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OPERATOR-NORM APPROXIMATIONS OF HOLOMORPHIC ONE-PARAMETER SEMIGROUPS OF CONTRACTIONS IN HILBERT SPACES

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Dedicated to the memory of Professor A. G. Kostyuchenko

Abstract. We establish the operator-norm convergence of the Iosida and Dunford-Segal approximation formulas for one-parameter semigroups of the class C_0, generated by maximal sectorial generators in separable Hilbert spaces. Our approach is essentially based on the Crouzeix-Delyon theorem [8] related to the generalization of the von Neumann inequality.

1. Introduction

In this section we briefly describe notations, the basic objects, and the main goal of this paper.

1.1. **Notations.** In what follows $\mathcal{L}(\mathfrak{H})$ denotes the Banach algebra of bounded linear operators on a complex separable Hilbert space $\mathfrak{H}$ with identity operator I. We also denote by $\operatorname{dom} T$, $\operatorname{ran} T$, and $\ker T$, respectively the domain, the range, and the null-space of the linear operator T, $\rho(T)$ is the set of all regular points of T.

1.2. **One-parameter semigroups and their generators.** Let $\mathfrak{H}$ be a complex separable Hilbert space and let $\{T(t)\}_{t\geq 0}$ be a C_0 one-parameter semigroup of contractions in $\mathfrak{H}$ ($||T(t)|| \leq 1,\ t \in \mathbb{R}_+$). Then its generator

$$(1.1)\qquad \begin{aligned} \operatorname{dom} A &= \left\{u \in \mathfrak{H} : \lim_{t\to+0} \frac{(I-T(t))u}{t} \quad \text{exists}\right\},\\ Au &:= \lim_{t\to+0} \frac{(I-T(t))u}{t}\,, \quad u \in \operatorname{dom} A \end{aligned}$$

is maximal accretive (shortly m-accretive) operator in $\mathfrak{H}$. This means that $\operatorname{Re}(Au,u) \geq 0$ for all $u \in \operatorname{dom} A$ and $\rho(A) \supseteq \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. Moreover, if a linear operator A is m-accretive operator, then there is a unique C_0-semigroup $\{T(t)\}_{t\geq 0}$ with generator A [13]. By means of the generator the semigroup can be recovered as the limit in the strong operator topology. We consider the following approximations

$$(1.2)\qquad T(t) = s - \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n}, \quad t \geq 0 \quad \text{(the Euler approximation)},$$

$$(1.3)\quad T(t) = s - \lim_{\lambda\to+\infty} \exp\left(-tA\left(I + \frac{A}{\lambda}\right)^{-1}\right), \quad t \geq 0 \quad \text{(the Iosida approximation)}.$$

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One more approximation formula for $\{T(t)\}$ is obtained by Dunford and Segal in [10] (see also [12]) and takes the form

(1.4) $$T(t) = s-\lim_{\eta\to+0}\exp\left(-t\,\frac{I-T(\eta)}{\eta}\right),\quad t\ge 0\quad \text{(the Dunford-Segal approximation)}.$$

Here in (1.3) and in (1.4) the exponent of a bounded operator B is defined as follows

$$\exp(B) = \sum_{n=1}^{\infty}\frac{B^n}{n!}.$$

We note that due to the estimate

$$||(A-zI)^{-1}|| \le \frac{1}{|\mathrm{Re}\,z|}\,,\quad \mathrm{Re}\,z<0$$

the operators

$$\left\{A\left(I+\frac{A}{\lambda}\right)^{-1}\right\}_{\lambda>0}$$

are bounded and m-accretive. Moreover, for an arbitrary m-accretive operator A one has

$$Au = \lim_{\lambda\to+\infty}A\left(I+\frac{A}{\lambda}\right)^{-1}u,\quad u\in\mathrm{dom}\,A.$$

Since $||T(\eta)||\le 1$, $\eta\ge 0$ the operators $\left\{\eta^{-1}(I-T(\eta)\right\}_{\eta>0}$ are bounded and accretive as well and

$$Au = \lim_{\eta\downarrow 0}\eta^{-1}(I-T(\eta))u,\quad u\in\mathrm{dom}\,A.$$

So, the generators of the semigroups in the right hand sides of (1.3) and (1.4) approximate generator A on $\mathrm{dom}\,A$ and, therefore, formulas (1.2), (1.3), and (1.4) justify that a C_0-semigroup $T(t)$ can be written as the operator exponent: $T(t)=\exp(-tA)$, $t\ge 0$. We note that according to the functional calculus for contractions in Hilbert spaces [16] if T is a contraction in $\mathfrak{H}$, $\ker(I+T)=\{0\}$, and $A:=(I-T)(I+T)^{-1}$, then for C_0-semigroup of contractions $T(t)$ with generator A, the equality $T(t)=f_t(T)$ holds with $f_t(z):=\exp(-t(1-z)(1+z)^{-1})$, $t\ge 0$ (see [16, Theorem III.8.1]).

1.3. **Holomorphic contractive semigroups and m-sectorial operators.**

Definition 1.1. [13]. Let $\alpha\in[0,\pi/2)$ and let

$$\mathcal{S}(\alpha) := \{z\in\mathbb{C}: |\arg z|\le\alpha\}$$

be a sector in the complex plane $\mathbb{C}$ with the vertex at the origin and the semi-angle α. Then a linear operator A in a Hilbert space $\mathfrak{H}$ is called sectorial with vertex at $z=0$ and the semi-angle α, if its numerical range

$$W(A) := \{(Au,u)\in\mathbb{C}:\ u\in\mathrm{dom}\,A,\ ||u||=1\}$$

is contained in $\mathcal{S}(\alpha)$, i.e.,

$$|\mathrm{Im}\,(Au,u)|\le\tan\alpha\,\mathrm{Re}\,(Au,u),\quad u\in\mathrm{dom}\,A.$$

An m-accretive sectorial operator with vertex at $z=0$ and with semi-angle α we call for short m-α-sectorial. The resolvent set of m-α-sectorial operator A contains the set $\mathbb{C}\setminus\mathcal{S}(\alpha)$ and

$$||(A-zI)^{-1}||\le\frac{1}{\mathrm{dist}\,(z,\mathcal{S}(\alpha))}\,,\quad z\in\mathbb{C}\setminus\mathcal{S}(\alpha).$$

Theorem 1.2. [13]. *A C_0-semigroup of contractions in the Hilbert space admits a holomorphic contractive continuation inside the sector $\mathcal{S}(\phi)$, $\phi\in(0,\pi/2]$ if and only if its generator is $m-(\pi/2-\phi)$-sectorial.*

Let $\alpha \in (0, \pi/2)$. We say that $T \in \mathcal{L}(\mathfrak{H})$ belongs to the class $C_{\mathfrak{H}}(\alpha)$ [1] if

$$||T \sin\alpha \pm iI\cos\alpha|| \leq 1. \tag{1.5}$$

By virtue of (1.5) one immediately gets that

$$T \in C_{\mathfrak{H}}(\alpha) \iff -T \in C_{\mathfrak{H}}(\alpha) \iff T^* \in C_{\mathfrak{H}}(\alpha),$$

and that condition (1.5) is equivalent to the inequalities

$$(||f||^2 - ||Tf||^2)\tan\alpha \geq 2|\mathrm{Im}\,(Tf, f)|, \quad f \in \mathfrak{H}.$$

This inequality implies that operator T is a contraction. In addition if A is m-α-sectorial operator, then

$$T = (I - A)(I + A)^{-1}$$

belongs to the class $C_{\mathfrak{H}}(\alpha)$. Conversely, if $T \in C_{\mathfrak{H}}(\alpha)$, and $\ker(I + T) = \{0\}$, then $A = (I - T)(I + T)^{-1}$ is m-α-sectorial operator. Properties of operators from the class $C_{\mathfrak{H}}(\alpha)$ were studied in [1] and [2]. In particular the next theorem gives a connection between the class $C_{\mathfrak{H}}(\alpha)$, $m-\alpha$-sectorial operators and corresponding one-parameter semigroups.

Theorem 1.3. [1]. *If A is m-α-sectorial operator in a Hilbert space $\mathfrak{H}$, then $T(t) = \exp(-tA) \in C_{\mathfrak{H}}(\alpha)$ for all $t \geq 0$. Conversely, let $T(t) = \exp(-tA)$ for $t \geq 0$ be a C_0-semigroup on a Hilbert space $\mathfrak{H}$. If $T(t) \in C_{\mathfrak{H}}(\alpha)$ for non-negative t in a neighborhood of $t = 0$, then the generator A is an m-α-sectorial operator.*

Notice that the proof Theorem 1.3 in [1] is based on the Euler approximation formula (1.2). We mention that the following statement is true [14]: if ϕ is analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\phi(z) = \overline{\phi(\bar{z})}$ and $|\phi(z)| < 1$ for $z \in \mathbb{D}$, then

$$T \in C_{\mathfrak{H}}(\alpha) \Rightarrow \phi(T) \in C_{\mathfrak{H}}(\alpha).$$

1.4. **Sectorial forms.** Recall some definitions and results from [13]. Let $\tau[\cdot,\cdot]$ be a sesquilinear form in a Hilbert space $\mathcal{H}$ defined on a linear manifold $\mathrm{dom}\,\tau$. The form τ is called symmetric if $\tau[u, v] = \overline{\tau[v, u]}$ for all $u, v \in \mathrm{dom}\,\tau$ and nonnegative if $\tau[u] := \tau[u, u] \geq 0$ for all $u \in \mathrm{dom}\,\tau$. The form $\tau^*[u, v] := \overline{\tau[v, u]}$ is called the adjoint to τ, and the forms

$$\tau_{\mathrm{R}}[u, v] := \frac{1}{2}\left(\tau[u, v] + \tau^*[u, v]\right), \quad \tau_{\mathrm{I}}[u, v] := \frac{1}{2i}\left(\tau[u, v] - \tau^*[u, v]\right), \quad u, v \in \mathrm{dom}\,\tau$$

are called the real and the imaginary parts of τ, respectively.

The form τ is called sectorial with the vertex at the origin and a semi-angle $\alpha \in [0, \pi/2)$ (α-sectorial) if its numerical range

$$W(\tau) = \{\tau[u] : \ u \in \mathrm{dom}\,\tau, \ ||u|| = 1\}$$

is contained in the sector $\mathcal{S}(\alpha)$, i.e.,

$$|\mathrm{Im}\,\tau[u]| \leq \tan\alpha\,\mathrm{Re}\,\tau[u], \quad u \in \mathrm{dom}\,\tau.$$

A sequence $\{u_n\}$ is called τ-convergent to the vector $u \in \mathfrak{H}$ if

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n,m\to\infty} \tau[u_n - u_m] = 0.$$

The form τ is called closed if for every sequence $\{u_n\}$ τ- convergent to a vector u it follows that $u \in \mathrm{dom}\,\tau$ and $\lim\limits_{n\to\infty} \tau[u - u_n] = 0$. A sectorial form τ with vertex at the origin is closed if and only if the linear manifold $\mathrm{dom}\,\tau$ is a Hilbert space with the inner product $(u, v)_\tau = \tau_{\mathrm{R}}[u, v] + (u, v)$ [13]. The form τ is called closable if it has a closed extension; in this case the closure of τ is the smallest closed extension of τ. If τ is a closed, densely defined sectorial form, then according to First Representation Theorem [13] there exists a unique m-sectorial operator A in $\mathfrak{H}$, associated with τ, i.e.,

$$(Au,v)=\tau[u,v] \quad \text{for all} \quad u\in \operatorname{dom} A \quad \text{and} \quad \text{for all} \quad v\in\operatorname{dom}\tau.$$

In this case the operator A^* is associated with the adjoint form $\tau^*[u,v]$. The nonnegative self-adjoint operator, denoted by A_R, associated with the real part $\tau_R[u,v]$ of the form τ, is called the "real part" of A. According to Second Representation Theorem [13] the identities hold

$$\operatorname{dom}\tau=\operatorname{dom}A_R^{1/2}.$$

A closed α-sectorial form τ has the representation

$$\tau[u,v]=\left((I+iM)A_R^{\frac{1}{2}}u, A_R^{\frac{1}{2}}v\right), \quad u,v\in\operatorname{dom}\tau, \tag{1.6}$$

where M is a bounded selfadjoint operator in the subspace $\overline{\operatorname{ran}}\,A_R$ and $||M||\le\tan\alpha$. For A one obtains

$$\operatorname{dom}A=\left\{u\in\operatorname{dom}\tau:(I+iM)A_R^{1/2}u\in\operatorname{dom}\tau\right\},\quad Au=A_R^{1/2}(I+iM)A_R^{1/2}u. \tag{1.7}$$

If A is a sectorial operator, then the form $\tau[u,v]=(Au,v),\ u,v\in\operatorname{dom}A$ is closable. The domain of its closure $A[\cdot,\cdot]$ we denote by $\mathcal{D}[A]$. The next theorem provides connections between m-sectorial operator A, its closed associated form $A[\cdot,\cdot]$ and one parameter semigroup $T(t)=\exp(-tA)$, $t\ge 0$.

Theorem 1.4. [3]. *Let A be m-sectorial operator in $\mathfrak{H}$ and let $T(t)$, $t\ge 0$ be C_0 semigroup with generator $-A$. Then the following conditions are equivalent:*

(i) $u\in\mathcal{D}[A]$,

(ii) $\sup\limits_{t>0}\left\{t^{-1}|((I-T(t))u,u)|\right\}<\infty$,

(iii) *the derivative*

$$\frac{d}{dt}(T(t)u,u)\Big|_{t=+0}$$

exists.

If $u,v\in\mathcal{D}[A]$, then

$$\frac{d}{dt}(T(t)u,v)\Big|_{t=+0}=\lim_{t\to+0}\frac{((T(t)-I)u,v)}{t}=-A[u,v].$$

1.5. **Crouzeix-Delyon theorem.** In the sequel we will essentially rely on the result established by M. Crouzeix and B. Delyon and related to the generalization of the von Neumann inequality.

Theorem 1.5. [8], [11]. *Let $\mathcal{S}_\alpha$ be a convex open sector of the complex plane with angle 2α, where $0\le\alpha\le\pi/2$. Then for any bounded operator $B\in\mathcal{L}(\mathfrak{H})$ with numerical range $W(B)\subseteq\overline{\mathcal{S}_\alpha}$ there is a best α-dependent constant $K_\alpha>0$ such that the inequality*

$$||R(B)||\le K_\alpha\sup_{z\in\mathcal{S}_\alpha}|R(z)| \tag{1.8}$$

holds for all rational functions $R:\mathbb{C}\to\mathbb{C}$ bounded in $\mathcal{S}_\alpha$. Furthermore, K_α is a continuous and decreasing function of $\alpha\in[0,\pi/2]$ satisfying the estimates

$$\frac{\pi\sin\alpha}{2\alpha}\le K_\alpha\le\min\left(\frac{\pi-\alpha}{\alpha},K_0\right)\quad\text{and}\quad\frac{\pi}{2}\le K_0\le 2+\frac{2}{\sqrt{3}}\ . \tag{1.9}$$

The inequality (1.8) is still valid if R is a holomorphic function in $\mathcal{S}_\alpha$, which is continuous and bounded in $\overline{\mathcal{S}_\alpha}$.

Using Theorem 1.5, in [4] the following statement concerning operator-norm convergence in the Euler approximation has been obtained.

Theorem 1.6. [4]. *Let A be an m-α-sectorial operator in a Hilbert space $\mathfrak{H}$. Then*

$$\left\|(I+tA/n)^{-n}-\exp(-tA)\right\| \leq \frac{K_\alpha}{\cos^2\alpha}\,\frac{1}{n}\ , \quad t\geq 0\ , \quad n\in\mathbb{N}. \tag{1.10}$$

The norm convergence rate $O(1/n)$ in the Euler formula firstly is established in [9] and holds for every generator of a semigroup holomorphic in a sector in arbitrary Banach spaces, i.e., also for m-α-sectorial operator in a Hilbert space $\mathfrak{H}$. In [6] the authors proved the $O(\ln n/n)$ rate using is results about quasi-sectorial contractions, established in [7]. Later the estimate $O(1/n)$ is obtained in [15] (see also [5]) by improving the probability theory methods originally developed in [6].

1.6. **The goal of this paper.** In this paper using the Crouzeix-Delyon theorem we show the operator-norm convergence of the rate $O(1/\lambda),\ \lambda\to+\infty$ in the Iosida and of the rate $O(\eta),\ \eta\to+0$ in Dunford-Segal approximations for holomorphic contractive semigroup at each point $t>0$ (see Theorem 3.1 and Theorem 3.4).

According to Theorem 1.3 the operators $T(\eta)=\exp(-A\eta),\ \eta\geq 0$ belong to the class $C_{\mathfrak{H}}(\alpha)$, when A is $m-\alpha$-sectorial. Therefore the operators $\eta^{-1}(I-T(\eta))$ are bounded $m-\alpha$-sectorial. Theorem 1.4 means that the forms

$$\omega_\eta[u,v]=(\eta^{-1}(I-T(\eta))u,v),\quad u,v\in\mathcal{D}[A],$$

associated with generators in Dunford-Segal formula, approximate the form $A[u,v]$ when $\eta\to+0$. In Theorem 2.1 of this paper we establish the convergence of sesquilinear forms

$$\tau_\lambda[u,v]=\left(A\left(I+\frac{A}{\lambda}\right)^{-1}u,v\right),\quad u,v\in\mathcal{D}[A],$$

associated with generators in the Iosida formula, to $A[u,v]$ when $\lambda\to+\infty$.

2. Convergence of closed sectorial forms

In this section we show that the sesquilinear forms associated with generators in Iosida formula converge to the closed form associated with the generator A (A is $m-\alpha$-sectorial operator) when $\lambda\to+\infty$.

Theorem 2.1. *Let A be m-sectorial operator in $\mathfrak{H}$. Then $u\in\mathcal{D}[A]$ if and only if the function*

$$\tau_\lambda[u]:=\left(\lambda A(A+\lambda I)^{-1}u,u\right)$$

is bounded in λ on the interval $(0,+\infty)$. Moreover,

$$\lim_{\lambda\to+\infty}(\lambda A(A+\lambda I)^{-1}u,v)=A[u,v],\quad u,v\in\mathcal{D}[A]. \tag{2.1}$$

Proof. Assume that A is $m-\alpha$-sectorial. Then the operator

$$S_\lambda:=\lambda A(A+\lambda I)^{-1}$$

is bounded and $m-\alpha$ sectorial for each $\lambda>0$. Indeed for $u\in\mathfrak{H}$ one has

$$(S_\lambda u,u)=\frac{1}{\lambda}(A\psi_\lambda,(A+\lambda I)\psi_\lambda)=\frac{1}{\lambda}||A\psi_\lambda||^2+(A\psi_\lambda,\psi_\lambda)\in\mathcal{S}(\alpha),$$

where $\psi_\lambda=\lambda(A+\lambda I)^{-1}u\in\operatorname{dom}A$.

Suppose that for some $u\in\mathfrak{H}$

$$|\tau_\lambda[u]|\leq C\quad\text{for all}\quad\lambda>0$$

with some positive constant C. Then from $\tau_\lambda[u]=(S_\lambda u,u)$ and $\operatorname{Re}(A\psi_\lambda,\psi_\lambda)\geq 0$ we get that $\operatorname{Re}(A\psi_\lambda,\psi_\lambda)\leq C$ for all $\lambda>0$. The inequality $|\operatorname{Im}(A\psi_\lambda,\psi_\lambda)|\leq\tan\alpha\operatorname{Re}(A\psi_\lambda,\psi_\lambda)$ yields

$$|(A\psi_\lambda,\psi_\lambda)|\leq\frac{1}{\cos\alpha}\operatorname{Re}(A\psi_\lambda,\psi_\lambda).$$

Hence

$$|(A\psi_\lambda,\psi_\lambda)|\le \frac{C}{\cos\alpha}.$$

Because $s-\lim\limits_{\lambda\to+\infty}\lambda(A+\lambda I_{\mathfrak H})^{-1}=I_{\mathfrak H}$, we have $\lim\limits_{\lambda\to+\infty}\psi_\lambda=u$. Applying [13, Theorem VI.1.16], we obtain $u\in\mathcal{D}[A]$.

Let us prove (2.1). For $A[u,v]$ and A we will use the representations (1.6) and (1.7)

$$\begin{aligned}A[u,v]&=\left((I+iM)A_R^{1/2}u,A_R^{1/2}v\right),\quad u,v\in\mathcal{D}[A]=\operatorname{dom}A_R^{1/2},\\ A&=A_R^{1/2}(I+iM)A_R^{1/2}.\end{aligned}$$

For $\lambda>0$ put

$$M_\lambda=\left(A_R(A_R+\lambda I)^{-1}\right)^{1/2}M\left(A_R(A_R+\lambda I)^{-1}\right)^{1/2}.$$

Then

$$\begin{aligned}(A+\lambda I)f&=(A_R+\lambda I)^{1/2}(I+iM_\lambda)(A_R+\lambda I)^{1/2}f,\quad f\in\operatorname{dom}A,\\ (A+\lambda I)^{-1}&=(A_R+\lambda I)^{-1/2}(I+iM_\lambda)^{-1}(A_R+\lambda I)^{-1/2}.\end{aligned}$$

Let $u,v\in\mathcal{D}[A]$. Then

$$\begin{aligned}&A[u,v]-(\lambda A(A+\lambda I)^{-1}u,v)=((I+iM)A_R^{1/2}u,A_R^{1/2}v)\\ &\quad-\lambda((I+iM)A_R^{1/2}(A_R+\lambda I)^{-1/2}(I+iM_\lambda)^{-1}(A_R+\lambda I)^{-1/2}u,A_R^{1/2}v)\\ &\quad=\left((I+iM)A_R^{1/2}\left(I-\lambda(A_R+\lambda I)^{-1/2}(I+iM_\lambda)^{-1}(A_R+\lambda I)^{-1/2}\right)u,A_R^{1/2}v\right).\end{aligned}$$

Next we use that if $-1\in\rho(CD)$, then $-1\in\rho(DC)$ and

$$C(I+DC)^{-1}=(I+CD)^{-1}C,\tag{2.2}$$

$$(CD+I)^{-1}+C(DC+I)^{-1}D=I\tag{2.3}$$

for bounded operators C and D in $\mathfrak H$. By virtue of (2.2) for

$$C=A_R^{1/2}(A_R+\lambda I)^{-1/2},\quad D=iA_R^{1/2}(A_R+\lambda I)^{-1/2}M$$

we get

$$\begin{aligned}&A_R^{1/2}(A_R+\lambda I)^{-1/2}(I+iM_\lambda)^{-1}\\ &\quad=\left(A_R(A_R+\lambda I)^{-1}\right)^{1/2}\left(I+i\left(A_R(A_R+\lambda I)^{-1}\right)^{1/2}M\left(A_R(A_R+\lambda I)^{-1}\right)^{1/2}\right)^{-1}\\ &\quad=\left(I+i\left(A_R(A_R+\lambda I)^{-1}M\right)\right)^{-1}\left(A_R(A_R+\lambda I)^{-1}\right)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}&A_R^{1/2}(A_R+\lambda I)^{-1/2}(I+iM_\lambda)^{-1}(A_R+\lambda I)^{-1/2}u\\ &\quad=\left(I+i\left(A_R(A_R+\lambda I)^{-1}M\right)\right)^{-1}(A_R+\lambda I)^{-1}A_R^{1/2}u,\quad u\in\mathcal{D}[A].\end{aligned}$$

Therefore

$$\begin{aligned}&A[u,v]-(\lambda A(A+\lambda I)^{-1}u,v)\\ &\quad=\left((I+iM)\left(A_R^{1/2}-\left(I+i\left(A_R(A_R+\lambda I)^{-1}M\right)\right)^{-1}\right.\right.\\ &\quad\times\left.\left.\left(\lambda(A_R+\lambda I)^{-1}\right)A_R^{1/2}u\right),A_R^{1/2}v\right).\end{aligned}\tag{2.4}$$

We have

$$||(A_R + \lambda I)^{-1}|| \leq \frac{1}{\lambda},\ ||A_R(A_R + \lambda I)^{-1}|| \leq 1,$$
$$s - \lim_{\lambda\to+\infty} A_R(A_R + \lambda I)^{-1} = 0,$$
$$s - \lim_{\lambda\to+\infty} \lambda(A_R + \lambda I)^{-1} = I_{\mathfrak{H}}.$$

Since $DC = M_\lambda$, is a self-adjoint operator, from (2.3) we get

$$\left\| \left(I + i\,(A_R(A_R + \lambda I)^{-1}\,M\right)^{-1} \right\| \leq 1 + ||M|| \leq 1 + \tan\alpha.$$

It follows that

$$\lim_{\lambda\to+\infty} \left(I + i\,(A_R(A_R + \lambda I)^{-1}\,M\right)^{-1} \left(\lambda(A_R + \lambda I)^{-1}f - f - iA_R(A_R + \lambda I)^{-1}Mf\right) = 0.$$

Therefore, from (2.4) we get (2.1). Since

$$s - \lim_{\lambda\to+0} \lambda A(A + \lambda I)^{-1} = 0,$$

the function $\tau_\lambda[u] = (S_\lambda u, u)$ is bounded on $(0, +\infty)$ for all $u \in \mathcal{D}[A]$. □

3. Operator-norm convergence

In this section we established the convergence in the operator-norm topology in the Iosida and Dunford-Segal approximation formulas for $m-\alpha$-sectorial generators.

3.1. **The Iosida approximation.**

Theorem 3.1. *Let A be $m-\alpha$-sectorial operator in the Hilbert space $\mathfrak{H}$ and let $\{T(t) = \exp(-tA)\}_{t\geq 0}$ be the semigroup with generator A. Then*

$$\left\|T(t) - \exp\left(-t\lambda A(A + \lambda I)^{-1}\right)\right\| \leq \frac{2e^{-1}\,K_\alpha}{\cos^2\alpha}\,\frac{1}{t\lambda} \quad \textit{for all} \quad t > 0, \quad \lambda > 0, \tag{3.1}$$

where K_α satisfies (1.9).

Proof. We will use Theorem 1.5. We need to estimate

$$\sup_{z\in\mathcal{S}(\alpha)} \left|\exp(-tz) - \exp(-tz\lambda(\lambda + z)^{-1})\right| = \sup_{z\in\partial\mathcal{S}(\alpha)} \left|\exp(-tz) - \exp(-tz\lambda(\lambda + z)^{-1})\right|$$

for $t > 0$ and $\lambda > 0$. Let $x > 0$ and let $z = xe^{i\alpha} \in \partial\mathcal{S}(\alpha)$. Then

$$\begin{aligned}&\exp(-txe^{i\alpha}) - \exp(-txe^{i\alpha}\lambda(\lambda + xe^{i\alpha})^{-1})\\ &= -\int_0^x \frac{d}{ds}\left(\exp(-tse^{i\alpha}\lambda(\lambda + se^{i\alpha})^{-1})\exp(-t(x - s)e^{i\alpha})\right) ds.\end{aligned}$$

Calculating the derivative one obtains

$$\begin{aligned}&\frac{d}{ds}\left(\exp(-tse^{i\alpha}\lambda(\lambda + se^{i\alpha})^{-1})\exp(-t(x - s)e^{i\alpha})\right)\\ &= te^{i2\alpha}\,\frac{s(se^{i\alpha} + 2\lambda)}{(\lambda + se^{i\alpha})^2}\,\exp(-tse^{i\alpha}\lambda(\lambda + se^{i\alpha})^{-1})\exp(-t(x - s)e^{i\alpha}).\end{aligned}$$

Then

$$\begin{aligned}&\left|\frac{d}{ds}\left(\exp(-tse^{i\alpha}\lambda(\lambda + se^{i\alpha})^{-1})\exp(-t(x - s)e^{i\alpha})\right)\right|\\ &= t\exp\left(-t\lambda\frac{s(s + \lambda\cos\alpha)}{s^2 + \lambda^2 + 2s\lambda\cos\alpha}\right)\exp(-t(x - s)\cos\alpha)\,\frac{s\sqrt{s^2 + 4\lambda^2 + 4s\lambda\cos\alpha}}{s^2 + \lambda^2 + 2s\lambda\cos\alpha}.\end{aligned}$$

□

Since

$$\max_{y\geq 0}\{y\exp(-ay)\}=\frac{1}{a}\,e^{-1},\quad a>0,$$

and

$$\frac{\sqrt{s^2+4\lambda^2+4s\lambda\cos\alpha}}{s+\lambda\cos\alpha}\leq\frac{2}{\cos\alpha}$$

we get

$$\begin{aligned}&\exp\Big(-t\lambda\frac{s(s+\lambda\cos\alpha)}{s^2+\lambda^2+2s\lambda\cos\alpha}\Big)\frac{s\sqrt{s^2+4\lambda^2+4s\lambda\cos\alpha}}{s^2+\lambda^2+2s\lambda\cos\alpha}\\&=\Big(\frac{s(s+\lambda\cos\alpha)}{s^2+\lambda^2+2s\lambda\cos\alpha}\exp\Big(-t\lambda\frac{s(s+\lambda\cos\alpha)}{s^2+\lambda^2+2s\lambda\cos\alpha}\Big)\Big)\frac{\sqrt{s^2+4\lambda^2+4s\lambda\cos\alpha}}{s+\lambda\cos\alpha}\\&\leq\frac{e^{-1}}{t\lambda}\,\frac{2}{\cos\alpha}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\exp(-txe^{i\alpha})-\exp(-txe^{i\alpha}\lambda(\lambda+xe^{i\alpha})^{-1})\right|\\&\leq\int_0^x\Big|\frac{d}{ds}\big(\exp(-tse^{i\alpha}\lambda(\lambda+se^{i\alpha})^{-1})\exp(-t(x-s)e^{i\alpha})\big)\Big|\,ds\\&\leq t\,\frac{e^{-1}}{t\lambda}\,\frac{2}{\cos\alpha}\int_0^x\exp(-t(x-s)\cos\alpha)ds=\frac{1}{t\lambda}\,\frac{2e^{-1}}{\cos^2\alpha}(1-\exp(-tx\cos\alpha))\\&\leq\frac{1}{t\lambda}\,\frac{2e^{-1}}{\cos^2\alpha}.\end{aligned}$$

Clearly, on the ray $z=xe^{-i\alpha}$, $x>0$ the same estimate is valid. Thus

$$\sup_{z\in\partial\mathcal{S}(\alpha)}\left|\exp(-tz)-\exp(-tz\lambda(\lambda+z)^{-1})\right|\leq\frac{1}{t\lambda}\,\frac{2e^{-1}}{\cos^2\alpha}.$$

Now, applying Theorem 1.5, we arrive at (3.1).

Corollary 3.2. *For* $t>0$

$$\lim_{\lambda\to+\infty}\left\|\exp\left(-t\lambda A(A+\lambda I)^{-1}\right)-\exp(-tA)\right\|=0.$$

Theorem 3.1 allows to prove analyticity of the function $\exp(-zA)$ for m-sectorial generator A without using the Cauchy integral formula [13].

Corollary 3.3. *Let A be $m-\alpha$-sectorial operator. Then the C_0-semigroup $\{T(t)=\exp(-tA)\}_{t\geq 0}$ admits holomorphic contractive continuation inside the sector $\mathcal{S}(\pi/2-\alpha)$.*

Proof. Let $z=xe^{i\varphi}$, $x>0$, $|\varphi|<\pi/2-\alpha$. Then the operator zA is $m-(\alpha+|\varphi|)$-sectorial. By Theorem 3.1

$$\left\|\exp(-zA)-\exp\left(-\lambda zA(zA+\lambda I)^{-1}\right)\right\|\leq\frac{2e^{-1}\,K_{\alpha+|\varphi|}}{\cos^2(\alpha+|\varphi|)}\,\frac{1}{\lambda}\quad\text{for all}\quad\lambda>0,$$

It follows that the family $\{\exp\left(-\lambda zA(zA+\lambda I)^{-1}\right)\}$ convergence to $\exp(-zA)$ in the operator-norm topology, when $\lambda\to+\infty$, uniformly with respect to $z\in M$ for any compact set $M\subset\mathrm{Int}\mathcal{S}(\pi/2-\alpha)$. Therefore, by the Weierstrass theorem, $T(z)=\exp-(zA)$ is holomorphic inside $\mathcal{S}(\pi/2-\alpha)$, and $||T(z)||\leq 1$. Besides, $T(z_1+z_2)=T(z_1)T(z_2)$, $z_1,z_2\in\mathcal{S}(\alpha)$. □

3.2. The Dunford-Segal approximation.

Theorem 3.4. *Let* $\{T(t)\}_{t\geq 0}$ *be a contractive semigroup with* $m-\alpha$*-sectorial generator in the Hilbert space* $\mathfrak{H}$*. Then*

$$\left\|T(t)-\exp\Big(-t\,\frac{I-T(\eta)}{\eta}\Big)\right\|\leq\frac{e^{-1}K_\alpha}{\cos^2\alpha}\,\frac{\eta}{t}\qquad \textit{for all}\quad t>0,\quad \eta>0, \tag{3.2}$$

where K_α *satisfies* (1.9).

Proof. Let $z=xe^{i\alpha}$, $x>0$. Then

$$\begin{aligned}&\exp(-txe^{i\alpha})-\exp\Big(t\eta^{-1}(e^{-\eta xe^{i\alpha}}-1)\Big)\\&=-\int\limits_0^x\frac{d}{ds}\Big(\exp(-t(x-s)e^{i\alpha})\,\exp\Big(t\eta^{-1}(e^{-\eta se^{i\alpha}}-1)\Big)\Big)\,ds\\&=-t\int\limits_0^x\Big(e^{i\alpha}\exp(-t(x-s)e^{i\alpha})\exp\Big(t\eta^{-1}(e^{-\eta se^{i\alpha}}-1\Big)\left(1-\exp(-\eta se^{i\alpha})\right)\Big)\,ds.\end{aligned}$$

Further we shall estimate from above the absolute value

$$\begin{aligned}&\left|\exp\Big(t\eta^{-1}(e^{-\eta se^{i\alpha}}-1\Big)\left(1-\exp(-\eta se^{i\alpha})\right)\right|\\&=\exp\left(t\eta^{-1}\left(e^{-\eta s\cos\alpha}\cos(\eta s\sin\alpha)-1\right)\right)\\&\times\sqrt{1+e^{-2\eta s\cos\alpha}-2e^{-\eta s\cos\alpha}\cos(\eta s\sin\alpha)}.\end{aligned}$$

Using the inequality $\exp(-y)\leq e^{-1}y^{-1}$ for $y>0$, we have

$$\exp\left(-t\eta^{-1}\left(1-e^{-\eta s\cos\alpha}\cos(\eta s\sin\alpha)\right)\right)\leq\frac{\eta}{t}\,\frac{e^{-1}}{1-e^{-\eta s\cos\alpha}\cos(\eta s\sin\alpha)},$$

$$\begin{aligned}&\sqrt{\frac{1+e^{-2\eta s\cos\alpha}-2e^{-\eta s\cos\alpha}\cos(\eta s\sin\alpha)}{(1-e^{-\eta s\cos\alpha}\cos(\eta s\sin\alpha))^2}}\\&=\sqrt{\frac{e^{2\eta s\cos\alpha}-2e^{\eta s\cos\alpha}\cos(\eta s\sin\alpha)+1}{(e^{\eta s\cos\alpha}-\cos(\eta s\sin\alpha))^2}}=\sqrt{1+\frac{\sin^2(\eta s\sin\alpha)}{(e^{\eta s\cos\alpha}-\cos(\eta s\sin\alpha))^2}}\\&\leq\sqrt{1+\frac{\sin^2(\eta s\sin\alpha)}{(\eta s)^2\cos^2\alpha}}=\sqrt{1+\frac{\sin^2(\eta s\sin\alpha)}{(\eta s)^2\sin^2\alpha}\tan^2\alpha}\leq\sqrt{1+\tan^2\alpha}=\frac{1}{\cos\alpha}.\end{aligned}$$

Here we have used the inequalities

$$e^{\eta s\cos\alpha}-\cos(\eta s\sin\alpha)\geq e^{\eta s\cos\alpha}-1\geq\eta s\cos\alpha.$$

Hence, we have

$$\left|\exp\Big(t\eta^{-1}(e^{-\eta se^{i\alpha}}-1\Big)\left(1-\exp(-\eta se^{i\alpha})\right)\right|\leq\frac{\eta}{t}\,\frac{e^{-1}}{\cos\alpha}.$$

Finally,

$$\begin{aligned}&\left|\exp(-txe^{i\alpha})-\exp\Big(t\eta^{-1}(e^{-\eta xe^{i\alpha}}-1)\Big)\right|\\&\leq\frac{\eta}{\cos\alpha}\int\limits_0^x\exp(-t(x-s)\cos\alpha)ds\leq\frac{\eta}{t}\,\frac{e^{-1}}{\cos^2\alpha}.\end{aligned}$$

Similarly

$$\left|\exp(-txe^{-i\alpha})-\exp\Big(t\eta^{-1}(e^{-\eta xe^{-i\alpha}}-1)\Big)\right|\leq\frac{\eta}{t}\,\frac{e^{-1}}{\cos^2\alpha}.$$

Therefore

$$\sup_{z\in\mathcal{S}(\alpha)}\left|\exp(-tz)-\exp\left(t\eta^{-1}(e^{-\eta z}-1)\right)\right|$$

$$=\sup_{z\in\partial\mathcal{S}(\alpha)}\left|\exp(-tz)-\exp\left(t\eta^{-1}(e^{-\eta z}-1)\right)\right|\le\frac{\eta}{t}\,\frac{e^{-1}}{\cos^2\alpha},\quad t>0,\quad \eta>0.$$

To complete the proof we use Theorem 1.5. □

Corollary 3.5.

$$\lim_{\eta\to+0}\left\|\exp\left(-t\,\frac{I-T(\eta)}{\eta}\right)-T(t)\right\|=0$$

for all $t>0$.

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