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LINEARIZATION OF DOUBLE-INFINITE TODA LATTICE BY MEANS OF INVERSE SPECTRAL PROBLEM

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Dedicated to the fond memory of dear Tolya Kostyuchenko

Abstract. The author earlier in [3, 4, 6, 7] proposed some way of integration the Cauchy problem for semi-infinite Toda lattices using the inverse spectral problem for Jacobi matrices. Such a way for double-infinite Toda lattices is more complicated and was proposed in [9]. This article is devoted to a detailed account of the result [3, 4, 6, 7, 9]. It is necessary to note that in the case of double-infinite lattices we cannot give a general solution of the corresponding linear system of differential equations for spectral matrix. Therefore, in this case the corresponding results can only be understood as a procedure of finding the solution of the Toda lattice.

1. Introduction

The classical double-infinite Toda lattice [60] is a nonlinear difference-differential equation of the form

$$\begin{aligned}
&\dot{\alpha}_n(t) = \frac{1}{2}\alpha_n(t)(\beta_{n+1}(t) - \beta_n(t)),\\
&\dot{\beta}_n(t) = \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}, \quad t \in [0, T), \quad T > 0.
\end{aligned} \tag{1}$$

Here the unknowns $\alpha_n(t), \beta_n(t)$ are real continuously differentiable functions, $\cdot = \frac{d}{dt}$. For (1) it is possible to pose a Cauchy problem as follows: for given initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{Z}$, it is necessary to find solutions $\alpha_n(t), \beta_n(t), n \in \mathbb{Z}, t \in [0, T)$.

Note, that this system in fact is some Hamiltonian system describing the dynamics of a chain of particles $q_n(t), n \in \mathbb{Z}$, on a straight line with exponential interactions. To be more specific, $\alpha_n(t), \beta_n(t)$ are some coordinates of $q_n(t)$ and $\dot{q}_n(t)$ (Flashka variables [34, 35]). To stufy a Cauchy problem for (1) is an essential problem with large physical literature. We mention some of corresponding articles, which are essential for us.

In the case of a finite number of equations in (1), where $\mathbb{Z}$ is replaced with the finite set $\{0, \dots, N\}$, the initial essential for us result were obtained in [44, 55].

For semi-infinite case, where $\mathbb{Z}$ is replaced with $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, there were results similar to the classical method of integration of the Cauchy problem for the KdV equation on $(x, t) \in [0, \infty) \times [0, T)$ by means of the inverse spectral problem for the Sturm-Liouville equation for $x \in [0, \infty)$. In our case, instead of the Sturm-Liouville equation, we take its difference analogue, — the classical Jacobi matrix. Corresponding results are published in [3, 4, 6, 7]. Now, the unknowns $\alpha_n(t)$ are supposed to be positive and the functions $\alpha_n(t), \beta_n(t)$ are bounded uniformly w.r.t. $n \in \mathbb{N}_0$.

Let us explain the main idea of this approach. The equation (1) is connected with a classical Jacobi matrix $J(t)$ having $(\beta_n(t))_{n=0}^{\infty}$ in the main diagonal and $(\alpha_n(t))_{n=0}^{\infty}$ in two neighboring equal diagonals. The knowledge of a solution of our Cauchy problem

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is equivalent to the knowledge of the matrix $J(t)$. The matrix $J(t)$ acts in the space $\ell_2 = \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \dots$ (i.e. ℓ_2 on $\mathbb{N}_0$) and generates a bounded selfadjoint operator $\mathbf{J}(t)$ (its boundness follows from our assumption: we consider only bounded solutions of (1)).

The essential point is that the evolution in time $t \in [0, T)$ of the matrix $J(t)$ is complicated and is given by equation (1), but the evolution of its spectral measure $d\rho(\lambda; t)$ is simple, namely

$$d\rho(\lambda; t) = e^{\lambda t} d\rho(\lambda; 0), \quad \lambda \in \mathbb{R}, \quad t \in [0, T). \tag{2}$$

This fact is a main exception of Toda system. Note that, from (2), it follows that the spectrum of the operator $\mathbf{J}(t)$ does not depend on t. Let us also explain that using (1) we can find, for $\rho(\lambda; t)$, a simple differential equation w.r.t. t and (2) is its solution.

Now finding a solution of our Cauchy problem is very simple, — using the initial data $\alpha_0(0), \beta_0(0), n \in \mathbb{N}_0$, i.e., the matrix $J(0)$ we find its spectral measure $d\rho(\lambda; 0)$. Then using (2) we know the spectral measure $d\rho(\lambda; t)$ of the matrix $J(t)$. This knowledge gives the knowledge of $J(t)$ by classical formulas (it is necessary to use the orthogonalization of powers $1, \lambda, \lambda^2, \dots$ etc, i.e., to solve the inverse spectral problem for the Jacobi matrix). So, we find a solution of our Cauchy problem.

Some articles give generalizations of this approach to semi-infinite case. So, [58] investigates solutions of a Toda lattice, which are not necessarily bounded (then the Hermitian operators $\mathbf{J}(t)$ are in general unbounded). Lattices more general than (1) in the case where the formula (2) is more complicated, were considered in [12, 17, 57, 18, 53, 54] (a "nonisospectral case" when the spectrum of $\mathbf{J}(t)$ changes in time). A case of matrix (or operator) equations of type (1) were considered in [36, 15]; see also [22, 23]. Now the corresponding operators $\mathbf{J}(t)$ act on the space $\mathbb{C}^d \oplus \mathbb{C}^d \oplus \dots$, where $d > 1$ is the dimension of matrix-values of the solutions (or in an orthogonal sum of the fixed Hilbert space in which the values of our solutions lie). A spectral theory for the corresponding Jacobi-type matrices was developed in [50, 2, 10, 11]. New classes of Toda soliton solutions were found in [39]. For some other results on Toda lattices, which are connected with this inverse spectral approach, see [45, 26, 24, 40, 48, 49, 59, 33, 38]. The application of such an approach to Schur flows was given in [41].

Let us now pass to the double-infinite Toda lattices. Note that a direct application of the spectral theory of classical Jacobi matrices now is impossible since n must range over $\mathbb{Z}$ and cannot serve as indexes m, n of elements of the Jacobi matrix $(a_{m,n})_{m,n=0}^{\infty}$. The situation with such Toda lattices is essentially more complicated as compared with the case $\mathbb{N}_0$. At first we note some essential results which are not directly connected with our inverse spectral problem approach.

In the classical work [60], Toda has applied, to the integration of a Cauchy problem for (1), the difference analogue of the inverse scattering problem for the Sturm-Liouville equation on the whole axis $\mathbb{R} \ni x$ with the operator equal to a discrete analog of the Laplacian. This approach gives a possibility to find the set of corresponding solutions. Earlier works in this direction are [52, 34, 35].

In the work [51] the case of periodic solutions was investigated by using theta functions (also see the previous work [44]). In the series of works [59, 19, 20, 21, 27, 28, 29, 30, 31, 32], finding solutions of Cauchy problem for (1) was given in terms of a difference analogue of the scattering theory for the Schroedinger equation with periodic potential; also there was considered the case where the potential tends to different constants when $n \to +\infty$ and $n \to -\infty$. Such an approach was generalized in the above mentioned works as to find solutions connected with the case more general than the periodic one. Namely, finite-zones potentials of the corresponding underlying operators were investigated. Let us explain that an application of the scattering theory for finding solutions of (1) is based on the fact that if the underlying potential changes in according to (1), then scattering data changes in a simple way (similar to (2) in our approach) .

Essential results concerning finding of the solutions of a Cauchy problem for double-infinite equation (1) were obtained in [63, 64, 65]. In particular, solutions that strongly tend to zero as $|n| \to \infty$ were founded. These results are intimately connected with our approach.

The problem of integrating a Cauchy problem for (1) without some essential conditions on the structure of solutions was open. In the work [13] the authors tried to find general solutions of this problem for (1) using an interpretation of equation (1) as a $2\times2-$matrix equation in the space $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots$. So, we get some matrix-valued Toda system, but the noncommutativity of matrices gives a complicated differential equation for the matrix $\rho(\lambda;t)$.

In this approach, the authors used the standard duplication method, — the double infinite vector $\xi = (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots)$, $\xi_n \in \mathbb{C}^1$, is understood as the vector $(x_0, x_1, \ldots) \in \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots$, where $x_n = (\xi_n, \xi_{-n-1}) \in \mathbb{C}^2$, i.e. $\xi \leftrightarrow (x_0 = (\xi_0, \xi_{-1}), x_1 = (\xi_1, \xi_{-2}), \ldots)$.

But, it is possible to make another more convenient duplication,

$$\xi = (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots) \leftrightarrow (x_0 = \xi_0, x_1 = (\xi_1, \xi_{-1}), x_2 = (\xi_2, \xi_{-2}), \ldots). \tag{3}$$

In this duplication, vectors ξ transfer into vectors from the space

$$\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots . \tag{4}$$

In our duplication, a Toda lattice transfers into an equation relatively the unknowns with values in the space (4). To apply the approach [3, 4, 6, 7] in our case, it is necessary to develop a spectral theory of Jacobi-type matrices acting on the space (4). Note that this theory is essentially a spectral theory of ordinary double-infinite Jacobi matrices in the space ℓ_2 on $\mathbb{Z}$.

It is possible to make such a generalization, and the corresponding results are published in the article [9]. Note that the spectral theory of such Jacobi-type matrices is nonstandard, its development connect with some construction found in articles [8, 43].

It is necessary to say that the above mentioned spectral theory of selfadjoint Jacobi-type matrices in the space (4) can be developed thanks to the general theory of generalized eigenvectors of selfadjoint operators and expansions with respect to such eigenvectors. Such notions were at first introduced in the article [37, 1]; we used the corresponding results from the books [2, 14, 16].

The present article is devoted to a complete exposition of the above mentioned results from [9] (the last article also contains many other facts but we will touch only the results connected with equation (1)). It is also necessary to say that some of the results from [3, 4, 6, 7, 9] were proved in a formal way (for example, deduction of differential equations for the spectral measure). In the present article, we give a mathematically strong deduction of the corresponding results.

We start with a semi-infinite Toda lattice (Section 2) and give a mathematically strong account of these results. Note that we do not include some examples of finding a solution of the corresponding Cauchy problem, some of them are contained in [6, 7]. Section 3 is devoted to a proof of notation of Toda lattice (1) as a Lax equation with some coefficient matrix $A(t)$ in the space (4); for us is essential the properties of $A(t)$. In the Section 4 we develop the direct and inverse spectral theory of Jacobi-type matrices in the space (4). Note, that by construction of such theory there are some exceptions since the first space in the sum (4) is $\mathbb{C}^1$, different from other spaces $\mathbb{C}^2$. Now the spectral measure $d\rho(\lambda)$ is $2 \times 2-$matrix measure, corresponding inverse problem is more difficult, as for the space $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots$.

In Section 5 we have derived a differential equation on t for elements of a spectral matrix, $d\rho(\lambda;t) = (d\rho_{\alpha,\beta}(\lambda;t))_{\alpha,\beta=0}^1$, of the corresponding operator $\mathbf{J}(t)$ acting on the space (4). Now $\rho_{0,1}(\lambda;t) = \rho_{1,0}(\lambda;t)$, our differential equation is a linear system of equations of the first order w.r.t. the real functions $r_{0,0}(\lambda;t), r_{0,1}(\lambda;t), r_{1,1}(\lambda;t)$, which are derivatives of $d\rho_{\alpha,\beta}(\lambda;t)$ with respect to some scalar measure $d\sigma(\lambda)$: $r_{\alpha,\beta}(\lambda;t) = \frac{d\rho_{\alpha,\beta}(\lambda;t)}{d\sigma(\lambda)}$. Here $t \in [0,T)$, λ is a parameter, coefficients of the system un a simple way depend on $\alpha_0(t)$, $\beta_0(t)$, and λ, but, unfortunately, we cannot give a general solution of this system. This is the main difference between the semi-infinite and the double-infinite cases for Toda lattice: in the first case we can find a general solution of the corresponding (one dimensional) differential equation, it gives formula (2), in our second case we have only a procedure for finding a solution, so we get only a linearization of our problem, not its solution.

Section 6 contains the main Theorem 11 of this work, — we give a procedure for finding a solution of the Cauchy problem for (1).

This section also contains some applications obtained with results on differential equations. Hamiltonian system in the form of a second order equation connected with (1) has the form

$$\ddot{x}_n(t) = e^{x_{n-1}(t)-x_n(t)} - e^{x_n(t)-x_{n+1}(t)}, \quad n \in \mathbb{Z}, \quad t \in [0,T) \tag{5}$$

(the connection between (5) and (1) is given by the Flashka change of variables, $\alpha_n(t) = e^{1/2(x_n(t)-x_{n+1}(t))}$, $\beta_n(t) = -\dot{x}_n(t)$). The Cauchy problem for (5) with the given initial data $x_n(0), \dot{x}_n(0), n \in \mathbb{Z}$, by the Flashka change of variables, transfers into our Cauchy problem for (1) and Theorem 11 gives a possibility to investigate it.

The last part of this Section is devoted to the shock problem for (5). This problem following [42] (see also [25]) is, for the equation (5) with $n \in \mathbb{N}_0$, to find a solution of the mixed Cauchy problem if $x_n(0), \dot{x}_n(0), n \in \mathbb{N}_0$, and $x_{-1}(t)$ are given (the "mixed semi-infinite Toda lattice"). Some approaches to the investigation of this and similar problems were given in the works [46, 47, 5, 56, 61, 62, 25]. We consider one version of this problem, that is, it is assumed that the following is given: $x_n(0), \dot{x}_n(0), n \in \mathbb{N}_0$, and $x_{-1}(t) - x_0(t), t \in [0,T)$ (instead of $x_{-1}(t)$). In the article, we show that a solution of such Toda shock problem is connected with finding a solution of some Riccati equation. If in the above $x_{-1}(t) = 0$, then we have a problem similar to [42] but moved one step further. Note that these results connected with the shock problem use only the material of Section 2.

2. Semi-infinite Toda lattice

In this Section we present old base results [3, 4, 6, 7] for a Toda lattice on $\mathbb{N}_0 = \{0, 1, \ldots\}$, in a little more accurate mathematical fashion. This lattice has the form

$$\begin{aligned} \dot{\alpha}_n(t) &= \frac{1}{2}\alpha(t)(\beta_{n+1}(t) - \beta_n(t)), \\ \dot{\beta}_n(t) &= \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n \in \mathbb{N}_0, \quad \alpha_{-1}(t) = 0, \end{aligned} \tag{6}$$

where $\alpha_n(t), \beta_n(t)$ are real continuously differentiable functions of $t \in [0,T), T \le \infty$; $\cdot = \frac{d}{dt}$. System (6) is a systems of differential-difference nonlinear equations, for (6) it is possible to consider the following Cauchy problem: from given initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{N}_0$, it is necessary to find a solution $\alpha_n(t), \beta_n(t), n \in \mathbb{N}_0$, for $t \in [0,T)$.

We will assume that the all the functions $\alpha_n(t), \beta_n(t)$ are bounded uniformly w.r.t. $n \in \mathbb{N}_0$, and

$$\alpha_n(t) > 0, \quad t \in [0,T), \quad n \in \mathbb{N}_0. \tag{7}$$

For our method of the inverse spectral problem, to integrate (6) it is necessary to rewrite (6) as a Lax equation for operators in the ordinary space ℓ_2 of sequences $f = (f_n)_{n=0}^{\infty}, f_n \in \mathbb{C}$.

Using the unknowns $\alpha_n(t), \beta_n(t)$ from (6) introduce the classical Jacobi matrix $J(t)$ that depends on t,

$$J(t) = \begin{bmatrix} \beta_0(t) & \alpha_0(t) & 0 & 0 & 0 & \dots \\ \alpha_0(t) & \beta_1(t) & \alpha_1(t) & 0 & 0 & \dots \\ 0 & \alpha_1(t) & \beta_2(t) & \alpha_2(t) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad t \in [0, T). \tag{8}$$

Matrix (8), for every t, generates a Hermitian operator $\mathbf{J}(t)$ in the space ℓ_2 in a classical way; it is the closure of the Hermitian operator $\ell_{fin} \ni f = (f_n)_{n=0}^{\infty} \mapsto ((J(t)f)_n)_{n=0}^{\infty} \in \ell_2$. The boundness of $\alpha_n(t), \beta_n(t)$ and their differentiability gives that this operator $\mathbf{J}(t) \forall t$ is bounded selfadjoint and weakly differentiable w.r.t. $t \in [0, T)$.

Introduce the matrix

$$A(t) = \frac{1}{2}\begin{bmatrix} 0 & -\alpha_0(t) & 0 & 0 & 0 & \dots \\ \alpha_0(t) & 0 & -\alpha_1(t) & 0 & 0 & \dots \\ 0 & \alpha_1(t) & 0 & -\alpha_2(t) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad t \in [0, T), \tag{9}$$

which generates in ℓ_2, similar to $J(t)$, an antisymmetric bounded operator $\mathbf{A}(t)$, weakly differentiable w.r.t. $t \in [0, T)$.

The following classical results is well known: *the Toda equation* (6) *and the corresponding Lax equation*

$$\dot{J}(t) = J(t)A(t) - A(t)J(t) \quad \text{or} \quad \dot{\mathbf{J}}(t) = \mathbf{J}(t)\mathbf{A}(t) - \mathbf{A}(t)\mathbf{J}(t), \quad t \in [0, T), \tag{10}$$

are equivalent.

The proof of this fact is very simple. We calculate $J(t)A(t) - A(t)J(t)$ using (8) and (9). Then the first equality in (10) is the same as (6).

Further we will consider another linear operators in the space ℓ_2. Introduce some notations. In this space we will use the standard orthonormal basis

$$\varepsilon_n = (0, \dots, 0, \underbrace{1}_{n \text{ place}}, 0, 0, \dots), \quad n \in \mathbb{N}_0. \tag{11}$$

Then for every bounded linear operator $\mathbf{A} \colon \ell_2 \to \ell_2$ we can construct its matrix $A = (A_{j,k})_{j,k=0}^{\infty}$, where $A_{j,k} = (\mathbf{A}\varepsilon_k, \varepsilon_j)_{\ell_2} \in \mathbb{C}$. Then

$$(\mathbf{A}f)_j = \sum_{k=0}^{\infty} A_{j,k} f_k, \quad f = (f_n)_{n=0}^{\infty} \in \ell_2, \quad j \in \mathbb{N}_0. \tag{12}$$

Conversely, every matrix $A = (A_{j,k})_{j,k=0}^{\infty}$ with uniformly bounded $A_{j,k} \in \mathbb{C}$ generates by (12) a corresponding operator $\mathbf{A}$.

Consider the resolvent $\mathbf{R}_z(t) = (\mathbf{J}(t) - z\mathbf{1})^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, $t \in [0, T)$, of our selfadjoint bounded operator $\mathbf{J}(t)$. According to (12) this resolvent has the matrix

$$R_z(t) = (R_{z;j,k}(t))_{j,k=0}^{\infty}, \quad R_z(t) = (\mathbf{R}_z(t)\varepsilon_k, \varepsilon_j)_{\ell_2}. \tag{13}$$

Our nearest aim is to deduce some differential equation w.r.t. t for the zero matrix element of (13) (the “Weyl function”),

$$m(z;t) = R_{z;0,0}(t) = (\mathbf{R}_z(t)\varepsilon_0, \varepsilon_0)_{\ell_2}, \quad z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T); \quad m(\bar{z};t) = \overline{m(z;t)}. \tag{14}$$

Theorem 1. *The Weyl function $m(z;t)$ satisfies the following differential equation:*

$$\dot{m}(z;t) = (z - \beta_0(t))m(z;t) + 1, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T). \tag{15}$$

Proof. We will use the well known formula for the derivative with respect to t of the inverse of the smooth operator-valued function $[0, T) \ni t \mapsto \mathbf{C}(t)$, namely,

$$(\mathbf{C}^{-1}(t))^{\cdot} = -\mathbf{C}^{-1}(t)\dot{\mathbf{C}}(t)\mathbf{C}^{-1}(t), \quad t \in [0, T). \tag{16}$$

According to (10) we get $\forall z \in \mathbb{C} \setminus \mathbb{R}$

$$(\mathbf{J}(t) - z\mathbf{1})^{\cdot} = \dot{\mathbf{J}}(t) = \mathbf{J}(t)\mathbf{A}(t) - \mathbf{A}(t)\mathbf{J}(t) = (\mathbf{J}(t) - z\mathbf{1})\mathbf{A}(t) - \mathbf{A}(t)(\mathbf{J}(t) - z\mathbf{1}), \tag{17}$$

therefore using (16) and (17) we have $\forall z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned} \dot{\mathbf{R}}_z(t) &= ((\mathbf{J}(t) - z\mathbf{1})^{-1})^{\cdot} = -\mathbf{R}_z(t)(\mathbf{J}(t) - z\mathbf{1})^{\cdot}\mathbf{R}_z(t) \\ &= -\mathbf{R}_z(t)((\mathbf{J}(t) - z\mathbf{1})\mathbf{A}(t) - \mathbf{A}(t)(\mathbf{J}(t) - z\mathbf{1}))\mathbf{R}_z(t) \\ &= \mathbf{R}_z(t)\mathbf{A}(t) - \mathbf{A}(t)\mathbf{R}_z(t) = [\mathbf{R}_z(t), \mathbf{A}(t)], \quad \text{or} \\ \dot{R}_z(t) &= [R_z(t), A(t)], \quad t \in [0, T) \end{aligned} \tag{18}$$

(we use in the article the standard notation for the commutator, $[B, C] = BC - CB$).

Let us calculate $\dot{m}(z;t)$. Observe that $\mathbf{A}^*(t) = -\mathbf{A}(t)$, therefore using (18) $\forall z \in \mathbb{C} \setminus \mathbb{R},\ t \in [0, T)$ we get

$$\begin{aligned} \dot{m}(z;t) &= (\dot{\mathbf{R}}_z(t)\varepsilon_0, \varepsilon_0)_{\ell_2} = ((\mathbf{R}_z(t)\mathbf{A}(t) - \mathbf{A}(t)\mathbf{R}_z(t))\varepsilon_0, \varepsilon_0)_{\ell_2} \\ &= (\mathbf{R}_z(t)\mathbf{A}(t)\varepsilon_0, \varepsilon_0)_{\ell_2} + (\mathbf{R}_z(t)\varepsilon_0, \mathbf{A}(t)\varepsilon_0)_{\ell_2}. \end{aligned} \tag{19}$$

From (9) we conclude that

$$\mathbf{A}(t)\varepsilon_0 = (0, \frac{1}{2}\alpha_0(t), 0, 0, \dots) = \frac{1}{2}\alpha_0(t)\varepsilon_1. \tag{20}$$

The equalities (19), (20) give

$$\begin{aligned} \dot{m}(z;t) &= \frac{1}{2}\alpha_0(t)(\mathbf{R}_z(t)\varepsilon_1, \varepsilon_0)_{\ell_2} + \frac{1}{2}\alpha_0(t)(\mathbf{R}_z(t)\varepsilon_0, \varepsilon_1)_{\ell_2} \\ &= \frac{1}{2}\alpha_0(t)((\mathbf{R}_z(t)\varepsilon_0)_1 + \overline{(\mathbf{R}_{\bar{z}}(t)\varepsilon_0)_1}), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T). \end{aligned} \tag{21}$$

But $\mathbf{1} = (\mathbf{J}(t) - z\mathbf{1})\mathbf{R}_z(t)$, therefore,

$$1 = \sum_{k=0}^{\infty} (J(t) - z1)_{0,k}(R_z(t))_{0,k} = (\beta_0(t) - z)m(z;t) + \alpha_0(t)(\mathbf{R}_z(t)\varepsilon_0)_1.$$

So, we have

$$(\mathbf{R}_z(t)\varepsilon_0)_1 = \alpha_0^{-1}(t)((z - \beta_0(t))m(z;t) + 1), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T). \tag{22}$$

Using (22), we conclude from (21) and (14) that

$$\dot{m}(z;t) = (z - \beta_0(t))m(z;t) + 1, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T). \qquad \square$$

Further we will need some facts from the classical theory of Jacobi matrices (see, for example, [2], Ch. 7 and also [16], Ch. 15). Let us recall these facts.

Consider a Jacobi matrix J of type (8) with constant coefficients $\alpha_n > 0, \beta_n \in \mathbb{R}$, $n \in \mathbb{N}_0$. For simplicity we assume that α_n, β_n are bounded w.r.t. n. Such a matrix J generates in the space ℓ_2 (as above) a bounded selfadjoint operator $\mathbf{J}$. For this operator, the Borel spectral measure $\mathfrak{B}(\mathbb{R}) \ni \triangle \mapsto \rho(\triangle)$ (or $d\rho(\lambda)$) exists. This measure has bounded support (equals to the spectrum of $\mathbf{J}$) and is a probability measure, $\rho(\mathbb{R}) = 1$. It is also convenient to use corresponding spectral function $\rho(\lambda) = \int_{-\infty}^{\lambda} d\rho(\lambda) = \rho((-\infty, \lambda))$, $\lambda \in \mathbb{R}$.

The expansion of the space ℓ_2 in the continual linear combination of generalized eigenvector of $\mathbf{J}$ has the following form. Consider the solution $P(\lambda) = (P_n(\lambda))_{n=0}^{\infty}$, $\lambda \in \mathbb{R}$, of the following difference equation with the initial data indicated below,

$$\begin{gathered} (JP(\lambda))_n = \alpha_{n-1}P_{n-1}(\lambda) + \beta_n P_n(\lambda) + \alpha_n P_{n+1}(\lambda) = \lambda P_n(\lambda), \\ n \in \mathbb{N}_0, \quad P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1; \quad \lambda \in \mathbb{R}. \end{gathered} \tag{23}$$

Every $P_n(\lambda)$ is a polynomial of degree n with real coefficients (the "polynomial of the first kind").

The sequence $P(\lambda) = (P_n(\lambda))_{n=0}^{\infty}$ is a generalized eigenvector of the operator $\mathbf{J}$ corresponding to the "eigenvalue λ". If λ belongs to discrete spectrum of $\mathbf{J}$, then $P(\lambda) \in \ell_2$

is an ordinary eigenvector of $\mathbf{J}$ with the eigenvalue λ. For a general λ, the sequence $P(\lambda)$ is arbitrary, i.e. belongs to $\ell := \mathbb{C}^\infty$. More exactly, for a given matrix J it is possible to construct some weighted space $\ell_2(h^{-1})$ (the "negative space", i.e., the space of sequences $u = (u_n)_{n=0}^\infty, u_n \in \mathbb{C}$ for which $\sum_{n=0}^\infty |u_n|^2 h_n^{-1} < \infty$, where $h = (h_n)_{n=0}^\infty, h_n \geq 1$). The construction is such that $P(\lambda) \in \ell_2(h^{-1}) \subset \ell$.

The generalized eigenvector expansion for our $\mathbf{J}$ has the following form. For a finite sequence $f = (f_n)_{n=0}^\infty, f_n \in \mathbb{C}$ (the set of such sequences is denoted by ℓ_{fin}) we construct the Fourier transform

$$\widehat{f}(\lambda) = \sum_{n=0}^\infty f_n P_n(\lambda), \quad \lambda \in \mathbb{R}. \tag{24}$$

The generalized eigenvectors expansion now is equivalent to existence of the Parseval equality, for arbitrary $f, g \in \ell_{\text{fin}}$,

$$(f, g)_{\ell_2} = \int_{\mathbb{R}} \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)} d\rho(\lambda). \tag{25}$$

Extending it by continuity, the equality (25) is possible to extend to arbitrary $f, g \in \ell_2$. In this case, $\widehat{f}(\lambda)$ is the limit, in the space $L^2(\mathbb{R}, d\rho(\lambda))$, expressions (24) for sequence $f^{(n)} \in \ell_{\text{fin}}, n \in \mathbb{N}$, which tends to f in ℓ_2.

From (24), (25) it follows that the polynomials $P_n(\lambda)$ are orthonormal in the space $L^2(\mathbb{R}, d\rho(\lambda))$ and form a basis in this space. This basis can be constructed following the Gramm-Schmidt procedure of orthogonalization of the functions $1, \lambda, \lambda^2, \ldots$ in the space $L^2(\mathbb{R}, d\rho(\lambda))$. It is easy to understand that for elements α_n, β_n of the matrix J can be calculated as

$$\alpha_n = \int_{\mathbb{R}} \lambda P_n(\lambda) P_{n+1}(\lambda) d\rho(\lambda), \quad \beta_n = \int_{\mathbb{R}} \lambda P_n^2(\lambda) d\rho(\lambda), \quad n \in \mathbb{N}_0. \tag{26}$$

We also note that for the Weyl function $m(z) = (\mathbf{R}_z \varepsilon_0, \varepsilon_0)_{\ell_2}$, where $\mathbf{R}_z$ is the resolvent of operator $\mathbf{J}$, we have the representation

$$m(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{27}$$

The latter is a general fact. Let $E(\triangle), \triangle \in \mathfrak{B}(\mathbb{R})$, be an expansion of the identity for the operator $\mathbf{J}$. Then it is possible to write

$$E(\triangle) = \int_{\triangle} \Phi(\lambda) d\rho(\lambda), \quad \triangle \in \mathfrak{B}(\mathbb{R}), \tag{28}$$

where $\Phi(\lambda)$ is an operator-valued function of λ the values of which are bounded linear operators acting from some dense linear subset of ℓ_2 into some other linear subset in ℓ (more exactly, from the space $\ell_2(h) \subset \ell_2$ with "positive norm" into the space with negative norm, $\ell_2(h^{-1}) \subset \ell$). The operator $\Phi(\lambda)$ is called the generalized projection operator. It is possible to write it as $\Phi(\lambda) = \frac{dE(\lambda)}{d\rho(\lambda)}$, it projects the vectors from positive space into its images in negative space.

After these remindings we return to our matrices and operators $J(t), \mathbf{J}(t)$, depending on $t \in [0, T)$. For fixed t, to these objects, it is possible to apply the above mentioned theory. Now the spectral measure $d\rho(\lambda; t)$, the generalized projection operator $\Phi(\lambda; t)$ etc., all of them depend on t.

It is easy to understand, that we can choose a spectral measure $d\sigma(\lambda)$ (a joint spectral measure), which is common for every operator $\mathbf{J}(t), t \in [0, T)$. Namely, introduce the following measure:

$$\mathfrak{B}(\mathbb{R}) \ni \triangle \mapsto \int_0^T \rho(\triangle; t) dt =: \sigma(\triangle) \geq 0. \tag{29}$$

This definition is correct, since $\forall t \in [0,T)\, \rho(\mathbb{R};t) = 1$. It is obvious that every measure $d\rho(\lambda;t)$ is absolutely continuous w.r.t. $d\sigma(\lambda)$.

For the operators $\mathbf{J}(t), t \in [0,T)$, all relations (23)–(28) take place for every fixed t if we use the corresponding spectral measure $d\rho(\lambda;t)$. Our nearest aim is to rewrite the equation (15) as an equation for a function closely connected with the spectral measure. For this we introduce $\forall t \in [0,T)$ the derivative of the spectral measure,

$$r(\lambda;t) = \frac{d\rho(\lambda;t)}{d\sigma(\lambda)}, \quad \lambda \in \mathbb{R}, \quad \text{i.e.} \quad \rho(\triangle;t) = \int_{\triangle} r(\lambda;t)d\sigma(\lambda), \quad \triangle \in \mathfrak{B}(\mathbb{R}). \tag{30}$$

We will rewrite (15) in terms $r(\lambda;t)$.

Theorem 2. *For $d\sigma(\lambda)-$almost all $\lambda \in \mathbb{R}$, the function $r(\lambda;t)$ is continuously differentiable w.r.t. t on $[0,T)$ and is a solution of the following differential equation:*

$$\dot{r}(\lambda;t) = (\lambda - \beta_0(t))r(\lambda;t), \quad t \in [0,T). \tag{31}$$

Proof. From the definition (30) of the function $r(\lambda;t)$, it is impossible to directly conclude that this function is smooth w.r.t. t. Therefore it is necessary to use the notion of a generalized solution of equation (31).

Let $L^2([0,T),dt)$ be the ordinary space of functions on $[0,T)$, constructed from the Lebesgue measure dt; $(u,v)_{L^2}$ is the corresponding scalar product. For arbitrary infinitely differentiable function $[0,T) \ni t \mapsto u(t) \in \mathbb{C}$, vanishing in some neighborhoods of 0 and T, we have according to (30) and (27), (15) the following: $\forall z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned}
-\int_{\mathbb{R}} \frac{1}{\lambda - z}(r(\lambda;\cdot),\dot{u}(\cdot))_{L^2}d\sigma(\lambda) &= -\int_{\mathbb{R}} \frac{1}{\lambda - z} d(\rho(\lambda;\cdot),\dot{u}(\cdot))_{L^2} \\
&= -(m(z;\cdot),\dot{u}(\cdot))_{L^2} = (\dot{m}(z;\cdot),u(\cdot))_{L^2} = ((z-\beta_0(\cdot))m(z;\cdot)+1,u(\cdot))_{L^2} \\
&= ((\lambda-\beta_0(\cdot))m(z;\cdot),u(\cdot))_{L^2} = \int_{\mathbb{R}} \frac{1}{\lambda - z}((\lambda-\beta_0(\cdot))d\rho(\lambda;\cdot),u(\cdot))_{L^2} \\
&= \int_{\mathbb{R}} \frac{1}{\lambda - z}((\lambda-\beta_0(\cdot))r(\lambda;\cdot),u(\cdot))_{L^2}d\sigma(\lambda).
\end{aligned} \tag{32}$$

We will now use the well-known fact: if $d\omega(\lambda)$ is a charge on $\mathfrak{B}(\mathbb{R})$ with bounded support, then the condition $\int_{\mathbb{R}}(\lambda - z)^{-1}d\omega(\lambda) = 0$, $z \in \mathbb{C} \setminus \mathbb{R}$, is equivalent to the equality $d\omega(\lambda) = 0$. Using this fact, we conclude from (7) that, for $d\sigma(\lambda)-$almost all $\lambda \in \mathbb{R}$,

$$-(r(\lambda;\cdot),\dot{u}(\cdot))_{L^2} = ((\lambda-\beta_0(\cdot))r(\lambda;\cdot),u(\cdot))_{L^2}. \tag{33}$$

Relation (33) means that $r(\lambda;t)$ is a generalized solution of equation (31). But for ordinary differential equations every generalized solution is a smooth solution up to the point 0 (see, for example, [16], Ch. 16, Section 6). Therefore $r(\lambda;t)$ is a smooth solution of equation (31) on $[0,T)$. $\square$

For equation (31) with arbitrary $\lambda \in \mathbb{R}$ it is easily to write a solution of the corresponding Cauchy problem,

$$r(\lambda;t) = e^{\int_0^t(\lambda-\beta_0(s))ds}r(\lambda;0) = e^{\lambda t}e^{-\int_0^t \beta_0(s)ds}r(\lambda;0), \quad t \in [0,T), \quad \lambda \in \mathbb{R}. \tag{34}$$

Using (30) and (34) we calculate that for the spectral function $\rho(\lambda;t) = \rho((-\infty,\lambda);t)$, $\lambda \in \mathbb{R}$, of the operator $\mathbf{J}(t)$, $t \in [0,T)$, we have

$$\begin{aligned}
\rho(\lambda;t) &= \int_{-\infty}^{\lambda} r(\mu;t)d\sigma(\mu) = e^{-\int_0^t \beta_0(s)ds} \int_{-\infty}^{\lambda} e^{\mu t}r(\mu;0)d\sigma(\mu) \\
&= e^{-\int_0^t \beta_0(s)ds} \int_{-\infty}^{\lambda} e^{\mu t}d\rho(\mu;0), \quad \text{i.e.}
\end{aligned}$$

$$d\rho(\lambda;t) = e^{-\int_0^t \beta_0(s)ds}e^{\lambda t}d\rho(\lambda;0), \quad \lambda \in \mathbb{R}, \quad t \in [0,T). \tag{35}$$

This formula gives a rule to calculate the spectral measure of the operator $\mathbf{J}(t)$ from such measure of the initial operator $\mathbf{J}(0)$.

The function $\beta_0(t)$ can not be arbitrary; since $\forall t \in [0,T)\, \rho(\mathbb{R};t) = 1$, from (35) we conclude that

$$e^{\int_0^t \beta_0(s)ds} = \int_{\mathbb{R}} e^{\lambda t} d\rho(\lambda;0), \quad t \in [0,T).$$

Therefore,

$$\beta_0(t) = \Big(\int_{\mathbb{R}} e^{\lambda t} d\rho(\lambda;0)\Big)^{-1} \int_{\mathbb{R}} \lambda e^{\lambda t} d\rho(\lambda;0), \quad t \in [0,T). \tag{36}$$

We can formulate now the main theorem of this section, which is a simple consequence of the previous consideration.

Theorem 3. *Consider the Cauchy problem for the semi-infinite Toda lattice* (6)*. It is necessary to find a solution $\alpha_n(t) > 0, \beta_n(t), t \in [0,T), n \in \mathbb{N}_0$, of* (6) *from the given initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{N}_0$.*

The procedure of finding this solution is the following: using the initial data we construct the Jacobi matrix $J(0)$ (8) *and then find its spectral measure $d\rho(\lambda;0)$. Using formulas* (35) *and* (36) *we find the spectral measure $d\rho(\lambda;t)$ of the matrix $J(t)$ and then find the solution by the formulas*

$$\begin{aligned} \alpha_n(t) &= \int_{\mathbb{R}} \lambda P_n(\lambda;t) P_{n+1}(\lambda;t) d\rho(\lambda;t), \\ \beta_n(t) &= \int_{\mathbb{R}} \lambda P_n^2(\lambda;t) d\rho(\lambda;t), \quad n \in \mathbb{N}_0, \quad t \in [0,T). \end{aligned} \tag{37}$$

In (37)*, $P_n(\lambda;t)$ are polynomials of the first kind connected with the measure $d\rho(\lambda;t)$ and constructed from the orthogonalization procedure applied to $1, \lambda, \lambda^2, \dots$ in the space $L^2(\mathbb{R}, d\rho(\lambda;t))$ (note that $\beta_0(t)$ from* (37) *is equal to* (36)*).*

Of course, to actually find a solution of the Cauchy problem for (6) following this procedure is very hard, but it is possible to find such initial data for which the calculations can be carried out.

In the conclusion of this Section we will make two remarks.

Remark 1. Using (31) it is possible to deduce a similar equation for the spectral function $\rho(\lambda;t) = \rho((-\infty,\lambda);t)$, $\lambda \in \mathbb{R}$, $t \in [0,T)$.

Remark 2. Instead of Theorem 3 it is possible to get some another result of such type. Namely, using equation (15), we can find some equations for moments of the measure $d\rho(\lambda;t)$, i.e., for the functions $s_n = \int_{\mathbb{R}} \lambda^n d\rho(\lambda;t)$, $n \in \mathbb{N}_0$, $t \in [0,T)$. We can find solutions of these equations and then find the elements $\alpha_n(t), \beta_n(t)$ of the corresponding Jacobi matrix $J(t)$ using the classical formulas of the theory of moments. More detailed explanation of this approach will be given in Section 6, Theorem 12.

3. The Toda lattice as the Lax equation for block Jacobi matrix

In this Section we will rewrite the Cauchy problem for double-infinite Toda lattice as corresponding problem for Lax equation for block Jacobi matrices.

Unlike the constructions of Section 2, we will use instead of the ordinary space ℓ_2 on $\mathbb{N}_0$, the following space:

$$\begin{aligned} &\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_0 = \mathbb{C}^1, \quad \mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathbb{C}^2; \\ &\mathbf{l}_2 \ni f = (f_n)_{n=0}^{\infty}, \quad f_0 =: f_{0,0} \in \mathbb{C}^1, \quad f_n = (f_{n,0}, f_{n,1}) \in \mathbb{C}^2. \end{aligned} \tag{38}$$

Standard orthonormal basis in this space is

$$
\begin{aligned}
&(\varepsilon_{n,v_n})_{n=0}^{\infty}, \quad v_n = 0,1, \quad \text{where} \quad \varepsilon_{0,0} = (1,0,0,\ldots),\\
&\varepsilon_{n,0} = (0,\ldots,0,\underbrace{(1,0)}_{n \text{ place}},0,0,\ldots), \quad \varepsilon_{n,1} = (0,\ldots,0,\underbrace{(0,1)}_{n \text{ place}},0,0,\ldots), \quad n \in \mathbb{N};\\
&(1,0) =: e_0, \quad (0,1) =: e_1
\end{aligned}
\tag{39}
$$

(*in* (39) *for* $n = 0$ v_0 *always takes the only value* 0*; in other words, we put* $f_0 = (f_{0,0}, 0)$). So, every vector from $\mathbf{l}_2$ is a linear combination (finite or not) of the vectors ε_{n,v_n}.

Every bounded operator $\mathbf{A}\colon \mathbf{l}_2 \to \mathbf{l}_2$ can be written in the basis (39) as a block infinite matrix

$$
A = \begin{bmatrix} A_{0;0} & A_{0;1} & A_{0;2} & \ldots \\ A_{1;0} & A_{1;1} & A_{1;2} & \ldots \\ A_{2;0} & A_{2;1} & A_{2;2} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},
\tag{40}
$$

where the matrices $A_{j;k}$ from (40) act in the following way:

$$
\begin{aligned}
&A_{0;0}\colon \mathbb{C}^1 \to \mathbb{C}^1; \quad A_{0;k}\colon \mathbb{C}^2 \to \mathbb{C}^1, \quad k \in \mathbb{N}; \quad A_{j;0}\colon \mathbb{C}^1 \to \mathbb{C}^2, \quad j \in \mathbb{N};\\
&A_{j;k}\colon \mathbb{C}^2 \to \mathbb{C}^2, \quad j,k \in \mathbb{N}.
\end{aligned}
\tag{41}
$$

These matrices $A_{j;k}$, $j,k \in \mathbb{N}_0$ have the forms

$$
\begin{aligned}
&A_{j;k} = \begin{bmatrix} A_{j,0;k,0} & A_{j,0;k,1} \\ A_{j,1;k,0} & A_{j,1;k,1} \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2, \quad j,k \in \mathbb{N};\\
&A_{0;k} = \begin{bmatrix} A_{0,0;k,0} & A_{0,0;k,1} \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^1, \quad A_{j;0} = \begin{bmatrix} A_{j,0;0,0} \\ A_{j,1;0,0} \end{bmatrix} : \mathbb{C}^1 \to \mathbb{C}^2, \quad j,k \in \mathbb{N};\\
&A_{0;0} = [A_{0,0;0,0}]\colon \mathbb{C}^1 \to \mathbb{C}^1;\\
&\forall j, \quad k \in \mathbb{N}_0, \quad v_j, v_k \in \{0,1\} \quad A_{j,v_j;k,v_k} = (\mathbf{A}\varepsilon_{k,v_k}, \varepsilon_{j,v_j})_{\mathbf{l}_2}.
\end{aligned}
\tag{42}
$$

The notation for the action of $\mathbf{A}$ by (40), (42) is the following: $\forall f = (f_k)_{k=0}^{\infty} \in \mathbf{l}_2$

$$
(\mathbf{A}f)_j = \sum_{k=0}^{\infty} A_{j;k} f_k = \sum_{k=0}^{\infty} \begin{bmatrix} A_{j,0;k,0} & A_{j,0;k,1} \\ A_{j,1;k,0} & A_{j,1;k,1} \end{bmatrix} (f_{k,0}, f_{k,1}).
\tag{43}
$$

Note that for the adjoint operator $\mathbf{A}^*\colon \mathbf{l}_2 \to \mathbf{l}_2$ we have

$$
(\mathbf{A}^*\varepsilon_{k,v_k}, \varepsilon_{j,v_j})_{\mathbf{l}_2} = (\varepsilon_{k,v_k}, \mathbf{A}\varepsilon_{j,v_j})_{\mathbf{l}_2} = \overline{(\mathbf{A}\varepsilon_{j,v_j}, \varepsilon_{k,v_k})}_{\mathbf{l}_2},
$$

i.e., its matrix A^* is also of type (40), namely, it is a block matrix adjoint to (40) (on every place with indices j,k, the matrix $A^*_{k;j}$ is located).

Consider the double infinite Toda lattices. Such a lattice has the form (compare with (6))

$$
\dot{\alpha}_n = \frac{1}{2}\alpha_n(\beta_{n+1} - \beta_n), \quad \dot{\beta}_n = \alpha_n^2 - \alpha_{n-1}^2, \quad n \in \mathbb{Z} = \{\ldots,-1,0,1,\ldots\},
\tag{44}
$$

where $\alpha_n = \alpha_n(t), \beta_n = \beta_n(t)$ are real continuously differentiable function of $t \in [0,T)$. Expression (44) is a system of differential-difference nonlinear equations and for (44) it is possible to consider the following Cauchy problem: from given initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{Z}$, find a solution $\alpha_n(t), \beta_n(t), n \in \mathbb{Z}$, for $t \in [0,T)$.

We will assume, as in (7), that all the functions $\alpha_n(t), \beta_n(t)$ are bounded uniformly w.r.t. n and

$$
\alpha_n(t) > 0, \quad t \in [0,T), \quad n \in \mathbb{Z}.
\tag{45}
$$

In the article [9] it was shown that the above formulated Cauchy problem for (44), (45) can be rewritten as a Cauchy problem for some *semi-infinite block* Toda lattice.

This lattice can be rewritten as a Lax equation to which we can apply the approach of type [3, 4, 6, 7].

Now we will actually repeat the considerations of [9] and directly prove that the Cauchy problem (44), (45) can be put in the Lax form for block matrices in the space $\mathbf{l}_2$.

Using the functions $\alpha_n(t), \beta_n(t), n \in \mathbb{Z}, t \in [0, T)$ we construct $\forall t \in [0, T)$ the following block Jacobi matrices of type (40) acting in the space $\mathbf{l}_2$ (38):

(46)

$$J(t) = \begin{bmatrix} b_0(t) & a_0^*(t) & 0 & 0 & 0 & \dots \\ a_0(t) & b_1(t) & a_1(t) & 0 & 0 & \dots \\ 0 & a_1(t) & b_2(t) & a_2(t) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}; \quad \begin{aligned} & b_0(t) = [\beta_0(t)] \colon \mathbb{C}^1 \to \mathbb{C}^1, \\ & a_0(t) = \begin{bmatrix} \alpha_0(t) \\ \alpha_{-1}(t) \end{bmatrix} \colon \mathbb{C}^1 \to \mathbb{C}^2, \\ & a_0^*(t) = [\alpha_0(t)\ \alpha_{-1}(t)] \colon \mathbb{C}^2 \to \mathbb{C}^1; \end{aligned}$$

$$a_n(t) = \begin{bmatrix} \alpha_n(t) & 0 \\ 0 & \alpha_{-n-1}(t) \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2, \quad b_n(t) = \begin{bmatrix} \beta_n(t) & 0 \\ 0 & \beta_{-n}(t) \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2, \quad n \in \mathbb{N};$$

i.e.

$$J(t) = \left[\begin{array}{c|cc|cc|cc|cc|c} \beta_0 & \alpha_0 & \alpha_{-1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline \alpha_0 & \beta_1 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \alpha_{-1} & 0 & \beta_{-1} & 0 & \alpha_{-2} & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & \alpha_1 & 0 & \beta_2 & 0 & \alpha_2 & 0 & 0 & 0 & \dots \\ 0 & 0 & \alpha_{-2} & 0 & \beta_{-2} & 0 & \alpha_{-3} & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & \alpha_2 & 0 & \beta_3 & 0 & \alpha_3 & 0 & \dots \\ 0 & 0 & 0 & 0 & \alpha_{-3} & 0 & \beta_{-3} & 0 & \alpha_{-4} & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right].$$

The action of (46) on a vector $f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ is three-diagonal, $\forall t \in [0, T)$

$$\begin{aligned} & (J(t)f)_n = a_{n-1}(t) f_{n-1} + b_n(t) f_n + a_n(t) f_{n+1}, \quad n \in \mathbb{N}, \\ & (J(t)f)_0 = b_0(t) f_0 + a_0^*(t) f_1, \quad f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2. \end{aligned} \tag{47}$$

The condition (45) on the coefficients of the matrix $J(t)$ gives that the corresponding operator $\mathbf{J}(t)$ in the space $\mathbf{l}_2$ is bounded, real, and selfadjoint.

The knowledge of the block matrix $J(t)$ (46) is equivalent to the knowledge of all functions $\alpha_n(t), \beta_n(t), n \in \mathbb{Z}$, from (44). The Lax equation for the matrix $J(t)$ (46) has the form

$$\begin{aligned} & \dot{J}(t) = J(t)A(t) - A(t)J(t) = [J(t), A(t)] \quad \text{or} \\ & \dot{\mathbf{J}}(t) = \mathbf{J}(t)\mathbf{A}(t) - \mathbf{A}(t)\mathbf{J}(t), \quad t \in [0, T), \end{aligned} \tag{48}$$

where $A(t)$ is some coefficient matrix of type (40) (depending on t); $\mathbf{J}(t), \mathbf{A}(t)$ the bounded operators in $\mathbf{l}_2$, corresponding to (46) and $A(t)$.

The aim of this Section is to show that for a special matrix $A(t)$ the equations (44) and (48) are the same.

Introduce the following matrix $A(t)$ of type (46): $\forall t \in [0, T)$

$$A(t) = \begin{bmatrix} \widetilde{b}_0 & \widetilde{c}_0 & 0 & 0 & \dots \\ \widetilde{a}_0 & \widetilde{b}_1 & \widetilde{c}_1 & 0 & \dots \\ 0 & \widetilde{a}_1 & \widetilde{b}_2 & \widetilde{c}_2 & \dots \\ 0 & 0 & \widetilde{a}_2 & \widetilde{b}_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{49}$$

$$:= \frac{1}{2}\left[\begin{array}{c|cc|cc|cc|c}
0 & -\alpha_0 & \alpha_{-1} & 0 & 0 & 0 & 0 & \dots \\ \hline
\alpha_0 & 0 & 0 & -\alpha_1 & 0 & 0 & 0 & \dots \\
-\alpha_{-1} & 0 & 0 & 0 & \alpha_{-2} & 0 & 0 & \dots \\ \hline
0 & \alpha_1 & 0 & 0 & 0 & -\alpha_2 & 0 & \dots \\
0 & 0 & -\alpha_{-2} & 0 & 0 & 0 & \alpha_{-3} & \dots \\ \hline
0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & -\alpha_{-3} & 0 & 0 & \dots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]$$

(its elements are $\alpha_n, n \in \mathbb{Z}$, and the matrices $\widetilde{a}_n, \widetilde{c}_n, n \in \mathbb{N}_0$ depend on t).

Theorem 4. *The Toda equation* (44) *with conditions* (45) *and the Lax equation* (48) *with three-diagonal block matrices $J(t)$* (46) *and $A(t)$* (49) *are equivalent.*

Proof. For the block matrices A, B of type (40) we have the ordinary rule to calculate the elements of their product,

$$(AB)_{j;k} = \sum_{l=0}^{\infty} A_{j;l} B_{l;k}, \quad j, k \in \mathbb{N}_0.$$

Using this rule it is easy to calculate that for the matrices J and A, (46), (49) with the notations $c_0 = a_0^*, c_n = a_n, n \in \mathbb{N}$, we have for arbitrary $n \in \mathbb{N}_0$

$$\begin{aligned}
(JA)_{n-2;n} &= c_{n-2}\widetilde{c}_{n-1}, \\
(JA)_{n-1;n} &= b_{n-1}\widetilde{c}_{n-1} + c_{n-1}\widetilde{b}_n, \\
(JA)_{n;n} &= a_{n-1}\widetilde{c}_{n-1} + b_n\widetilde{b}_n + c_n\widetilde{a}_n, \\
(JA)_{n+1;n} &= a_n\widetilde{b}_n + b_{n+1}\widetilde{a}_n, \\
(JA)_{n+2;n} &= a_{n+1}\widetilde{a}_n,
\end{aligned} \tag{50}$$

where every element with negative index is assumed to be equal to zero (see also [15], equalities (20)).

For AJ equalities similar to (50) are true, if a_n, b_n, c_n are replaces with $\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_n$ and, conversely, $\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_n$ with a_n, b_n, c_n.

Comparing the elements in the left- and right-sides of equality (48) and using the formulas (50) and analogical formulas for AJ we get for $\forall n \in \mathbb{N}$ that

$$\begin{aligned}
0 &= c_n\widetilde{c}_{n+1} - \widetilde{c}_n c_{n+1}, \\
\dot{c}_n &= b_n\widetilde{c}_n + c_n\widetilde{b}_{n+1} - \widetilde{b}_n c_n - \widetilde{c}_n b_{n+1}, \\
\dot{b}_n &= a_{n-1}\widetilde{c}_{n-1} + b_n\widetilde{b}_n + c_n\widetilde{a}_n - \widetilde{a}_{n-1}c_{n-1} - \widetilde{b}_n b_n - \widetilde{c}_n a_n, \\
\dot{a}_n &= a_n\widetilde{b}_n + b_{n+1}\widetilde{a}_n - \widetilde{a}_n b_n - \widetilde{b}_{n+1}a_n, \\
0 &= a_{n+1}\widetilde{a}_n - \widetilde{a}_{n+1}a_n
\end{aligned} \tag{51}$$

(as before, every element from (51) with negative index is equal to zero).

According to (49) we have

$$\begin{aligned}
&\widetilde{a}_0 = \frac{1}{2}\begin{bmatrix} \alpha_0 \\ -\alpha_{-1} \end{bmatrix}, \quad \widetilde{a}_n = \frac{1}{2}\begin{bmatrix} \alpha_n & 0 \\ 0 & -\alpha_{-n-1} \end{bmatrix}, \quad n \in \mathbb{N}; \\
&\widetilde{c}_0 = \frac{1}{2}\begin{bmatrix} -\alpha_0 & \alpha_{-1} \end{bmatrix}, \quad \widetilde{c}_n = \frac{1}{2}\begin{bmatrix} -\alpha_n & 0 \\ 0 & \alpha_{-n-1} \end{bmatrix}, \quad n \in \mathbb{N}; \\
&\widetilde{b}_n = 0, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{52}$$

The second equality and (46), (52) gives

$$\dot c_0 = \dot a_0^* = \begin{bmatrix}\dot\alpha_0 & \dot\alpha_{-1}\end{bmatrix} = \frac{1}{2}\beta_0 \begin{bmatrix}-\alpha_0 & \alpha_{-1}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-\alpha_0 & \alpha_{-1}\end{bmatrix}\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_{-1}\end{bmatrix},$$

$$\text{i.e.}\quad \dot\alpha_0 = \frac{1}{2}\alpha_0(\beta_1-\beta_0),\quad \dot\alpha_{-1} = \frac{1}{2}\alpha_{-1}(\beta_0-\beta_{-1});$$

$$\dot c_n = \dot a_n = \begin{bmatrix}\dot\alpha_n & 0\\ 0 & \dot\alpha_{-n-1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\beta_n & 0\\ 0 & \beta_{-n}\end{bmatrix}\begin{bmatrix}-\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix}\begin{bmatrix}\beta_{n+1} & 0\\ 0 & \beta_{-n-1}\end{bmatrix},$$

$$\text{i.e.}\quad \dot\alpha_n = \frac{1}{2}\alpha_n(\beta_{n+1}-\beta_n),\quad \dot\alpha_{-n-1} = \frac{1}{2}\alpha_{-n-1}(\beta_{-n}-\beta_{-n-1}),\quad n\in\mathbb{N}.$$

So, we see that the second equality in (51) is equivalent to the first equation in (44).

The third equalities in (51) and (46), (52) give

$$\dot b_0 = [\dot\beta_0] = c_0\widetilde a_0 - \widetilde c_0 a_0 = \frac{1}{2}[\alpha_0 \quad \alpha_{-1}]\begin{bmatrix}\alpha_0\\ -\alpha_{-1}\end{bmatrix} - \frac{1}{2}[-\alpha_0 \quad \alpha_{-1}]\begin{bmatrix}\alpha_0\\ \alpha_{-1}\end{bmatrix},$$

$$\text{i.e.}\quad \dot\beta_0 = \alpha_0^2 - \alpha_{-1}^2;$$

$$\dot b_1 = \begin{bmatrix}\dot\beta_1 & 0\\ 0 & \dot\beta_{-1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\alpha_0\\ \alpha_{-1}\end{bmatrix}[-\alpha_0 \quad \alpha_{-1}] + \frac{1}{2}\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_{-2}\end{bmatrix}\begin{bmatrix}\alpha_1 & 0\\ 0 & -\alpha_{-2}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\alpha_0\\ -\alpha_{-1}\end{bmatrix}\begin{bmatrix}\alpha_0 & \alpha_{-1}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-\alpha_1 & 0\\ 0 & \alpha_{-2}\end{bmatrix}\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_{-2}\end{bmatrix},$$

$$\text{i.e.}\quad \dot\beta_1 = \alpha_1^2-\alpha_0^2,\quad \dot\beta_{-1} = \alpha_{-1}^2 - \alpha_{-2}^2;$$

$$\dot b_n = \begin{bmatrix}\dot\beta_n & 0\\ 0 & \dot\beta_{-n}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\alpha_{n-1} & 0\\ 0 & \alpha_{-n}\end{bmatrix}\begin{bmatrix}-\alpha_{n-1} & 0\\ 0 & \alpha_{-n}\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix}\begin{bmatrix}\alpha_n & 0\\ 0 & -\alpha_{-n-1}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\alpha_{n-1} & 0\\ 0 & -\alpha_{-n}\end{bmatrix}\begin{bmatrix}\alpha_{n-1} & 0\\ 0 & \alpha_{-n}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix}\begin{bmatrix}\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix},$$

$$\text{i.e.}\quad \dot\beta_n = \alpha_n^2 - \alpha_{n-1}^2,\quad \dot\beta_{-n} = \alpha_{-n}^2,\quad n=2,3,\dots.$$

These calculations shows, that the third equation in (51) is equivalent to the second equation in (44).

It is easy to calculate, that the forth equality in (51) for considered matrices is also equivalent to the first equation in (44). Namely, we have

$$\dot a_0 = \begin{bmatrix}\dot\alpha_0\\ \dot\alpha_{-1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_{-1}\end{bmatrix}\begin{bmatrix}\alpha_0\\ -\alpha_{-1}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\alpha_0\\ -\alpha_{-1}\end{bmatrix}[\beta_0],$$

$$\text{i.e.}\quad \dot\alpha_0 = \frac{1}{2}\alpha_0(\beta_1-\beta_0),\quad \dot\alpha_{-1} = \frac{1}{2}\alpha_{-1}(\beta_0-\beta_{-1});$$

$$\dot a_n = \begin{bmatrix}\dot\alpha_n & 0\\ 0 & \dot\alpha_{-n-1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\beta_{n+1} & 0\\ 0 & \beta_{-n-1}\end{bmatrix}\begin{bmatrix}\alpha_n & 0\\ 0 & -\alpha_{-n-1}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\alpha_n & 0\\ 0 & -\alpha_{-n-1}\end{bmatrix}\begin{bmatrix}\beta_n & 0\\ 0 & \beta_{-n}\end{bmatrix},$$

$$\text{i.e.}\quad \dot\alpha_n = \frac{1}{2}\alpha_n(\beta_{n+1}-\beta_n),\quad \dot\alpha_{-n-1} = \frac{1}{2}\alpha_{-n-1}(\beta_{-n}-\beta_{-n-1}),\quad n\in\mathbb{N}.$$

The first and the last equalities in (51) for considered matrices are automatically fulfilled. So, for the first equality we have

$$c_n\widetilde c_{n+1} - \widetilde c_n c_{n+1} = \frac{1}{2}\begin{bmatrix}\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix}\begin{bmatrix}-\alpha_{n+1} & 0\\ 0 & \alpha_{-n-2}\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix}\begin{bmatrix}\alpha_{n+1} & 0\\ 0 & \alpha_{-n-2}\end{bmatrix} = 0,\quad n\in\mathbb{N}.$$

Consider the last equality. We have

$$a_1\widetilde{a}_0-\widetilde{a}_1a_0=\frac{1}{2}\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_{-2}\end{bmatrix}\begin{bmatrix}\alpha_0\\ -\alpha_{-1}\end{bmatrix}-\frac{1}{2}\begin{bmatrix}\alpha_1 & 0\\ 0 & -\alpha_{-2}\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_{-1}\end{bmatrix}=0;$$

$$\begin{aligned}a_{n+1}\widetilde{a}_n-\widetilde{a}_{n+1}a_n&=\frac{1}{2}\begin{bmatrix}\alpha_{n+1} & 0\\ 0 & \alpha_{-n-2}\end{bmatrix}\begin{bmatrix}\alpha_n & 0\\ 0 & -\alpha_{-n-1}\end{bmatrix}\\ &\quad-\frac{1}{2}\begin{bmatrix}\alpha_{n+1} & 0\\ 0 & -\alpha_{-n-2}\end{bmatrix}\begin{bmatrix}\alpha_n & 0\\ 0 & \alpha_{-n-1}\end{bmatrix}=0.\end{aligned}$$

As a result, we have proved that the Toda equation (44) and the Lax equation (48) are equivalent. $\square$

4. Spectral theory of the block Jacobi matrix in the space $\mathbf{l}_2$.

In this Section we construct a spectral theory of an arbitrary block Jacobi matrix J acting in the space $\mathbf{l}_2$ (38) and not depending on $t\in[0,T)$. According to (40), (46) this matrix has the form

$$J=\begin{bmatrix}b_0 & a_0^* & 0 & 0 & 0 & \dots\\ a_0 & b_1 & a_1 & 0 & 0 & \dots\\ 0 & a_1 & b_2 & a_2 & 0 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix};\qquad \begin{aligned}&b_0=[\beta_0]\colon\mathbb{C}^1\to\mathbb{C}^1,\\ &a_0=\begin{bmatrix}\alpha_0\\ \alpha_{-1}\end{bmatrix}\colon\mathbb{C}^1\to\mathbb{C}^2,\\ &a_0^*=[\alpha_0\ \alpha_{-1}]\colon\mathbb{C}^2\to\mathbb{C}^1;\end{aligned}\tag{53}$$
$$a_n=a_n^*\colon\mathbb{C}^2\to\mathbb{C}^2,\quad b_n=b_n^*\colon\mathbb{C}^2\to\mathbb{C}^2,\quad n\in\mathbb{N}.$$

We assume that all elements of the matrices $a_n, b_n, n\in\mathbb{N}_0$, are real bounded numbers, $\alpha_{-1}>0$ and $a_n^{-1}, n\in\mathbb{N}$, exist. The matrix J generate, in a usual way, in the space $\mathbf{l}_2$ a bounded selfadjoint operator $\mathbf{J}$.

For us it is necessary to construct an expansion in generalize eigenvectors of this operator using the scheme of books [2], Ch. 5 and [16], Ch. 15. For this we consider the quasinuclear rigging of the space $\mathbf{l}_2$,

$$\mathbf{l}=(\mathbf{l}_{\mathrm{fin}})'\supset\mathbf{l}_2(p^{-1})\supset\mathbf{l}_2\supset\mathbf{l}_2(p)\supset\mathbf{l}_{\mathrm{fin}}.\tag{54}$$

In (54) $\mathbf{l}_2(p)$ denotes the space $\mathbf{l}_2$ with the weight $p=(p_n)_{n=0}^\infty, p\geq 1, \sum_{n=0}^\infty p_n^{-1}<\infty$ (i.e. $\|f\|^2_{\mathbf{l}_2(p)}=\|f_0\|^2_{\mathbb{C}^1}p_0+\|f_1\|^2_{\mathbb{C}^2}p_1+\|f_2\|^2_{\mathbb{C}^2}p_2+\ldots$), $\mathbf{l}_2(p^{-1})$ is a similar space with p replaced with $p^{-1}=(p_n^{-1})_{n=0}^\infty$. Here $\mathbf{l}$ is a space of all sequences $f=(f_n)_{n=0}^\infty, f_0\in\mathbb{C}^1$, $\forall n\in\mathbb{N}\ f_n\in\mathbb{C}^2$; $\mathbf{l}_{\mathrm{fin}}$ is a space of finite sequences.

Then for our operator $\mathbf{J}$ and its resolution of identity $E(\triangle)$ a probability Borel measure $d\rho_b(\lambda)$ on $\mathbb{R}$ exists (a base spectral measure), for which

$$E(\triangle)f=\int_\triangle\mathbf{\Phi}(\lambda)\,d\rho_b(\lambda)f,\quad \mathbf{J}f=\int_{\mathbb{R}}\lambda\mathbf{\Phi}(\lambda)\,d\rho_b(\lambda),\quad f\in\mathbf{l}_2(p).\tag{55}$$

Here $\triangle$ is an arbitrary Borel set from $\mathbb{R}$, i.e., $\triangle\in\mathfrak{B}(\mathbb{R})$. For every $\lambda\in\mathbb{R}$, $\mathbf{\Phi}(\lambda)$ is a bounded operator acting from $\mathbf{l}_2(p)$ into $\mathbf{l}_2(p^{-1})$ (a generalized projection operator): $\forall f\in\mathbf{l}_2(p)\ \mathbf{\Phi}(\lambda)f$ is a generalized eigenvector of the operator $\mathbf{J}$. The operator $\mathbf{\Phi}(\lambda)$ is positive in the following sense: $\forall f\in\mathbf{l}_2(p)\ (\mathbf{\Phi}(\lambda)f,f)\geq 0$. Note also that the base spectral measure $d\rho_b(\lambda)$ is defined not uniquely.

For our operator $\mathbf{J}$ and rigging (54), the sequence $\varphi(\lambda)=(\varphi_n(\lambda))_{n=0}^\infty$, where $\varphi_0(\lambda)=\varphi_{0,0}(\lambda)\in\mathbb{C}^1$, $\forall n\in\mathbb{N}\ \varphi_n(\lambda)=(\varphi_{n,0}(\lambda),\varphi_{n,1}(\lambda))\in\mathbb{C}^2$, is a generalized eigenvector of the operator $\mathbf{J}$ with an eigenvalue λ if $(\varphi(\lambda),\mathbf{J}u)_{\mathbf{l}_2}=\lambda(\varphi(\lambda),u)_{\mathbf{l}_2}$ for arbitrary $u\in\mathbf{l}_2(p)\supset\mathbf{l}_{\mathrm{fin}}$. Using (53) we conclude that the last equality means: $\forall\lambda\in\mathbb{R}$

$$\begin{aligned}&\beta_0\varphi_{0,0}(\lambda)+\alpha_0\varphi_{1,0}(\lambda)+\alpha_{-1}\varphi_{1,1}(\lambda)=b_0\varphi_0(\lambda)+a_0^*\varphi_1(\lambda)=\lambda\varphi_0(\lambda),\\ &a_{n-1}\varphi_{n-1}(\lambda)+b_n\varphi_n(\lambda)+a_n\varphi_{n+1}(\lambda)=\lambda\varphi_n(\lambda),\quad n\in\mathbb{N}.\end{aligned}\tag{56}$$

Consider the difference equation (56). Since $\alpha_{-1} > 0$ and every matrix $a_n, n \in \mathbb{N}$ is invertible, we can find $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}$ step by step from given initial two scalar data

$$\varphi_{0,0}(\lambda) = c_{0,0}, \quad \varphi_{1,0}(\lambda) = c_{1,0}. \tag{57}$$

Note that now the situation is another as for the classical Jacobi matrix, when the analogous solution is defined by one scalar data.

For $\alpha = 0, 1$ let $\theta^{(\alpha)}(\lambda) = (\theta_n^{(\alpha)})_{n=0}^{\infty} = (\theta_{n,v_n}^{(\alpha)}(\lambda))_{n=0,v_n=0,1}^{\infty}$ be two solutions of equations (56) with the following initial data (57):

$$\theta_{0,0}^{(0)}(\lambda) = 1, \quad \theta_{1,0}^{(0)}(\lambda) = 0; \quad \theta_{0,0}^{(1)}(\lambda) = 0, \quad \theta_{1,0}^{(1)}(\lambda) = 1. \tag{58}$$

From linearity of system (56) it is easy to conclude that every solution $\varphi(\lambda)$ of (56) is a linear combination of solutions $\theta^{(0)}(\lambda)$ and $\theta^{(1)}(\lambda)$

$$\varphi(\lambda) = \varphi_0(\lambda)\theta^{(0)}(\lambda) + \varphi_{1,0}(\lambda)\theta^{(1)}(\lambda). \tag{59}$$

Note, that in accordance with (39), we will often write $f_0 = f_{0,0}$ for the zero coordinate of the vector $f = (f_n)_{n=0}^{\infty} \in \mathbf{l}$.

For us it is essential to get a representation of type (59) for the matrix of the operator $\mathbf{\Phi}(\lambda)$ which we will now understand as an operator $\mathbf{l}_{\text{fin}} \to \mathbf{l}$.

At first we note that for every linear operator $\mathbf{A}\colon \mathbf{l}_{\text{fin}} \to \mathbf{l}$ we can introduce its block matrix of type (40), elements of this block matrix A can be introduced by rules (41)–(43).

Let

$$\begin{aligned} &\Phi(\lambda) = (\Phi_{j;k}(\lambda))_{j,k=0}^{\infty}, \quad \Phi_{j;k}(\lambda) = (\Phi_{j,v_j;k,v_k}(\lambda))_{v_j,v_k=0,1}, \\ &\text{i.e.} \quad \Phi(\lambda) = (\Phi_{j,v_j;k,v_k}(\lambda))_{j,k=0;v_j,v_k=0,1}^{\infty}, \end{aligned} \tag{60}$$

be a matrix of type (40) for the operator $\mathbf{\Phi}(\lambda)$. The formula of type (59) for $\Phi(\lambda)$ is the following.

Lemma 1. *The elements of the matrix $\Phi(\lambda)$ (60) have the following representation:*

$$\begin{aligned} \Phi_{j,v_j;k,v_k}(\lambda) &= \Phi_{0,0;0,0}(\lambda)\theta_{j,v_j}^{(0)}(\lambda)\theta_{k,v_k}^{(0)}(\lambda) + \Phi_{0,0;1,0}(\lambda)\theta_{j,v_j}^{(0)}(\lambda)\theta_{k,v_k}^{(1)}(\lambda) \\ &+ \Phi_{1,0;0,0}(\lambda)\theta_{j,v_j}^{(1)}(\lambda)\theta_{k,v_k}^{(0)}(\lambda) + \Phi_{1,0;1,0}(\lambda)\theta_{j,v_j}^{(1)}(\lambda)\theta_{k,v_k}^{(1)}(\lambda), \\ &j, k \in \mathbb{N}_0, \quad v_j, v_k = 0, 1; \quad \lambda \in \mathbb{R}. \end{aligned} \tag{61}$$

Proof. Using (42) for $\mathbf{\Phi}(\lambda)$ we conclude that $\forall j, k \in \mathbb{N}_0, v_j, v_k = 0, 1$,

$$\Phi_{j,v_j;k,v_k}(\lambda) = (\mathbf{\Phi}(\lambda)\varepsilon_{k,v_k}, \varepsilon_{j,v_j})_{\mathbf{l}_2} = (\mathbf{\Phi}(\lambda)\varepsilon_{k,v_k})_{j,v_j}, \tag{62}$$

and, therefore, for fixed k, v_k (62) is a vector from $\mathbf{l}$ which is a generalized eigenvector of the operator $\mathbf{J}$ with eigenvalue λ. Therefore it is a solution of the difference equations (56) with initial data $\Phi_{0,0;k,v_k}(\lambda)$ and $\Phi_{1,0;k,v_k}(\lambda)$, and according to (59) we can write

$$\Phi_{j,v_j;k,v_k}(\lambda) = \Phi_{0,0;k,v_k}(\lambda)\theta_{j,v_j}^{(0)}(\lambda) + \Phi_{1,0;k,v_k}(\lambda)\theta_{j,v_j}^{(1)}(\lambda), \quad j \in \mathbb{N}_0, \quad v_j = 0, 1. \tag{63}$$

In a general case, the generalized projection operator $\mathbf{\Phi}(\lambda)$ is formally Hermitian, therefore our matrix (60) is Hermitian. Since the operator $\mathbf{J}$ is real, so is $\mathbf{\Phi}(\lambda)$ and, therefore, its matrix (60). As a result, the matrix (60) is real and symmetric.

Applying this fact similarly to (63) we get

$$\Phi_{0,0;k,v_k}(\lambda) = \Phi_{k,v_k;0,0}(\lambda) = \Phi_{0,0;0,0}(\lambda)\theta_{k,v_k}^{(0)}(\lambda) + \Phi_{1,0;0,0}(\lambda)\theta_{k,v_k}^{(1)}(\lambda),$$

$$\Phi_{1,0;k,v_k}(\lambda) = \Phi_{k,v_k;1,0}(\lambda) = \Phi_{0,0;1,0}(\lambda)\theta_{k,v_k}^{(0)}(\lambda) + \Phi_{1,0;1,0}(\lambda)\theta_{k,v_k}^{(1)}(\lambda).$$

Substituting these expressions into (63) we get (61). □

Let us return to the first equality in (55). For $f, g \in \mathbf{l}_{\text{fin}}$ and $\triangle = \mathbb{R}$, this and the matrix (60) give

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} (\mathbf{\Phi}(\lambda)f, g)_{\mathbf{l}_2} d\rho_b(\lambda) = \int_{\mathbb{R}} \Big[\sum_{j,k=0,v_j,v_k=0,1}^{\infty} \Phi_{j,v_j;k,v_k}(\lambda) f_{k,v_k} \overline{g}_{j,v_j} \Big] d\rho_b(\lambda). \tag{64}$$

Rewrite this equality in a more convenient form. To this end introduce the Fourier transform (w.r.t. generalized eigenvectors of the operator $\mathbf{J}$): for $f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_{\text{fin}}$ we put $\forall \lambda \in \mathbb{R}$

$$\widehat{f}(\lambda) = (\widehat{f}_0(\lambda), \widehat{f}_1(\lambda)) \in \mathbb{C}^2, \quad \widehat{f}_\alpha(\lambda) = \sum_{n=0, v_n=0,1}^{\infty} \theta_{n,v_n}^{(\alpha)}(\lambda) f_{n,v_n}, \quad \alpha = 0, 1. \tag{65}$$

Using the representation (61) and (65), we get

$$\begin{aligned}
&\sum_{j,k=0, v_j, v_k=0,1}^{\infty} \Phi_{j,v_j;k,v_k}(\lambda) f_{k,v_k} \overline{g}_{j,v_j} = \Phi_{0,0;0,0}(\lambda)\widehat{f}_0(\lambda)\overline{\widehat{g}_0(\lambda)} \\
&\quad + \Phi_{0,0;1,0}(\lambda)\widehat{f}_1(\lambda)\overline{\widehat{g}_0(\lambda)} + \Phi_{1,0;0,0}(\lambda)\widehat{f}_0(\lambda)\overline{\widehat{g}_1(\lambda)} + \Phi_{1,0;1,0}(\lambda)\widehat{f}_1(\lambda)\overline{\widehat{g}_1(\lambda)} \\
&\quad = (C(\lambda)\widehat{f}(\lambda), \widehat{g}(\lambda))_{\mathbb{C}^2}; \\
&C(\lambda) := \begin{bmatrix} \Phi_{0,0;0,0}(\lambda) & \Phi_{0,0;1,0}(\lambda) \\ \Phi_{1,0;0,0}(\lambda) & \Phi_{1,0;1,0}(\lambda) \end{bmatrix}.
\end{aligned} \tag{66}$$

Therefore we can rewrite (64) in the form: $\forall f, g \in \mathbf{l}_{\text{fin}}$

$$\begin{aligned}
&(f,g)_{\mathbf{l}_2} = \int_{\mathbb{R}} (C(\lambda)\widehat{f}(\lambda), \widehat{g}(\lambda))_{\mathbb{C}^2} \, d\rho_b(\lambda) = \int_{\mathbb{R}} (d\rho(\lambda)\widehat{f}(\lambda), \widehat{g}(\lambda))_{\mathbb{C}^2}, \quad \text{where} \\
&\mathfrak{B}(\mathbb{R}) \ni \triangle \mapsto \rho(\triangle) = \int_{\triangle} C(\lambda)\, d\rho_b(\lambda) = \int_{\triangle} \begin{bmatrix} \Phi_{0,0;0,0}(\lambda) & \Phi_{0,0;1,0}(\lambda) \\ \Phi_{1,0;0,0}(\lambda) & \Phi_{1,0;1,0}(\lambda) \end{bmatrix} d\rho_b(\lambda).
\end{aligned} \tag{67}$$

This $2 \times 2-$matrix measure $d\rho(\lambda) = (d\rho_{\mu,\nu}(\lambda))_{\mu,\nu=0,1}$ is called a matrix (standard) spectral measure of the operator $\mathbf{J}$; the first equality in (67) is the corresponding Parseval equality. This measure has the following properties.

Lemma 2. *For every $\triangle \in \mathfrak{B}(\mathbb{R})$, the matrix $\rho(\lambda)$ is real, symmetric, and positive definite, i.e., the matrix spectral measure $d\rho(\lambda)$ is real nonnegative. It is a probability measure, $\rho(\mathbb{R}) = 1$.*

Proof. As was said above, the matrix $\Phi(\lambda)$ (60) is real and symmetric, therefore the matrix $C(\lambda)$ from (66) is also real and symmetric. The definition (67) gives that such properties take place also for $\rho(\triangle)$.

The nonnegativity of the measure $d\rho(\lambda)$ follows from positivity of the operator $\mathbf{\Phi}(\lambda)$. Namely, $\forall x = (x_0, x_1) \in \mathbb{C}^2$ we construct the vector $f = (x_0, (x_1, 0), 0, 0, \ldots) = x_0\varepsilon_{0,0} + x_1\varepsilon_{1,0} \in \mathbf{l}_{\text{fin}}$. We have

$$\begin{aligned}
(C(\lambda)x, x)_{\mathbb{C}^2} &= \Phi_{0,0;0,0}(\lambda)x_0\overline{x}_0 + \Phi_{0,0;1,0}(\lambda)x_1\overline{x}_0 + \Phi_{1,0;0,0}(\lambda)x_0\overline{x}_1 + \Phi_{1,0;1,0}(\lambda)x_1\overline{x}_1 \\
&= (\mathbf{\Phi}(\lambda)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_2} x_0\overline{x}_0 + (\mathbf{\Phi}(\lambda)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_2} x_1\overline{x}_0 + (\mathbf{\Phi}(\lambda)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_2} x_0\overline{x}_1 \\
&\quad + (\mathbf{\Phi}(\lambda)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_2} x_1\overline{x}_1 = (\mathbf{\Phi}(\lambda)f, f)_{\mathbf{l}_2} \geq 0.
\end{aligned}$$

It remains to prove that $d\rho(\lambda)$ is a probability measure. We will use the vectors $\varepsilon_{0,0}$ and $\varepsilon_{1,0}$, their Fourier transform, according to (65), (58), are $\widehat{\varepsilon_{0,0}}(\lambda) = (1,0), \widehat{\varepsilon_{1,0}}(\lambda) = (0,1)$. Therefore, the Parseval equality (67) gives

$$\begin{aligned}
1 = (\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_2} = \rho_{0,0}(\mathbb{R}), \quad 0 = (\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_2} = \rho_{1,0}(\mathbb{R}), \\
1 = (\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_2} = \rho_{1,1}(\mathbb{R}), \quad \text{i.e.} \quad \rho(\mathbb{R}) = 1 \colon \mathbb{C}^2 \to \mathbb{C}^2.
\end{aligned}$$

$\square$

In investigation of the spectral problem for our matrix J (53), it is convenient to go from the generalized eigenvectors $\theta^{(0)}(\lambda)$ and $\theta^{(1)}(\lambda)$ to the corresponding matrix-valued solutions of systems (56) (as in the case of difference equations with operator-valued coefficients, see [2], Ch. 7, Section 2). At first we will give a simple, but essential remark.

Remark 3. It is also convenient to understand our operator $\mathbf{J}$ as an operator acting in the space

$$\ell_2(\mathbb{C}^2) = \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots \tag{68}$$

instead of the space $\mathbf{l}_2$ (38). For this we consider the matrix J (53) as a $2\times 2-$block Jacobi matrix, where a_n, b_n for $n \in \mathbb{N}$ are old, but

$$b_0 = \begin{bmatrix} \beta_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad a_0 = \begin{bmatrix} \alpha_0 & 0 \\ \alpha_{-1} & 0 \end{bmatrix}, \quad a_0^* = \begin{bmatrix} \alpha_0 & \alpha_{-1} \\ 0 & 0 \end{bmatrix}. \tag{69}$$

Such matrix generates in $\ell_2(\mathbb{C}^2)$, on finite vectors from (68) $\ell_{\text{fin}}(\mathbb{C}^2)$, an operator that we will denote, as earlier, by $\mathbf{J}$. This operator (and its closure) acts in $\ell_2(\mathbb{C}^2)$ but its $Ran\,\mathbf{J} \subset \mathbf{l}_2 \subset \ell_2(\mathbb{C}^2)$. *We will not to introduce new notations for such modified J and $\mathbf{J}$.*

In short,, we can say that we have a situation of difference equations with operator $\mathbb{C}^2 \mapsto \mathbb{C}^2$ coefficients, stated in [2], Ch. 7, Section 2, but with following difference: the *operator a_0 has the form* (69) *instead of being selfadjoint and invertible.*

Now for us it is essential to rewrite the notion of $\theta^{(0)}(\lambda), \theta^{(1)}(\lambda)$ as matrix-solutions. We put (see (58))

$$\begin{aligned} P_n(\lambda) &= \begin{bmatrix} P_{n;0,0}(\lambda) & P_{n;0,1}(\lambda) \\ P_{n;1,0}(\lambda) & P_{n;1,1}(\lambda) \end{bmatrix} = \begin{bmatrix} \theta^{(0)}_{n,0}(\lambda) & \theta^{(1)}_{n,0}(\lambda) \\ \theta^{(0)}_{n,1}(\lambda) & \theta^{(1)}_{n,1}(\lambda) \end{bmatrix}, \quad n \in \mathbb{N} \quad \text{and} \\ P_0(\lambda) &= \begin{bmatrix} \theta^{(0)}_{0,0}(\lambda) & \theta^{(1)}_{0,0}(\lambda) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{or } [1\ 0]). \end{aligned} \tag{70}$$

The following formal result holds true.

Lemma 3. *The matrices $P_n(\lambda), n \in \mathbb{N}_0$, satisfy the following equations:* $\forall \lambda \in \mathbb{R}$

$$\begin{aligned} &b_0 P_0(\lambda) + a_0^* P_1(\lambda) = \lambda P_0(\lambda), \\ &a_{n-1} P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda P_n(\lambda), \quad n \in \mathbb{N}. \end{aligned} \tag{71}$$

Conversely, the solution $P(\lambda) = (P_n(\lambda))_{n=0}^\infty$ of equation (71) *with the initial data*

$$P_{0;0,0}(\lambda) = 1, \quad P_{0;0,1}(\lambda) = 0; \quad P_{1;0,0}(\lambda) = 0, \quad P_{1;0,1}(\lambda) = 1 \tag{72}$$

is given by (70).

Proof. Let apply the left-hand side of the second equality in (71) to the vector $(1,0) \in \mathbb{C}^2$. Using (70), we get

$$a_{n-1}(\theta^{(0)}_{n-1,0}(\lambda), \theta^{(0)}_{n-1,1}(\lambda)) + b_n(\theta^{(0)}_{n,0}(\lambda), \theta^{(0)}_{n,1}(\lambda)) + a_n(\theta^{(0)}_{n+1,0}(\lambda), \theta^{(0)}_{n+1,1}(\lambda)), \tag{73}$$

i.e., the left-hand side of the second equality in (56) for solution $\varphi(\lambda) = \theta^{(0)}(\lambda)$. Therefore, (73) is, according to (71), equals to $\lambda(\theta^{(0)}_{n,0}(\lambda), \theta^{(0)}_{n,1}(\lambda)) = \lambda P_n(\lambda)(1,0)$. As a result, the second equality in (71), after this application to vector $(0,1)$, takes place.

Similarly we prove that the second equality in (71), after applying it to the vector $(0,1)$, also takes place (now $\varphi(\lambda) = \theta^{(1)}(\lambda)$). Since the vectors $(1,0), (0,1)$ make a basis in $\mathbb{C}^2$, the second equality in (71) takes place.

The first equality is also true: application of its left side to $(1,0)$ gives $b_0(\theta^{(0)}_{0,0}(\lambda), 0) + a_0^*(\theta^{(0)}_{1,0}(\lambda), \theta^{(0)}_{1,1}(\lambda))$, i.e., the left-hand side of the first equality in (56) for $\varphi(\lambda) = \theta^{(0)}(\lambda)$. Such an application to $(0,1)$ gives zero, i.e., $\lambda P_0(\lambda)(0,1)$.

The inverse result of the lemma is obvious; since equations (71) and (56) are equivalent, the solutions $\varphi(\lambda) = (P_{0;0,0}(\lambda), (P_{1;0,0}(\lambda), P_{1;1,0}(\lambda)), (P_{2;0,0}(\lambda), P_{2;1,0}(\lambda)), \ldots)$ of (56) has the first initial data from (58) and $\varphi(\lambda) = (P_{0;0,1}(\lambda), (P_{1;0,1}(\lambda), P_{1;1,1}(\lambda)), (P_{2;0,1}(\lambda), P_{2;1,1}(\lambda)), \ldots)$ has the second initial data. From this the representation (70) follows. $\square$

We can formulate the above obtained results in a form of the following theorem about direct spectral problem for operator $\mathbf{J}$.

Theorem 5. *Consider a Jacobi matrix J of the form* (53) *with conditions on its elements formulated at the beginning of Section 4. This matrix gives rise to a selfadjoint bounded operator* $\mathbf{J}$ *on the space* $\mathbf{l}_2$ (38). *With* $\mathbf{J}$ *we connect the rigging* (54).

The direct spectral problem for $\mathbf{J}$ *consists of the following. The complete system of generalized eigenvectors of the operator* $\mathbf{J}$ *is the following:* $\forall\lambda \in \mathbb{R}$

$$\varphi_\alpha(\lambda) = ((P_{n;0,\alpha}(\lambda), P_{n;1,\alpha}(\lambda))_{n=0}^\infty) \in \mathbf{l}, \quad \alpha = 0, 1, \tag{74}$$

where the "matrix polynomials"

$$P_n(\lambda) = \begin{bmatrix} P_{n;0,0}(\lambda) & P_{n;0,1}(\lambda) \\ P_{n;1,0}(\lambda) & P_{n;1,1}(\lambda) \end{bmatrix} \tag{75}$$

are solutions of system (71) *with the initial conditions* (72) *(the last index in* $P_{n;\alpha,\beta}(\lambda)$ *is the number of the solution).*

This system is complete in the following sense: if we introduce $\forall f \in \mathbf{l}_{\mathrm{fin}}$ *and* $\lambda \in \mathbb{R}$ *the Fourier transform*

$$\widehat{f}(\lambda) = (\widehat{f}_0(\lambda), \widehat{f}_1(\lambda)) \in \mathbb{C}^2, \quad \widehat{f}(\lambda) = \sum_{n=0}^{\infty} P_n^*(\lambda) f_n, \tag{76}$$

then we have the Parseval equality

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} (d\rho(\lambda)\widehat{f}(\lambda), \widehat{g}(\lambda))_{\mathbb{C}^2}, \quad f, g \in \mathbf{l}_{\mathrm{fin}}. \tag{77}$$

Here $d\rho(\lambda) = (d\rho_{\alpha,\beta}(\lambda))_{\alpha,\beta=0}^1$ *is a* $2\times 2-$*matrix nonnegative spectral measure of the operator* $\mathbf{J}$; *it is a probability measure,* $\rho(\mathbb{R}) = 1$. *The mapping* $\mathbf{l}_{\mathrm{fin}} \ni f \mapsto \widehat{f}(\lambda)$, *after its extension by closure, is a unitary operator between the spaces* $\mathbf{l}_2$ *and* $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$.

Note that here $L^2(\mathbb{C}, \mathbb{R}, d\rho(\lambda))$ denotes the L^2-Hilbert space of functions $\mathbb{R} \ni \lambda \mapsto F(\lambda) \in \mathbb{C}^2$, which is constructed using the scalar product

$$(F, G)_{L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda))} = \int_{\mathbb{R}} (d\rho(\lambda)F(\lambda), G(\lambda))_{\mathbb{C}^2} \tag{78}$$

in a classical manner, — we introduce at first an integral in (78) for simple functions $F(\lambda), G(\lambda)$ (i.e. finite combinations of $\mathbb{C}^2-$ characteristic functions) and then we take the completion (for a more detailed account, see [2], Ch. 7, Section 2).

Proof. The first part of the theorem follows from Lemma 3. To prove the second part, it is necessary first to note that definitions (65) and (76) are the same, see formulas (70). Further it is necessary to apply (67). The density of $\widehat{f}(\lambda), f \in \mathbf{l}_2$, we be discussed later, when we will consider the inverse spectral problem for $\mathbf{J}$. □

So, we pass to the inverse spectral problem for $\mathbf{J}$, i.e., the problem of reconstructing the matrix J from its spectral measure $d\rho(\lambda)$.

At first, we give some simple consequences from the Parseval equality (77), which we will understand to be extended to $f, g \in \mathbf{l}_2$. Let in (77) $g = \varepsilon_{n,v_n}$, then according to (76),

$$(\widehat{\varepsilon}_{n,v_n})(\lambda) = (P_{n;v_n,0}(\lambda), P_{n;v_n,1}(\lambda)) = P_n^*(\lambda)e_{v_n}, \quad n \in \mathbb{N}_0, \quad v_n = 0, 1. \tag{79}$$

Therefore we have the following formula for reconstruction of f from its Fourier transform:

$$f_{n,v_n} = \int_{\mathbb{R}} (d\rho(\lambda)\widehat{f}(\lambda), (P_{n;v_n,0}(\lambda), P_{n;v_n,1}(\lambda)))_{\mathbb{C}^2} = \int_{\mathbb{R}} (P_n(\lambda)d\rho(\lambda)\widehat{f}(\lambda), e_{v_n})_{\mathbb{C}^2}. \tag{80}$$

Taking $f = \varepsilon_{j,v_j}$ in (70) and using (79) we get the following conditions for orthogonality of $P_{n;\alpha,\beta}(\lambda)$:

$$\begin{aligned} &\int_{\mathbb{R}} (d\rho(\lambda)(P_{j;\alpha,0}(\lambda), P_{j;\alpha,1}(\lambda)), (P_{k;\beta,0}(\lambda), P_{k;\beta,1}(\lambda)))_{\mathbb{C}^2} \\ &\quad = (\varepsilon_{j,\alpha})_{k,\beta} = \delta_{j,k}\delta_{\alpha,\beta}, \quad j, k \in \mathbb{N}_0, \quad \alpha, \beta = 0, 1 \end{aligned} \tag{81}$$

(here $\delta_{m,n}$ is ordinary Kronecker symbol).

From (81) it follows that the following $\mathbb{R}^2-$valued functions of $\lambda \in \mathbb{R}$ are orthonormal:

$$
\begin{aligned}
&(P_{0;0,0}(\lambda), P_{0;0,1}(\lambda)); (P_{1;0,0}(\lambda), P_{1;0,1}(\lambda)), (P_{1;1,0}(\lambda), P_{1;1,1}(\lambda));\\
&(P_{2;0,0}(\lambda), P_{2;0,1}(\lambda)), (P_{2;1,0}(\lambda), P_{2;1,1}(\lambda)); \dots .
\end{aligned} \tag{82}
$$

Using equalities (71) and the initial data (72) it is easy to calculate $P_n(\lambda), n \in \mathbb{N}_0$ step by step. So, we have

$$
P_0(\lambda) = [1 \quad 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_1(\lambda) = \begin{bmatrix} 0 & 1 \\ \frac{1}{\alpha_{-1}}(\lambda - \beta_0) & -\frac{\alpha_0}{\alpha_1} \end{bmatrix}. \tag{83}
$$

The next matrix polynomials $P_2(\lambda), P_3(\lambda), \dots$ have a more complicated distribution of powers of λ among the elements of the matrix $P_n(\lambda)$. *We will later consider the most simple case, when all the matrices $a_n, b_n, n \in \mathbb{N}$, in* (53) *are diagonal.* This case is sufficient for our purpose to investigate the Toda lattice. So, we have the following.

Lemma 4. *Let all the matrices $a_n, b_n, n \in \mathbb{N}$, in* (53) *be diagonal. Then for $n \geq 2$ in the matrix $P_n(\lambda)$, its elements are polynomials of the following degrees: $P_{n;0,0}(\lambda)$ of $n-2$, $P_{n;0,1}(\lambda)$ and $P_{n;1,1}(\lambda)$ of $n-1$ and $P_{n;1,0}(\lambda)$ of n. Their leading coefficients are nonzero.*

Proof. At first we note that if c is a diagonal $2\times 2-$matrix with nonzero diagonal elements $c_{0,0}, c_{1,1} \in \mathbb{C}$ then its multiplication by the matrix $d = (d_{\alpha,\beta})_{\alpha,\beta=0}^1$ is a matrix with elements equal to the products of $d_{\alpha,\beta}$ by the scalars $c_{0,0}$ or $c_{1,1}$. Therefore the structure of matrix d after its multiplication on the left by c remains the same.

Let $n = 2$. Them from (71) we conclude that $P_2(\lambda) = a_1^{-1}(\lambda P_1(\lambda) - b_1 P_1(\lambda) - a_0 P_0(\lambda))$ and it has, according to (83), the structure indicated in the lemma.

For $n = 3, 4, \dots$, we conclude from (71) step by step, as above, that the structure of the matrix $P_n(\lambda)$ is as required. ☐

Denote by $R_m(\lambda)$ some ordinary polynomial of degree $m \in \mathbb{N}_0$ with real coefficients, with the leading coefficient being nonzero. Then from Lemma 4 we can conclude that the sequence (82) of $\mathbb{R}^2-$polynomials has the form

$$
\begin{aligned}
&(1,0); (0, R_0(\lambda)), (R_1(\lambda), R_0(\lambda)); (R_0(\lambda), R_1(\lambda)), (R_2(\lambda), R_1(\lambda));\\
&(R_1(\lambda), R_2(\lambda)), (R_3(\lambda), R_2(\lambda)); \dots ; (R_n(\lambda), R_{n+1}(\lambda)), (R_{n+2}(\lambda), R_{n+1}(\lambda)); \dots
\end{aligned} \tag{84}
$$

(with different polynomials $R_n(\lambda)$ with some index n, but standing in different places).

It is easy to construct elementary $\mathbb{R}^2-$polynomials linear combinations of which give the $\mathbb{R}^2-$polynomials from (84).

Consider the following sequence of basis $\mathbb{R}^2-$polynomials

$$
(1,0); (0,1), (\lambda, 0); (0, \lambda), (\lambda^2, 0); \dots ; (0, \lambda^n), (\lambda^{n+1}, 0); \dots . \tag{85}
$$

Then every $\mathbb{R}^2-$polynomials $(P(\lambda), Q(\lambda))$ where $P(\lambda), Q(\lambda)$ are some real polynomials of degree $m, n \in \mathbb{N}_0$ corresponding, is some linear combination of the basis polynomials (85). Namely, let $P(\lambda) = \sum_{j=0}^m p_j \lambda^j, Q(\lambda) = \sum_{k=0}^n q_k \lambda^k$, then evidently $(P(\lambda), Q(\lambda)) = \sum_{j=0}^m p_j(\lambda^j, 0) + \sum_{k=0}^n q_k(0, \lambda^k)$.

It is also possible to say that if $m \in \mathbb{N}_0$ and $n = m+1$, then the corresponding $\mathbb{R}^2-$polynomial $(P(\lambda), Q(\lambda))$ is a linear combination of the first $\mathbb{R}^2-$polynomials from (85) up to $(0, \lambda^{m+1})$. If $m \in \mathbb{N}$ and $n = m-1$, then for $(P(\lambda), Q(\lambda))$ it is necessary to take, in (85), such polynomials up to $(\lambda^m, 0)$.

To solve the inverse spectral problem, it will be necessary to take the orthogonalization of $\mathbb{R}^2-$polynomials (85) in the space $L^2(\mathbb{R}^2, \mathbb{R}, d\rho(\lambda)) \subset L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$. All $\mathbb{R}^2-$polynomials belong to the space $L^2(\mathbb{R}^2, \mathbb{R}, d\rho(\lambda))$ and they are linearly independent in this space if *the measure $d\rho(\lambda)$ has bounded support and is not concentrated on finitely* *many points from* $\mathbb{R}$ (the proof of this assertion immediately follows from an analogous fact for ordinary polynomials and a scalar measure on $\mathbb{R}$).

From above mentioned facts it is easy to prove the following theorem.

Theorem 6. *Let* $\mathbf{J}$ *be a bounded selfadjoint operator in the space* $\mathbf{l}_2$*, generated by the matrix* J (53)*. Its spectral measure* $d\rho(\lambda)$ *has bounded support and is not concentrated on finitely many points from* $\mathbb{R}$*. The generalized eigenvectors* (82) *corresponding to* $\mathbf{J}$ *can be obtained by means of the classical Gramm-Schmidt orthogonalization procedure in the space* $L^2(\mathbb{R}^2, \mathbb{R}, d\rho(\lambda))$*, which is applied to the sequence* (85)*.*

Proof. At first we explain the existence of above mentioned properties of $d\rho(\lambda)$. Its support is bounded, since operator $\mathbf{J}$ is bounded. Assume that the measure $d\rho(\lambda)$ concentrated on a finite number of points λ from $\mathbb{R}$. Then from (78) it follows that the space $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$ must be finite-dimensional, but this is in contradiction with isometry (77) between $\mathbf{l}_2$ and $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$.

Consider the generalized eigenvectors (82) of the operator $\mathbf{J}$. The first and the second are $(1,0), (0,1)$, i.e., the first two vectors from (85). Further we apply the Gramm-Schmidt procedure in the real space $L^2(\mathbb{R}^2, \mathbb{R}, d\rho(\lambda))$. Lemma 4 and orthonormality of the vectors from (82) show that our assertion takes place. $\square$

Also note that from boundedness of the support of $d\rho(\lambda)$, it follows that the set of all $\mathbb{R}^2-$polynomials is dense in $L^2(\mathbb{R}^2, \mathbb{R}, d\rho(\lambda))$. Therefore the Theorem 6 gives that *the set* $\{\widehat{f}(\lambda), f \in \mathbf{l}_{\text{fin}}\}$ *is dense in* $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$ (this fact has been used in the proof of Theorem 5).

The elements of the matrix J can be expressed in terms of its spectral measure $d\rho(\lambda)$ of $\mathbf{J}$, which is similar to the case of classical Jacobi matrices. For this, it is convenient to introduce the integral of the type

$$\int_{\mathbb{R}} A(\lambda)\, d\rho(\lambda) B(\lambda) = C, \tag{86}$$

where $\mathbb{R} \ni \lambda \mapsto A(\lambda), B(\lambda)$ are $2 \times 2-$matrix-valued functions. By the definition, the matrix C is such that $\forall x, y \in \mathbb{C}^2$

$$(Cx, y)_{\mathbb{C}^2} = \int_{\mathbb{R}} (d\rho(\lambda) B(\lambda) x, A^*(\lambda) y)_{\mathbb{C}^2}.$$

Using this definition (86) and (75) it is easy to rewrite the orthogonal conditions (81) in the form

$$\begin{aligned} &\int_{\mathbb{R}} P_j(\lambda)\, d\rho(\lambda) P_k^*(\lambda) = \delta_{j,k} 1, \quad j,k \in \mathbb{N}_0, \quad (j,k) \neq (0,0); \\ &\int_{\mathbb{R}} P_0(\lambda)\, d\rho(\lambda) P_0^*(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{87}$$

Lemma 5. *The following equalities take place:*

$$\begin{aligned} &a_n^* = \int_{\mathbb{R}} \lambda P_n(\lambda)\, d\rho(\lambda) P_{n+1}^*(\lambda), \quad b_n = \int_{\mathbb{R}} \lambda P_n(\lambda)\, d\rho(\lambda) P_n^*(\lambda), \quad n \in \mathbb{N}_0; \\ &a_n^* = a_n, \quad n \in \mathbb{N}. \end{aligned} \tag{88}$$

Proof. Let $n \in \mathbb{N}$. We consider the second equality in (71), multiply it by $P_{n+1}^*(\lambda)$ and integrate over $\mathbb{R}$. The conditions (87) gives the first formula in (88), since $a_n = a_n^*$.

For obtaining the second formula in (88), it is necessary to repeat the above calculation replacing $P_{n+1}^*(\lambda)$ with $P_n^*(\lambda)$.

Let $n = 0$. We will use the matrices (69), (83), and the first equality in (71). Then, as above, we get formulas (88) for $n = 0$. $\square$

The results formulated above give the following theorem.

Theorem 7. *Let J (53) be a block Jacobi matrix with properties indicated at the beginning of this Section with diagonal matrices $a_n, b_n, n \in \mathbb{N}$. Then the elements of the matrix J are reconstructed by formulas* (88).

It is also possible to construct the matrix J by formulas (88) for an apriori given $2\times 2-$matrix measure $d\rho(\lambda)$ with the above mentioned properties. At first it is necessary to construct the matrix polynomials $P(\lambda), n \in \mathbb{N}_0$. To construct them, we apply the Gramm-Schmidt orthogonalization procedure in the space $L^2(\mathbb{R}^2, \mathbb{R}, d\rho(\lambda))$ to the $\mathbb{R}^2-$polynomials (85) and get the sequence (82). The formulas (75) give $P_n(\lambda), n \in \mathbb{N}_0$.

Then it is possible to prove that the matrix J constructed in such way has necessary properties and $a_n, b_n, n \in \mathbb{N}$ are diagonal matrices; apriory given $d\rho(\lambda)$ is a spectral measure of the corresponding operator $\mathbf{J}$. But we will not investigate this results in this article in detail.

In the last part of this Section we will give some additional constructions connected with the direct spectral problem for the matrix J (53) and the corresponding operator $\mathbf{J}$.

I. Recall that matrix polynomials (75) are an analog of the polynomials $P_n(\lambda)$ of the first kind connected with the classical Jacobi matrix. We construct now an analog of polynomials $Q_n(\lambda)$ of the second kind (see [2], Ch. 7, Sections 1,2). The matrix J now is general: $\forall n \in \mathbb{N}$ the matrices a_n, b_n are not necessarily diagonal.

From (55), (60), (61), and (65), it follows that $\forall z \in \mathbb{C}\setminus\mathbb{R}$ the resolvent $\mathbf{R}_z$ of operator $\mathbf{J}$ can be written as

$$(\widehat{\mathbf{R}_z f})(\lambda) = (\lambda - z)^{-1}\widehat{f}(\lambda), \quad \lambda \in \mathbb{R}, \quad f \in \mathbf{l}_2. \tag{89}$$

Therefore (79) gives

$$\begin{aligned}&(\widehat{\mathbf{R}_z\varepsilon_{n,\beta}})(\lambda) = (\lambda - z)^{-1}(P_{m;\beta,0}(\lambda), P_{m;\beta,1}(\lambda)) = (\lambda - z)^{-1}P_m^*(\lambda)e_\beta,\\ &\beta = 0,1, \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad \lambda \in \mathbb{R}.\end{aligned} \tag{90}$$

Using (80) we conclude from (90): $\forall z \in \mathbb{C}\setminus\mathbb{R}$ and $\alpha = 0, 1$,

$$\begin{aligned}(\mathbf{R}_z\varepsilon_{m,\beta})_{n,\alpha} &= \int_{\mathbb{R}} \frac{1}{\lambda - z}(P_n(\lambda)\, d\rho(\lambda)\widehat{\varepsilon}_{m,\beta}(\lambda), e_\alpha)_{\mathbb{C}^2}, \quad \text{i.e.}\\ (\mathbf{R}_z\varepsilon_{m,\beta})_n &= \int_{\mathbb{R}} \frac{1}{\lambda - z}P_n(\lambda)\, d\rho(\lambda)\widehat{\varepsilon}_{m,\beta}(\lambda), \quad m, n \in \mathbb{N}_0, \quad \beta = 0,1.\end{aligned} \tag{91}$$

Using last equality we get

$$\begin{aligned}(\mathbf{R}_z\varepsilon_{m,\beta})_n = &\int_{\mathbb{R}}(\lambda - z)^{-1}(P_n(\lambda) - P_n(z))\, d\rho(\lambda)\widehat{\varepsilon}_{m,\beta}(\lambda)\\ &+ P_n(z)\int_{\mathbb{R}}(\lambda - z)^{-1}d\rho(\lambda)\widehat{\varepsilon}_{m,\beta}(\lambda),\\ &m, n \in \mathbb{N}_0, \quad \beta = 0,1, \quad z \in \mathbb{C}\setminus\mathbb{R}.\end{aligned} \tag{92}$$

Let, in (92), $m = 0, \beta = 0$ and $m = 1, \beta = 0$. According to (79) and (83), we have $\forall n \in \mathbb{N}_0$

$$\begin{aligned}(\mathbf{R}_z\varepsilon_{0,0})_n = &\int_{\mathbb{R}}(\lambda - z)^{-1}(P_n(\lambda) - P_n(z))\, d\rho(\lambda)(1,0)\\ &+ P_n(z)\int_{\mathbb{R}}(\lambda - z)^{-1}d\rho(\lambda)(1,0),\\ (\mathbf{R}_z\varepsilon_{1,0})_n = &\int_{\mathbb{R}}(\lambda - z)^{-1}(P_n(\lambda) - P_n(z))\, d\rho(\lambda)(0,1)\\ &+ P_n(z)\int_{\mathbb{R}}(\lambda - z)^{-1}d\rho(\lambda)(0,1).\end{aligned} \tag{93}$$

Let $x = (x_0, x_1)$, $y = (y_0, y_1)$ be two arbitrary vectors from $\mathbb{C}^2$; introduce the 2×2-matrix

$$\begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} =: [x \quad y]. \tag{94}$$

With these notations, the two equalities (93) we can be rewritten in the form

$$\begin{aligned} \big[(\mathbf{R}_z\varepsilon_{0,0})_n \ (\mathbf{R}_z\varepsilon_{1,0})_n\big] &= \int_{\mathbb{R}} (\lambda - z)^{-1}(P_n(\lambda) - P_n(z))\, d\rho(\lambda) \\ &+ P_n(z)\int_{\mathbb{R}} (\lambda - z)^{-1} d\rho(\lambda), \quad n \in \mathbb{N}_0, \quad z \in \mathbb{C}\setminus\mathbb{R} \end{aligned} \tag{95}$$

(we used above the symmetry of the matrix $d\rho(\lambda)$).

As for the classical Jacobi matrices, we will say that the matrix

$$Q_n(z) = \int_{\mathbb{R}} (\lambda - z)^{-1}(P_n(\lambda) - P_n(z))\, d\rho(\lambda), \quad n \in \mathbb{N}_0, \quad z \in \mathbb{C}\setminus\mathbb{R} \tag{96}$$

is a matrix polynomial of the second kind. The matrix-valued function

$$m(z) = \int_{\mathbb{R}} (\lambda - z)^{-1} d\rho(\lambda), \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{97}$$

is, by definition, the matrix Weyl function. The relation (95) has now the form:

$$\big[(\mathbf{R}_z\varepsilon_{0,0})_n \ (\mathbf{R}_z\varepsilon_{1,0})_n\big] = Q_n(z) + P_n(z)m(z), \quad n \in \mathbb{N}_0, \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{98}$$

So, *we have proved the essential relation* (98).

We will show now that (as in the classical theory) the sequence $(Q_n(z))_{n=0}^\infty$ is a solution of the difference equations (71) for $n \in \mathbb{N}$ but with initial data other than for $(P_n(\lambda))_{n=0}^\infty$.

Lemma 6. *The sequence $Q(z) := (Q_n(z))_{n=0}^\infty, z \in \mathbb{C}$, is well defined and is a solution of the difference equations* (71) *for $n \in \mathbb{N}$ with λ replaced by z and the initial data*

$$Q_0(z) = 0, \quad Q_1(z) = \begin{bmatrix} 0 & 0 \\ \alpha_{-1}^{-1} & 0 \end{bmatrix}. \tag{99}$$

Proof. At first we note that every element of the matrix $P_n(\lambda)$ is a polynomial, therefore the expression $(\lambda - z)^{-1}(P_n(\lambda) - P_n(z))$ is well defined for every $z \in \mathbb{C}$ and (96) takes place for all $z \in \mathbb{C}$. If we apply the difference expression $\mathcal{L}$ from the second equality in the left-hand side of (71) to (96) we get: $\forall n \in \mathbb{N}$ and $z \in \mathbb{C}$

$$\begin{aligned} (\mathcal{L}Q(t))_n &= \int_{\mathbb{R}} (\lambda - z)^{-1}((\mathcal{L}P(\lambda))_n - (\mathcal{L}P(t))_n)\, d\rho(\lambda) \\ &= \int_{\mathbb{R}} (\lambda - z)^{-1}(\lambda P_n(\lambda) - zP_n(z))\, d\rho(\lambda) = z\int_{\mathbb{R}} (\lambda - z)^{-1}(P_n(\lambda) - P_n(z))\, d\rho(\lambda) \\ &+ \int_{\mathbb{R}} P_n(\lambda)\, d\rho(\lambda) = zQ_n(z) \end{aligned}$$

(the equality $\int_{\mathbb{R}} P_n(\lambda) d\rho(\lambda) = 0$ follows from (87) and $\rho(\mathbb{R}) = 1$).

The initial data (99) is obtained from (83) and $\rho(\mathbb{R}) = 1$. Note that we can find a solution of the second equality in (71) by replacing $P_n(\lambda)$ with $Q_n(\lambda)$ step by step for $n \in \mathbb{N}$ since such a_n are invertible. □

Remark 4. It is necessary to note that the above stated spectral theory takes place also for general Hermitian matrices J (53) when the corresponding operator $\mathbf{J}$ in the space $\mathbf{l}_2$ is not bounded. But it is Hermitian with equal defect numbers (since J is real). Therefore it has a selfadjoint extension in the space $\mathbf{l}_2$. These extensions can be described and representation (98) plays an essential role in such a description.

II. The construction of Subsection I allow, in some cases, to find the Weyl function $m(z)$ (97) and consequently the spectral measure $d\rho(\lambda)$. Let us explain this approach.

For our matrix J (53) and its matrix polynomials $P_n(z)$ (75), consider the series

$$\sum_{n=0}^{\infty} P_n^*(z)P_n(z), \quad z \in \mathbb{C}. \tag{100}$$

It is easy to understand that *the series* (100) *is divergent in the following sense:* $\forall x \in \mathbb{C}^2, x \neq 0$,

$$\lim_{N\to\infty} \sum_{n=0}^{N} (P_n^*(z)P_n(z)x, x)_{\mathbb{C}^2} = \sum_{n=0}^{\infty} \|P_n(z)x\|^2_{\mathbb{C}^2} = \infty, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{101}$$

In fact, it follows from (71) that in the case of a finite sum in (101), the vector $\varphi(z) = (P_n(z)x)_{n=0}^{\infty}$ belongs to $\mathbf{l}_2$ and is a nonzero eigenvector of the operator $\mathbf{J}$ with complex eigenvalue z. But this is impossible since $\mathbf{J}$ is selfadjoint.

Remark 5. Such situation takes place also for J with unbounded elements but in the case when the corresponding operator $\mathbf{J}$ is selfadjoint in $\mathbf{l}_2$.

We can now prove a simple but, in some cases, useful theorem.

Theorem 8. *Assume that we can find a* $2 \times 2-$*matrix function* $\mathbb{C} \setminus \mathbb{R} \ni z \mapsto f(z)$ *for which* $\forall x \in \mathbb{C}^2$

$$\sum_{n=0}^{\infty} \|(Q_n(z) + P_n(z)f(z))x\|^2_{\mathbb{C}^2} < \infty. \tag{102}$$

Then $m(z) = f(z), z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. At first we note that the condition (102) for $f(z) = m(z)$ takes place. Namely, using (98) and definition (94) we conclude that $\forall x = (x_0, x_1) \in \mathbb{C}^2$

$$\begin{aligned}(Q_n(z) + P_n(z)m(z))x =& ((\mathbf{R}_z\varepsilon_{0,0})_{n,0}x_0 + (\mathbf{R}_z\varepsilon_{1,0})_{n,0}x_1,\\ &(\mathbf{R}_z\varepsilon_{0,0})_{n,1}x_0 + (\mathbf{R}_z\varepsilon_{1,0})_{n,1}x_1), \quad n \in \mathbb{N}_0,\end{aligned}$$

and the last sequence of vectors from $\mathbb{C}^2$ belongs to $\mathbf{l}_2$.

From (102) for $m(z)$ and $f(z)$, by subtraction, we get

$$\begin{aligned}\Big(\sum_{n=0}^{\infty} \|P_n(z)(m(z) - f(z))x\|^2_{\mathbb{C}^2}\Big)^{1/2} &\leq \Big(\sum_{n=0}^{\infty} \|(Q_n(z) + P_n(z)m(z))x\|^2_{\mathbb{C}^2}\Big)^{1/2}\\ &+ \Big(\sum_{n=0}^{\infty} \|(Q_n(z) + P_n(z)f(z))x\|^2_{\mathbb{C}^2}\Big)^{1/2} < \infty.\end{aligned} \tag{103}$$

If $m(z) \neq f(z)$ for some $z \in \mathbb{C} \setminus \mathbb{R}$, then $\exists x \in \mathbb{C}^2$ for which $(m(z) - f(z))x \neq 0$ and inequality (103) contradicts to (101). □

5. The equations for Weyl function and spectral matrix

In the Section 3 we have shown that the double-infinite Toda equation can be written as a Lax equation (48) for block type Jacobi matrices $J(t), A(t)$. The corresponding operator $\mathbf{J}(t)$ in the space $\mathbf{l}_2$ (38) for every fixed $t \in [0, T)$ is a block Jacobi operator of the type investigated in the Section 4. Our aim now is to deduce from (48), equations on elements of the Weyl matrix function and derivatives of the spectral measure. These results are similar to Theorems 1, 2.

We will start with deducing the corresponding differential equation for the Weyl matrix function $m(z;t)$ of the operator $\mathbf{J}(t)$ in the space $\mathbf{l}_2$. According to (97), this $2\times 2-$matrix function has the form

$$m(z;t)=\begin{bmatrix}m_{0,0}(z;t) & m_{0,1}(z;t)\\ m_{1,0}(z;t) & m_{1,1}(z;t)\end{bmatrix}=\int_{\mathbb{R}}\frac{1}{\lambda-z}d\rho(\lambda;t)$$
$$=\begin{bmatrix}(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2} & (\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2}\\ (\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,0})_{\mathbf{l}_2} & (\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{1,0})_{\mathbf{l}_2}\end{bmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T),\tag{104}$$

where $d\rho(\lambda;t)$ is the matrix spectral measure of the operator $\mathbf{J}(t)$, $\mathbf{R}_z(t)$ its resolvent and (see (39))

$$\varepsilon_{0,0}=(1,0,0,\ldots),\quad \varepsilon_{1,0}=(0,(1,0),0,0,\ldots).\tag{105}$$

Note that selfadjointness of the operator $\mathbf{J}(t)$ and it being real gives that $\mathbf{R}_z^*(t)=\mathbf{R}_{\bar z}(t)=\overline{\mathbf{R}_z(t)}$ where "—" denotes the complex conjugation in the space $\mathbf{l}_2$ and its extension on operators in $\mathbf{l}_2$. The last equality, since ε_{n,v_n} is real, gives that

$$(\mathbf{R}_z(t)\varepsilon_{k,v_k},\varepsilon_{j,v_j})_{\mathbf{l}_2}=(\varepsilon_{k,v_k},\overline{\mathbf{R}_z(t)\varepsilon_{j,v_j}})_{\mathbf{l}_2}=(\mathbf{R}_z(t)\varepsilon_{j,v_j},\varepsilon_{k,v_k})_{\mathbf{l}_2}.$$

So, we have $\forall j,k\in\mathbb{N}_0, v_j,v_k=0,1$

$$(\mathbf{R}_z(t)\varepsilon_{k,v_k},\varepsilon_{j,v_j})_{\mathbf{l}_2}=(\mathbf{R}_z(t)\varepsilon_{j,v_j},\varepsilon_{k,v_k})_{\mathbf{l}_2},\quad\text{in particular}$$
$$m_{0,1}(z;t)=m_{1,0}(z;t),\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T).\tag{106}$$

To find a system of differential equations w.r.t. t for the functions $m_{\alpha,\beta}(z;t)$, we will use the idea of deduction of the equation (15) from the Lax equation, but now this deduction is more complicated as in Section 2 since our block matrices $R_z(t)$ (corresponding to the operators $\mathbf{R}_z(t)$) and $A(t)$ of the form (49) are more cumbersome.

To our operator $\mathbf{R}_z(t)$ we can apply the general constructions (40) – (43). So, for the corresponding matrix $R_z(t)$ with fixed t we have, using (106),

$$R_z(t)=(R_{z;j;k})_{j,k=0}^{\infty}=(R_{z;j,v_j;k,v_k})_{j,k=0,v_j,v_k=0,1}^{\infty},$$
$$R_{z;j,v_j;k,v_k}=(\mathbf{R}_z\varepsilon_{k,v_k},\varepsilon_{j,v_j})_{\mathbf{l}_2}=(\mathbf{R}_z\varepsilon_{j,v_j},\varepsilon_{k,v_k})_{\mathbf{l}_2}.\tag{107}$$

In particular $\forall z\in\mathbb{C}\setminus\mathbb{R}, t\in[0,T)$

$$R_{z;0,0;0,0}=m_{0,0}(z;t),\quad R_{z;0,0;1,0}=m_{0,1}(z;t),$$
$$R_{z;1,0;0,0}=m_{1,0}(z;t)=m_{0,1}(z;t),\quad R_{z;1,0;1,0}=m_{1,1}(z;t).\tag{108}$$

So, we start from the Lax equation (48) and to do calculations of the type performed in Section 2. For the resolvent $\mathbf{R}_z(t)$ of the operator $\mathbf{J}(t)$, from the second equality in (48), we easy get

$$\dot{\mathbf{R}}_z(t)=-\mathbf{R}_z(t)(\mathbf{J}(t)-z\mathbf{1})^{\cdot}\mathbf{R}_z(t)=-\mathbf{R}_z(t)\dot{\mathbf{J}}(t)\mathbf{R}_z(t)=[\mathbf{R}_z(t),\mathbf{A}(t)]\quad\text{or}$$
$$\dot R_z(t)=[R_z(t),A(t)]=R_z(t)A(t)-A(t)R_z(t),\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T).\tag{109}$$

Calculate $\dot m_{\mu,\nu}(z;t)$ for $\mu,\nu=0,1$. If we observe that $\mathbf{A}^*(t)=-\mathbf{A}(t)$, we get

$$\dot m_{0,0}(z;t)=(\dot{\mathbf{R}}_z(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2}=((\mathbf{R}_z(t)\mathbf{A}(t)-\mathbf{A}(t)\mathbf{R}_z(t))\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2}$$
$$=(\mathbf{R}_z(t)\mathbf{A}(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2}+(\mathbf{R}_z(t)\varepsilon_{0,0},\mathbf{A}(t)\varepsilon_{0,0})_{\mathbf{l}_2}.\tag{110}$$

According to (49) and (43), (107),

$$\mathbf{A}(t)\varepsilon_{0,0}=\Big(0,\Big(\frac12\alpha_0,-\frac12\alpha_{-1}\Big),0,0,\ldots\Big)=\frac12\alpha_0\varepsilon_{1,0}-\frac12\alpha_{-1}\varepsilon_{1,1};\tag{111}$$

$$(\mathbf{R}_z(t)f)_j=\sum_{k=0}^{\infty}R_{z;j;k}(t)f_k,\quad R_{z;j,v_j;k,v_k}(t)=(\mathbf{R}_z(t)\varepsilon_{k,v_k},\varepsilon_{j,v_j}),\tag{112}$$
$$v_j,v_k=0,1,\quad f\in\mathbf{l}_2.$$

Therefore,

$$
\begin{aligned}
(\mathbf{R}_z(t)\mathbf{A}(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2} &= (\mathbf{R}_z(t)(\frac{1}{2}\alpha_0\varepsilon_{1,0}-\frac{1}{2}\alpha_{-1}\varepsilon_{1,1}),\varepsilon_{0,0})_{\mathbf{l}_2}\\
&= \frac{1}{2}\alpha_0(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2}-\frac{1}{2}\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{0,0})_{\mathbf{l}_2};\\
(\mathbf{R}_z(t)\varepsilon_{0,0},\mathbf{A}(t)\varepsilon_{0,0})_{\mathbf{l}_2} &= (\mathbf{R}_z(t)\varepsilon_{0,0},\frac{1}{2}\alpha_0\varepsilon_{1,0}-\frac{1}{2}\alpha_{-1}\varepsilon_{1,1})_{\mathbf{l}_2}\\
&= \frac{1}{2}\alpha_0(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,0})_{\mathbf{l}_2}-\frac{1}{2}\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,1})_{\mathbf{l}_2}.
\end{aligned}
$$

From the last two equalities, (110) and (106) *we get*

$$
\begin{aligned}
\dot m_{0,0}(z;t) &= \alpha_0 m_{0,1}(z;t)-\frac{1}{2}\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,1})_{\mathbf{l}_2}-\frac{1}{2}\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{0,0})_{\mathbf{l}_2}\\
&= \alpha_0 m_{0,1}(z;t)-\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,1})_{\mathbf{l}_2},\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T).
\end{aligned}\tag{113}
$$

Consider $m_{0,1}(z;t)$. Analogously to (110) we have from (109) that

$$
\dot m_{0,1}(z;t) = (\dot{\mathbf{R}}_z(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2} = (\mathbf{R}_z(t)\mathbf{A}(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2}+(\mathbf{R}_z(t)\varepsilon_{1,0},\mathbf{A}(t)\varepsilon_{0,0})_{\mathbf{l}_2}.\tag{114}
$$

According to (49) and (43), (107),

$$
\mathbf{A}(t)\varepsilon_{1,0} = (-\frac{1}{2}\alpha_0,(0,0),(\frac{1}{2}\alpha_1,0),0,0,\dots) = -\frac{1}{2}\alpha_0\varepsilon_{0,0}+\frac{1}{2}\alpha_1\varepsilon_{2,0};\tag{115}
$$

$$
\begin{aligned}
(\mathbf{R}_z(t)\mathbf{A}(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2} &= (\mathbf{R}_z(t)(-\frac{1}{2}\alpha_0\varepsilon_{0,0}+\frac{1}{2}\alpha_1\varepsilon_{2,0}),\varepsilon_{0,0})_{\mathbf{l}_2}\\
&= -\frac{1}{2}\alpha_0(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2}+\frac{1}{2}\alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{0,0})_{\mathbf{l}_2}.
\end{aligned}\tag{116}
$$

For last term in (114), using (112), (111), we have

$$
\begin{aligned}
(\mathbf{R}_z(t)\varepsilon_{1,0},\mathbf{A}(t)\varepsilon_{0,0})_{\mathbf{l}_2} &= (\mathbf{R}_z(t)\varepsilon_{1,0},\frac{1}{2}\alpha_0\varepsilon_{1,0}-\frac{1}{2}\alpha_{-1}\varepsilon_{1,1})_{\mathbf{l}_2}\\
&= \frac{1}{2}\alpha_0(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{1,0})_{\mathbf{l}_2}-\frac{1}{2}\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{1,1})_{\mathbf{l}_2}.
\end{aligned}\tag{117}
$$

From (114), (116), and (117), *we get*

$$
\begin{aligned}
\dot m_{0,1}(z;t) &= -\frac{1}{2}\alpha_0 m_{0,0}(z;t)+\frac{1}{2}\alpha_0 m_{1,1}(z;t)-\frac{1}{2}\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{1,1})_{\mathbf{l}_2}\\
&\quad+\frac{1}{2}\alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{0,0})_{\mathbf{l}_2},\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T).
\end{aligned}\tag{118}
$$

Consider $m_{1,1}(z;t)$. Analogously to (110) and (114) we have from (109)

$$
\dot m_{1,1}(z;t) = (\dot{\mathbf{R}}_z(t)\varepsilon_{1,0},\varepsilon_{1,0})_{\mathbf{l}_2} = (\mathbf{R}_z(t)\mathbf{A}(t)\varepsilon_{1,0},\varepsilon_{1,0})_{\mathbf{l}_2}+(\mathbf{R}_z(t)\varepsilon_{1,0},\mathbf{A}(t)\varepsilon_{1,0})_{\mathbf{l}_2}.\tag{119}
$$

Calculate the first term in this sum. Using the expression (115) we get

$$
\begin{aligned}
(\mathbf{R}_z(t)\mathbf{A}(t)\varepsilon_{1,0},\varepsilon_{1,0})_{\mathbf{l}_2} &= (\mathbf{R}_z(t)(-\frac{1}{2}\alpha_0\varepsilon_{0,0}+\frac{1}{2}\alpha_1\varepsilon_{2,0}),\varepsilon_{1,0})_{\mathbf{l}_2}\\
&= -\frac{1}{2}\alpha_0(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,0})_{\mathbf{l}_2}+\frac{1}{2}\alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{1,0})_{\mathbf{l}_2}.
\end{aligned}\tag{120}
$$

For the second term we conclude from (115) that

$$
\begin{aligned}
(\mathbf{R}_z(t)\varepsilon_{1,0},\mathbf{A}(t)\varepsilon_{1,0})_{\mathbf{l}_2} &= (\mathbf{R}_z(t)\varepsilon_{1,0},-\frac{1}{2}\alpha_0\varepsilon_{0,0}+\frac{1}{2}\alpha_1\varepsilon_{2,0})_{\mathbf{l}_2}\\
&= -\frac{1}{2}\alpha_0(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2}+\frac{1}{2}\alpha_1(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{2,0})_{\mathbf{l}_2}.
\end{aligned}\tag{121}
$$

From (119), (120), (121), and (106), *we get*

$$
\begin{aligned}
\dot m_{1,1}(z;t) &= -\alpha_0 m_{0,1}(z;t)+\frac{1}{2}\alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{1,0})_{\mathbf{l}_2}+\frac{1}{2}\alpha_1(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{2,0})_{\mathbf{l}_2}\\
&= -\alpha_0 m_{0,1}(z;t)+\alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{1,0})_{\mathbf{l}_2},\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T).
\end{aligned}\tag{122}
$$

As a result of these calculations we have the three expressions for $\dot m_{0,0}(z;t), \dot m_{0,1}(z;t)$, $\dot m_{1,1}(z;t)$, namely (113), (118), (122). To deduce a system of differential equations with respect to the unknowns $m_{0,0}(z;t), m_{0,1}(z;t), m_{1,1}(z;t)$ and time t it is necessary to express the terms $(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,1})_{\mathbf{l}_2}$, $(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{1,1})_{\mathbf{l}_2}$, $(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{0,0})_{\mathbf{l}_2}$, $(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{1,0})_{\mathbf{l}_2}$ in (113), (118), (122) in terms of these unknowns and elements of the matrices $J(t)$ and $A(t)$.

To this end, we will use the obvious identity

$$\mathbf{1} + z\mathbf{R}_z(t) = \mathbf{R}_z(t)\mathbf{J}(t), \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad t\in[0,T). \tag{123}$$

At first, transfer the relation (113) into an equation. From (123) we have

$$1 + zm_{0,0}(z;t) = ((\mathbf{1}+z\mathbf{R}_z(t))\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2} = (\mathbf{R}_z(t)\mathbf{J}(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2}. \tag{124}$$

According to (46),

$$\mathbf{J}(t)\varepsilon_{0,0} = \beta_0\varepsilon_{0,0} + \alpha_0\varepsilon_{1,0} + \alpha_{-1}\varepsilon_{1,1}, \tag{125}$$

therefore, (124) gives

$$\begin{aligned}1 + zm_{0,0}(z;t) &= (\mathbf{R}_z(t)(\beta_0\varepsilon_{0,0} + \alpha_0\varepsilon_{1,0} + \alpha_{-1}\varepsilon_{1,1}),\varepsilon_{0,0})_{\mathbf{l}_2}\\ &= \beta_0(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2} + \alpha_0(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2} + \alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{0,0})_{\mathbf{l}_2}\\ &= \beta_0 m_{0,0}(z;t) + \alpha_0 m_{0,1}(z;t) + \alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{0,0})_{\mathbf{l}_2}.\end{aligned}$$

We conclude

$$\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{0,0})_{\mathbf{l}_2} = 1 + (z-\beta_0)m_{0,0}(z;t) - \alpha_0 m_{0,1}(z;t).$$

Using (106) and substituting this expression into (113), we get

$$\dot m_{0,0}(z;t) = 2\alpha_0 m_{0,1}(z;t) + (\beta_0 - z)m_{0,0}(z;t) - 1, \quad z\in\mathbb{C}\setminus\mathbb{R}, \quad t\in[0,T). \tag{126}$$

Transfer (118) into an equation. From (123) and (46) we have

$$\begin{aligned}&zm_{0,1}(z;t) = ((\mathbf{1}+z\mathbf{R}_z(t))\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2} = (\mathbf{R}_z(t)\mathbf{J}(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2};\\ &\mathbf{J}(t)\varepsilon_{1,0} = \alpha_0\varepsilon_{0,0} + \beta_1\varepsilon_{1,0} + \alpha_1\varepsilon_{2,0}.\end{aligned} \tag{127}$$

Therefore,

$$\begin{aligned}zm_{0,1}(z;t) &= (\mathbf{R}_z(t)(\alpha_0\varepsilon_{0,0} + \beta_1\varepsilon_{1,0} + \alpha_1\varepsilon_{2,0}),\varepsilon_{0,0})_{\mathbf{l}_2}\\ &= \alpha_0(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{0,0})_{\mathbf{l}_2} + \beta_1(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{0,0})_{\mathbf{l}_2} + \alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{0,0})_{\mathbf{l}_2}\\ &= \alpha_0 m_{0,0}(z;t) + \beta_1 m_{0,1}(z;t) + \alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{0,0})_{\mathbf{l}_2}.\end{aligned}$$

We conclude

$$\alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0},\varepsilon_{0,0})_{\mathbf{l}_2} = (z-\beta_1)m_{0,1}(z;t) - \alpha_0 m_{0,0}(z;t). \tag{128}$$

So, we have found the required expression for the fourth term in (118). Consider the third term. Rewrite expression (127) (using (125)) in following way:

$$\begin{aligned}zm_{0,1}(z;t) &= ((\mathbf{1}+z\mathbf{R}_z(t))\varepsilon_{0,0},\varepsilon_{1,0})_{\mathbf{l}_2} = (\mathbf{R}_z(t)\mathbf{J}(t)\varepsilon_{0,0},\varepsilon_{1,0})_{\mathbf{l}_2}\\ &= (\mathbf{R}_z(t)(\beta_0\varepsilon_{0,0} + \alpha_0\varepsilon_{1,0} + \alpha_{-1}\varepsilon_{1,1}),\varepsilon_{1,0})_{\mathbf{l}_2}\\ &= \beta_0(\mathbf{R}_z(t)\varepsilon_{0,0},\varepsilon_{1,0})_{\mathbf{l}_2} + \alpha_0(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{1,0})_{\mathbf{l}_2} + \alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{1,0})_{\mathbf{l}_2}\\ &= \beta_0 m_{0,1}(z;t) + \alpha_0 m_{1,1}(z;t) + \alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{1,0})_{\mathbf{l}_2}.\end{aligned}$$

Using (106) we conclude

$$\alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,0},\varepsilon_{1,1})_{\mathbf{l}_2} = \alpha_{-1}(\mathbf{R}_z(t)\varepsilon_{1,1},\varepsilon_{1,0})_{\mathbf{l}_2} = (z-\beta_0)m_{0,1}(z;t) - \alpha_0 m_{1,1}(z;t). \tag{129}$$

Using (129) and (128) we can rewrite (118):

$$\begin{aligned}\dot m_{0,1}(z;t) &= -\frac{1}{2}\alpha_0 m_{0,0}(z;t) + \frac{1}{2}\alpha_0 m_{1,1}(z;t) - \frac{1}{2}(z-\beta_0)m_{0,1}(z;t) + \frac{1}{2}\alpha_0 m_{0,0}(z;t)\\ &\quad + \frac{1}{2}(z-\beta_1)m_{0,1}(z;t) - \frac{1}{2}\alpha_0 m_{0,0}(z;t)\\ &= -\alpha_0 m_{0,0}(z;t) + \alpha_0 m_{1,1}(z;t) + \frac{1}{2}(\beta_0-\beta_1)m_{0,1}(z;t).\end{aligned}$$

As a result, we get the equation

$$\dot m_{0,1}(z;t) = -\alpha_0 m_{0,0}(z;t) + \alpha_0 m_{1,1}(z;t) + \frac{1}{2}(\beta_0 - \beta_1) m_{0,1}(z;t), \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad t \in [0,T). \tag{130}$$

Transfer (122) into an equation. From (123) and (127) we have using (106) that

$$\begin{aligned} 1 + z m_{1,1}(z;t) &= ((\mathbf{1} + z\mathbf{R}_z(t))\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_2} = (\mathbf{R}_z(t)\mathbf{J}(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_2} \\ &= (\mathbf{R}_z(t)(\alpha_0\varepsilon_{0,0} + \beta_1\varepsilon_{1,0} + \alpha_1\varepsilon_{2,0}), \varepsilon_{1,0})_{\mathbf{l}_2} \\ &= \alpha_0(\mathbf{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_2} + \beta_1(\mathbf{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_2} + \alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_2} \\ &= \alpha_0 m_{0,1}(z;t) + \beta_1 m_{1,1}(z;t) + \alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_2}. \end{aligned}$$

We conclude

$$\alpha_1(\mathbf{R}_z(t)\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_2} = 1 + (z - \beta_1) m_{1,1}(z;t) - \alpha_0 m_{0,1}(z;t).$$

Substituting this expression into (122), we get

$$\dot m_{1,1}(z;t) = -2\alpha_0 m_{0,1}(z;t) - (\beta_1 - z) m_{1,1}(z;t) + 1, \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad t \in [0,T). \tag{131}$$

As a result, we got, for the unknown functions $m_{0,0}(z;t)$, $m_{0,1}(z;t)$, $m_{1,1}(z;t)$, three differential equations (126), (130) and (131). So, we have proved the following result.

Theorem 9. *The elements $m_{0,0}(z;t)$, $m_{0,1}(z;t)$, and $m_{1,1}(z;t)$ of the Weyl matrix* (104) *satisfy the following system of differential equations:*

$$\begin{aligned} &\dot m_{0,0}(z;t) = (\beta_0(t) - z) m_{0,0}(z;t) + 2\alpha_0(t) m_{0,1}(z;t) - 1, \\ &\dot m_{0,1}(z;t) = -\alpha_0(t) m_{0,0}(z;t) + \frac{1}{2}(\beta_0(t) - \beta_1(t)) m_{0,1}(z;t) + \alpha_0(t) m_{1,1}(z;t), \\ &\dot m_{1,1}(z;t) = -2\alpha_0(t) m_{0,1}(z;t) - (\beta_1(t) - z) m_{1,1}(z;t) + 1; \\ &m_{1,0}(z;t) = m_{0,1}(z;t), \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad t \in [0,T). \end{aligned} \tag{132}$$

As in Section 2, it is possible to rewrite the system (132) in the terms of the unknowns measures (charges) $d\rho_{\alpha,\beta}(\lambda;t), \alpha,\beta = 0,1$, the elements of the $2\times 2-$matrix spectral measure $d\rho(\lambda;t)$; more exactly, we will rewrite (132) in terms of some of their derivatives. These charges are connected with $m_{\alpha,\beta}(z;t)$ by equality (104); for these measures we preserve the standard notations $\forall t \in [0,T)$

$$\mathfrak{B}(\mathbb{R}) \ni \triangle \mapsto \rho_{\alpha,\beta}(\triangle;t), \rho(\triangle;t); \quad \alpha,\beta = 0,1. \tag{133}$$

At first, it is easy to understand that such a measure $d\sigma(\lambda)$, independent of t, exists for which all the measures $d\rho(\lambda;t), t \in [0,T)$, are absolutely continuous w.r.t. $d\sigma(\lambda)$. Namely, the following result takes place (compare with (29)):

Lemma 7. *Consider the following mapping:*

$$\mathfrak{B}(\mathbb{R}) \ni \triangle \mapsto \int_0^T \mathrm{Tr}\ \rho(\triangle;t)dt =: \sigma(\triangle) \geq 0. \tag{134}$$

The mapping (134) *is some nonnegative finite measure on $\mathbb{R}$ (a joint scalar spectral measure) and all measures* (133) *are absolutely continuous w.r.t. $d\sigma(\lambda)$.*

Proof. Since $\forall t \in [0,T)$

$$\mathfrak{B}(\mathbb{R}) \ni \triangle \mapsto \mathrm{Tr}\ \rho(\triangle;t)dt = \rho_{0,0}(\triangle;t) + \rho_{1,1}(\triangle;t) \leq 2$$

is a nonnegative measure, then using the passage to the limit under integral we conclude that (134) is a nonnegative measure.

Let, for $\triangle \in \mathfrak{B}(\mathbb{R})$, $\sigma(\triangle) = 0$, then from (134) we conclude that Tr $\rho(\triangle;t) = 0$ for every $t \in [0,T)$. Since the matrix $\rho(\triangle;t)$ is nonnegative, from the last equality it follows that $\rho(\triangle;t) = 0$, i.e., the measure $\rho(\triangle;t) = 0$. □

Denote the Radon-Nikodym derivative $d\rho(\lambda;t)/d\sigma(\lambda)$ by $r(\lambda;t) = (r_{\alpha,\beta}(\lambda;t))_{\alpha,\beta=0}^1$. So we have

$$\rho(\triangle;t) = \int_\triangle r(\lambda;t)d\sigma(\lambda), \quad \rho_{\alpha,\beta}(\triangle;t) = \int_\triangle r_{\alpha,\beta}(\lambda;t)d\sigma(\lambda); \tag{135}$$
$$\triangle\in \mathfrak{B}(\mathbb{R}), \quad t\in[0,T), \quad \alpha,\beta=0,1.$$

We will also use the spectral functions (matrix and scalar)

$$\rho(\lambda;t) = \rho((-\infty,\lambda);t), \quad \rho_{\alpha,\beta}(\lambda;t) = \rho_{\alpha,\beta}((-\infty,\lambda);t); \tag{136}$$
$$\lambda\in\mathbb{R}, \quad t\in[0,T), \quad \alpha,\beta=0,1.$$

As in Section 2 it is essential to rewrite the system (132) as a system for the unknowns $r_{\alpha,\beta}(\lambda;t)$, now a similar system is also useful. The following result takes place.

Theorem 10. *The system* (132) *is equivalent to the following system of differential equations for the unknowns $r_{\alpha,\beta}(\lambda;t)$: for almost all $\lambda\in\mathbb{R}$ with respect to the measure $d\sigma(\lambda)$,*

$$\begin{aligned} &\dot r_{0,0}1925 = (\beta_0(t)-\lambda)r_{0,0}(\lambda;t) + 2\alpha_0(t)r_{0,1}(\lambda;t),\\ &\dot r_{0,1}(\lambda;t) = -\alpha_0(t)r_{0,0}(\lambda;t) + \frac{1}{2}(\beta_0(t)-\beta_1(t))r_{0,1}(\lambda;t) + \alpha_0(t)r_{1,1}(\lambda;t),\\ &\dot r_{1,1}(\lambda;t) = -2\alpha_0(t)r_{0,1}(\lambda;t) - (\beta_1(t)-\lambda)r_{1,1}(\lambda;t);\\ &r_{1,0}(\lambda;t) = r_{0,1}(\lambda;t), \quad t\in[0,T). \end{aligned} \tag{137}$$

The functions $r_{\alpha,\beta}(\lambda;t)$ from (137) *are one time continuously differentiable in* $[0,T)$ *for $d\sigma(\lambda)-$almost all $\lambda\in\mathbb{R}$.*

Proof. It is similar to the proof of Theorem 2. Namely, let $(u,v)_{L^2}$ be the scalar product in the ordinary space $L^2([0,T),dt)\ni u,v$. Using the first equality in (132) and (135) we get $\forall z\in\mathbb{C}\setminus\mathbb{R}$ and for arbitrary infinite differentiable complex valued functions $u_{0,0}(t), t\in[0,T)$, vanishing in some neighborhoods of 0 and T, that

$$\begin{aligned} -\int_{\mathbb{R}} \frac{1}{\lambda-z}(r_{0,0}(\lambda;\cdot),\dot u_{0,0}(\cdot))_{L^2}d\sigma(\lambda) &= -\int_{\mathbb{R}}\frac{1}{\lambda-z}d(\rho_{0,0}(\lambda;\cdot),\dot u_{0,0}(\cdot))_{L^2}\\ &= -(m_{0,0}(z;\cdot),\dot u_{0,0}(\cdot))_{L^2} = (\dot m_{0,0}(z;\cdot),u_{0,0}(\cdot))_{L^2}\\ &= ((\beta_0(\cdot)-z)m_{0,0}(z;\cdot) + 2\alpha_0(\cdot)m_{0,1}(z;\cdot) - 1, u_{0,0}(\cdot))_{L^2}\\ &= \int_{\mathbb{R}}\frac{1}{\lambda-z}((\beta_0(\cdot)-\lambda)d\rho_{0,0}(\lambda;\cdot) + 2\alpha_0(\cdot)d\rho_{0,1}(\lambda;\cdot), u_{0,0}(\cdot))_{L^2}\\ &= \int_{\mathbb{R}}\frac{1}{\lambda-z}((\beta_0(\cdot)-\lambda)r_{0,0}(\lambda;\cdot) + 2\alpha_0(\cdot)r_{0,1}(\lambda;\cdot), u_{0,0}(\cdot))_{L^2}d\sigma(\lambda). \end{aligned} \tag{138}$$

Here we have used he equality

$$\begin{aligned} (\beta_0(t)-z)m_{0,0}(z;t) + 2\alpha_0(t)m_{0,1}(z;t) - 1 = \int_{\mathbb{R}}\frac{1}{\lambda-z}((\beta_0(t)-\lambda)d\rho_{0,0}(\lambda;t)\\ + \int_{\mathbb{R}}\frac{1}{\lambda-z}2\alpha_0(t)d\rho_{0,1}(\lambda;t), \quad z\in\mathbb{C}\setminus\mathbb{R}, \quad t\in[0,T). \end{aligned}$$

This equality takes place since $\forall t\in[0,T)$ $\rho_{0,0}(\mathbb{R};t)=1$ and therefore $-1=\int_{\mathbb{R}}(z-\lambda)(\lambda-z)^{-1}d\rho_{0,0}(\lambda;t)$.

Similarly, to the proof of Theorem 2, we conclude from (133) that, for $d\sigma(\lambda)-$almost all $\lambda\in\mathbb{R}$,

$$-(r(\lambda,\cdot),\dot u_{0,0}(\cdot))_{L^2} = ((\beta_0(\cdot)-\lambda)r_{0,0}(\lambda;\cdot) + 2\alpha_0(\cdot)r_{0,1}(\lambda;\cdot), u_{0,0}(\cdot))_{L^2}. \tag{139}$$

Analogous equalities can be obtained using the second and the third equalities from (132) (now it is necessary to recall that $\rho_{1,1}(\mathbb{R};t)=1, t\in[0,t))$. These equalities will be the following: for $d\sigma(\lambda)-$almost all $\lambda \in \mathbb{R}$,

$$(140)\qquad \begin{aligned}(-r_{0,1}(\lambda;\cdot),\dot{u}_{0,1}(\cdot))_{L^2} &= (-\alpha_0(\cdot)r_{0,0}(\lambda;\cdot)+\frac{1}{2}(\beta_0(\cdot)-\beta_1(\cdot))r_{0,1}(\lambda;\cdot)\\ &\quad+\alpha_0(\cdot)r_{1,1}(\lambda;\cdot),u_{0,1}(\cdot))_{L^2},\\ (-r_{1,1}(\lambda;\cdot),\dot{u}_{1,1}(\cdot))_{L^2} &= (-2\alpha_0(\cdot)r_{0,1}(\lambda;\cdot)-(\beta_1(\cdot)-\lambda)r_{1,1}(\lambda;\cdot),u_{1,1}(\cdot))_{L^2},\end{aligned}$$

where $u_{0,1}(t)$ and $u_{1,1}(t)$ are arbitrary functions of type $u_{0,0}(t)$.

Let us stress that the functions $u_{0,0}(t)$, $u_{0,1}(t)$ and $u_{1,1}(t)$ in (139), (140) are arbitrary smooth functions which are annulated in some neighborhoods of points $0, T$. Therefore $r_{0,0}(\lambda;t)$, $r_{0,1}(\lambda;t)$ and $r_{1,1}(\lambda;t)$ (for $d\sigma(\lambda)-$almost all $\lambda \in \mathbb{R}$) are generalized solutions (in terms of the classical theory of generalized functions) of system (137). But for a system of linear ordinary differential equations with smooth coefficients there is a well known, — every generalized solution of such an equation is, in fact, a smooth solution ([16], Ch. 16, Section 6). Therefore we can assert that the vector $(r_{0,0}(\lambda;t), r_{0,1}(\lambda;t), r_{1,1}(\lambda;t))$ is one time continuously differentiable in $[0,T)$ and is a solution of a Cauchy problem for (137) with the initial data $(r_{0,0}(\lambda;0), r_{0,1}(\lambda;0), r_{1,1}(\lambda;0))$. □

It is useful to rewrite systems (132) and (137) using the Toda lattice (44). In these two systems, the coefficients $\alpha_0(t)$, $\beta_0(t)$, $\beta_1(t)$ are the same as those in (44). Therefore we have

$$(141)\qquad \frac{1}{2}(\beta_0(t)-\beta_1(t)) = -\frac{\dot{\alpha}_0(t)}{\alpha_0(t)},\quad \beta_1(t)=\beta_0(t)+2\frac{\dot{\alpha}_0(t)}{\alpha_0(t)},\quad t\in[0,T).$$

Using these equalities we can rewrite, for example, the equations (137) in the following form: for $d\sigma(\lambda)-$almost all $\lambda \in \mathbb{R}$,

$$(142)\qquad \begin{aligned}\dot{r}_{0,0}(\lambda;t) &= (\beta_0(t)-\lambda)r_{0,0}(\lambda;t)+2\alpha_0(t)r_{0,1}(\lambda;t),\\ \dot{r}_{0,1}(\lambda;t) &= -\alpha_0(t)r_{0,0}(\lambda;t)-\frac{\dot{\alpha}_0(t)}{\alpha_0(t)}r_{0,1}(\lambda;t)+\alpha_0(t)r_{1,1}(\lambda;t),\\ \dot{r}_{1,1}(\lambda;t) &= -2\alpha_0(t)r_{0,1}(\lambda;t)-(\beta_0(t)-\lambda+2\frac{\dot{\alpha}_0(t)}{\alpha_0(t)})r_{1,1}(\lambda;t);\\ r_{1,0}(\lambda;t) &= r_{0,1}(\lambda;t),\quad t\in[0,T).\end{aligned}$$

If, under some additional assumptions, we can prove that $\alpha_0(t)$ is constant, $\alpha_0(t) = \alpha_0(0)$, $t \in [0,T)$, then the system (142) has a more simple form,

$$(143)\qquad \begin{aligned}\dot{r}_{0,0}(\lambda;t) &= (\beta_0(t)-\lambda)r_{0,0}(\lambda;t)+2\varkappa r_{0,1}(\lambda;t),\\ \dot{r}_{0,1}(\lambda;t) &= -\varkappa r_{0,0}(\lambda;t)+\varkappa r_{1,1}(\lambda;t),\\ \dot{r}_{1,1}(\lambda;t) &= -2\varkappa r_{0,1}(\lambda;t)-(\beta_0(t)-\lambda)r_{1,1}(\lambda;t);\\ r_{1,0}(\lambda;t) &= r_{0,1}(\lambda;t),\quad t\in[0,T),\quad \varkappa=\alpha_0(t).\end{aligned}$$

Systems (132), (137), (142) and (143) are similar to equations (15), (31), but unlike to Section 2 we cannot calculate, in a general case, solutions of the Cauchy problems for (137), (142) and even for (143). Therefore we can only say, that *if for some concrete coefficients of systems* (142), (143) *we can calculate the solutions* $r_{0,0}(\lambda;t)$, $r_{0,1}(\lambda;t)$, $r_{1,1}(\lambda;t)$ *of the Cauchy problem with the initial data* $r_{0,0}(\lambda;0)$, $r_{0,1}(\lambda;0)$, $r_{1,1}(\lambda;0)$, *then using formulas* (135), (136) *we can find the spectral measure* $d\rho(\lambda;t)$ *of our operators* $\mathbf{J}(t)$, $t \in [0,T)$.

Note also that in the last situation we can apply the representation for solution of the Cauchy problem for (142), (143) of type (34), (35) and therefore we can avoid the use of the measure $d\sigma(\lambda)$. The functions $\alpha_0(t)$, $\beta_0(t)$, $t \in [0,T)$, are found from the condition that $d\rho(\lambda;t)$ is a matrix probability measure for every $t \in [0,T)$.

Remark 6. It is necessary to note some useful facts concerning the matrix $r(\lambda;t)$ from (135). Namely, denote by $D(\lambda;t)$ its determinant,

$$D(\lambda;t) = \mathrm{Det} r(\lambda;t) = r_{0,0}(\lambda;t) r_{1,1}(\lambda;t) - r_{0,1}^2(\lambda;t), \quad \lambda \in \mathbb{R}, \quad t \in [0,T). \tag{144}$$

Then the following equality takes place: for $d\sigma(\lambda)-$almost all $\lambda \in \mathbb{R}$,

$$D(\lambda;t) = \frac{\alpha_0^2(0)}{\alpha_0^2(t)} D(\lambda;0), \quad t \in [0,T). \tag{145}$$

For the proof of (145) it is necessary calculate $\dot{D}(\lambda;t)$. Using (144) and (142) after simple calculation we get for $d\sigma(\lambda)-$almost all $\lambda \in \mathbb{R}$ that

$$\dot{D}(\lambda;t) = -2 \frac{\dot{\alpha}_0(t)}{\alpha_0(t)} D(\lambda;t), \quad t \in [0,T). \tag{146}$$

Integration of differential equation (146) gives (145).

If from some additional information about solutions of the Cauchy problem for (44) we know that $D(\lambda;t)$ for λ from the set of some positive measure $d\sigma(\lambda)$ does not depend on t, then from (145) we conclude that $\alpha_0(t) = \alpha_0(0)$, $t \in [0,T)$, and *the system* (142) *has a more simple form* (143).

Note also, that the operators $\mathbf{J}(t)$ are isospectral (see e.g. [59], Ch. 12), therefore their spectrums are located on an fixed interval $[a,b] \subset \mathbb{R}$.

6. The main theorem and some applications to Hamiltonian systems

In this Section we will give a procedure for finding a solution of the Cauchy problem for a double-infinite Toda lattice, and also consider the corresponding problem for differential equations connected with the Toda lattice. The constructions are similar to the case of the semi-infinite Toda lattices, see Section 2 and the corresponding articles [3, 4, 6, 7].

So, consider the double-infinite Toda lattice (44),

$$\begin{aligned} \dot{\alpha}_n(t) &= \frac{1}{2} \alpha_n(t)(\beta_{n+1}(t) - \beta_n(t)), \\ \dot{\beta}_n(t) &= \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n \in \mathbb{Z}, \quad t \in [0,T); \end{aligned} \tag{147}$$

here $\alpha_n(t) > 0, \beta_n(t)$ are real continuously differentiable functions on $[0,T)$. For (147) we pose a Cauchy problem as follows: to find a solution $\alpha_n(t), \beta_n(t)$ from the initial data $\alpha_n(0)$, $\beta_n(0)$, $n \in \mathbb{Z}$, $t \in [0,T)$.

Theorem 11. *We will find a solution, bounded uniformly in $n \in \mathbb{Z}$ and $t \in [0,T)$, of the posed above Cauchy problem for* (147)*. Such a solution exists and can be found by the following procedure:*

1) Using the initial data construct the matrix of type (46),

$$J(0) = \left|\begin{array}{c|cc|cc|cc|cc|c}
\hline
\beta_0(0) & \alpha_0(0) & \alpha_{-1}(0) & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
\alpha_0(0) & \beta_1(0) & 0 & \alpha_1(0) & 0 & 0 & 0 & 0 & 0 & \dots \\
\alpha_{-1}(0) & 0 & \beta_{-1}(0) & 0 & \alpha_{-2}(0) & 0 & 0 & 0 & 0 & \dots \\
\hline
0 & \alpha_1(0) & 0 & \beta_2(0) & 0 & \alpha_2(0) & 0 & 0 & 0 & \dots \\
0 & 0 & \alpha_{-2}(0) & 0 & \beta_{-2}(0) & 0 & \alpha_{-3}(0) & 0 & 0 & \dots \\
\hline
0 & 0 & 0 & \alpha_2(0) & 0 & \beta_3(0) & 0 & \alpha_3(0) & 0 & \dots \\
0 & 0 & 0 & 0 & \alpha_{-3}(0) & 0 & \beta_{-3}(0) & 0 & \alpha_{-4}(0) & \dots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right. \tag{148}$$

and find its $2 \times 2-$matrix spectral measure,

$$d\rho(\lambda;0) = \begin{bmatrix} d\rho_{0,0}(\lambda;0) & d\rho_{0,1}(\lambda;0) \\ d\rho_{0,1}(\lambda;0) & d\rho_{1,1}(\lambda;0) \end{bmatrix}, \quad \lambda \in \mathbb{R}. \tag{149}$$

2) Consider on $[0,T)$ *the system* (142) *(or* (143)*) of differential equations w.r.t. the unknown continuously differentiable functions* $r_{\alpha,\beta}(\lambda;t)$*. Here* $\alpha_0(t)>0$ *and* $\beta_0(t)\in\mathbb{R}$ *are fixed continuously differentiable functions;* $\lambda\in\mathbb{R}$, $t\in[0,T)$.

The connection between $r_{\alpha,\beta}(\lambda;t)$ *and spectral measure* $d\rho(\lambda;t)$ *and function* $\rho(\lambda;t)$ *is given by formulas* (135) *and* (136).

Find a solution of the Cauchy problem for (142), (143) *with initial data* (149) *in terms of* $r_{\alpha,\beta}(\lambda;0)$*. Using this solution* $\rho_{\alpha_0,\beta_0}(\lambda;t)$ *find the functions* $\alpha_0(t)$, $\beta_0(t)$ *from the condition* $\rho_{\alpha_0,\beta_0}(\mathbb{R};t)=1, t\in[0,T)$*. Such a solution is denoted in terms* $\rho_{\alpha,\beta}(\lambda;t)$ *by* $\rho(\lambda;t)$.

3) For fixed $t\in[0,T)$*, in the Hilbert space* $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$ *of type* (78) *of* $\mathbb{C}^2-$*vector valued functions of* λ*, consider the sequence (see* (85)*)*

$$(1,0);((0,1),(\lambda,0));((0,\lambda),(\lambda^2,0));\ldots;((0,\lambda^n),(\lambda^{n+1},0));\ldots. \tag{150}$$

Apply to sequence (150) *the Gramm-Schmidt orthogonalization procedure. As a result, we get a sequence of vectors from* $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$ *(see* (82)*),*

$$\begin{aligned}&(P_{0;0,0}(\lambda;t),P_{0;0,1}(\lambda;t));(P_{1;0,0}(\lambda;t),P_{1;0,1}(\lambda;t)),(P_{1;1,0}(\lambda;t),P_{1;1,1}(\lambda;t));\\&(P_{2;0,0}(\lambda;t),P_{2;0,1}(\lambda;t)),(P_{2;1,0}(\lambda;t),P_{2;1,1}(\lambda;t));\ldots.\end{aligned} \tag{151}$$

4) The sought solutions of our Cauchy problem can be found using the formulas $\forall t\in[0,T)$

$$\begin{aligned}&\begin{bmatrix}\alpha_n(t) & 0\\ 0 & \alpha_{-n-1}(t)\end{bmatrix}=\int_{\mathbb{R}}\lambda P_n(\lambda;t)d\rho(\lambda;t)P^*_{n+1}(\lambda;t),\quad n\in\mathbb{N},\\
&\begin{bmatrix}\alpha_0(t) & \alpha_{-1}(t)\\ 0 & 0\end{bmatrix}=\int_{\mathbb{R}}\lambda P_0(\lambda;t)d\rho(\lambda;t)P^*_1(\lambda;t),\\
&\begin{bmatrix}\beta_n(t) & 0\\ 0 & \beta_{-n}(t)\end{bmatrix}=\int_{\mathbb{R}}\lambda P_n(\lambda;t)d\rho(\lambda;t)P^*_n(\lambda;t),\quad n\in\mathbb{N},\\
&\beta_0(t)=\int_{\mathbb{R}}\lambda d\rho_{0,0}(\lambda;t),\qquad \textit{where}\\
&P_n(\lambda;t)=\begin{bmatrix}P_{n;0,0}(\lambda;t) & P_{n;0,1}(\lambda;t)\\ P_{n;0,1}(\lambda;t) & P_{n;1,1}(\lambda;t)\end{bmatrix},\quad n\in\mathbb{N}_0,\quad\lambda\in\mathbb{R},\\
&(P_{0;0,1}(\lambda;t)=P_{0;1,1}(\lambda;t)=0).\end{aligned}$$

Proof. It follows from the results of Sections 3–5. □

The Toda lattice represents a Hamilton system describing the dynamics of a infinite chain of particles $q_n(t), n\in\mathbb{Z}$, on a straight line $\mathbb{R}$ with exponential interaction. The corresponding Hamiltonian has the form

$$H(p,q)=\frac{1}{2}\sum_{n=-\infty}^{\infty}p_n^2+\sum_{n=-\infty}^{\infty}e^{q_n-q_{n+1}}, \tag{152}$$

where $q_n=q_n(t)$ is the coordinate of the $n-$th particle, and $p_n=p_n(t)$ is its momentum, $t\in[0,T)$. The Hamiltonian equations $\dot q_n=\partial H/\partial p_n=p_n$, $\dot p_n=-\partial H/\partial q_n=-e^{q_n-q_{n+1}}+e^{q_{n-1}-q_n}, n\in\mathbb{Z}$, can be rewritten as the following system:

$$\ddot x_n(t)=e^{x_{n-1}(t)-x_n(t)}-e^{x_n(t)-x_{n+1}(t)},\quad n\in\mathbb{Z},\quad t\in[0,T). \tag{153}$$

Here $x_n(t)\in\mathbb{R}$ is the $n-$th coordinate of the point $q_n(t)$, $\dot x_n(t)$ is its momentum $p_n(t)$.

So, we have an infinite system (153) of nonlinear second order differential equations. For this system we can pose the following Cauchy problem: for given initial data $x_n(0),\dot x_n(0), n\in\mathbb{Z}$, it is necessary to find a solution $x_n(t), n\in\mathbb{Z}, t\in[0,T)$.

Introduce the Flashka change of variables

$$\alpha_n(t)=e^{\frac{1}{2}(x_n(t)-x_{n+1}(t))},\quad \beta_n(t)=-\dot x_n(t),\quad n\in\mathbb{Z},\quad t\in[0,T). \tag{154}$$

Then our system (153) transfers into the double-infinite Toda lattice (147) and our Cauchy problem into such a problem for (147). We can apply now Theorem 11 to our case taking into account that our condition about uniform boundedness of $\alpha_n(t), \beta_n(t)$ means in terms $x_n(t)$ the boundedness

$$|x_n(t) - x_{n+1}(t)|, \quad |\dot{x}_n(t)|, \quad n \in \mathbb{Z}, \quad t \in [0, T). \tag{155}$$

So, we can formulate the following corollary.

Corollary 1. *The above stated Cauchy problem for* (153) *with uniform boundedness of* (155) *is transformed by* (154) *to Theorem 11. The procedure of Theorem 11 gives a possibility to find a solution of the Cauchy problem for* (153)*.*

Let us now pass to the Toda shock problem in the formulation of [42]. Consider the system (153) but for $n \in \mathbb{N}_0$

$$\ddot{x}_n(t) = e^{x_{n-1}(t) - x_n(t)} - e^{x_n(t) - x_{n+1}(t)}, \quad n \in \mathbb{N}_0, \quad t \in [0, T). \tag{156}$$

For this system it is possible to formulate a shock problem, i.e., the following mixed problem: it is necessary to find a solution of (156) $x_n, n \in \mathbb{N}_0, t \in [0, T)$, if we know the initial data $x_n(0), \dot{x}_n(0), n \in \mathbb{N}_0$, and the boundary data $x_{-1}(t) = f(t), t \in [0, T)$, where $f(t)$ is a given real continuous function.

Below we will consider some version of this problem (" a generalized shock problem"): it is necessary to find a bounded solution of (156) if we know $x_n(0), \dot{x}_n(0), n \in \mathbb{N}_0$, and $x_{-1}(t) - x_0(t) = f(t), t \in [0, T)$ (instead of $x_{-1}(t)$). If, above, $x_{-1}(t) = 0, t \in [0, T)$, then we have a problem similar to [42] but moved one step to the right.

Using now the Flashka variables (154), but for $n \in \mathbb{N}_0$, we get from (154), (153): $\forall t \in [0, T)$ that

$$\begin{aligned}
&\dot{\alpha}_n(t) = \frac{1}{2}\alpha_n(t)(\beta_{n+1}(t) - \beta_n(t)), \quad n \in \mathbb{N}_0;\\
&\dot{\beta}_n(t) = -\ddot{x}_n(t) = \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n \in \mathbb{N};\\
&\dot{\beta}_0(t) = -\ddot{x}_0(t) = \alpha_0^2(t) - e^{x_{-1}(t) - x_0(t)} = \alpha_0^2(t) + \varphi(t), \quad \varphi(t) := -e^{f(t)}.
\end{aligned} \tag{157}$$

Rewrite the equality (157) in the form of a Lax equation type with an ordinary Jacobi matrix (8) and the matrix (9). Consider the matrix $J(t)$ (8) the elements of which are solutions of system (157). Using the construction of Section 2 it is easy to understand that the system (157) is equivalent to the following nonhomogeneous Lax equation: $\forall t \in [0, T)$

$$\begin{aligned}
&\dot{J}(t) = J(t)A(t) - A(t)J(t) + \varphi(t)B,\\
&B = (b_{\alpha,\beta})_{\alpha,\beta=0}^{\infty}, \quad b_{0,0} = 1, \quad b_{\alpha,\beta} = 0 \quad \text{if} \quad \alpha + \beta > 0.
\end{aligned} \tag{158}$$

Let $\mathbf{J}(t)$ be the bounded selfadjoint operator in ℓ_2 generated by $J(t)$ and $\mathbf{R}_z(t)$ its resolvent; $\mathbf{A}(t), \mathbf{B} = \mathbf{B}^2$ be operators generated by $A(t)$ and B. Then according to (16), (158), and (18) we have

$$\begin{aligned}
\dot{\mathbf{R}}_z(t) &= ((\mathbf{J}(t) - z\mathbf{1})^{-1})^{\cdot} = -\mathbf{R}_z(t)(\mathbf{J}(t) - z\mathbf{1})^{\cdot}\mathbf{R}_z(t)\\
&= -\mathbf{R}_z(t)\dot{\mathbf{J}}(t)\mathbf{R}_z(t) = -\mathbf{R}_z(t)([\mathbf{J}(t), \mathbf{A}(t)] + \varphi(t)\mathbf{B})\mathbf{R}_z(t)\\
&= -\mathbf{R}_z(t)[\mathbf{J}(t) - z\mathbf{1}, \mathbf{A}(t)]\mathbf{R}_z(t) - \varphi(t)\mathbf{R}_z(t)\mathbf{B}\mathbf{R}_z(t)\\
&= [\mathbf{R}_z(t), \mathbf{A}(t)] - \varphi(t)\mathbf{R}_z(t)\mathbf{B}\mathbf{R}_z(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T).
\end{aligned} \tag{159}$$

Using (159) instead of (18) we deduce an analog of equation (15) for $m(z;t)$ (14) in our case. Let $\varepsilon_0 = (1, 0, 0, \ldots)$, then from (159) we conclude $\forall z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$

$$\dot{m}(z;t) = (\dot{\mathbf{R}}_z(t)\varepsilon_0, \varepsilon_0)_{\ell_2} = ([\mathbf{R}_z(t), \mathbf{A}(t)]\varepsilon_0, \varepsilon_0)_{\ell_2} - \varphi(t)(\mathbf{R}_z(t)\mathbf{B}\mathbf{R}_z(t)\varepsilon_0, \varepsilon_0)_{\ell_2}. \tag{160}$$

For the first summand in the right-hand side of (160) we can repeat the calculation (19)–(22) and get $(z - \beta_0(t))m(z;t) + 1$.

For the second summand we have

$$(\mathbf{R}_z(t)\mathbf{B}\mathbf{R}_z(t)\varepsilon_0,\varepsilon_0)_{\ell_2}=(\mathbf{B}\mathbf{R}_z(t)\varepsilon_0,\mathbf{R}_z^*(t)\varepsilon_0)_{\ell_2}=(\mathbf{B}\mathbf{R}_z(t)\varepsilon_0,\mathbf{B}\mathbf{R}_z^*(t)\varepsilon_0)_{\ell_2}$$
$$=(\mathbf{R}_z(t)\varepsilon_0,\varepsilon_0)_{\ell_2}\overline{(\mathbf{R}_z^*(t)\varepsilon_0,\varepsilon_0)}_{\ell_2}=(\mathbf{R}_z(t)\varepsilon_0,\varepsilon_0)_{\ell_2}^2=m^2(z;t).$$

Therefore (160) gives the following equation of Riccati type for $m(z;t)$:

$$\dot m(z;t)=(z-\beta_0(t))m(z;t)+1-\varphi(t)m^2(z;t),\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T). \tag{161}$$

It is possible to give some procedure of integration of equation (161). For this we note that, according to (14) and (27), we have

$$m(z;t)=\int_{\mathbb{R}}\frac{1}{\lambda-z}d\rho(\lambda;t),\quad z\in\mathbb{Z}\setminus\mathbb{R},\quad t\in[0,T), \tag{162}$$

where $d\rho(\lambda;t)$ is a spectral measure of the operator $\mathbf{J}(t)$. Since the norms of these operators are bounded uniformly in t, the support of $d\rho(\lambda;t)$ for every $t\in[0,T)$ belongs to some ball $\{\lambda\in\mathbb{R}||\lambda|\le R\}$. Therefore for $|z|>R,t\in[0,T)$, we can write

$$\begin{aligned}m(z;t)&=\int_{\mathbb{R}}\frac{1}{\lambda-z}d\rho(\lambda;t)=-\frac{1}{z}\int_{\mathbb{R}}\frac{1}{1-\lambda/z}d\rho(\lambda;t)\\&=-\frac{1}{z}\int_{\mathbb{R}}\Big(\sum_{n=0}^{\infty}\Big(\frac{\lambda}{z}\Big)^n\Big)d\rho(\lambda;t)=-\sum_{n=0}^{\infty}\frac{s_n(t)}{z^{n+1}},\quad s_n(t)=\int_{\mathbb{R}}\lambda^n d\rho(\lambda;t).\end{aligned} \tag{163}$$

Using the approach of Section 2 it is possible to prove that the moments $s_n(t),n\in\mathbb{N}_0$, are continuously differentiable and $\forall t\in[0,T)$

$$\begin{aligned}\dot s_n(t)&=\int_{\mathbb{R}}\lambda^n d\dot\rho(\lambda;t),\quad n\in\mathbb{N}_0;\\ \dot m(z;t)&=-\sum_{n=0}^{\infty}\frac{\dot s_n(t)}{z^{n+1}},\quad |z|>R.\end{aligned} \tag{164}$$

According to (163), (164) we can rewrite (161) in the form of an equation for the moments.

Namely, at first we note that equation (161) is equivalent to the following: $\forall z\in\mathbb{C}\setminus\mathbb{R},t\in[0,T)$

$$\dot m(z;t)=\int_{\mathbb{R}}\frac{\lambda-\beta_0(t)}{\lambda-z}d\rho(\lambda;t)-\varphi(t)m^2(z;t). \tag{165}$$

Using (162)–(164) we have, instead of (165), that

$$\sum_{n=0}^{\infty}\frac{\dot s_n(t)}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}(s_{n+1}(t)-\beta_0(t)s_n(t))+\varphi(t)\sum_{j,k=0}^{\infty}\frac{s_j(t)s_k(t)}{z^{j+k+2}},\quad |z|>R. \tag{166}$$

We will compare the coefficients in (166) at the powers $\frac{1}{z^{n+1}}$ for $n=0,1,\dots$. As a result, using the equality $\rho(\mathbb{R};t)=1,t\in[0,T)$, we get

$$\begin{aligned}&s_1(t)=\beta_0(t),\dot s_1(t)=s_2(t)-\beta_0(t)s_1(t)+\varphi(t),\\&\dot s_2(t)=s_3(t)-\beta_0(t)s_2(t)+2\varphi(t)s_1(t),\dots,\\&\dot s_n(t)=s_{n+1}(t)-\beta_0(t)s_n(t)+\varphi(t)\sum_{j,k=0;j+k=n-1}^{\infty}s_j(t)s_k(t),\quad n\in\mathbb{N},\quad t\in[0,T).\end{aligned} \tag{167}$$

As a result, formulas (167) give a procedure for calculating the moments $s_n(t)$ of the spectral measure $d\rho(\lambda;t)$ of the Jacobi operator $\mathbf{J}(t)$. Therefore, by means of classical formulas for the expression of elements of the Jacobi matrix $J(t)$ in terms of the moments of its spectral measure, we can get formulas for solution of our generalized shock problem. We can formulate the following result.

Theorem 12. *Consider the generalized Toda shock problem, where it is necessary to find a solution of system* (156) *if we know* $x_n(0),\dot x_n(0),n\in\mathbb{N}_0$, *and* $x_{-1}(t)-x_0(t)=f(t),t\in[0,T)$. *For its solution it is necessary to pass to the Flashka variables* $\alpha_n(t)=$

$e^{\frac{1}{2}(x_n(t)-x_{n+1}(t))}$, $n \in \mathbb{N}_0$, $\beta_n(t) = -\dot{x}_n(t), n \in \mathbb{N}$, $\beta_0(t) = \alpha_0^2(t) - e^{f(t)}$; $t \in [0, T)$. *After this it is necessary to calculate* $\forall t \in [0, T)$ *the moments* $s_n(t), n \in \mathbb{N}$, *of the spectral measure* $d\rho(\lambda; t)$ *of the Jacobi matrix* $J(t)(s_0(t) = 1)$ *using the formulas* (167) *where* $\varphi(t) = e^{-f(t)}$. *Then the solution of our problem in the Flashka variables,* $\alpha_n(t), \beta_n(t), n \in \mathbb{N}_0$, *are given by the classical formulas used in the calculation of elements of the Jacobi matrix from its moments.*

Remark 7. Using Theorem 12 it is possible to find a solution of the Cauchy problem for a semi-infinite Toda lattice by means of a way other than that in Section 2 (see Remark 2). In this case, the function appearing above, $\varphi(t)$, is equal to 0, i.e., $f(t) = -\infty$.

Remark 8. It is possible for our main problem, the Cauchy problem for the double-infinite Toda lattice, to apply the type of path of Remark 7. To do this, it is necessary to develop at first a moment theory of the "classical type" but connected with Jacobi matrices from Section 4.

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