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INVERSE EIGENVALUE PROBLEMS FOR NONLOCAL STURM-LIOUVILLE OPERATORS ON A STAR GRAPH

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To the memory of A. G. Kostyuchenko.

ABSTRACT. We solve the inverse spectral problem for a class of Sturm–Liouville operators with singular nonlocal potentials and nonlocal boundary conditions on a star graph.

1. INTRODUCTION

Mathematical theory of differential equations on graphs is one of topical areas in modern mathematical physics [1–4]. This is due to both the internal need of the development of the theory and solving particular applied problems related to communication, power, or transportation networks. During the last decade, the theory of quantum graphs undergoes a rapid development. Here different mathematical methods and approaches to solving important problems in quantum theory to create modern technologies, quantum computers, etc. intertwine and enrich each other [5–7]. The most interesting for the theory of quantum graphs is studying, on metric graphs, wave and diffusion equations, as well as Schrödinger equations [7–10]. Construction of self-adjoint Schrödinger operators on metric graphs with corresponding boundary-value conditions is well known. This allows to formulate and to solve a number of direct and inverse spectral problems and scattering problems for quantum graphs [11–18].

In this paper, we study direct and inverse spectral problems for a Schrödinger operator with nonlocal potential on a star graph. Inverse spectral problems on a finite length interval for the Sturm-Liouville problem with nonlocal potential and various boundary-value conditions were treated in detail in [19–22].

Let the center vertex of a star graph Γ is located at the origin, and its m edges that have the origin as a vertex have lengths l_j, $j = 1, 2, \dots, m$. We will assume that there is a function $\psi_j(x) \in W_2^2(0, l_j)$ defined on each j-th edge. Let us consider the following eigenvalue boundary-value problem with complex–valued nonlocal potentials $v_j(x) \in L_2(0, l_j)$:

$$-\frac{d^2\psi_j(x)}{dx^2} + v_j(x)\psi_j(0) = \lambda\psi_j(x), \quad 0 < x < l_j, \quad j = 1, 2, \dots, m, \tag{1}$$

with the boundary-value conditions

$$\psi_j(l_j) = 0, \quad j = 1, 2, \dots, m; \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_m(0); \tag{2}$$

and the nonlocal boundary-value conditions

$$\sum_{j=1}^{m} \left[\psi_j'(0) - \int_0^{l_j} \psi_j(x)\overline{v_j(x)}\, dx\right] = 0. \tag{3}$$

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This problem can be associated with an operator A on the space $L_2(\Gamma) = \bigoplus_{j=1}^{m} L_2(0, l_j)$ that acts on the vector-valued function

$$\psi(x) = \operatorname{col}(\psi_1(x), \psi_2(x), \dots, \psi_m(x))$$

as follows:

$$A\psi = \operatorname{col}(-\psi_1''(x) + v_1(x)\psi_1(0), \dots, -\psi_m''(x) + v_m(x)\psi_m(0)). \tag{4}$$

The domain of the operator A is defined to be all the functions

$$\psi(x) = \operatorname{col}(\psi_1(x), \psi_2(x), \dots, \psi_m(x))$$

with the components $\psi_j \in W_2^2(0, l_j)$ all of which satisfy the boundary-value conditions (2), (3).

Theorem 1. *Let the complex–valued nonlocal potentials satisfy $v_j(x) \in L_2(0, l_j)$, $j = 1, 2, \dots, m$. Then the operator A is a self-adjoint operator on the space $L_2(\Gamma)$, its spectrum consists of finite multiplicity eigenvalues λ_k tending to infinity as $k \to \infty$.*

The main goal of this paper is to solve the inverse spectral problem that consists in recovering the nonlocal potentials v_j, $j = 1, 2, \dots, m$, from the set $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ of all eigenvalues of the problem (1)–(3), counting the multiplicities.

2. Dirichlet problem

Let the functions ψ_j in problem (1)–(2) satisfy the boundary Dirichlet conditions at the center of the star graph,

$$\psi_1(0) = \psi_2(0) = \dots = \psi_m(0) = 0. \tag{5}$$

Then problem (1)-(2)-(5) splits into m Dirichlet problems for each function ψ_j,

$$-\psi_j''(x) = \lambda\psi_j(x), \quad \psi_j(0) = \psi_j(l_j) = 0, \quad j = 1, \dots, m. \tag{6}$$

Spectrum of problem (1)-(2)-(5) is a union of the spectrums of the m problems of type (6), that is, it consists of a union of the numbers $n_j \frac{\pi}{l_j}$ as j ranges from 1 to m, where n_j are naturals. Here the number $\lambda = (n_j \frac{\pi}{l_j})^2$ is an eigenvalue of problem (1)-(2)-(5) of multiplicity $k \le m$ if and only if there exist, in the star graph Γ, k rays, which have lengths $l_{j_1}, l_{j_2}, \dots, l_{j_k}$ and integers $n_{j_1}, n_{j_2}, \dots, n_{j_k}$ such that $n_{j_\nu} \frac{\pi}{l_{j_\nu}} = n_j \frac{\pi}{l_j}$, $\nu = 1, 2, \dots, k$. Denote by $\{\mu_n\}_{n=1}^{\infty}$ an increasingly ordered sequence of the numbers $n \frac{\pi}{l_j}$ with different l_j, $j = 1, \dots, m$, and integer n corresponding to their multiplicities. Naturally, $\mu_1 = \frac{\pi}{l_{\max}}$, where $l_{\max} = \max(l_1, \dots, l_m)$ and $\mu_n = n\pi(l_1 + \dots + l_m)^{-1} + O(1)$ as $n \to \infty$. The set $\{\mu_n^2\}_{n=1}^{\infty} = \Lambda_D$ is spectrum of the Dirichlet problem (6), i.e., an increasingly ordered set of all eigenvalues of this problem, counting the multiplicities.

Proposition 1. *Let the lengths l_j of rays of a star graph Γ be rationally multiples of l, that is,*

$$l_j = \frac{p_j}{q_j} l, \quad j = 1, \dots, m, \tag{7}$$

where the integers p_j and q_j are mutually prime. Then the sequence $\{\mu_n\}_{n=1}^{\infty}$ has the following periodicity property: there is an integer R such that

$$\mu_{R+k} = \mu_R + \mu_k. \tag{8}$$

Proof. If condition (7) is satisfied, then there exist the least integers n_j such that

$$\frac{n_1 q_1}{p_1} \frac{\pi}{l} = \dots = \frac{n_m q_m}{p_m} \frac{\pi}{l} = \mu_R.$$

Here, $n_j = \frac{Q}{P} \frac{p_j}{q_j}$, $R = n_1 + \dots + n_m$, where Q is the least common multiple of the numbers $(q_1, \dots, q_m)$ and P is the greatest common divisor of the numbers $(p_1, \dots, p_m)$. Since $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of all the numbers $n \frac{\pi}{l_j}$ increasingly ordered, we have (8). $\square$

Definition 1. Let $a_1 \le a_2 \le \cdots \le a_n \le \cdots$ be an infinite sequence of real numbers. A number $k \ge 2$ will be called asymptotic multiplicity of the sequence $\{a_n\}_{n=1}^{\infty}$ if for any $\varepsilon > 0$ there exists an infinite number of nonintersecting intervals of length ε containing k numbers of the sequence $\{a_k\}_{k=1}^{\infty}$, and the number k is maximal.

Proposition 2. *Let a star graph Γ have $m \ge 2$ rays. Then the asymptotic multiplicity of the sequence $\{\mu_n\}_{n=1}^{\infty}$ equals m.*

Proof. If the lengths l_j satisfy conditions of Proposition 1, then the numbers $\mu_R, \mu_{2R}, \ldots$ are m-multiples in the sequence $\{\mu_n\}_{n=1}^{\infty}$. Hence, the sequence $\{\mu_n\}_{n=1}^{\infty}$ has asymptotic multiplicity equal to m. If l_j, $j = 1, \ldots, m$, are arbitrary, we can use the Dirichlet theorem [23] stating that for arbitrary real numbers $\alpha_1, \ldots, \alpha_m$ and a natural $Q > 1$ there exist integers $q, n_1, \ldots, n_m$ such that

$$\max(|\alpha_1 q - n_1|, \ldots, |\alpha_m q - n_m|) < Q^{-\frac{1}{m}}, \quad 0 < q \le Q. \tag{9}$$

Assume that the sequence $\{\mu_n\}_{n=1}^{\infty}$ does not have asymptotic multiplicity m. Then there exist $\varepsilon_0 > 0$ and a number A such that any interval I_{ε_0} of length ε_0 and lying to the right of the point A can not contain m numbers of the sequence $\{\mu\}_{n=1}^{\infty}$. This contradicts the inequalities (9). Indeed, by setting $\alpha_j = \frac{l_j}{\pi}(A+1)$ in (9) and choosing the number Q sufficiently large so that $\frac{\pi}{l_j} Q^{-\frac{1}{m}} < \frac{\varepsilon_0}{2}$, we obtain from (9) that $|(A+1)q - n_j \frac{\pi}{l_j}| < \frac{\varepsilon_0}{2}$, $j = 1, \ldots, m$. This means that the m numbers $n_j \frac{\pi}{l_j} \in \{\mu\}_{n=1}^{\infty}$ lie in the interval $I_{\varepsilon_0}(A) = [(A+1)q - \frac{\varepsilon_0}{2}, (A+1)q + \frac{\varepsilon_0}{2}]$. The interval $I_{\varepsilon_0}(A)$ itself is located to the right of the point A. □

The fact that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is a union of m arithmetic sequences $\frac{n\pi}{l_j}$, $n = 1, 2, \ldots$, $j = 1, \ldots, m$, allows to recover the entire sequence $\{\mu_n\}_{n=1}^{\infty}$ from its asymptotic behavior.

Proposition 3. *Let $\{\mu_n^2\}_{n=1}^{\infty} = \Lambda_D$ be spectrum of Dirichlet problem (6) on a graph Γ with $m \ge 2$ rays. Let a sequence $\{a_n\}_{n=1}^{\infty}$ be asymptotically equivalent to a sequence $\{\mu_n\}_{n=1}^{\infty}$, that is,*

$$\lim_{n\to\infty} (a_n - \mu_n) = 0. \tag{10}$$

Then the sequence $\{\mu_n\}_{n=1}^{\infty}$ is uniquely defined by the sequence $\{a_n\}_{n=1}^{\infty}$.

Proof. Using Proposition 2 we see, since the asymptotic multiplicity of the sequence $\{\mu_n\}_{n=1}^{\infty}$ equals m, that the asymptotic multiplicity of the sequence $\{a_n\}_{n=1}^{\infty}$ is also m. Let $\varepsilon > 0$ be chosen and $\tilde{I}_\varepsilon = [q - \varepsilon, q]$ be a line segment containing m numbers of the sequence $\{a_n\}_{n=1}^{\infty}$ with the minimal value being $q(\varepsilon) \ge \varepsilon^{-1}$. Let $z_k(\varepsilon)$ be the k-th number of the sequence $\{a_n\}_{n=1}^{\infty}$ located to the right of the line segment $\tilde{I}_\varepsilon$. Then

$$\mu_k = \lim_{\varepsilon\to 0} [z_k(\varepsilon) - q(\varepsilon)]. \tag{11}$$

Indeed, identity (11) holds if the sequence $\{a_n\}_{n=1}^{\infty}$ is taken to be $\{\mu_n\}_{n=1}^{\infty}$, since $\{\mu_n\}_{n=1}^{\infty}$ consists of an increasingly ordered union of numbers of the form $n\frac{\pi}{l_j}$, where n are integers and $\{l_j\}_{j=1}^{m}$ are lengths of rays of the graph Γ. By (10), identity (11) remains true if $\{\mu_n\}_{n=1}^{\infty}$ is replaced with $\{a_n\}_{n=1}^{\infty}$. □

Remark 1. Let $\{a_n^2\}_{n=1}^{\infty}$ be a known spectrum of the perturbed problem (6). Then Proposition 3 implies that the spectrum $\{\mu_n^2\}_{n=1}^{\infty}$ of Dirichlet problem (6) can be uniquely determined, hence the same is true for all the lengths $l_1, \ldots, l_m$.

3. Spectral analysis of unperturbed problem

Consider the unperturbed problem (1)–(3), that is a problem with zero nonlocal potentials, $v_1 \equiv v_2 \equiv \cdots \equiv v_m \equiv 0$,

$$-\psi_j'' = \lambda\psi_j, \quad \psi_j(l_j) = 0, \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_m(0), \quad \sum_{k=1}^{m} \psi_k'(0) = 0. \tag{12}$$

Theorem 2. *The eigenvalues λ of problem (12), which are different from the eigenvalues $\{\mu_n^2\}_{n=1}^{\infty} = \Lambda_D$ of the Dirichlet problem, are simple. A number $\lambda = \mu_n^2$ is a multiplicity k eigenvalue of the unperturbed problem (12) if and only if μ_n^2 is a multiplicity $k+1$ eigenvalue of a Dirichlet problem. All eigenvalues $\{\nu_n^2\}_{n=1}^{\infty}$ of the unperturbed problem (12), indexed in an increasing order counting their multiplicities, are positive and weakly alternate with eigenvalues of the Dirichlet problem*

$$\nu_n \le \mu_n \le \nu_{n+1}. \tag{13}$$

The numbers ν_n are all positive zeros, counting the multiplicities, of the characteristic function

$$\chi_0(z) = \sum_{j=1}^{m} \cos zl_j \prod_{k\neq j} \sin zl_k. \tag{14}$$

Proof. Let λ be a multiple eigenvalue of the initial problem (1)–(3). Then there exist at least two linearly independent eigenfunctions corresponding to this eigenvalue. But then there is a nonzero linear combination of these functions such that all ψ_j become zero in $x = 0$. This means that λ is an eigenvalue of Dirichlet problem (6).

Let λ be a multiplicity $k+1$ eigenvalue of the Dirichlet problem. This means that the graph Γ has $k+1$ rays with lengths $l_{j_1}, \dots, l_{j_{k+1}}$ and there are integers $n_{j_1}, \dots, n_{j_{k+1}}$ such that $\lambda = (n_{j_p}\frac{\pi}{l_{j_p}})^2$ for any $p = 1, \dots, k+1$. But then this number λ is a multiplicity k eigenvalue of problem (12). The corresponding linearly independent eigenfunctions can be written explicitly as

$$\psi_j(x) = \sin\sqrt{\lambda}(l_{j_1} - x)\cdot(-1)^{n_{j_p}}, \quad \psi_{j_p}(x) = \sin\sqrt{\lambda}(l_{j_p} - x)\cdot(-1)^{n_{j_1}},$$
$$\psi_j(x) \equiv 0, \quad \text{if} \quad j \neq j_1, \quad j \neq j_p, \quad p = 2, \dots, k+1.$$

If the number $\lambda = (n_{j_p}\frac{\pi}{l_{j_p}})^2$ is not a multiple eigenvalue of the Dirichlet problem, then problem (12) does not have a nontrivial solution. All eigenfunctions of the Dirichlet problem, which correspond to the eigenvalue λ, can not be eigenfunctions of problem (12). Hence, a k-multiple eigenvalue $\lambda = z^2$ of problem (12) is a $k+1$-multiple eigenvalue of the Dirichlet problem, and z is a k-multiple zero of the characteristic function $\chi_0(z)$ of the form (14). If $\lambda = \nu_n^2 \notin \{\mu_n^2\}_{n=1}^{\infty}$, then this eigenvalue of problem (12) has multiplicity one. The corresponding eigenfunction $\psi = \operatorname{col}(\psi_1, \dots, \psi_m)$ has the components $\psi_j = \frac{\sin\nu_n(l_j - x)}{\sin\nu_n l_j}$, $j = 1, \dots, m$. The boundary-value condition $\sum_{j=1}^{m} \psi_j'(0) = 0$ implies that $z = \nu_n$ is a simple zero of the characteristic function $\chi_0(z)$ of the form (14). $\square$

Definition 2. Real numbers $\alpha_1, \dots, \alpha_m$, $m \ge 2$, are called rationally independent if the identity $\sum_{j=1}^{m} n_j\alpha_j = 0$ with integer $n_j \in Z$ implies that $n_1 = n_2 = \cdots = n_m = 0$.

Theorem 3. *Let a star graph Γ have all the lengths $l_1, \dots, l_m$ of its rays rationally independent. Then the following is true.*

1. *All eigenvalues of Dirichlet problem (6) have multiplicities one.*
2. *All eigenvalues of the unperturbed problem (12) have multiplicities one.*
3. *Eigenvalues of the unperturbed problems alternate with eigenvalues of the Dirichlet problem,*

$$\nu_1 < \mu_1 < \nu_2 < \cdots < \nu_n < \mu_n < \nu_{n+1} < \cdots$$

4. *All positive zeros of the characteristic function $\chi_0(z)$ of the form (14) are simple and coincide with $\{\nu_n\}_{n=1}^{\infty}$. The characteristic function $\chi_0(z)$ takes values of opposed signs on two subsequent members of the sequence $\{\mu_n\}_{n=1}^{\infty}$,*

$$\chi_0(\mu_n)\chi_0(\mu_{n+1}) < 0. \tag{15}$$

There exists a sequence $\mu_{n_k} \to \infty$ such that $\lim_{k\to\infty} \chi_0(\mu_{n_k}) = 1$.
5. *The sequence $\{\nu_n\}_{n=1}^{\infty}$ has asymptotic multiplicity $m-1$.*
6. *The sequence $\{\nu_n\}_{n=1}^{\infty}$ can be uniquely determined from an asymptotically equivalent sequence $\{b_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}(b_n - \nu_n) = 0$.*
7. *If $m \geq 3$, the spectrum $\{\nu_n^2\}_{n=1}^{\infty}$ of the unperturbed problem (12) uniquely defines the number of rays in the graph Γ and their lengths $l_1, \dots, l_m$.*

Proof. Property 1 follows from the explicit form of the numbers $\{\nu_n\}_{n=1}^{\infty}$ that are a union of different numbers of the form $n_j \frac{\pi}{l_j}$. These numbers are distinct, since the identity $n_j \frac{\pi}{l_j} = n_k \frac{\pi}{l_k}$ with integer n_j and n_k is equivalent to the identity $n_j l_k - n_k l_j = 0$ which is impossible due to the assumption of rational independence of the lengths $l_1, \dots, l_m$.

Property 2 follows from Theorem 2 and the proved Property 1. In the case under consideration, none of the numbers ν_n can coincide with any of the numbers μ_k for, otherwise, the number μ_k would have multiplicity 2, which is impossible in virtue of 1. The numbers ν_n^2 are eigenvalues of the unperturbed problem (12) and they differ from eigenvalues of the Dirichlet problem. The eigenfunction corresponding to this eigenvalue is $\psi(x) = \operatorname{col}(\psi_1(x), \psi_2(x), \dots, \psi_m(x))$ with the components of the form $\psi_j(x) = \frac{\sin \nu_n (x - l_j)}{\sin \nu_n l_j}$, $j = 1, \dots, m$, and must satisfy the boundary-value condition $\sum_{j=1}^{m} \psi_j'(0) = 0$. This leads to a characteristic equation for ν_n,

$$\sum_{j=1}^{m} \frac{\cos \nu_n l_j}{\sin \nu_n l_j} = 0. \tag{16}$$

It immediately follows from (16) that positive solutions ν_n of this equation lie between neighboring zeros of all the denominators, i.e., between the numbers $\{\mu_n\}_{n=1}^{\infty}$. Also, there is exactly one solution of characteristic equation (16) between two consecutive values μ_n and μ_{n+1}. This gives Property 3.

Since positive roots of equation (16) coincide with positive roots of the characteristic function $\chi_0(z) = \sum_{j=1}^{m} \cos z l_j \prod_{k\neq j} \sin z l_k$, we obtain the first part of property 4. Let us now prove (15). Let $\mu_n = n_j \frac{\pi}{l_j}$ and $\mu_{n+1} = n_k \frac{\pi}{l_k}$ be two consecutive values in $\{\mu_n\}_{n=1}^{\infty}$. As z ranges over the interval (μ_n, μ_{n+1}), all the functions $\sin z l_p$, $p = 1, \dots, m$, preserve their signs. From the explicit form (14) of the characteristic function, we see that $\chi_0(\mu_n) = \cos \mu_n l_j \prod_{p\neq j} \sin \mu_n l_p$ and $\chi_0(\mu_{n+1}) = \cos \mu_{n+1} l_k \prod_{p \neq k} \sin \mu_{n+1} l_p$. This gives (15). Using the Dirichlet theorem in the form of (9), one can show that there is a sequence $\mu_{n_k} \to \infty$ such that $\lim_{k\to\infty} \chi_0(\mu_{n_k}) = 1$.

Property 5 holds, since asymptotic multiplicity of $\{\nu_n\}_{n=1}^{\infty}$ is always less then or equal to $m-1$. Numbers in the sequence $\{\nu_n\}_{n=1}^{\infty}$ alternate with the numbers $\{\mu_n\}_{n=1}^{\infty}$, and their asymptotic multiplicity equals m by Proposition 2.

To prove Property 6, we use the construction in Proposition 3. Let us show that

$$\nu_k = \lim_{\varepsilon\to 0} [b_k(\varepsilon) - p(\varepsilon)], \tag{17}$$

where $b_k(\varepsilon)$ is the k-th member of the sequence $\{b_n\}_{n=1}^{\infty}$ after the interval $\tilde{I}_\varepsilon \in [p-\varepsilon, p]$. Here, the interval $\tilde{I}_\varepsilon$ contains $m-1$ consecutive numbers from $\{b_n\}_{n=1}^{\infty}$. Since

$$\lim_{n\to\infty} (b_n - \nu_n) = 0,$$

we can assume that the interval $\tilde{I}_\varepsilon$ itself contains $m-1$ consecutive numbers from $\{\nu_n\}_{n=1}^{\infty}$, and the number p is a multiple of π. This can be achieved by shifting the interval $\tilde{I}_\varepsilon$ to the right. Let $\delta > 0$ be arbitrary. Choose ε to be so small that the first k zeros of the equation

$$\sum_{j=1}^{m} \cot z(l_j - \delta_j) = 0$$

with $|\delta_j| < \varepsilon$ would differ from the corresponding first k zeros of the characteristic equation (16) by less than $\frac{\delta}{2}$ and $|b_n - \nu_n| < \frac{\delta}{2}$ for all b_n lying on the right of the interval $\tilde{I}_\varepsilon$. Then $|\nu_k - (b_k(\varepsilon) - p(\varepsilon))| < \delta$. Hence, the limit in (17) always exists, and it gives a method for constructing $\{\nu_n\}_{n=1}^{\infty}$ from $\{b_n\}_{n=1}^{\infty}$. □

Theorem 4. *Let a star graph* Γ *have* $m \geq 3$ *rays with rationally independent lengths* $l_1, \dots, l_m$. *Let a given sequence* $\{b_n\}_{n=1}^{\infty}$ *be asymptotically equivalent to a sequence* $\{\nu_n\}_{n=1}^{\infty}$. *Then the sequence* $\{b_n\}_{n=1}^{\infty}$ *uniquely determines the metric on the graph* Γ, *i.e., the lengths* $l_1, \dots, l_m$ *of all the rays.*

Proof. Using Properties 6 and 4 in Theorem 3 construct the entire sequence $\{\nu_n\}_{n=1}^{\infty}$ from $\{b_n\}_{n=1}^{\infty}$. Since ν_n are zeros of the characteristic function $\chi_0(z)$, which is an analytic function of exponential type 1 and has zeros symmetric with respect to the point $z = 0$, and the zeros have the asymptotics $\nu_n = \frac{n\pi}{l_1 + \cdots + l_m} + o(1)$ as $n \to \infty$, the function $\chi_0(z)$ can be represented as an infinite product

$$\chi_0(z) = C_0 z^{m-1} \prod_{k=1}^{\infty} \Big(1 - \frac{z^2}{\nu_k^2}\Big). \tag{18}$$

The constant $C_0 = \prod_{k=1}^{m} l_k \sum_{j=1}^{m} l_j^{-1}$ and the integer m are uniquely defined, since it follows from Property 4 in Theorem 3 that $\lim_{k\to\infty} \chi_0(\mu_{n_k}) = 1$. The explicit form (14) for the characteristic function $\chi_0(x)$ shows that it is almost periodic with a finite number of periods, that is,

$$\chi_0(x) = \sum_{k} C_k e^{i\omega_k x}, \tag{19}$$

where $k = (k_1, \dots, k_m)$, $k_j = 0, 1$, $|k| = \sum_{j=1}^{m} k_j$, $\omega_k = \sum_{j=1}^{m} (-1)^{k_j} l_j$, and $C_k = \frac{1}{2^m i^{m-1}}(m - 2|k|)$. Since all the lengths $l_1, \dots, l_m$ are rationally independent, all the frequencies ω_k are distinct and can be uniquely determined from χ_0, since

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_0(x) e^{-i\omega x}\, dx \neq 0$$

only for $\omega = \omega_k$.

But then all ω_k uniquely define all lengths $l_1, \dots, l_m$ of the rays of the graph Γ. □

4. Spectral analysis of the problem with nonlocal potential

We will be studying problem (1)–(3) only in the case where the lengths l_j of the rays of the graph are rationally independent. Having nonlocal potentials $v_j(x) \in L_2(0, l_j)$, $j = 1, 2, \dots, m$, in problem (1)–(3), it is useful to represent them in the form of a Fourier series,

$$v_j(x) = \sum_{n=1}^{\infty} v_{j,n} \sin\Big[\frac{n\pi}{l_j}(l_j - x)\Big], \tag{20}$$

where the coefficients are expressed in terms of the potentials,

$$v_{j,n}(x) = \frac{2}{l_j} \int_0^{l_j} v_j(x) \sin\Big[\frac{n\pi}{l_j}(l_j - x)\Big] dx. \tag{21}$$

Theorem 5. *Let, in problem (1)–(3), the nonlocal potentials satisfy* $v_j(x) \in L_2(0, l_j)$ *and the lengths* $l_1, \dots, l_m$ *of rays of the star graph* Γ *be rationally independent. Then the following holds true.*

1. *All eigenvalues of problem (1)–(3), distinct from* $(\frac{n\pi}{l_j})^2$, $n \in N$, *are simple.*

2. *The number* $\lambda = (\frac{n\pi}{l_j})^2$ *is an eigenvalue of problem (1)–(3) if and only if*

$$v_{j,n} = (-1)^{n+1}\frac{2n\pi}{l_j^2}. \tag{22}$$

3. *The number* $\lambda = (\frac{n\pi}{l_j})^2$ *is a double eigenvalue of problem (1)–(3) if and only if, in addition to (22), we have*

$$\frac{3}{2l_j} + \sum_{k\neq n}\frac{\alpha_{j,k}(-1)^k\frac{k\pi}{l_j}}{(\frac{n\pi}{l_j})^2-(\frac{k\pi}{l_j})^2} + \sum_{p\neq j}\Big(\frac{n\pi}{l_j}\cot\frac{n\pi l_p}{l_j} + \sum_{k=1}^{\infty}\frac{\alpha_{p,k}(-1)^k\frac{k\pi}{l_p}}{(\frac{n\pi}{l_j})^2-(\frac{k\pi}{l_p})^2}\Big) = 0, \tag{23}$$

where

$$\alpha_{j,k} = v_{j,k} + \overline{v_{j,k}} + (-1)^k\frac{l_j^2}{2k\pi}|v_{j,k}|^2.$$

4. *The number* $\lambda = 0$ *is an eigenvalue of problem (1)–(3) if and only if*

$$\sum_{j=1}^{m}\Big(\frac{1}{l_j} - \sum_{k=1}^{\infty}\frac{\alpha_{j,k}(-1)^k}{\frac{k\pi}{l_j}}\Big) = 0. \tag{24}$$

5. *Problem (1)–(3) has no eigenvalues with multiplicity exceeding* 2.

Proof. The proof of Proposition 1 was given at the beginning of the proof of Theorem 2. Let $\lambda = (\frac{n\pi}{l_j})^2$ be an eigenvalue of problem (1)–(3). Then nontrivial solutions of the equation $-\psi_j''(x) + v_j(x)\psi_j(0) = \lambda\psi_j(x)$ satisfying the condition $\psi_j(l_j) = 0$ become either zero in $x = 0$ and then they are multiples of $\sin[\frac{n\pi}{l_j}(l_j - x)]$ or they are distinct from zero. In the latter case, solutions exist only if conditions (22) are satisfied. This solution is a multiple of the function $\hat{\psi}_j$,

$$\hat{\psi}_j = (-1)^n\frac{(l_j - x)}{l_j}\cos\Big[\frac{n\pi}{l_j}(l_j - x)\Big] + \sum_{k\neq n}\frac{v_{j,k}\sin[\frac{k\pi}{l_j}(l_j - x)]}{(\frac{n\pi}{l_j})^2 - (\frac{k\pi}{l_j})^2}. \tag{25}$$

If $\psi_j(x) = \sin[\frac{n\pi}{l_j}(l_j - x)]$, then all $\psi_k(x) \equiv 0$ for $k \neq j$ in problem (1)–(2), since the lengths l_k can not be rationally expressed in terms of l_j. Substituting these functions into the boundary-value condition (3) leads to (23). If, in problem (1)–(2), $\psi_j(x) = \hat{\psi}_j$ given by (25), then $\psi_j(0) = 1$ and all $\psi_p(x)$, for $p \neq j$, can be represented as

$$\hat{\psi}_p = \frac{\sin[\frac{n\pi}{l_j}(l_p - x)]}{\sin\frac{n\pi}{l_j}l_p} + \sum_k\frac{v_{p,k}\sin[\frac{k\pi}{l_p}(l_p - x)]}{(\frac{n\pi}{l_j})^2 - (\frac{k\pi}{l_p})^2}, \quad p \neq j. \tag{26}$$

By substituting such a solution into the boundary-condition (3) we obtain condition (23). It is easy to see that the functions $\psi = (\hat{\psi}_1, \dots \hat{\psi}_m)$, where $\hat{\psi}_j$, $\hat{\psi}_p$, are given by identities (25), (26), become solutions of problem (1)–(2) with $\lambda = 0$, if we formally make $n \to 0$. Substituting this solution into the boundary-value condition (3) leads to identity (24), which is a condition that this function is an eigenfunction of problem (1)–(3) corresponding to the eigenvalue $\lambda = 0$.

It was shown above that problem (1)–(2) can not have two linearly independent solutions, hence problem (1)-(2)-(3) can not have eigenvalues with multiplicities exceeding 2. □

To give an exact description of the distribution of eigenvalues of problem (1)–(3), it is convenient to show that the eigenvalues are connected with zeros of an analytic function, which is a characteristic function of problem (1)–(3).

To this end, consider a special solution of equation (1) with $\lambda = z^2$, satisfying condition (2),

$$\varphi_j(x; z) = \Big(\sin z(l_j - x) + \sin zl_j\sum_k\frac{v_{j,k}\sin[\frac{k\pi}{l_j}(l_j - x)]}{z^2 - (\frac{k\pi}{l_j})^2}\Big)\prod_{k\neq j}\sin zl_k. \tag{27}$$

The function φ is an eigenfunction of problem (1)-(3) if it satisfies the boundary-value condition (3). This gives the characteristic equation $\chi(z) = 0$, where the characteristic function $\chi(z)$ is defined by $\chi(z) = \sum_{j=1}^{m} [\varphi_j'(0) - (\varphi_j, v_j)]$ and has the form

$$\chi(z) = \sum_{j=1}^{m} \Big(\cos zl_j + \frac{\sin zl_j}{z} \sum_{n=1}^{\infty} \frac{\alpha_{j,n}(-1)^n \frac{n\pi}{l_j}}{z^2 - (\frac{n\pi}{l_j})^2}\Big) \prod_{k\neq j} \sin zl_k, \tag{28}$$

where

$$\alpha_{j,n} = v_{j,n} + \overline{v_{j,n}} + (-1)^n \frac{l_j^2}{2n\pi} |v_{j,n}|^2. \tag{29}$$

Lemma 1. *The characteristic function $\chi(z)$ of the form (28) is an entire analytic function of z and, for $z = \frac{n\pi}{l_j}$, $n \in N$, takes the values*

$$\chi\Big(\frac{n\pi}{l_j}\Big) = \Big|1 + (-1)^n \frac{l_j^2}{2n\pi} v_{j,n}\Big|^2 \chi_0\Big(\frac{n\pi}{l_j}\Big). \tag{30}$$

Proof. The proof is carried out by direct computations using the explicit form (28) of the characteristic function. □

Theorem 6. *The number $\lambda = z^2 \neq 0$ is an eigenvalue of problem (1)-(3) if and only if z is a zero of the characteristic function $\chi(z)$. The number $\lambda = z^2 \neq 0$ is a double eigenvalue of the problem (1), (2) if and only if z is a double zero of the characteristic function. All zeros $z \neq 0$, $z^2 \notin \Lambda_D$, of the characteristic function are simple. The characteristic function does not have zeros of multiplicities greater than 2, distinct from $z = 0$. The number $\lambda = 0$ is an eigenvalue of problem (1)–(3) if and only if $z = 0$ and is a zero of the characteristic function $\chi(z)$ with multiplicity $m + 1$.*

Proof. It follows that the squares $z^2 \notin \{\mu_n^2\}_{n=1}^{\infty}$ of zeros of the characteristic function $\chi(z)$ are eigenvalues from the fact that the special solution (27) is an eigenfunction and vice versa. For $z^2 = (\frac{n\pi}{l_j})^2 = \lambda$, the number $\lambda = z^2$ is an eigenvalue if and only if (22) is satisfied which, by (30), is equivalent to $\chi(\frac{n\pi}{l_j}) = 0$. It is easy to check that condition (23) is equivalent to $\dot\chi(\frac{n\pi}{l_j}) = 0$ which, in its turn, together with $\chi(\frac{n\pi}{l_j}) = 0$, is equivalent to that $\frac{n\pi}{l_j}$ is a double eigenvalue. Condition (24), is equivalent to $z = 0$ is a zero of the characteristic function $\chi(z)$ with multiplicity $m + 1$. Hence, the eigenvalues, counting multiplicities, coincide with squares of zeros of $\chi(z)$, counting multiplicities.

If the function $\chi(z)$ had a multiple root $z_0 \neq \frac{n\pi}{l_j}$, this would imply that $\frac{\partial}{\partial z}\varphi(x; z)|_{z=z_0}$ were a generalized eigenfunction, which is impossible since the operator A is self-adjoint. In the same way, we can prove that there are no zeros of $\chi(z)$ that have multiplicities greater than 2 for $z \in \{\mu_n\}_{n=1}^{\infty}$. □

Since eigenvalues of problem (1)–(3) are squares of zeros of the characteristic function $\chi(z)$, it is important to describe the distribution of zeros of the function $\chi(z)$ having form (28). Write the characteristic function in (28), $\chi(z)$, as

$$\chi(z) = \sum_{j=1}^{m} \Big(\cos zl_j + \frac{1}{z} \int_0^{l_j} \alpha_j(t) \sin z(l_j - t)\, dt\Big) \prod_{k\neq j} \sin(zl_k), \tag{31}$$

where $\alpha_j(t) \in L_2(0, l_j)$, and the numbers $\alpha_{j,n}$ in (29) are coefficients in the Fourier sine expansion of the function α_j.

Representation (31) for the characteristic function χ permits to recover the function χ from the set of its zeros. We use the following analogue of a result due to Marchenko, see [24], Lemma 3.4.2.

Lemma 2. *For an entire function of the form (31) to admit the representation*

$$\chi(z) = Cz^{m-1} \prod_{k=1}^{\infty} \frac{z_k^2 - z^2}{\nu_k^2}, \tag{32}$$

it is necessary and sufficient that

$$z_k^2 = \nu_k^2 + \alpha_k,$$

where $\{\alpha_k\}_{k=1}^{\infty} \in l_2$ *and* $\{\beta_n\}_{n=1}^{\infty} \in l_2,\ \beta_n = \sum_{k=-m}^{m} \alpha_{n+k}(\nu_{n+k} - \mu_n)^{-1}$.

Theorem 7. *The increasingly ordered sequence* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \lambda_{n+1} \le \cdots$ *of all eigenvalues of problem (1)–(3), counting multiplicities, has the following properties:*

1. *the sequence weakly alternates with the sequence* $\{\mu_n^2\}_{n=1}^{\infty} = \Lambda_D$ *of eigenvalues of the Dirichlet problem,*

$$\lambda_n \le \mu_n^2 \le \lambda_{n+1}, \quad n \in N; \tag{33}$$

2. *there is an asymptotic representation,*

$$\lambda_n = \nu_n^2 + \alpha_n, \tag{34}$$

where $\{\nu_n^2\}_{n=1}^{\infty}$ *is spectrum of the unperturbed problem (12),* $\sum_{j=1}^{\infty} \alpha_j^2 < +\infty$, $\sum_{n=1}^{\infty} \beta_n^2 < +\infty$ *and* $\beta_n = \sum_{k=-m}^{m} \alpha_{n+k}(\nu_{n+k} - \mu_n)^{-1}$.

Conditions 1 and 2 are necessary and sufficient for a sequence $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ *to be an ordered sequence of all eigenvalues of problem (1)–(3) with nonlocal potentials* $v_j \in L_2(0, l_j)$.

Proof. By Property 4 in Theorem 3, there exists a sequence μ_{n_k} such that

$$\lim_{n_k \to \infty} \chi_0(\mu_{n_k}) = 1. \tag{35}$$

Indeed, by the Rouché theorem, the entire function $\chi(z)$ and the function $\chi_0(z)$ have the same number of zeros, counting multiplicities, in the strip $-\mu_{n_k} < \operatorname{Re} z < \mu_{n_k}$ for large n_k. Hence, by Theorem 6, the eigenvalues λ_n of problem (1)–(3) are squares of zeros of the function $\chi(z)$. The functions $\chi(z)$ and $\chi_0(z)$ have zeros that are symmetric with respect to the point $z = 0$. Hence, the number of eigenvalues λ_n in problem (1)–(3) satisfying the condition $\lambda_n < \mu_{n_k}^2$ equals the number of eigenvalues of the unperturbed problem, ν_n^2, satisfying the condition $\nu_n^2 < \mu_{n_k}$. By Theorem 3, the number of them is exactly n_k. This gives the inequality $\lambda_n < \mu_{n_k}^2$ for $n_k \to \infty$.

On the other hand, if

$$v_{j,n} \ne (-1)^{n+1} \frac{2n\pi}{l_j^2}, \tag{36}$$

then, by Theorem 5, $\chi(\mu_n) \ne 0$ for any n. Hence, in every interval (μ_n, μ_{n+1}), the characteristic function χ has at least one zero z_0, since both the function $\chi(z)$ and the function $\chi_0(z)$ take different values at the endpoints of the interval (μ_n, μ_{n+1}) by (15) and (30). Consequently, there is one eigenvalue $\lambda = z^2$ in the interval $I_n = (\mu_n^2, \mu_{n+1}^2)$. The assumption that at least one interval I_n contains more than one eigenvalue leads to a contradiction with the estimate $\lambda_n < \mu_{n_k}^2$. Hence, if conditions (38) are satisfied, inequality (34) holds,

$$\lambda_n < \mu_n^2 < \lambda_{n+1}, \quad n \in N. \tag{37}$$

Since condition (38) can be satisfied by an arbitrary small change of the potential, passing to the limit in (38) we get (34).

The asymptotic representations (34) follows from the asymptotic representation of zeros of the characteristic function $\chi(z)$ defined by Lemma 2.

Sufficiency of Properties 1 and 2 in Theorem 6 for a sequence $\{\lambda_n\}_{n=1}^{\infty}$ to be a sequence of eigenvalues of problem (1)–(3) follows from Lemma 2. The sequence $\{\lambda_n\}_{n=1}^{\infty}$ uniquely defines the functions $\alpha_j \in L_2(0, l_j)$ and their Fourier coefficients $\alpha_{j,n}$. But then equation (29) permits to determine $\{v_{j,n}\}_{n=1}^{\infty} \in l_2$ and, consequently, the nonlocal potentials $v_j \in L_2(0, l_j)$ as Fourier series (20). Here, $\{\lambda_n\}_{n=1}^{\infty}$ will be spectrum of problem (1)–(3) with such nonlocal potentials. ∎

5. Inverse spectral problem

The Sturm-Liouville nonlocal inverse spectral problem on a star graph, i.e., the problem inverse to the eigenvalue problem (1)–(3) consists in finding the number m of rays of the graph Γ, the lengths l_j of the rays, and the local potentials $v_j(x)$, $j = 1, \dots, m$, from the entire set $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ of all eigenvalues of problem (1)–(3), counting their multiplicities. If the number of rays of the graph is $m \geq 3$ and all their lengths are rationally independent, Theorem 6 gives a description of the set Λ and formulas (30) and (32) allow to formulate an algorithm for solving the inverse problem in the form of the following 6 steps.

Step 1. From the spectrum $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$, using Theorem 4, construct a sequence $\{\nu_n\}_{n=1}^{\infty}$ the squares of which give spectrum of the unperturbed problem, and their asymptotic multiplicity, $m-1$, defines the number of rays of the graph Γ.

Step 2. Using the sequence $\{\nu_n\}_{n=1}^{\infty}$ determine the characteristic function $\chi_0(z)$ of the unperturbed problem in the form of an infinite product, $\chi_0(z) = C_0 z^{m-1} \prod_{k=1}^{\infty} (1 - \frac{z^2}{\nu_k^2})$. Using the function $\chi_0(z)$, we determine all lengths $l_1 > l_2 > \cdots > l_m$ of the rays of the graph and the numbers $\{\mu_n\}_{n=1}^{\infty}$.

Step 3. Construct the characteristic function $\chi(z)$ for problem (1)–(3) in the form of an infinite product

$$\chi(z) = C_0 z^{m-1} \prod_{k=1}^{\infty} \frac{\lambda_k^2 - z^2}{\nu_k^2}.$$

Step 4. Find the values $\chi(\frac{n\pi}{l_j})$, $j = 1, \dots, m$, $n \in N$.

Step 5. Solving a quadratic equation for $v_{j,n}$,

$$\Big|1 + \frac{(-1)^n l_j^2}{2n\pi} v_{j,n}\Big|^2 = \chi\Big(\frac{n\pi}{l_j}\Big)\chi_0^{-1}\Big(\frac{n\pi}{l_j}\Big), \tag{38}$$

find the value of $v_{j,n}$ having the least modulus.

Step 6. The nonlocal potentials $v_j(x)$ are given by their Fourier series (20) with the determined Fourier coefficients $v_{j,n}$.

If the graph Γ has two rays with the lengths l_1 and l_2 being rationally independent and given, then one can find the nonlocal potentials v_1 and v_2 from the spectrum Λ of problem (1)–(3) using the above algorithm starting with step 3.

Example 1. Let the lengths $l_1, l_2, \dots, l_m$ of the rays of the graph Γ be known and rationally independent. Assume we are given eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of problem (1)–(3) and, starting with $k = n+1$, they coincide with eigenvalues of the unperturbed problem, $\lambda_k = \nu_k^2$, $k > n$. Then the nonlocal potentials $v_j(x)$ in problem (1)–(3) can be found using the above algorithm starting with step 3. In this case, the characteristic function $\chi(z)$ differs from $\chi_0(z)$ only by a rational function $R(z)$,

$$R(z) = \prod_{k=1}^{n} \frac{\lambda_k - z^2}{\nu_k^2 - z^2}.$$

Hence, identity (38) becomes

$$\Big|1 + \frac{(-1)^n l_j^2}{2n\pi} v_{j,n}\Big|^2 = R\Big(\frac{n\pi}{l_j}\Big),$$

which gives $v_{j,n}$.

Remark 2. For star graphs, as in the case of a finite interval, there exist isospectral nonlocal potentials. This is connected with non-uniqueness of a solution of the quadratic equations (38). However, for star graphs as well, one can give an effective description of isospectral nonlocal potentials and describe large classes of potentials for which the inverse spectral problem has a unique solution, similarly as it is done in [19–22].

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