary results

2.1. **General results.** Here we provide some general results from [13] and [16]. For brevity we adapt them only for the case of 2×2 systems (1.6), (1.12) investigated below.

Note that the line $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) = \mathrm{Re}(b\lambda)\}$ divides the complex plane in two halfplanes. Denote them by S_1 and S_2. Namely, $S_1 = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < \mathrm{Re}(b\lambda)\}$ and $S_2 = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \mathrm{Re}(b\lambda)\}$. Clearly, each of the halfplanes S_p, $p \in \{1, 2\}$ is of the form $S_p = \{\lambda \in \mathbb{C} : \varphi_{1p} < \arg\lambda < \varphi_{2p}\}$. Fix $p \in \{1, 2\}$ and denote by S the sector strictly embedded into S_p, i.e.,

$$\begin{aligned} S &:= \{\lambda : \varphi_{1p} + \varepsilon < \arg\lambda < \varphi_{2p} - \varepsilon\}, \quad \varepsilon > 0; \\ S_R &:= \{\lambda \in S : |\lambda| > R\}. \end{aligned} \tag{2.1}$$

Theorem 2.1. [15, 3]. *Let S be the sector of the form* (2.1). *Then for a sufficiently large R, system* (1.6), (1.12) *has the fundamental system of matrix solutions*

$$Y_1(x;\lambda) = \begin{pmatrix} (1+o(1))e^{i\lambda x} \\ o(1)e^{i\lambda x} \end{pmatrix}, \quad Y_2(x;\lambda) = \begin{pmatrix} o(1)e^{ib\lambda x} \\ (1+o(1))e^{ib\lambda x} \end{pmatrix}, \quad \lambda \to \infty, \quad \lambda \in S_R, \tag{2.2}$$

which is analytic in $\lambda \in S_R$ and has the asymptotic behavior uniformly in x.

Suppose that $\Phi(x;\lambda)$ is a fundamental 2×2 matrix solution of equation (1.6), satisfying the initial condition $\Phi(0;\lambda) = I_2$ ($I_2 (\in \mathbb{C}^{2\times 2})$ is the identity matrix), i.e.

$$\Phi(x;\lambda) := \begin{pmatrix} \Phi_1(x;\lambda) & \Phi_2(x;\lambda) \end{pmatrix}, \quad \Phi_j(x;\lambda) := \begin{pmatrix} \varphi_{1j}(x;\lambda) \\ \varphi_{2j}(x;\lambda) \end{pmatrix}, \quad j \in \{1, 2\},$$

and $\Phi_1(0;\lambda) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\Phi_2(0;\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Lemma 2.2. [16, Theorem 2.1, step (vi)]. *The system $\{\Phi_j(\cdot;\lambda) : j \in \{1,2\}, \lambda \in \mathbb{C}\}$ of solutions of equation* (1.6) *is complete in $L^2([0,1];\mathbb{C}^2)$, i.e., the only $f \in L^2([0,1];\mathbb{C}^2)$ satisfying*

$$\int_0^1 \langle \Phi_j(x;\lambda), f(x)\rangle\, dx = 0, \quad \lambda \in \mathbb{C}, \quad j \in \{1, 2\},$$

is trivial, $f(\cdot) \equiv 0$.

2.2. **Transformation operators.**

Lemma 2.3. [13]. *Assume that $e_\pm(\cdot;\lambda)$ are solutions of system* (1.6), (1.12), *corresponding to the initial conditions* $e_+(0;\lambda) = \binom{1}{1}$, $e_-(0;\lambda) = \binom{1}{-1}$. *Then solutions $e_\pm(\cdot;\lambda)$ admit the representations*

$$e_\pm(x;\lambda) = (I + K^\pm)e^0_\pm(x;\lambda) = e^0_\pm(x;\lambda) + \int_0^x K^\pm(x,t)e^0_\pm(t;\lambda)\, dt, \tag{2.3}$$

where

$$e^0_\pm(x;\lambda) = \begin{pmatrix} e^{ib_1\lambda x} \\ \pm e^{ib_2\lambda x} \end{pmatrix}, \quad K^\pm(x,t) = \left(K^\pm_{jk}(x,t)\right)^2_{j,k=1},$$

and $K^\pm_{ij}(\cdot,\cdot) \in A(\Omega)$, $\Omega = \{(x,t) : 0 \le t \le x \le 1\}$.

Introduce the following functions

$$R_{jk}^{\pm}(t) := K_{jk}^{+}(1,t) \pm K_{jk}^{-}(1,t), \quad j,k \in \{1,2\}. \tag{2.4}$$

Lemma 2.4. [15, 16]. *Let $Q(\cdot) \in A(\mathbb{C}) \otimes \mathbb{C}^{2\times 2}$, and let $K^{\pm}(\cdot,\cdot)$ be the kernels of the transformation operators given by (2.3). Then the following relations hold:*

$$R_{11}^{-}(1) = K_{11}^{+}(1,1) - K_{11}^{-}(1,1) = 2i(b-1)^{-1}Q_{12}(0), \tag{2.5}$$
$$R_{21}^{+}(1) = K_{21}^{+}(1,1) + K_{21}^{-}(1,1) = 2ib(b-1)^{-1}Q_{21}(1). \tag{2.6}$$
$$R_{12}^{+}(1) = K_{12}^{+}(1,1) + K_{12}^{-}(1,1) = 2ib(1-b)^{-1}Q_{12}(1), \tag{2.7}$$
$$R_{22}^{-}(1) = K_{22}^{+}(1,1) - K_{22}^{-}(1,1) = 2ib^2(1-b)^{-1}Q_{21}(0). \tag{2.8}$$

Lemma 2.5. *Under the assumptions of Lemma 2.4 the following relation holds:*

$$\left(R_{21}^{-}\right)'(1) = \frac{2b}{(b-1)^2}Q_{12}(0)Q_{21}(1). \tag{2.9}$$

Proof. According to [13], the kernels $K^{\pm}(x,t)$ satisfy the equations

$$BD_xK^{\pm}(x,t) + D_tK^{\pm}(x,t)B = -iQ(x)K^{\pm}(x,t), \quad (x,t) \in \Omega, \tag{2.10}$$

and the boundary conditions

$$K_{12}^{\pm}(x,x) = \frac{ib}{1-b}\, Q_{12}(x), \quad K_{21}^{\pm}(x,x) = \frac{ib}{b-1}\, Q_{21}(x), \tag{2.11}$$

$$bK_{11}^{\pm}(x,0) \pm K_{12}^{\pm}(x,0) = 0, \quad bK_{21}^{\pm}(x,0) \pm K_{22}^{\pm}(x,0) = 0. \tag{2.12}$$

It follows from (2.10) that

$$b^{-1}\frac{\partial}{\partial x}\left(K_{21}^{\pm}(x,t)\right) + \frac{\partial}{\partial t}\left(K_{21}^{\pm}(x,t)\right) = -iQ_{21}(x)K_{11}^{\pm}(x,t). \tag{2.13}$$

Using the identity

$$\frac{du(x,x)}{dx} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{t=x} + \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=x}, \quad u \in C^1(\Omega),$$

it follows from (2.13) that

$$b^{-1}\frac{d}{dx}\,K_{21}^{\pm}(x,x) + (1-b^{-1})\left.\frac{\partial K_{21}^{\pm}(x,t)}{\partial t}\right|_{t=x} = -iQ_{21}(x)K_{11}^{\pm}(x,x).$$

Taking into account the second relation in (2.11) we obtain

$$\left.\frac{\partial K_{21}^{\pm}(x,t)}{\partial t}\right|_{t=x} = \frac{ib}{1-b}\, Q_{21}(x)K_{11}^{\pm}(x,x) - \frac{ib}{(b-1)^2}\, Q_{21}'(x). \tag{2.14}$$

Setting $x=1$ in (2.14) and taking into account (2.5) and the identity

$$\frac{d}{dt}R_{21}^{-}(t) = \frac{d}{dt}\left(K_{21}^{+}(1,t) - K_{21}^{-}(1,t)\right)$$

we arrive at the following expression for $(R_{21}^{-})'(1)$:

$$\left.\left(\frac{d}{dt}R_{21}^{-}(t)\right)\right|_{t=1} = \frac{ib}{1-b}Q_{21}(1)R_{11}^{-}(1) = \frac{2b}{(b-1)^2}\, Q_{12}(0)Q_{21}(1).$$

Lemma is proved. □

2.3. **Boundary value problem.** Next we show that the problem (1.6), (1.10) with an arbitrary potential matrix $Q(\cdot) = \bigl(Q_{jk}(\cdot)\bigr)_{j,k=1}^2 \in L^1[0,1] \otimes \mathbb{C}^{2\times 2}$ can be reduced to a similar problem with an off-diagonal potential matrix.

Lemma 2.6. *The problem* (1.6), (1.10) *with a potential matrix* $Q(\cdot) = \bigl(Q_{jk}(\cdot)\bigr)_{j,k=1}^2$ *is equivalent to the problem*

$$-iBy' + \widetilde{Q}(x)y = \lambda y, \quad x \in [0,1], \tag{2.15}$$

$$\begin{cases} y_1(0) = 0, \\ \widetilde{a}_{22}y_2(0) + \widetilde{a}_{23}y_1(1) + \widetilde{a}_{24}y_2(1) = 0 \end{cases} \tag{2.16}$$

with the off-diagonal potential matrix $\widetilde{Q} = \begin{pmatrix} 0 & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & 0 \end{pmatrix}$. *Moreover, for any* $j \in \{2,3,4\}$ *the following equivalence holds:* $\widetilde{a}_{2j} = 0$ *if and only if* $a_{2j} = 0$.

Proof. Denote by $W(\cdot)$ the fundamental 2×2 matrix solution of the Cauchy problem

$$iBW'(x) = Q_1(x)W(x), \quad W(0) = I_2, \tag{2.17}$$

where the 2×2 matrix function $Q_1(\cdot)$ is diagonal ,

$$Q_1(x) = \operatorname{diag}\bigl(Q_{11}(x), Q_{22}(x)\bigr). \tag{2.18}$$

Since $BQ_1(x) = Q_1(x)B$ for any $x \in [0,1]$, the matrix functions $W_1(\cdot) = BW(\cdot)$ and $W_2(\cdot) = W(\cdot)B$ satisfy equation (2.17) and common initial conditions

$$iBW_j'(x) = Q_1(x)W_j(x), \quad W_j(0) = B, \quad j \in \{1,2\}. \tag{2.19}$$

According to the Cauchy uniqueness theorem $W_1(x) = W_2(x)$ for $x \in [0,1]$, i.e.

$$W(x)B - BW(x) = 0, \quad x \in [0,1]. \tag{2.20}$$

Letting $\widetilde{L} = (I\otimes W)^{-1}L(I\otimes W)$ we deduce from (1.6), (2.17) and (2.20) that for any $f \in C^1[0,1]\otimes \mathbb{C}^2$

$$\begin{aligned} \widetilde{L}f - \lambda f &= W^{-1}(x)(-iB)W(x)f' + W^{-1}(x)(-iB)W'(x)f \\ &\quad + W^{-1}(x)Q(x)W(x)f - \lambda f = -iB\frac{d}{dx}f + \widetilde{Q}(x)f - \lambda f, \end{aligned} \tag{2.21}$$

where

$$\widetilde{Q}(x) := W^{-1}(x)\bigl(Q(x) - Q_1(x)\bigr)W(x). \tag{2.22}$$

It follows from (2.20) that the matrix function $W(\cdot)$ is diagonal

$$W(x) = \operatorname{diag}\bigl(W_{11}(x), W_{22}(x)\bigr). \tag{2.23}$$

It follows from (2.22) and (2.23) that $\widetilde{Q}(\cdot)$ is off-diagonal

$$\widetilde{Q}(x) = \begin{pmatrix} 0 & \widetilde{Q}_{12}(x) \\ \widetilde{Q}_{21}(x) & 0 \end{pmatrix}, \quad x \in [0,1].$$

Thus, the problem (1.6), (1.10) is transformed into the problem (2.15), (2.16), where $\widetilde{a}_{22} = a_{22}$, $\widetilde{a}_{23} = a_{23}W_{11}(1)$, $\widetilde{a}_{24} = a_{24}W_{22}(1)$. Since $W(\cdot)$ is the fundamental matrix solution of (2.17), $\det W(x) = W_{11}(x)W_{22}(x) \neq 0$ for $x\in[0,1]$. Hence $W_{11}(1)W_{22}(1) \neq 0$, that implies the desired equivalence: $a_{2j} \neq 0 \Leftrightarrow \widetilde{a}_{2j} \neq 0$. □

In what follows we investigate the problem (1.6), (1.10) assuming that conditions (1.10) are not weakly regular, i.e., conditions (1.9) are violated. In the following lemma we describe all possible types of such boundary conditions.

Lemma 2.7. *Assume that the conditions* (1.9) *are violated. Then boundary conditions* (1.8) *are equivalent either to conditions* (1.10) *or to conditions* (1.11).

Proof. Let us consider the cases, when conditions (1.9) are violated. Then there are four distinct possibilities: a) $J_{14} = J_{12} = 0$, b) $J_{14} = J_{34} = 0$, c) $J_{23} = J_{12} = 0$, d) $J_{23} = J_{34} = 0$. The linear transform $T_1 : \binom{y_1}{y_2} \mapsto \binom{y_2}{y_1}$ reduces the case a) to the case d), the case b) to the case c) and vice versa. The linear transform $T_2 :\ y(x) \mapsto y(1-x)$ reduces the case b) to the case d) and vice versa. Therefore, it suffices to consider only the case b).

In turn, the latter case splits into two subcases:

$$J_{14} = J_{34} = J_{13} = 0 \quad \text{and} \quad J_{14} = J_{34} = 0, \quad J_{13} \neq 0.$$

At first let $J_{14} = J_{34} = J_{13} = 0$. Then

$$\operatorname{rank}\begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \end{pmatrix} = 1$$

and hence boundary conditions (1.8) are equivalent to the conditions with the coefficient matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \widehat{a}_{21} & 0 & \widehat{a}_{23} & \widehat{a}_{24} \end{pmatrix}.$$

Applying transform T_2 we arrive at boundary conditions (1.10).

Further, let $J_{14} = J_{34} = 0,\ J_{13} \neq 0$. Multiplying the coefficient matrix of the linear algebraic system (1.8) by the matrix A_{13}^{-1} from the left, we obtain

$$\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} = \begin{pmatrix} 1 & \widetilde{a}_{12} & 0 & \widetilde{a}_{14} \\ 0 & \widetilde{a}_{22} & 1 & \widetilde{a}_{24} \end{pmatrix}, \tag{2.24}$$

where $\widetilde{a}_{12} = J_{13}^{-1}J_{23}$, $\widetilde{a}_{22} = J_{13}^{-1}J_{12}$, $\widetilde{a}_{14} = -J_{13}^{-1}J_{34}$ and $\widetilde{a}_{24} = J_{13}^{-1}J_{14}$. Since $J_{14} = J_{34} = 0$, one has $\widetilde{a}_{14} = -J_{13}^{-1}J_{34} = 0$, $\widetilde{a}_{24} = J_{13}^{-1}J_{14} = 0$ and we arrive at boundary conditions (1.11). Further, if $\widetilde{a}_{12}\widetilde{a}_{22} \neq 0$ we arrive at conditions (1.11). If $\widetilde{a}_{12} = 0$ we obtain conditions (1.10). Finally, if $\widetilde{a}_{22} = 0$ we apply transformation T_2 and get boundary conditions (1.10). □

The eigenvalues of problem (1.6), (1.10) are the zeros of the characteristic determinant

$$\Delta(\lambda) := \det U(\lambda),$$

where

$$U(\lambda) := \begin{pmatrix} U_1(\Phi_1(\cdot;\lambda)) & U_1(\Phi_2(\cdot;\lambda)) \\ U_2(\Phi_1(\cdot;\lambda)) & U_2(\Phi_2(\cdot;\lambda)) \end{pmatrix} =: \begin{pmatrix} u_{11}(\lambda) & u_{12}(\lambda) \\ u_{21}(\lambda) & u_{22}(\lambda) \end{pmatrix}. \tag{2.25}$$

The characteristic determinant of the problem (1.6), (1.10) with $Q(\cdot) \equiv 0$ is denoted by $\Delta_0(\lambda)$.

Simple computations show that the characteristic determinant of the problem (1.6), (1.10) is

$$\Delta(\lambda) = a_{22} + a_{23}\varphi_{12}(\lambda) + a_{24}\varphi_{22}(\lambda), \tag{2.26}$$

where $\varphi_{jk}(\lambda) := \varphi_{jk}(1,\lambda),\ j,k \in \{1,2\}$. Moreover, if $Q = 0$ we have

$$\Delta_0(\lambda) = a_{22} + a_{24}e^{ib\lambda}. \tag{2.27}$$

Next we introduce a concept of degenerate and non-degenerate BVP (BC).

Definition 2.8. *(i) Boundary conditions* (1.8) *are called degenerate if either* $\Delta_0(\cdot) \equiv 0$ *or it has no zeros.*

(ii) Boundary value problem (1.6), (1.8) *is called degenerate if either* $\Delta(\cdot) \equiv 0$ *or it has no zeros.*

Otherwise boundary conditions (1.8) *(problem* (1.6), (1.8)*) are called non-degenerate.*

By formula (2.27), boundary conditions (1.10) are non-degenerate if and only if $a_{22}a_{24} \neq 0$. Hence the boundary conditions described in each of Theorems 1.4 and 1.5 are degenerate for some coefficients and non-degenerate for the rest of coefficients. On the other hand, all boundary conditions that covered by Theorem 1.6 are degenerate.

Note, in addition, that in the case $a_{23} = a_{24} = 0$ problem (1.6), (1.8) is degenerate: $\Delta(\lambda) \equiv \Delta_0(\lambda) = a_{22} = \text{const}$. It means that Theorems 1.4, 1.5, 1.6 cover all non-degenarate BVP (1.6), (1.12), (1.10).

Note also that boundary conditions for Sturm-Liouville operator is degenerate if and only if $\Delta_0(\cdot) = \text{const}$. It is not the case for system (1.6). For instance, if $a_{22} = 0$ and $a_{24} \neq 0$ then $\Delta_0(\lambda) = a_{24}e^{ib\lambda} \neq \text{const}$ but has no zeros.

3. Proofs of the main results

3.1. **Proof of Theorem 1.4.** We divide the proof in several steps.

(i) The spectrum $\sigma(L)$ of the operator L generated by problem (1.6), (1.12), (1.10) in $L_2\left([0,1];\mathbb{C}^2\right)$ coincides with the zero set of the determinant $\Delta(\lambda)$ and the multiplicity p_n of the zero λ_n of the entire function $\Delta(\lambda)$ coincides with the dimension of the root subspace

$$\mathcal{H}_n := \operatorname{span}\{\ker(L-\lambda_n)^k : \ k \in \mathbb{Z}_+\}, \quad \dim \mathcal{H}_n = p_n$$

(see [1, Sec. 5.6], [18]).

Introduce the solutions $w_1(x;\lambda)$ and $w_2(x;\lambda)$ of the equation (1.6) by setting

$$w_1(x;\lambda) := u_{22}(\lambda)\Phi_1 - u_{21}(\lambda)\Phi_2, \quad w_2(x;\lambda) := -u_{12}(\lambda)\Phi_1 + u_{11}(\lambda)\Phi_2. \tag{3.1}$$

It can easily be seen that $U_j(w_j) = \Delta(\lambda),\ U_1(w_2) = U_2(w_1) = 0,\ U_j(w_j(\cdot,\lambda_n)) = 0$. Further, the functions $w_j^{(k)}(x;\lambda) := D_\lambda^k w_j(x;\lambda)$ satisfy the equations

$$Lw_j^{(k)} = \lambda w_j^{(k)} + kw_j^{(k-1)}, \quad j \in \{1,2\}.$$

Since $U_\nu(D_\lambda^k w_j(\cdot;\lambda)) = D_\lambda^k(U_\nu(w_j(\cdot;\lambda)))$ and λ_n is the zero of $\Delta(\cdot)$ of multiplicity p_n, then the functions $w_j^{(k)}(x;\lambda_n)$, $k \in \{0,1,\dots,p_n-1\}$, satisfy boundary conditions (1.8) and hence belong to the root subspace $\mathcal{H}_n$.

Assuming that the root vector system of the operator L is incomplete in $L_2\left([0,1];\mathbb{C}^2\right)$, there exists a nonzero vector $f = \operatorname{col}(f_1,f_2)$, $f \in L_2\left([0,1];\mathbb{C}^2\right)$, orthogonal to this system. Then the entire functions

$$w_j(\lambda;f) := \int_0^1 \langle w_j(x;\lambda), f(x)\rangle\, dx, \quad j \in \{1,2\}, \tag{3.2}$$

have a zero of multiplicity $\geqslant p_n$ at every point $\lambda_n \in \sigma(L)$. Thus,

$$G_j(\lambda;f) := \frac{w_j(\lambda;f)}{\Delta(\lambda)}, \quad j \in \{1,2\}, \tag{3.3}$$

is the entire function. Since both $w_j(\cdot;f)$ and $\Delta(\cdot)$ are the entire functions of order not exceeding one, the order of $G_j(\cdot;f)$ does not exceed one as well ([11, Chapter 1, § 3.9]). In the next few steps we show that $G_j(\cdot;f) \equiv 0,\ j \in \{1,2\}$, by estimating it from above.

(ii) To estimate the determinant $\Delta(\cdot)$ from below we transform its expression using Lemmas 2.3, 2.4 and 2.5. Since $\Phi(0;\lambda) = I_2$ and $e_\pm(0;\lambda) = \binom{1}{\pm 1}$, we have

$$2\Phi_1(\cdot;\lambda) = e_+(\cdot;\lambda) + e_-(\cdot;\lambda), \quad 2\Phi_2(\cdot;\lambda) = e_+(\cdot;\lambda) - e_-(\cdot;\lambda).$$

Taking into account representations (2.3) for the solutions $e_{\pm}(\cdot;\lambda)$, we obtain

$$\varphi_{12}(1;\lambda) = \frac{1}{2}\int_0^1 R_{21}^{+}(t)e^{i\lambda t}dt + \frac{1}{2}\int_0^1 R_{22}^{-}(t)e^{ib\lambda t}dt, \tag{3.4}$$

$$\varphi_{22}(1;\lambda) = e^{ib\lambda} + \frac{1}{2}\int_0^1 R_{21}^{-}(t)e^{i\lambda t}dt + \frac{1}{2}\int_0^1 R_{22}^{+}(t)e^{ib\lambda t}dt. \tag{3.5}$$

Noting that $R_{jk}^{\pm}(\cdot) \in C^1[0,1],\ j,k \in \{1,2\}$, we integrate by parts in (3.4) and (3.5)

$$\begin{aligned}\varphi_{12}(\lambda) = \frac{R_{21}^{+}(1)}{2i}\ \frac{e^{i\lambda}}{\lambda} + \frac{R_{22}^{-}(1)}{2ib}\ \frac{e^{ib\lambda}}{\lambda} - \left(\frac{R_{21}^{+}(0)}{2i} + \frac{R_{22}^{-}(0)}{2ib}\right)\frac{1}{\lambda} \\ - \int_0^1 \frac{(R_{21}^{+})'(t)}{2i}\ \frac{e^{i\lambda t}}{\lambda}\ dt\ - \int_0^1 \frac{(R_{22}^{-})'(t)}{2ib}\ \frac{e^{ib\lambda t}}{\lambda}\ dt,\end{aligned} \tag{3.6}$$

$$\varphi_{22}(\lambda) = e^{ib\lambda} + \frac{1}{2}\int_0^1 R_{22}^{+}(t)e^{ib\lambda t}dt + \frac{R_{21}^{-}(1)}{2i}\ \frac{e^{i\lambda}}{\lambda} - \frac{R_{21}^{-}(0)}{2i\lambda} - \int_0^1 \frac{(R_{21}^{-})'(t)}{2i}\ \frac{e^{i\lambda t}}{\lambda}\ dt. \tag{3.7}$$

It follows from conditions (2.11) that $R_{21}^{-}(1) = K_{21}^{+}(1,1) - K_{21}^{-}(1,1) = 0$. Combining this relation with equalities (3.6), (3.7) we arrive at the following expression for the characteristic determinant (2.26)

$$\begin{aligned}\Delta(\lambda) = a_{22} + a_{24}\Big(e^{ib\lambda} + \frac{1}{2}\int_0^1 R_{22}^{+}(t)e^{ib\lambda t}dt\Big) + \frac{a_{23}R_{21}^{+}(1)}{2i}\ \frac{e^{i\lambda}}{\lambda} \\ + \frac{a_{23}R_{22}^{-}(1)}{2ib}\ \frac{e^{ib\lambda}}{\lambda} - \Big(\frac{a_{23}R_{21}^{+}(0) + a_{24}R_{21}^{-}(0)}{2i} + \frac{a_{23}R_{22}^{-}(0)}{2ib}\Big)\frac{1}{\lambda} \\ - \int_0^1 \frac{a_{23}(R_{22}^{-})'(t)}{2ib}\ \frac{e^{ib\lambda t}}{\lambda}\ dt\ - \int_0^1 \frac{a_{23}(R_{21}^{+})'(t) + a_{24}(R_{21}^{-})'(t)}{2i}\ \frac{e^{i\lambda t}}{\lambda}\ dt.\end{aligned} \tag{3.8}$$

Using (2.6) $\Delta(\cdot)$ can be rewritten in the form

$$\Delta(\lambda) = a_{22} + a_{24}e^{ib\lambda} + \frac{ba_{23}Q_{21}(1)}{b-1}\ \frac{e^{i\lambda}}{\lambda} + o(1) + o\left(e^{ib\lambda}\right) + o\left(\frac{e^{i\lambda}}{\lambda}\right), \quad |\lambda| \to +\infty. \tag{3.9}$$

(iii) In this step we estimate $\Delta(\cdot)$ from below. To this end we introduce the following sectors:

$$\begin{aligned} S_1 &= \{\lambda:\ \mathrm{Re}(ib\lambda) > 0,\ \mathrm{Re}(i\lambda) < 0,\ |\lambda| > R\}, \\ S_2 &= \{\lambda:\ \mathrm{Re}(i\lambda) > 0,\ \mathrm{Re}(ib\lambda) < 0,\ |\lambda| > R\}\end{aligned} \tag{3.10}$$

and denote by $S_{1,\varepsilon}$ and $S_{2,\varepsilon}$ the closed sectors strictly embedded into S_1 and S_2, respectively (cf. with (2.1)). Denote also by $S_{3,\varepsilon}$ and $S_{4,\varepsilon}$ the remaining sectors, i.e. $S_{1,\varepsilon} \cup S_{2,\varepsilon} \cup S_{3,\varepsilon} \cup S_{4,\varepsilon} = \mathbb{C}^2 \backslash \{\lambda:\ |\lambda| \leqslant R\}$.

First we estimate $\Delta(\cdot)$ in $S_{1,\varepsilon}$. Since $a_{24} \neq 0$, it follows from (3.9) that there exists a constant $C_1 > 0$ such that for sufficiently large $|\lambda|$ the following estimate holds:

$$|\Delta(\lambda)| \geqslant C_1|e^{ib\lambda}|, \quad \lambda \in S_{1,\varepsilon}. \tag{3.11}$$

Next, let us estimate $\Delta(\cdot)$ in $S_{2,\varepsilon}$. Since $a_{23} \neq 0$ and $Q_{21}(1) \neq 0$, representation (3.9) implies the following estimate for the characteristic determinant $\Delta(\cdot)$:

$$|\Delta(\lambda)| = \frac{|e^{i\lambda}|}{|\lambda|}\Big(\frac{ba_{23}Q_{21}(1)}{b-1} + o(1)\Big) \geqslant C_2\frac{|e^{i\lambda}|}{|\lambda|}, \quad \lambda \in S_{2,\varepsilon}, \tag{3.12}$$

with some $C_2 > 0$.

(iv) In this step we estimate the growth of $w_j(\cdot;f)$ from above in the sector $S_{1,\varepsilon}$. We put

$$\widetilde{U}(\lambda) := \big(\widetilde{u}_{jk}(\lambda)\big)_{j,k=1}^{2} := \big(U_j(Y_k(\cdot;\lambda))\big)_{j,k=1}^{2},$$

where $Y_j := \operatorname{col}(y_{1j}, y_{2j}),\ j \in \{1,2\}$, are the solutions of equation (1.6) with asymptotic behavior (2.2):

$$Y_1(x;\lambda) = \begin{pmatrix} (1+o(1))e^{i\lambda x} \\ o(1)e^{i\lambda x} \end{pmatrix}, \quad Y_2(x;\lambda) = \begin{pmatrix} o(1)e^{ib\lambda x} \\ (1+o(1))e^{ib\lambda x} \end{pmatrix}. \tag{3.13}$$

Alongside solutions (3.1) we introduce the solutions $V_1(x;\lambda)$ and $V_2(x;\lambda)$ by setting

$$V_1(x;\lambda) = \widetilde{u}_{22}(\lambda)Y_1 - \widetilde{u}_{21}(\lambda)Y_2, \quad V_2(x;\lambda) = -\widetilde{u}_{12}(\lambda)Y_1 + \widetilde{u}_{11}(\lambda)Y_2, \quad \lambda \in S_{1,\varepsilon}. \tag{3.14}$$

The fundamental matrices $\Phi(x,\cdot)$ and $Y(x,\cdot) := \big(Y_1(\cdot;\lambda)\ Y_2(x,\cdot)\big)$ of equation (1.6) as well as the matrices $U(\cdot)$ and $\widetilde{U}(\cdot)$ are connected by

$$\Phi(x;\lambda) = Y(x;\lambda)P(\lambda) \quad \text{and} \quad U(\lambda) = \widetilde{U}(\lambda)P(\lambda), \quad \lambda \in S_{1,\varepsilon}, \tag{3.15}$$

where $P(\cdot)$ is holomorphic in $S_{1,\varepsilon}$ and invertible 2×2 matrix function. Hence

$$w_j(x;\lambda) = V_j(x;\lambda)\det P(\lambda), \quad \lambda \in S_{1,\varepsilon}, \quad j \in \{1,2\}. \tag{3.16}$$

It follows from (3.13) that

$$\widetilde{U}(\lambda) = \begin{pmatrix} \widetilde{u}_{11}(\lambda) & \widetilde{u}_{12}(\lambda) \\ \widetilde{u}_{21}(\lambda) & \widetilde{u}_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 1+o(1) & o(1) \\ o(1) & \big(a_{24}+o(1)\big)e^{ib\lambda} \end{pmatrix}, \quad |\lambda| \to \infty. \tag{3.17}$$

Substituting matrix entries (3.17) into (3.14), we have

$$V_1(x;\lambda) = \begin{pmatrix} \big(a_{24}+o(1)\big)e^{ib\lambda + i\lambda x} + o\big(e^{ib\lambda x}\big) \\ o(1)e^{ib\lambda+i\lambda x} + o\big(e^{ib\lambda x}\big) \end{pmatrix} = \begin{pmatrix} a_{24}e^{ib\lambda+i\lambda x} + o\big(e^{ib\lambda}\big) \\ o\big(e^{ib\lambda}\big) \end{pmatrix}, \tag{3.18}$$

$$V_2(x;\lambda) = \begin{pmatrix} o\big(e^{i\lambda x}\big) + o\big(e^{ib\lambda x}\big) \\ o\big(e^{i\lambda x}\big) + (1+o(1))e^{ib\lambda x} \end{pmatrix} = \begin{pmatrix} o\big(e^{ib\lambda x}\big) \\ e^{ib\lambda x} + o\big(e^{ib\lambda x}\big) \end{pmatrix} \tag{3.19}$$

for $|\lambda| \to \infty,\ \lambda \in S_{1,\varepsilon}$. Setting $x = 0$ in the first of equalities (3.15) and taking into account (3.13) and $\Phi(0,\lambda) = I_2$, we get $P(\lambda) = I_2 + o_2(\lambda)$. Then the solutions $w_1(x;\lambda)$ and $w_2(x;\lambda)$ (see (3.16)) have the asymptotic behavior

$$w_1(x;\lambda) = \begin{pmatrix} a_{24}e^{ib\lambda+i\lambda x} + o\big(e^{ib\lambda}\big) \\ o\big(e^{ib\lambda}\big) \end{pmatrix}, \quad |\lambda| \to \infty, \quad \lambda \in S_{1,\varepsilon}, \tag{3.20}$$

$$w_2(x;\lambda) = \begin{pmatrix} o\big(e^{ib\lambda x}\big) \\ e^{ib\lambda x} + o\big(e^{ib\lambda x}\big) \end{pmatrix}, \quad |\lambda| \to \infty, \quad \lambda \in S_{1,\varepsilon}. \tag{3.21}$$

Now we can estimate $|w_j(\lambda;f)|$ in the sector $S_{1,\varepsilon}$

$$\begin{aligned} |w_1(\lambda;f)| &\leqslant \|f\|_{L_2([0,1];\mathbb{C}^2)} \|w_1(\cdot;\lambda)\|_{L_2([0,1];\mathbb{C}^2)} \leqslant \frac{\widetilde{C}_1|e^{ib\lambda}|}{\sqrt{|\operatorname{Im}\lambda|}} + o\big(|e^{ib\lambda}|\big) = o\big(e^{ib\lambda}\big), \\ |w_2(\lambda;f)| &\leqslant \frac{\widetilde{C}_2|e^{ib\lambda}|}{\sqrt{|\operatorname{Im}(b\lambda)|}} = o\big(e^{ib\lambda}\big), \quad \lambda \in S_{1,\varepsilon}, \end{aligned} \tag{3.22}$$

with some positive constants $\widetilde{C}_1, \widetilde{C}_2$.

(v) As in the previous step we obtain that the matrix function $\widetilde{U}(\lambda)$ from (3.17) and the solutions $w_1(x;\lambda)$ and $w_2(x;\lambda)$ have the following asymptotic behavior

$$\widetilde{U}(\lambda) = \begin{pmatrix} 1+o(1) & o(1) \\ a_{23}e^{i\lambda} + o(e^{i\lambda}) & a_{22}+o(1) \end{pmatrix}, \quad \lambda \to \infty, \quad \lambda \in S_{2,\varepsilon},$$

$$w_1(x;\lambda) = \begin{pmatrix} \big(a_{22}+o(1)\big)e^{i\lambda x} + o\big(e^{i\lambda}\big) \\ -\big(a_{23}+o(1)\big)e^{i\lambda+ib\lambda x} + o\big(e^{i\lambda}\big) \end{pmatrix} = \begin{pmatrix} a_{22}e^{i\lambda x} + o\big(e^{i\lambda}\big) \\ -a_{23}e^{i\lambda+ib\lambda x} + o\big(e^{i\lambda}\big) \end{pmatrix},$$
$$w_2(x;\lambda) = \begin{pmatrix} o\big(e^{i\lambda x}\big)+o\big(e^{ib\lambda x}\big) \\ o\big(e^{i\lambda x}\big)+\big(1+o(1)\big)e^{ib\lambda x} \end{pmatrix} = \begin{pmatrix} o\big(e^{i\lambda x}\big) \\ o\big(e^{i\lambda x}\big) \end{pmatrix}, \quad \lambda\to\infty, \quad \lambda\in S_{2,\varepsilon}.$$

For $|w_j(\lambda;f)|$ the following estimate holds:

$$|w_j(\lambda;f)| = o\big(e^{i\lambda}\big), \quad |\lambda|\to\infty, \quad \lambda\in S_{2,\varepsilon}, \quad j\in\{1,2\}. \tag{3.23}$$

(vi) In this step we prove that $G_j(\lambda;f)\equiv 0,\ j\in\{1,2\},\ \lambda\in\mathbb{C}$. Combining relations (3.22) with (3.11), we get

$$|G_j(\lambda;f)| = o(1), \quad |\lambda|\to\infty, \quad \lambda\in S_{1,\varepsilon} \quad j\in\{1,2\}. \tag{3.24}$$

Further, it follows from (3.12) and (3.23) that

$$\lim_{|\lambda|\to+\infty}\left|w_j(\lambda;f)\Delta^{-1}(\lambda)\lambda^{-1}\right| = 0, \quad \lambda\in S_{2,\varepsilon}, \quad j\in\{1,2\}. \tag{3.25}$$

The last relation yields

$$|G_j(\lambda;f)| = o(\lambda) \quad \text{for} \quad |\lambda|\to\infty, \quad \lambda\in S_{2,\varepsilon}, \quad j\in\{1,2\}. \tag{3.26}$$

Let's draw in the sectors $S_{1,\varepsilon}$ and $S_{2,\varepsilon}$ two lines passing through the origin. They divide the complex plane into four sectors of opening less than π. Both relations (3.24) and (3.26) hold at the boundary of these sectors. Applying to each of the sectors the Phragmén-Lindelöf theorem and taking into account that $G_j(\lambda;f)$ are functions of exponential type, we conclude that relations (3.26) hold in each of the sectors $S_{j,\varepsilon},\ j\in\{1,2,3,4\}$. Hence, $G_j(\lambda;f),\ j\in\{1,2\}$, are the constant functions. Taking into account relations (3.24), we obtain $G_j(\lambda;f)\equiv 0,\ j\in\{1,2\},\ \lambda\in\mathbb{C}$.

(vii) In this step we prove the completeness property. In the previous step we proved that $w_j(\lambda;f)\equiv 0,\ j\in\{1,2\},\ \lambda\in\mathbb{C}$. These equalities mean that $f(\cdot)$ is orthogonal to $w_j(\cdot;\lambda)$ for all $\lambda\in\mathbb{C},\ j\in\{1,2\}$. It follows from (3.1) that $f(\cdot)$ is orthogonal to two linear independent solutions $\Phi_1(\cdot,\lambda)$ and $\Phi_2(\cdot,\lambda)$ of Cauchy problem for all $\lambda\neq\lambda_n,\ \lambda_n\in\sigma(L)$. Since the corresponding integral is continuous in λ, $f(\cdot)$ is orthogonal to $\Phi_1(\cdot,\lambda)$ and $\Phi_2(\cdot,\lambda)$ for $\lambda=\lambda_n$ as well. Thus, by Lemma 2.2, $f(\cdot)\equiv 0$. Therefore the root vector system of problem (1.6), (1.12), (1.10) is complete in $L_2\big([0,1];\mathbb{C}^2\big)$.

The minimality is implied, for instance, by [16, Lemma 2.4].

(viii) Let us prove the completeness property of the adjoint operator $L_{C,D}(Q)^*$. It is defined by the differential expression

$$L^* = -iB^*\frac{d}{dx} + Q^*(x), \quad Q^* = \begin{pmatrix} 0 & \overline{Q}_{21} \\ \overline{Q}_{12} & 0 \end{pmatrix}, \tag{3.27}$$

and the boundary conditions

$$\begin{cases} \overline{a}_{24}y_2(0) + \overline{a}_{22}y_2(1) = 0, \\ \overline{a}_{24}\overline{b}y_1(1) - \overline{a}_{23}y_2(1) = 0. \end{cases} \tag{3.28}$$

If $a_{22}\neq 0$, these conditions are equivalent to the conditions (1.11) and the completeness property of $L_{C,D}(Q)^*$ is implied by Theorem 1.3 even without additional assumptions on the potential matrix $Q(\cdot)$.

If $a_{22}=0$, we apply the transformation T_1 to the problem (3.27), (3.28), and arrive at the boundary value problem (1.6), (1.10) with the potential matrix $\widetilde{Q} = \begin{pmatrix} 0 & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & 0 \end{pmatrix}$ in place of $Q(\cdot)$, and the boundary conditions (1.10) that satisfy the assumptions of Theorem 1.4. Clearly, $\widetilde{Q}_{21}(\cdot) = \overline{Q_{21}(\cdot)}$, hence $\widetilde{Q}(\cdot)$ satisfies the conditions of Theorem 1.4 which yields the completeness of the operator $L_{C,D}(Q)^*$.

3.2. **The proof of Theorem 1.5.** Now boundary conditions (1.10) take the form

$$\begin{aligned} U_1(y) &= y_1(0) = 0, \\ U_2(y) &= a_{22}y_2(0) + a_{24}y_2(1) = 0. \end{aligned} \tag{3.29}$$

We divide the proof in several steps.

(i) Introduce the solution $w(x;\lambda)$ of equation (1.6) by setting

$$w(x;\lambda) := u_{22}(\lambda)\Phi_1(x;\lambda) - u_{21}(\lambda)\Phi_2(x;\lambda). \tag{3.30}$$

Assume that the root vector system of the operator L is incomplete in $L^2\left([0,1];\mathbb{C}^2\right)$. Then there exists a vector $f = \mathrm{col}(f_1, f_2) \in L^2\left([0,1];\mathbb{C}^2\right) \setminus \{0\}$ orthogonal to this system. Setting

$$w(\lambda; f) := \int_0^1 \langle w(x;\lambda), f(x)\rangle\, dx,$$

we obtain an entire function $w(\lambda; f)$ that has a zero of multiplicity $\geqslant p_n$ at every point $\lambda_n \in \sigma(L)$. Thus, the function

$$G(\lambda) := G(\lambda; f) := \frac{w(\lambda; f)}{\Delta(\lambda)}$$

is also the entire function. By [11, §1.3.9], the order of $G(\lambda)$ does not exceed one.

(ii) In this step we transform the characteristic determinant $\Delta(\cdot)$. Since $a_{23} = 0$, it follows from (3.8) that

$$\Delta(\lambda) = a_{22} + a_{24}\Big(e^{ib\lambda} + \frac{1}{2}\int_0^1 a_{24}R_{22}^{+}(t)e^{ib\lambda t}dt\Big) - \frac{a_{24}R_{21}^{-}(0)}{2i\lambda} - \int_0^1 \frac{a_{24}(R_{21}^{-})'(t)}{2i}\ \frac{e^{i\lambda t}}{\lambda}\ dt. \tag{3.31}$$

The later expression can be rewritten as

$$\Delta(\lambda) = a_{22} + a_{24}e^{ib\lambda} - \int_0^1 \frac{a_{24}(R_{21}^{-})'(t)}{2i}\ \frac{e^{i\lambda t}}{\lambda}\ dt + o\,(1) + o\left(e^{ib\lambda}\right), \quad |\lambda| \to \infty. \tag{3.32}$$

(iii) Next we estimate $G(\cdot)$ in the sector $S_{1,\varepsilon}$. First we estimate $\Delta(\cdot)$ from below. Since $a_{24} \neq 0$ it follows from (3.32) and (3.10) that with some positive constant C_3

$$|\Delta(\lambda)| \geqslant C_3|e^{ib\lambda}|, \quad \lambda \in S_{1,\varepsilon}, \tag{3.33}$$

Alongside the solution $w(x;\lambda)$ defined by (3.30) we introduce a solution

$$V(x;\lambda) = \widetilde{u}_{22}(\lambda)Y_1(x;\lambda) - \widetilde{u}_{21}(\lambda)Y_2(x;\lambda), \tag{3.34}$$

where $\widetilde{u}_{2j}(\lambda) = U_2(Y_j),\ j \in \{1,2\}$. As in Theorem 1.4 the fundamental matrices $\Phi(x,\cdot)$ and $Y(x,\cdot) := \big(Y_1(x,\cdot)\ Y_2(x,\cdot)\big)$ of equation (1.6) as well as the matrix functions $U(\cdot)$ and $\widetilde{U}(\cdot)$ are connected by

$$\Phi(x;\lambda) = Y(x;\lambda)P(\lambda) \quad \text{and} \quad U(\lambda) = \widetilde{U}(\lambda)P(\lambda), \quad \lambda \in S_{1,\varepsilon}, \tag{3.35}$$

where the 2×2 matrix function $P(\cdot)$ is holomorphic and invertible in $S_{1,\varepsilon}$. Hence

$$w(x;\lambda) = V(x;\lambda)P(\lambda), \quad \lambda \in S_{1,\varepsilon}. \tag{3.36}$$

It follows from (3.13) that

$$\begin{aligned} \widetilde{U}(\lambda) &= \begin{pmatrix} \widetilde{u}_{11}(\lambda) & \widetilde{u}_{12}(\lambda) \\ \widetilde{u}_{21}(\lambda) & \widetilde{u}_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) + o\big(e^{i\lambda}\big) & a_{24}e^{ib\lambda} + a_{22} + o\big(e^{ib\lambda}\big) + o(1) \end{pmatrix} \\ &= \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & a_{24}e^{ib\lambda} + o\big(e^{ib\lambda}\big) \end{pmatrix}, \quad |\lambda| \to \infty, \quad \lambda \in S_{1,\varepsilon}. \end{aligned} \tag{3.37}$$

Combining (3.37) with (3.34) and (3.13), we get

$$V(x;\lambda) = \begin{pmatrix} a_{24}e^{ib\lambda+i\lambda x} + o\big(e^{ib\lambda}\big) \\ o\big(e^{ib\lambda}\big) \end{pmatrix}, \quad \lambda \in S_{1,\varepsilon}.$$

As in the proof of Theorem 1.4 we obtain

$$w(x;\lambda) = \begin{pmatrix} a_{24}e^{ib\lambda+i\lambda x} + o\big(e^{ib\lambda}\big) \\ o\big(e^{ib\lambda}\big) \end{pmatrix}, \quad |\lambda| \to \infty, \quad \lambda \in S_{1,\varepsilon}.$$

Let us estimate $|w(\cdot;f)|$ in the sector $S_{1,\varepsilon}$. We have

$$|w(\lambda;f)| \leqslant \|f\|_{L_2([0,1];\mathbb{C}^2)} \cdot \|w(\cdot;\lambda)\|_{L_2([0,1];\mathbb{C}^2)} \leqslant \frac{\widetilde{C}_3|e^{ib\lambda}|}{\sqrt{|\operatorname{Im}\lambda|}} + o\big(e^{ib\lambda}\big) = o\big(e^{ib\lambda}\big),$$

where $\widetilde{C}_3 > 0$. Combining this estimate with (3.33) we arrive at the desired estimate

$$|G(\lambda;f)| = o(1), \quad |\lambda| \to \infty, \quad \lambda \in S_{1,\varepsilon}. \tag{3.38}$$

(iv) In this step we estimate $G(\cdot)$ in the sector $S_{2,\varepsilon}$. To estimate $\Delta(\cdot)$ from below we integrate by parts the last summand in (3.31). We obtain

$$\int_0^1 \frac{a_{24}(R_{21}^-)'(t)}{2i}\ \frac{e^{i\lambda t}}{\lambda}\ dt = -\frac{a_{24}(R_{21}^-)'(1)}{2}\ \frac{e^{i\lambda}}{\lambda^2} + o\left(\frac{e^{i\lambda}}{\lambda^2}\right), \quad \lambda \in S_{2,\varepsilon}. \tag{3.39}$$

By Lemma 2.5, the expression (3.32) for characteristic determinant $\Delta(\cdot)$ takes the form

$$\Delta(\lambda) = \frac{a_{24}bQ_{12}(0)Q_{21}(1)}{(b-1)^2}\ \frac{e^{i\lambda}}{\lambda^2} + o\left(\frac{e^{i\lambda}}{\lambda^2}\right), \quad \lambda \in S_{2,\varepsilon}.$$

Since $a_{24} \neq 0$ and $Q_{12}(0)Q_{21}(1) \neq 0$, by the assumption, we get the estimate

$$|\Delta(\lambda)| \geqslant C_4\frac{|e^{i\lambda}|}{|\lambda|^2}, \quad \lambda \in S_{2,\varepsilon}, \quad |\lambda| \to \infty,$$

where C_4 is a positive constant.

As in the step (iii) we obtain that the matrix function $\widetilde{U}(\lambda)$, the solution $w(x;\lambda)$ and the function $w(\lambda;f)$ have the following asymptotic behavior as $|\lambda| \to \infty,\ \lambda \in S_{2,\varepsilon}$,

$$\widetilde{U}(\lambda) = \begin{pmatrix} 1+o(1) & o(1) \\ o(e^{i\lambda}) & a_{22}+o(1) \end{pmatrix},$$

$$w(x;\lambda) = \begin{pmatrix} a_{22}e^{i\lambda x} + o(e^{i\lambda x}) + o\big(e^{i\lambda}\big) \\ o(e^{i\lambda x}) + o(e^{i\lambda}) \end{pmatrix} = \begin{pmatrix} a_{22}e^{i\lambda x} + o\big(e^{i\lambda}\big) \\ o(e^{i\lambda}) \end{pmatrix},$$

$$|w(\lambda;f)| = o\big(e^{i\lambda}\big).$$

Hence

$$|G(\lambda;f)| = o(\lambda^2) \quad \text{for} \quad |\lambda| \to \infty, \quad \lambda \in S_{2,\varepsilon}. \tag{3.40}$$

(v) In this step we prove completeness and minimality. As in the proof of Theorem 1.4 combining estimates (3.38) and (3.40) with the Phragmén-Lindelöf theorem, yieldst $G(\lambda;f) = a_1\lambda + a_0$. Then condition (3.38) yields $G(\lambda;f) = w(\lambda;f) \equiv 0$.

It follows that the vector-function $f(\cdot)$ is orthogonal to all solutions of system (1.6) subject to the boundary conditions

$$\begin{aligned} &y_1(0) + y_2(0) + y_1(1) = 0, \\ &a_{22}y_2(0) + a_{24}y_2(1) = 0. \end{aligned} \tag{3.41}$$

If $a_{22} \neq 0$, then $J_{14}J_{23} \neq 0$ and the boundary conditions (3.41) are weakly regular. Therefore, by Theorem 1.2, the root vector system of problem (1.6), (1.12), (3.41) is complete in $L_2\left([0,1];\mathbb{C}^2\right)$.

Let $a_{22} = 0$. Then the boundary value problem obtained from (1.6), (3.41) by applying transformations T_1 and T_2 defined in Lemma 2.7 satisfy conditions of Theorem 1.4. Thus, the root vector system of the problem (1.6), (1.12), (3.41) is complete in $L_2\left([0,1];\mathbb{C}^2\right)$.

The minimality follows from [16, Lemma 2.4].

(vi) Let us prove the completeness of the adjoint operator $L_{C,D}(Q)^*$. It is defined by the differential expression (3.27) and the boundary conditions (3.28) with $a_{23} = 0$. Applying the transformation T_2 to the BVP corresponding to the operator $L_{C,D}(Q)^*$ we arrive at the boundary value problem (1.6), (1.10) with the potential matrix $\widetilde{Q} = \begin{pmatrix} 0 & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & 0 \end{pmatrix}$ in place of $Q(\cdot)$, and the boundary conditions (1.10) that satisfy the assumptions of Theorem 1.5. Since, $\widetilde{Q}_{21}(x) = \overline{Q_{12}(1-x)}$ and $\widetilde{Q}_{12}(x) = \overline{Q_{21}(1-x)}$, one has $\widetilde{Q}_{12}(0)\widetilde{Q}_{21}(1) = \overline{Q_{21}(1)Q_{12}(0)} \neq 0$, hence $\widetilde{Q}$ satisfies the conditions of Theorem 1.5. Thus, the root vector system of the operator $L_{C,D}(Q)^*$ is complete and minimal.

The proof of Theorem 1.6 is similar to that of Theorems 1.4 and 1.5 and is omitted.

Remark 3.1. *Note in conclusion, that Theorems 1.4, 1.5, 1.6 remain valid for boundary value problem* (1.6), (1.7), (1.10)*, with not necessarily off-diagonal potential matrix* $\widetilde{Q}(\cdot)$*. Indeed, by Lemma 2.6,*

$$\widetilde{Q}(x) = W^{-1}(x)\begin{pmatrix} 0 & Q_{12}(x) \\ Q_{21}(x) & 0 \end{pmatrix} W(x).$$

Hence $\widetilde{Q}_{12}(x) = W_{11}^{-1}(x)Q_{12}(x)W_{22}(x)$ *and* $\widetilde{Q}_{21}(x) = W_{22}^{-1}(x)Q_{21}(x)W_{11}(x)$*. Since* $W_{11}(x)W_{22}(x) \neq 0$ *for* $x \in [0,1]$*, the following equivalences hold:*

- $Q_{21}(1) \neq 0 \Leftrightarrow \widetilde{Q}_{21}(1) \neq 0$*;*
- $Q_{12}(0)Q_{21}(1) \neq 0 \Leftrightarrow \widetilde{Q}_{12}(0)\widetilde{Q}_{21}(1) \neq 0$*;*
- $Q_{21}(0)Q_{21}(1) \neq 0 \Leftrightarrow \widetilde{Q}_{21}(0)\widetilde{Q}_{21}(1) \neq 0$.

Remark 3.2. *Theorems 1.4, 1.5 and 1.6 demonstrate the following phenomenon similar to that for Sturm-Liouville operators with degenerate boundary conditions (cf.* [12]*).*

The completeness property does not preserve under weak perturbation. For instance, the operator $L_{C,D}(\varepsilon Q) = -iB\frac{d}{dx} + \varepsilon Q$ *subject to boundary conditions* (1.10) *is complete for any* $\varepsilon \neq 0$ *whenever* $Q(\cdot)$ *satisfies the assumptions of Corollary 1.7. At the same time, the unperturbed operator* $L_{C,D}(0) = u-\lim_{\varepsilon\to 0} L_{C,D}(\varepsilon Q)$*) is not complete. Indeed, the system of root functions of* $L_{C,D}(0)$ *is* $\Psi_n(x) := \mathrm{col}(e^{i2\pi nx}, 0)$, $n \in \mathbb{Z}$.

References

1. M. S. Agranovich, B. Z. Katsenelenbaum, A. N. Sivov, N. N. Voitovich, *Generalized Method of Eigenoscillation in Diffraction Theory*, Wiley-VCH, Berlin, 1999.
2. G. D. Birkhoff, *On the asymptotic character of the solutions of certain linear differential equations containing a parameter*, Trans. Amer. Math. Soc. **9** (1908), 219–231.
3. G. D. Birkhoff, R. E. Langer, *The boundary problems and developments associated with a system of ordinary differential equations of the first order*, Proc. Amer. Acad. Arts Sci. **58** (1923), 49–128.
4. P. Djakov and B. Mityagin, *Unconditional convergence of spectral decompositions of 1D Dirac operators with regular boundary conditions*, arXiv:1008.4095 (Submitted on 24 Aug 2010). Accepted in IUMJ.
5. P. Djakov and B. Mityagin, *Criteria for existence of Riesz bases consisting of root functions of Hill and 1D Dirac operators*, arXiv:1106.5774 (Submitted on 28 Jun 2011).
6. F. Gesztesy and V. Tkachenko, *A Schauder and Riesz basis criterion for non-selfadjoint Schrödinger operators with periodic and anti-periodic boundary conditions*, arXiv:1104.4846 (Submitted on 26 Apr 2011, last revised 4 Jun 2011).
7. M. V. Keldysh, *On the characteristic values and characteristic functions of certain classes of non-selfadjoint equations*, Dokl. Akad. Nauk SSSR **77** (1951), no. 1, 11–14. (Russian)

8. A. P. Khromov, *Finite-dimensional perturbations of Volterra operators*, Sovrem. Mat. Fundam. Napravl. **10** (2004), 3–163. (Russian); English transl. J. Math. Sci. (N.Y.) **138** (2006), no. 5, 5893–6066.
9. A. G. Kostyuchenko and G. V. Radzievsky, *On the summability of n-multiple decompositions by Abel's method*, Sib. Mat. Zh. **15** (1974), no. 4, 855–870. (Russian)
10. A. G. Kostyuchenko and A. A. Shkalikov, *Summability of eigenfunction expansions of differential operators and convolution operators*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 4, 24–40. (Russian); English transl. Funct. Anal. Appl. **12** (1978), no. 4, 262–276.
11. A. F. Leont'ev, *Entire Functions. Exponent Series*, Nauka, Moscow, 1983.
12. M. M. Malamud, *On the completeness of a system of root vectors of the Sturm-Liouville operator with general boundary conditions*, Funktsional. Anal. i Prilozhen. **42** (2008), no. 3, 45–52. (Russian); English transl. Funct. Anal. Appl. **42** (2008), no. 3, 198–204.
13. M. M. Malamud, *Questions of uniqueness in inverse problems for systems of differential equations on a finite interval*, Trudy Moskov. Mat. Obshch. **60** (1999), 199–258. (Russian); English transl. Trans. Moscow Math. Soc. **60** (1999), 173–224.
14. M. M. Malamud and L. L. Oridoroga, *Completeness theorems for systems of differential equations*, Funktsional. Anal. i Prilozhen. **34** (2000), no. 4, 88–90. (Russian); English transl. Funct. Anal. Appl. **34** (2000), no. 4, 308–310.
15. M. M. Malamud and L. L. Oridoroga, *On the completeness of the system of root vectors for first-order systems*, Doklady Akademii Nauk **435** (2010), no. 3, 1–6. (Russian); English transl. Doklady Mathematics **435** (2010), no. 3, 899-904.
16. M. M. Malamud and L. L. Oridoroga, *On the completeness of the system of root vectors for first-order systems*, arXiv:1109.3683.
17. V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Naukova Dumka, Kiev, 1977. (Russian); English transl. Operator Theory: Advances and Applications, vol. 22, Birkhauser Verlag, Basel, 1986.
18. M. A. Naimark, *Linear Differential Operators*, Nauka, Moscow, 1969. (Russian); English transl. Part I: *Elementary Theory of Linear Differential Operators*, Frederick Ungar Publishing Co., New York, 1967; Part II: *Linear Differential Operators in a Hilbert Space*, Ungar, New York, 1968.
19. V. S. Rykhlov, *Compleneness of the eigenfuctions of some classes of irregular differential operators*, Uchen. Zap. TNU. Ser. Mat. Mekh. Inform. Kibern. **16** (2003), no. 1, 176–182. (Russian)
20. A. A. Shkalikov, *The completeness of the eigen- and associated functions of an ordinary differential operator with nonregular splitting boundary conditions*, Funktsional. Anal. i Prilozen. **10** (1976), no. 4, 69–80. (Russian); English transl. Funct. Anal. Appl. **10** (1976), no. 4, 305–316.
21. Ya. D. Tamarkin, *On Some General Problems of the Theory of Ordinary Linear Differential Equations and on the Expansion of Arbitrary Functions in Series*, Petrograd, 1917. (Russian)

Donets'k National University, 24 Universytets'ka, Donets'k, 83055, Ukraine
E-mail address: aganette@rambler.ru

Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine, 74 R. Luxemburg, Donets'k, 83114, Ukraine
E-mail address: mmm.@telenet.dn.ua

Donets'k National University, 24 Universytets'ka, Donets'k, 83055, Ukraine
E-mail address: vremenny-orid@mail.ru

Received 21/11/2011