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THE COMPLEX MOMENT PROBLEM IN THE EXPONENTIAL FORM WITH DIRECT AND INVERSE SPECTRAL PROBLEMS FOR THE BLOCK JACOBI TYPE CORRESPONDENCE MATRICES

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This paper is dedicated to A. G. Kostyuchenko

Abstract. We present a new generalization of the connection of the classical power moment problem with spectral theory of Jacobi matrices. In the article we propose an analog of Jacobi matrices related to the complex moment problem in the case of exponential form and to the system of orthonormal polynomials with respect to some measure with the compact support on the complex plane. In our case we obtain two matrices that have block three-diagonal structure and acting in the space of l_2 type as commuting self-adjoint and unitary operators. With this connection we prove the one-to-one correspondence between the measures defined on a compact set in the complex plane and the couple of block three-diagonal Jacobi type matrices. For simplicity we consider in this article only a bounded self-adjoint operator.

1. Introduction

The article consists of two parts. In the first part (Introduction and Preliminaries) we describe the main idea and main results of the article. Here we give a comparison of this idea with the classical one, and repeat some results about the complex moment problem in the exponential form, which are necessary in the next parts of the article. The solution of the complex moment problem in the exponential form is contained in our article [8], but without this part it is not convenient to pursue further investigations.

The second part of the article (Sections 3 and 4) is a solution of direct and inverse spectral problems. This is a generalization of the classical problems for Jacobi matrices and orthogonal polynomials on the real axis $\mathbb{R}$ to the case of block Jacobi type commuting self-adjoint and unitary matrices and the corresponding orthogonal polynomials on complex plane $\mathbb{C}$. This part continues our previous articles [9] in which we investigate block tree-diagonal matrices of a unitary operator and orthogonal polynomials on the unit circle $\mathbb{T} \subset \mathbb{C}$, and [10] in which we investigate block tree-diagonal matrices of a bounded normal operator and the corresponding orthogonal polynomials on the complex plane $\mathbb{C}$.

An investigation of the complex moment problem in the exponential form is dealt with in few works, we mention here only [14, 25]. But our proof of the complex moment representation (in [8]) is based on the Yu. M. Berezansky's generalized eigenfunction expansion of the corresponding couple of commuting self-adjoint and unitary operators and was in detail described in [8]. This method goes back to the old works of M. G. Krein [19, 20].

The consideration of the complex moment problem in the exponential form follows the investigation of a couple of commuting self-adjoint and unitary operators. And the

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consideration of the usual power complex moment problem follows the investigation of a couple of two commuting but self-adjoint operators [1, 7, 11, 12, 16, 24, 23]. Hence these both problems are closely related to the two-dimensional and hence to the multi-dimensional moment problems in different settings [11, 12, 13, 14]. Let us note separately about some (non general) case of the multi-dimensional moment problem presented by A. G. Kostyuchenko and B. S. Mityagin in [17, 18] with an approach used in [3].

To understand the second part of this article it is necessary to recall [10] the direct and inverse spectral problems for classical Jacobi matrices and orthogonal polynomials on the axis $\mathbb{R}$ see, for example, [1, 3, 26]). In this classical theory, one studies the Hermitian Jacobi matrix

$$(1)\qquad J=\left[\begin{array}{cccccc} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right], \quad b_n\in\mathbb{R}, \quad a_n>0, \quad n\in\mathbb{N}_0=\{0,1,2,\dots\}.$$

This matrix defines an operator (also denoted by J) on finite sequences $f\in l_{\rm fin}\subset l_2$. This operator is Hermitian with defect numbers either $(0,0)$ or $(1,1)$ (and hence in any case has a selfadjoint extensions on l_2). Under some conditions on the coefficients of the matrix J the closure of J is a selfadjoint operator in l_2.

The direct spectral problem, i.e., the eigenfunction expansion for J (for simplicity, we will assume that J is selfadjoint), is described in the following way. We obtain a sequence of polynomials, $P(x)=(P_n(x))_{n=0}^\infty$, $\forall x\in\mathbb{R}$, as a solution (step by step) of the equation $JP(x)=xP(x)$ with the initial condition $P_0(x)=1$, i.e., $\forall n\in\mathbb{N}_0$

$$(2)\qquad a_{n-1}P_{n-1}(x)+b_nP_n(x)+a_nP_{n+1}(x)=xP_n(x), \quad P_0(x)=1, \quad P_{-1}(x)=0$$

(under the condition $a_n>0$).

The sequence $P(x)$ of polynomials belongs to (the real part of) $l=\mathbb{C}^\infty$ $\forall x\in\mathbb{R}$ and is a generalized eigenvector for J with the eigenvalue x (in the sense of some rigging of l_2). The corresponding Fourier transform (denoted here by $\widehat{\ }$) using the generalized eigenfunctions of J has the form

$$(3)\qquad l_2\supset l_{\rm fin}\ni f=(f_n)_{n=0}^\infty\longmapsto \hat f(x)=\sum_{n=0}^\infty f_nP_n(x)\in L^2(\mathbb{R},d\rho(x))=L^2.$$

It is an unitary operator (after taking the closure) from the whole space l_2 to the whole L^2. The image of J is the operator of multiplication by x on the space L^2. The polynomials $P_n(x)$ are orthonormal w.r.t. $d\rho(x)$.

The inverse problem in this classical case is the following. Let us have a probability Borel measure $d\rho(x)$ on $\mathbb{R}$ for which all moments exist,

$$(4)\qquad s_n=\int_{\mathbb{R}}x^n\,d\rho(x), \quad n\in\mathbb{N}_0$$

(and the support of $d\rho(x)$ contains an open interval). Is it possible to recover the corresponding Jacobi matrix J in such manner, that the initial (given) measure $d\rho(x)$ would be uniquely generated by the spectral measure for J? The answer is given in the following construction. It is necessary to take the sequence of functions from L^2,

$$(5)\qquad 1,x,x^2,\dots$$

(which are linearly independent (with respect to $d\rho(x)$ according to (4)) and form a total set in L^2) and apply to it the classical Schmidt orthogonalization procedure. As a result, we get a sequence of orthonormal polynomial, that is, the orthonormal basis in L^2,

$$(6)\qquad P_0(x)=1,\ P_1(x),\ P_2(x),\dots$$

Then the coefficients of the matrix J can be reconstructed by the formulas

$$a_n = \int_{\mathbb{R}} xP_n(x)P_{n+1}(x)\,d\rho(x), \quad b_n = \int_{\mathbb{R}} x(P_n(x))^2\,d\rho(x), \quad n \in \mathbb{N}_0. \tag{7}$$

The above mentioned connections between Jacobi matrices, the classical moment problem, and orthogonal polynomials is very fruitful for an investigation of these objects. Many mathematicians worked in this direction, including M. G. Krein, N. I. Achiezer, Yu. M. Berezansky, M. L. Gorbachuk, V. I. Gorbachuk ... But it is necessary to single out the references [1, 3, 20, 15, 22].

The main question in the second part of this articles is the following: in what way it is possible to obtain a generalization of the above mentioned classical theory to orthonormal polynomials on the complex plane $\mathbb{C}$ (or on some subset of $\mathbb{C}$). In this article, we will need to pass from the selfadjoint operator in l_2 to a couple of commuting selfadjoint and unitary operators acting on some space, similar to l_2.

More exactly, instead of the space $l_2 = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$, it is necessary to take the space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \text{where} \quad \mathcal{H}_n = \mathbb{C}^{2n+1}, \quad n \in \mathbb{N}_0 \tag{8}$$

and, instead of the scalar matrix (1), it is necessary to consider two following Jacobi type block matrices.

The first matrix has elements a_n, b_n, and c_n that are finite dimensional operators (matrices) and act between the corresponding spaces $\mathcal{H}_n$ in (8), namely:

$$J_A = \begin{bmatrix} b_0 & c_0 & 0 & 0 & \cdots \\ a_0 & b_1 & c_1 & 0 & \cdots \\ 0 & a_1 & b_2 & c_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{array}{rcrcl} a_n & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_{n+1}, \\ b_n & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_n, \\ c_n & : & \mathcal{H}_{n+1} & \longrightarrow & \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{9}$$

Such a matrix (9) on finite vectors in $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$ in a natural way generates an operator denoted by A on $\mathbf{l}_2$. For simplicity, *we will demand that the norms of all matrices a_n, b_n, and c_n be uniformly bounded and therefore the operator A is bounded on $\mathbf{l}_2$.*

For the selfadjoint operator we have the following form of the elements a_n, b_n and c_n with some conditions:

$$a_n = \left.\left[\underbrace{\begin{matrix} * & * & * & \dots & * \\ a_{n;1,0} & * & * & \dots & * \\ 0 & a_{n;2,1} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n;2n+1,2n} \\ 0 & 0 & 0 & \dots & 0 \end{matrix}}_{2n+1} \right]\right\}{\scriptstyle 2n+3}, \tag{10}$$

$$c_n = \left.\left[\underbrace{\begin{matrix} * & c_{n;0,1} & 0 & \dots & 0 & 0 & 0 \\ * & * & c_{n;1,2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & c_{n;2n,2n+1} & 0 \end{matrix}}_{2n+3} \right]\right\}{\scriptstyle 2n+1}, \tag{11}$$

$$\begin{gathered} a_{n;1,0}, a_{n;2,1}, \dots, a_{n;2n+1,2n} > 0, \quad c_{n;0,1}, c_{n;1,2}, \dots, c_{n;2n,2n+1} > 0, \\ a_{n;i,j} = c_{n;j,i}, \quad n, i, j \in \mathbb{N}_0; \end{gathered} \tag{12}$$

and the matrices b_n have an arbitrary structure, so that the matrix J_A is a Hermitian operator. Here and in the following $''*''$ denoted a possibly non zero element in the matrix.

The second matrix has elements u_n, w_n, and v_n, which are also finite dimensional operators (matrices) and act between the corresponding spaces $\mathcal{H}_n$ in (8), namely:

$$J_U = \begin{bmatrix} w_0 & v_0 & 0 & 0 & \cdots \\ u_0 & w_1 & v_1 & 0 & \cdots \\ 0 & u_1 & w_2 & v_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{array}{lcrcl} u_n & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_{n+1}, \\ w_n & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_n, \\ v_n & : & \mathcal{H}_{n+1} & \longrightarrow & \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{13}$$

Such a matrix (13) on finite vectors in $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$, in a natural way, generates an operator (denoted by U) on $\mathbf{l}_2$. It is clear that the norms of all the matrices u_n, w_n, and v_n are uniformly bounded and therefore the operator U is bounded on $\mathbf{l}_2$.

For the unitary operator we have the following form of elements u_n, w_n, and v_n with some conditions

$$u_n = \left[\underbrace{\begin{matrix} u_{n;0,0} & * & \ldots & * & * \\ 0 & u_{n;1,1} & \ldots & * & * \\ 0 & 0 & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & u_{n;n,n} \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{matrix}}_{n+1} \; \underbrace{\begin{matrix} * & \ldots & * \\ * & \ldots & * \\ * & \ldots & * \\ \vdots & \ddots & \vdots \\ * & \ldots & * \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{matrix}}_{n} \right] \begin{array}{l} \left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{matrix}}\right\} n+1 \\ \left.\vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}}\right\} n+2 \end{array}, \tag{14}$$

$$v_n = \left[\underbrace{\begin{matrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ * & \ldots & * \\ \vdots & \ddots & \vdots \\ * & \ldots & * \\ * & \ldots & * \\ * & \ldots & * \end{matrix}}_{n+2} \; \underbrace{\begin{matrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \\ v_{n;n,n+2} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \ldots & 0 & 0 \\ * & \ldots & v_{n;2n-1,2n+1} & 0 \\ * & \ldots & * & v_{n;2n,2n+2} \end{matrix}}_{n+1} \right] \begin{array}{l} \left.\vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{matrix}}\right\} n+1 \end{array}, \tag{15}$$

$$\begin{aligned} & u_{n;0,0}, u_{n;1,1}, \ldots, u_{n;n,n} > 0, \\ & v_{n;n,n+2}, v_{n;n+1,n+3}, \ldots, v_{n;2n,2n+2} > 0, \quad n \in \mathbb{N}_0; \end{aligned} \tag{16}$$

and the matrices w_n have an arbitrary structure, so that the matrix J_U is unitary (it is convenient to denote a vector $x \in \mathcal{H}_n = \mathbb{C}^{2n+1}$ by $x = (x_0, x_1, \ldots, x_n)$).

Under some conditions on a_n, b_n, c_n, and u_n, w_n, v_n, $n \in \mathbb{N}_0$, the matrix J_U is unitary and commutes with J_A.

Let $z \in \mathbb{C}$ belong to the joint spectrum of A and U and $P(z) = (P_{t,j}(z))$, $t \in \mathbb{N}_0, j \in \mathbb{Z}, z \in \mathbb{C}$, be the corresponding generalized eigenvectors of A and U. Here $P_n(z) \in \mathcal{H}_n$, $(z = re^{i\theta})$ is a vector-valued polynomial with respect to r and $e^{i\theta}$, i.e., its coordinates are some linear combinations of $r^t e^{i\theta j}$, $t \in \mathbb{N}_0, j \in \mathbb{Z}$. According to the generalized eigenvectors expansion theorem it is some solution of three equations of type (2) (but with matrix coefficients)

$$AP(z) = rP(z), \quad UP(z) = e^{i\theta} P(z), \quad (U^* P(z) = e^{-i\theta} P(z)). \tag{17}$$

The corresponding Fourier transform $\widehat{\ }$ for the operators A and U has the form

$$\mathbf{l}_2 \supset \mathbf{l}_{\text{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{\substack{t\in\mathbb{N}_0 \\ j\in\mathbb{Z}}} (f_{t,j}, P_{t,j}(z))_{\mathcal{H}_n} \in L^2(\mathbb{C}, d\rho(z) = L^2, \tag{18}$$

where $d\rho(z) = d\rho(r,\theta)$ is a spectral measure of A and U on the complex plane $\mathbb{C}$. The operator (18) is a unitary operator (after taking the closure) and acts from the whole $\mathbf{l}_2$ to the whole L_2. The polynomials $P_{t,j}(\lambda)$ are orthonormal with respect to $d\rho(r,\theta)$ and form a basis in the space L^2. Note that these results are formulated in Theorem 4, but for us it is convenient to denote here these polynomials by

$$\begin{aligned} P_n(z) =&(P_{n;0}(z), P_{n;1}(z), \ldots, P_{n;2n}(z)) = (\overline{Q_{n;0}(z)}, \overline{Q_{n;1}(z)}, \ldots, \overline{Q_{n;2n}(z)}) \\ =&(Q_{n;2n}(z), Q_{n;2n-1}(z), \ldots, Q_{n;0}(z)). \end{aligned}$$

So, the results described above make the content of the direct spectral problem for A and U of type (9) with (10), (11), (12), and (13) with (14), (15), (16).

The inverse spectral problem now is formulated in the following way. Suppose we have a Borel measure $d\rho(z) = d\rho(r,\theta)$, $z = re^{i\theta}$, with compact support on $\mathbb{C}$; assume that all complex moments

$$c_{t,j} = \int_0^{\infty}\int_0^{2\pi} r^t e^{ij\theta} d\rho(r,\theta), \quad t\in\mathbb{N}_0, \quad j\in\mathbb{Z}, \tag{19}$$

exist and the support of $d\rho(z)$ is such that all the functions $r^t e^{ij\theta}$, $t\in\mathbb{N}_0, j\in\mathbb{Z}$, belonging to L^2, are linearly independent in this space (for example, the support of $d\rho(z)$ contains some open subset of $\mathbb{C}$) and forms a total set in L^2. It is necessary to construct the Jacobi type block matrices (9) and (13) with properties (10), (11), (12), and (14), (15), (16), respectively, in such a way that for the commuting selfadjoint operator A and the unitary operator U, there spectral measure is equal to the initial measure.

As in the classical case, it is necessary to apply the standard Schmidt orthogonalization procedure to the sequence of functions

$$r^t e^{ij\theta} \in L^2, \quad t\in\mathbb{N}_0, \quad j\in\mathbb{Z} \tag{20}$$

(instead of (5)). But the sequence (20) has two indices, therefore, it is necessary to choose a convenient global (linear) order for (20). The order is proposed in Figure 1 and (23).

After such an orthogonalization, we get a sequence of polynomials,

$$P_n(z) = (P_{n;0}(z), P_{n;1}(z), \ldots, P_{n;2n}(z)), \quad n\in\mathbb{N}_0,$$

and the matrices (9) with the inner structure (10), (11), (12), and (13) with the inner structure (14), (15), (16) are reconstructed by using formulas of type (7).

All necessary references, connected with the projection spectral theorem, will be given in the next Sections. Let us recall that the theory of block Jacobi matrices that are either Hermitian or selfadjoint operators acting on the spaces $l_2(\mathcal{H}) = \mathcal{H}\oplus\mathcal{H}\oplus\cdots$, where $\mathcal{H}$ is any Hilbert space, was investigated at first in [21] in the case $\dim\mathcal{H} < \infty$ and in [2, 3] in the case $\dim\mathcal{H} \le \infty$. For families of commuting selfadjoint operators acting on a symmetric Fock space, see [6]. Note that the Fock space has the form (8) with $\mathcal{H}_n$ that are, for $n > 0$, n-particle infinite-dimensional Hilbert space.

Remark 1. It is very interesting to develop a spectral theory of block Jacobi type matrices (9) and (13) on the space $\mathbf{l}_2$ (8) in the case of an unbounded selfadjoint operator A (commuting with some unitary operator U). What are the conditions on elements of the matrix A which would guarantee that the operator A is essentially selfadjoint? In what terms would it possible to describe all selfadjoint extensions of A on $\mathbf{l}_2$ commuting with some unitary operator U?

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and let A be a selfadjoint operator defined on $\mathfrak{D}(A)$ in $\mathcal{H}$, U a unitary operator commuting in the strong resolvent sense with A. Consider a rigging of $\mathcal{H}$,

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset \mathcal{D}, \tag{21}$$

such that $\mathcal{H}_+$ is a Hilbert space topologically and quasinuclear embedded into $\mathcal{H}$ (topologically means densely and continuously; quasinuclear means that the inclusion operator is of Hilbert-Schmidt type); $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$ with respect to the space $\mathcal{H}$; $\mathcal{D}$ is a linear, topological space, topologically embedded into $\mathcal{H}_+$.

The operators A and U are called standardly connected with the chain (21) if $\mathcal{D} \subset \mathfrak{D}(A)$ and the restrictions $A \upharpoonright \mathcal{D}$, $U \upharpoonright \mathcal{D}$ act from $\mathcal{D}$ into $\mathcal{H}_+$ continuously.

Let us recall that a vector $\omega_0 \in \mathcal{D}$ is called a strong cyclic vector of the operators A and U if for $p \in \mathbb{N}_0, q \in \mathbb{Z}$ we have $\omega_0 \in \mathfrak{D}(A^p)$, $U^q A^p \omega_0 \in \mathcal{D}$, and the set of all these vectors with ω_0 is total in the space $\mathcal{H}_+$ (and, hence, also in $\mathcal{H}$).

Assuming that a strong cyclic vector exists we formulate a short version of the projection spectral theorem (see. [4], Ch. 3, Theorem 2.7, or [3], Ch. 5, [5], Ch. 15).

Theorem 1. *For a selfadjoint operator A commuting in the strong resolvent sense with a unitary operator U, with a strong cyclic vector in a separable Hilbert space $\mathcal{H}$, there exists a nonnegative finite Borel measure $d\rho(z)$ such that for ρ-almost every $z \in \mathbb{C}$ there exists a generalized joint eigenvector $\xi_z \in \mathcal{H}_-$, i.e.,*

$$(\xi_z, Af)_{\mathcal{H}} = r(\xi_z, f)_{\mathcal{H}}, \quad (\xi_z, Uf)_{\mathcal{H}} = e^{-i\theta}(\xi_z, f)_{\mathcal{H}}, \quad f \in \mathcal{D}, \quad \xi_z \neq 0,$$

where $z = re^{i\theta}$ is a two-parameter (r and θ) eigenvalue.

The corresponding Fourier transform F given by

$$\mathcal{H} \supset \mathcal{H}_+ \ni f \mapsto (Ff)(z) = \hat{f}(z) = (f, \xi_z)_{\mathcal{H}} \in L_2(\mathbb{C}, d\rho(z))$$

is a unitary operator (after taking the closure) acting from $\mathcal{H}$ into $L^2(\mathbb{C}, d\rho(z))$. Images of the operators A and U under F are the operators of multiplication by r and $e^{i\theta}$ in $L^2(\mathbb{C}, d\rho(z))$.

We use the solution of the complex moment problem in the exponential form from [8]:

Theorem 2. *A sequence of complex numbers $\{c_{t,j}\}$, $t \in \mathbb{N}_0, j \in \mathbb{Z}$ has representation (19) if and only if it is positive definite, i.e.,*

$$\sum_{\substack{t,q\in\mathbb{N}_0\\ j,k\in\mathbb{Z}}} f_{t,j}\bar{f}_{q,k}c_{t+q,j-k} \geq 0 \quad \text{and} \quad \sum_{\substack{t,q\in\mathbb{N}_0\\ j,k\in\mathbb{Z}}} f_{t,j}\bar{f}_{q,k}c_{t+q+1,j-k} \geq 0$$

for all finite sequences of complex numbers $(f_{t,j})$, $t \in \mathbb{N}_0, j \in \mathbb{Z}$, $(f_{t,j} \in \mathbb{C})$.

The representation (19) is unique if

$$\sum_{p=1}^{\infty} \frac{1}{\sqrt[2p]{|c_{2p,0}|}} = \infty.$$

3. Orthogonalization procedure and construction of a corresponding three-diagonal block Jacobi type matrices

Let $d\rho(z)$ be a Borel measure with a compact support on $\mathbb{C}$ and $L^2(\mathbb{C}, d\rho(z))$ the space of square integrable complex-valued functions defined on $\mathbb{C}$. We suppose that the functions $\mathbb{C} \ni z \longmapsto r^m e^{in\theta}$, $m \in \mathbb{N}_0, n \in \mathbb{Z}$, are linearly independent and form a total set in L^2, here $z = re^{i\theta}$, $r \geq 0$, $\theta \in [0, 2\pi)$.

In order to find analogs of the Jacobi matrix J like (9), there is a need to choose an order of the orthogonalization in L^2 applied to the following family of functions:

$$\{r^m e^{in\theta}\}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{Z}. \tag{22}$$

We use the following linear order for the orthogonalization according to Schmidt procedure:

$$\begin{tikzcd}
\cdots & r^0e^{-i2\theta} & r^0e^{-i\theta} & r^0e^{i0\theta} \arrow[r] & r^0e^{i\theta} \arrow[dl] & r^0e^{i2\theta} \arrow[dl] & \cdots \\
\cdots & r^1e^{-i2\theta} & r^1e^{-i\theta} \arrow[ul] & r^1e^{i0\theta} \arrow[ul] & r^1e^{i\theta} \arrow[dl] & r^1e^{i2\theta} & \cdots \\
\cdots & * & r^2e^{-i\theta} & r^2e^{i0\theta} \arrow[ul] & r^2e^{i\theta} & * & \cdots \\
\cdots & * & * & r^3e^{i0\theta} & * & * & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

FIGURE 1. The order of orthogonalization

According to Figure 1, we get

$$\begin{gathered} 1; \quad e^{i\theta}, r^1, e^{-i\theta}; \quad e^{i2\theta}, r^1e^{i\theta}, r^2, r^1e^{-i\theta}, e^{-i2\theta}; \quad \dots ; \\ e^{in\theta}, r^1e^{i(n-1)\theta}, \dots, r^{n-1}e^{i\theta}, r^n, r^{n-1}e^{-i\theta} \dots, r^1e^{-i(n-1)\theta}, e^{-in\theta}; \quad \dots, \end{gathered} \tag{23}$$

where we take into account $r^0 = e^{i0\theta} = 1$. That is the set of some "lines" with the beginning in the points $e^{in\theta}$ with the end in the points $e^{-in\theta}$ and "with the corners" in the points r^n.

Applying the Schmidt orthogonalization procedure to the sequence of functions (23) (see for example [5] Ch. 7), we obtain an orthonormal system of polynomials (each polynomial consists of $\{r^m e^{in\theta}\}$, $m \in \mathbb{N}_0$, $n \in \mathbb{Z}$) which we denote in the following way:

$$\begin{array}{llllll}
P_{0;0}(z); & P_{1;0}(z), & P_{2;0}(z), & \dots; & P_{n;0}(z), & \dots \\
 & P_{1;1}(z), & P_{2;1}(z), & & P_{n;1}(z), & \\
 & P_{1;2}(z); & P_{2;2}(z), & & P_{n;2}(z), & \\
 & & P_{2;3}(z), & & P_{n;3}(z), & \\
 & & P_{2;4}(z); & & P_{n;4}(z), & \\
 & & & & \dots & \\
 & & & & P_{n;2n}(z); &
\end{array} \tag{24}$$

where each polynomial has the following form for $n \in \mathbb{N}_0$:

$$P_{n;\alpha}(z) = k_{n;\alpha} r^{n-|\alpha-n|} e^{i(n-\alpha)\theta} + R_{n;\alpha}, \quad \alpha = 0, 1, \dots, 2n, \quad k_{n;\alpha} > 0, \tag{25}$$

here $R_{n;\alpha}$ denotes the preceding part of the corresponding polynomial; $P_{0;0}(z) = 1$. In such a way $P_{n;\alpha}$ is some linear combination of

$$\{1; \; e^{i\theta}, r^1, e^{-i\theta}; \; \dots; \; e^{in\theta}, r^1e^{i(n-1)\theta}, \dots, r^{n-|\alpha-n|}e^{i(n-\alpha)\theta}\}. \tag{26}$$

Since the family (22) is total in the space L^2, the sequence (24) forms an orthonormal basis in this space.

Let us denote by $\mathcal{P}_{n;\alpha}$ the subspace spanned by (26). Hence $\forall n \in \mathbb{N}$

$$\begin{aligned} \mathcal{P}_{0;0} \subset& \mathcal{P}_{1;0} \subset \mathcal{P}_{1;1} \subset \mathcal{P}_{1;2} \subset \mathcal{P}_{2;0} \subset \mathcal{P}_{2;1} \subset \mathcal{P}_{2;2} \subset \mathcal{P}_{2;3} \subset \mathcal{P}_{2;4} \subset \cdots \\ &\subset \mathcal{P}_{n;0} \subset \mathcal{P}_{n;1} \subset \cdots \subset \mathcal{P}_{n;2n} \subset \cdots, \\ \mathcal{P}_{n;\alpha} =& \{P_{0;0}(z)\} \oplus \{P_{1;0}(z)\} \oplus \{P_{1;1}(z)\} \oplus \{P_{1;2}(z)\} \oplus \\ &\oplus \{P_{2;0}(z)\} \oplus \{P_{2;1}(z)\} \oplus \{P_{2;2}(z)\} \oplus \{P_{2;3}(z)\} \oplus \{P_{2;4}(z)\} \oplus \cdots \\ &\oplus \{P_{n;0}(z)\} \oplus \{P_{n;1}(z)\} \oplus \cdots \oplus \{P_{n;\alpha}(z)\}, \end{aligned} \tag{27}$$

where $\{P_{n;\alpha}(z)\}$, $n \in \mathbb{N}$, $\alpha = 0, 1, \ldots, 2n$, denotes the one-dimensional space spanned by $P_{n;\alpha}(z)$; $\mathcal{P}_{0;0} = \mathbb{C}$.

For the next investigation we need the Hilbert space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{2n+1}, \quad n \in \mathbb{N}_0, \tag{28}$$

instead of the usual space l_2.

Each vector $f \in \mathbf{l}_2$ has the form $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$, and consequently

$$\|f\|^2_{\mathbf{l}_2} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} < \infty, \quad (f, g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n}, \quad \forall f, g \in \mathbf{l}_2.$$

For $n \in \mathbb{N}_0$, the coordinates of a vector $f_n \in \mathcal{H}_n$ in some orthonormal basis $\{e_{n;0}, e_{n;1}, e_{n;2}, \ldots, e_{n;2n}\}$ in the space $\mathbb{C}^{n+1}$ is denoted by $(f_{n;0}, f_{n;1}, f_{n;2}, \ldots, f_{n;2n})$ and, hence, $f_n = (f_{n;0}, f_{n;1}, f_{n;2}, \ldots, f_{n;2n})$. It is clear, that the space $\mathbf{l}_2$ is isometric to $l_2 \times l_2$.

Using the orthonormal system (24) one can define a mapping of $\mathbf{l}_2$ into L^2. We put $\forall n \in \mathbb{N}_0$ and $\forall z = re^{i\theta} \in \mathbb{C}$, $P_n(z) = (P_{n;0}, P_{n;1}(z), P_{n;2}(z), \ldots, P_{n;2n}) \in \mathcal{H}_n$, then

$$\mathbf{l_2} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{n=0}^{\infty} (f_n, P_n(z))_{\mathcal{H}_n} \in L^2. \tag{29}$$

Since for $n \in \mathbb{N}_0$ we get

$$(f_n, P_n(z))_{\mathcal{H}_n} = f_{n;0}\overline{P_{n;0}(z)} + f_{n;1}\overline{P_{n;1}(z)} + f_{n;2}\overline{P_{n;2}(z)} + \cdots + f_{n;2n}\overline{P_{n;2n}(z)}$$

and

$$\|f\|^2_{\mathbf{l}_2} = \|(f_{0;0}, f_{1;0}, f_{1;1}, f_{1;2}, \ldots, f_{n;0}, f_{n;1}, \ldots, f_{n;2n}, \ldots)\|^2_{l_2},$$

we see that (29) is a mapping of the space $l_2 \times l_2$ into L^2 and the use of the orthonormal system (24) shows that this mapping is isometric. The image of $\mathbf{l}_2$ under the mapping (29) coincides with the space L^2, because under our assumption the system (24) is an orthonormal basis in L^2. Therefore the mapping (29) is a unitary transformation (denoted by I) that acts from $\mathbf{l}_2$ onto L^2.

Let A be a bounded linear operator defined on the space $\mathbf{l}_2$. It is possible to construct an operator matrix $(a_{j,k})_{j,k=0}^{\infty}$, where for each $j, k \in \mathbb{N}_0$ the element $a_{j,k}$ is an operator from $\mathcal{H}_k$ into $\mathcal{H}_j$, so that $\forall f, g \in \mathbf{l}_2$ we have

$$(Af)_j = \sum_{k=0}^{\infty} a_{j,k} f_k, \ j \in \mathbb{N}_0, \quad (Af, g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} (a_{j,k} f_k, g_j)_{\mathcal{H}_j}. \tag{30}$$

For the proof of (30) we only need to write the usual matrix for the operator A in the space $l_2 \times l_2$ using the basis

$$(e_{0;0};\ e_{1;0}, e_{1;1}, e_{1;2};\ \ldots;\ e_{n;0}, e_{n;1}, \ldots, e_{n;2n};\ \ldots), \quad e_{0;0} = 1. \tag{31}$$

Then $a_{j,k}$ for each $j, k \in \mathbb{N}_0$ is an operator $\mathcal{H}_k \longrightarrow \mathcal{H}_j$ that has the matrix representation

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{\mathbf{l}_2}, \tag{32}$$

where $\alpha = 0, 1, \ldots, 2j$, $\beta = 0, 1, \ldots, 2k$. We will write $a_{j,k} = (a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$, including the cases

$$a_{0,0} = (a_{0,0;\alpha,\beta})_{\alpha,\beta=0}^{0,0} = a_{0,0;0,0}, \quad a_{0,1} = (a_{0,1;\alpha,\beta})_{\alpha,\beta=0}^{0,2}, \quad a_{1,0} = (a_{1,0;\alpha,\beta})_{\alpha,\beta=0}^{2,0}.$$

Note that the same representation (30) is also valid for a general operator A on the space $\mathbf{l}_2$ defined on $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$, where $\mathbf{l}_{\text{fin}}$ denotes the set of finite vectors from $\mathbf{l}_2$. In this case the first formula from (30) takes place for $f \in \mathbf{l}_{\text{fin}}$; the second formula is valid for $f \in \mathbf{l}_{\text{fin}}$, $g \in \mathbf{l}_2$.

Let us consider the image $\hat{A} = IAI^{-1} : L^2 \longrightarrow L^2$ of the above bounded operator $A : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$ with respect to the mapping (29). Its matrix in the basis (24),

$$(P_{0;0}(z);\ P_{1;0}(z), P_{1;1}(z), P_{1;2}(z);\ \ldots;\ P_{n;0}(z), P_{n;1}(z), \ldots, P_{n;2n}(z); \ldots),$$

is equal to the usual matrix of the operator A regarded as an operator $l_2 \times l_2 \longrightarrow l_2 \times l_2$ in the corresponding basis (31). Using (32) and the above mentioned procedure we get the operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $A : l_2 \times l_2 \longrightarrow l_2 \times l_2$. *By the definition, this matrix is also an operator matrix of* $\hat{A} : L^2 \longrightarrow L^2$.

It is clear that $\hat{A}$ can be an arbitrary linear bounded operator in L^2.

Lemma 1. *For the polynomials* $P_{n;\alpha}(z)$, *and the subspaces* $\mathcal{P}_{m,\beta}$, $n, m \in \mathbb{N}_0$, $\alpha = 0, 1, \ldots, 2n$, $\beta = 0, 1, \ldots, 2m$, *the following relations hold:*

$$rP_{n;\alpha}(z) \in \mathcal{P}_{n+1;\alpha+1}, \qquad \alpha = 0, 1, \ldots, 2n; \tag{33}$$

$$e^{i\theta}P_{n;\alpha}(z) \in \mathcal{P}_{n+1;\alpha}, \qquad \alpha = 0, 1, \ldots, n; \tag{34}$$

$$e^{i\theta}P_{n;\alpha}(z) \in \mathcal{P}_{n+1;n}, \qquad \alpha = n+1, n+2, \ldots, 2n; \tag{35}$$

$$e^{-i\theta}P_{n;\alpha}(z) \in \mathcal{P}_{n+1;\alpha+2}, \qquad \alpha = n, n+1, \ldots, 2n; \tag{36}$$

$$e^{-i\theta}P_{n;\alpha}(z) \in \mathcal{P}_{n;2n}, \qquad \alpha = 0, 1, \ldots, n-1, \quad n \in \mathbb{N}_0. \tag{37}$$

Proof. According to (24) the polynomial $P_{n;\alpha}(z)$, $n \in \mathbb{N}_0$, is equal to some linear combination of (26) i.e.

$$\{1;\ e^{i\theta}, r^1, e^{-i\theta};\ \ldots;\ e^{in\theta}, r^1e^{i(n-1)\theta}, \ldots, r^{n-|\alpha-n|}e^{i(n-\alpha)\theta}\}.$$

Hence, multiplying, by r, each element in (26) we obtain the set

$$\{r^1;\ r^1e^{i\theta}, r^2, r^1e^{-i\theta};\ \ldots;\ r^1e^{in\theta}, r^2e^{i(n-1)\theta}, \ldots, r^{n-|\alpha-n|+1}e^{i(n-\alpha)\theta}\}$$

and linear combinations of such elements belongs to $\mathcal{P}_{n+1;\alpha+1}$ for $\alpha = 0, 1, \ldots, 2n$ since for the last (leading according to the order (23)) element we have

$$r^{n-|\alpha-n|+1}e^{i(n-\alpha)\theta} = r^{(n+1)-|(\alpha+1)-(n+1)|}e^{i((n+1)-(\alpha+1))\theta}.$$

Hence we proved (33).

Multiplying each element in (26) by $e^{i\theta}$ we obtain the set

$$\{e^{i\theta};\ r^0e^{i2\theta}, r^1e^{i\theta}, 1;\ \ldots;\ e^{i(n+1)\theta}, r^1e^{i(n)\theta}, \ldots, r^{n-|\alpha-n|}e^{i(n-\alpha+1)\theta}\} \tag{38}$$

linear combinations of which belong to $\mathcal{P}_{n+1;\alpha}$, if $\alpha = 0, 1, \ldots, n$, since for the last (leading according to the order (23)) element we have

$$r^{n-|\alpha-n|}e^{i(n-\alpha+1)\theta} = r^{(n+1)-|n-\alpha|-1}e^{i((n+1)-\alpha)\theta} = r^{(n+1)-|\alpha-(n+1)|}e^{i((n+1)-\alpha)\theta}.$$

The last equality is true since $\alpha \le n$. Hence we proved (34). And the linear combinations of elements (38) belong in general to $\mathcal{P}_{n+1;n}$, if $\alpha = n+1, \ldots, 2n$, since for the last (leading according to the order (23)) element we have

$$r^{n-|\alpha-n|}e^{i(n-\alpha+1)\theta} = r^{(n+1)-|\alpha-(n+1)|}e^{i((n+1)-\alpha)\theta}.$$

Since $P_{n;\alpha}(z)$ for $\alpha = n+1, n+2, \ldots, 2n$ contains the element r^n, we see that $r^ne^{i\theta}$ is contained in $e^{i\theta}P_{n;\alpha}(z)$ and $r^ne^{i\theta} \in \mathcal{P}_{n+1;n}$. Hence we proved (35).

Analogously, multiplying each element in (26) by $e^{-i\theta}$ we obtain the set of elements

$$\{e^{-i\theta};\ 1, r^1e^{-i\theta}, e^{-i2\theta};\ \dots;\ e^{i(n-1)\theta}, r^1e^{i(n-2)\theta}, \dots, r^{n-|\alpha-n|}e^{i(n-\alpha-1)\theta}\}. \tag{39}$$

The linear combinations of elements from (39) belong to the subspace $\mathcal{P}_{n+1;\alpha+2}$ if $\alpha = n, n+1, \dots, 2n$, since for the (leading according to the order (23)) element we have

$$\begin{aligned} r^{n-|\alpha-n|}e^{i(n-\alpha-1)\theta} =& r^{(n+1)-|\alpha-n|+1}e^{i(n-\alpha-1)\theta} \\ =& r^{(n+1)-|(\alpha+1)-(n+1)|+1}e^{i((n+1)-(\alpha+2))\theta} \\ =& r^{(n+1)-|(\alpha+2)-(n+1)|}e^{i((n+1)-(\alpha+2))\theta} \\ =& r^{(n+1)-|(n+1)-(\alpha+2)|}e^{i((n+1)-(\alpha+2))\theta}. \end{aligned}$$

Hence we proved (36).

Linear combinations of elements from (39) belong in any case to $\mathcal{P}_{n;2n}$ if $\alpha = 0, 1, \dots, n-1$, since $e^{-i(n-1)\theta}$ is contained in $P_{n;\alpha}(z)$, $\forall\alpha = 0, 1, \dots, n-1$. Hence $e^{-i\theta}P_{n;\alpha}(z)$ for $\alpha = 0, 1, \dots, n-1$ contains $e^{-in\theta}$. Hence we proved (37). $\square$

Lemma 2. *Let $\hat{A}$ be the selfadjoint operator of multiplication by r on the space L^2*

$$L^2 \ni \varphi(z) \longmapsto (\hat{A}\varphi)(z) = r\varphi(z) \in L^2.$$

The operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $\hat{A}$ (i.e. of $A = I^{-1}\hat{A}I$) has a three-diagonal structure: $a_{j,k} = 0$ for $|j-k| > 1$.

Proof. Using (32) for $e_{n;\gamma} = I^{-1}P_{n;\gamma}(z)$, $n \in \mathbb{N}_0$; $\gamma = 0, 1, \dots, 2n$, we have $\forall j, k \in \mathbb{N}_0$

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{l_2} = \int_{\mathbb{C}} rP_{k;\beta}(z)\overline{P_{j;\alpha}(z)}\, d\rho(z), \tag{40}$$

where $\alpha = 0, 1, \dots, 2j$, $\beta = 0, 1, \dots, 2k$. From (33) we have $rP_{k;\alpha}(z) \in \mathcal{P}_{k+1;\alpha+1}$ for if $\alpha = 0, 1, \dots, 2j$. According to (27) the integral in (40) is equal to zero for $j > k+1$.

On another hand the integral in (40) has a form

$$a_{j,k;\alpha,\beta} = \overline{\int_{\mathbb{C}} \bar{r}P_{j;\alpha}(z)\overline{P_{k;\beta}(z)}\, d\rho(z)}. \tag{41}$$

From (33) we have also that $\bar{r}P_{j;\alpha}(z) = rP_{j;\alpha}(z) \in \mathcal{P}_{j+1;\alpha+1}$. According to (27) the last integral is equal to zero for $k > j+1$ and for each $\alpha = 0, 1, \dots, 2j$, $\beta = 0, 1, \dots, 2k$.

As a result the integral in (41), i.e. coefficients $a_{j,k;\alpha,\beta}$, $j, k \in \mathbb{N}_0$, are equal to zero for $|j-k| > 1$; $\alpha = 0, 1, \dots, 2j$, $\beta = 0, 1, \dots, 2k$. In the previous considerations it is necessary to take into account that $e_{0;0} = I^{-1}P_{0;0}(z) = 1$. $\square$

In such a way the matrix $(a_{j,k})_{j,k=0}^{\infty}$ of the operator $\hat{A}$ has the three-diagonal block structure

$$\begin{bmatrix} a_{0,0} & a_{0,1} & 0 & 0 & 0 \dots \\ a_{1,0} & a_{1,1} & a_{1,2} & 0 & 0 \dots \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \end{bmatrix}. \tag{42}$$

A more careful analysis of expressions (40) allows to know about the zero and non zero elements of matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{2j,2k}$ in each case for $|j-k| \leq 1$. We use actively also the permutation properties of matrix indexes j, k and α, β.

Let us denote by $((a^*)_{j,k})_{j,k=0}^{\infty}$ the operator matrix of the operator $(\hat{A})^*$ which is adjoint to $\hat{A}$. Remark that $(\hat{A})^* = \hat{A}$ is also the operator of multiplication by $r = \bar{r}$. Taking into account the expression (40) for $j, k \in \mathbb{N}_0$ we have

$$(a^*)_{j,k;\alpha,\beta} = \int_{\mathbb{C}} \bar{r}P_{k;\beta}(z)\overline{P_{j;\alpha}(z)}\, d\rho(z) = \overline{\int_{\mathbb{C}} rP_{j;\alpha}(z)\overline{P_{k;\beta}(z)}\, d\rho(z)} = \bar{a}_{k,j;\beta,\alpha}, \tag{43}$$

where $\alpha = 0, 1, \ldots, 2j$, $\beta = 0, 1, \ldots, 2k$. Since $\bar{r} = r$, then the matrix (42) is Hermitian ($a_{j,k;\alpha,\beta} = \bar{a}_{k,j;\beta,\alpha}$).

Lemma 3. *Let $(a_{j,k})_{j,k=0}^{\infty}$ be an operator matrix of the operator of multiplication by r in L^2, where $a_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$; $a_{j,k} = (a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{2j,2k}$ are matrices of operators $a_{j,k}$ in the corresponding orthonormal basis. Then*

$$\begin{aligned} &a_{j,j+1;\alpha,\beta} = 0, \quad \alpha = 0, 1, \ldots 2j, \quad \beta = \alpha + 2, \alpha + 3, \ldots, 2j + 2; \\ &a_{j+1,j;\alpha,\beta} = 0, \quad \beta = 0, 1, \ldots 2j, \quad \alpha = \beta + 2, \beta + 3, \ldots, 2j + 2, \quad j \in \mathbb{N}_0. \end{aligned} \tag{44}$$

If we choose inside of each direction "in the line with a corner"

$$\{e^{in\theta}, r^1 e^{i(n-1)\theta}, r^2 e^{i(n-2)\theta}, \ldots, r^n, \ldots, r^1 e^{-i(n-1)\theta}, e^{-in\theta}\},$$

(see. Figure 1 and comments after (23)) another order (preserving the order of lines) then Lemma 3 is true and it will be possible to describe the zeros of matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{2j,2k}$. Such matrices $(a_{j,k})_{j,k=0}^{\infty}$ also have three-diagonal block structure and has the zeros in the same places.

Proof. According to (40) and (33) for $\forall \alpha = 0, 1, \ldots, 2j$ and $\forall \beta = \alpha + 2, \alpha + 3, \ldots, 2j + 2$ for $j \in \mathbb{N}_0$ we have

$$a_{j,j+1;\alpha,\beta} = \int_{\mathbb{C}} r P_{j+1,\beta}(z) \overline{P_{j;\alpha}(z)}\, d\rho(z) = \overline{\int_{\mathbb{C}} r P_{j,\alpha}(z) \overline{P_{j+1;\beta}(z)}\, d\rho(z)},$$

where $r P_{j;\alpha}(z) \in \mathcal{P}_{j+1;\alpha+1}$. But according to (27) $P_{j+1;\beta}(z)$ is orthogonal to $\mathcal{P}_{j+1;\alpha+1}$ for $\beta > \alpha + 1$ and, hence, the last integral is equal to zero. This gives the first equalities in (44).

Analogously from (40) and (33) we have $\forall \beta = 0, 1, \ldots, 2j$ and $\forall \alpha = \beta + 2, \beta + 3, \ldots, 2j + 2$

$$a_{j+1,j;\alpha,\beta} = \int_{\mathbb{C}} r P_{j,\beta}(z) \overline{P_{j+1;\alpha}(z)}\, d\rho(z), \quad j \in \mathbb{N}_0,$$

where $r P_{j;\beta}(z) \in \mathcal{P}_{j+1;\beta+1}$. But according to (27) $P_{j+1;\alpha}(z)$ is orthogonal to $\mathcal{P}_{j+1;\beta+1}$ if $\alpha > \beta + 1$ and, hence, the last integral is also equal to zero. This gives the second equalities in (44). ☐

The above shows that in (42) for $\forall j \in \mathbb{N}$ the over right corner of the every $((2j+1) \times (2j+3))$-matrix $a_{j,j+1}$, $j \in \mathbb{N}_0$ (starting from the third diagonal) and the under left corner of the every $((2j+3) \times (2j+1))$-matrix $a_{j+1,j}$ (starting from the third diagonal) consist of zero elements. Taking into account (42) we can conclude that the Hermitian matrix of the multiplication operator by r is multi-diagonal usual scalar matrix in the usual basis of the space $l_2 \times l_2$.

Lemma 4. *The following elements of the matrix $(a_{j,k})_{j,k=0}^{\infty}$ from Lemma 3 are positive:*

$$a_{j,j+1;\alpha,\alpha+1},\ a_{j+1,j;\alpha+1,\alpha}; \quad \alpha = 0, 1, \ldots 2j, \quad j \in \mathbb{N}_0. \tag{45}$$

Proof. We start with the study of $a_{0,1;0,1}$. Denote by $P'_{1;1}(z)$ non normalized vector $P_{1;1}(z)$ obtained due to the Schmidt orthogonalization procedure taking the vector r. According to (23) and (24) we have

$$P'_{1;1}(z) = r - (r, P_{1;0}(z))_{L^2} P_{1;0}(z) - (r, 1)_{L^2},$$

where $1 = P_{0;0}(z)$. Therefore using (40) we get

$$\begin{aligned} a_{0,1;0,1} &= \int_{\mathbb{C}} rP_{1;1}(z)\,d\rho(z) = \|P'_{1;1}(z)\|^{-1}_{L^2}\int_{\mathbb{C}} rP'_{1;1}(z)\,d\rho(z) \\ &= \|P'_{1;1}(z)\|^{-1}_{L^2}\int_{\mathbb{C}} r(r-(r,P_{1;0}(z))_{L^2}P_{1;0}(z)-(r,1)_{L^2})\,d\rho(z) \\ &= \|P'_{1;1}(z)\|^{-1}_{L^2}(\|r\|^2_{L^2}-|(r,P_{1;0}(z))_{L^2}|^2-|(r,1)_{L^2}|^2). \end{aligned}\tag{46}$$

Also using (47), we conclude that the last expression is positive and therefore $a_{0,1;0,1} > 0$, since (22) is the total set in L^2 of linear independent vectors. The element $a_{1,0;1,0}$ is also positive since the matrix is Hermitian.

Positiveness (46) follows from the Parseval equality for the decomposition of the function $r \in L^2$ with respect to the orthonormal basis (24) on the space L^2:

$$|(r,1)_{L^2}|^2 + |(r,P_{1;0}(z))_{L^2}|^2 + |(r,P_{1;1}(z))_{L^2}|^2 + \cdots = \|r\|^2_{L^2}.\tag{47}$$

Let us pass to the proof of the positiveness of $a_{j,j+1;\alpha,\alpha+1}$, where $\alpha = 0,1,\ldots,2j$, $j \in \mathbb{N}$. From (40) we have

$$a_{j,j+1;\alpha,\alpha} = \int_{\mathbb{C}} rP_{j+1;\alpha+1}(z)\overline{P_{j;\alpha}(z)}\,d\rho(z).\tag{48}$$

According to (25) and (27)

$$P_{j;\alpha}(z) = k_{j;\alpha}r^{j-|\alpha-j|}e^{i(j-\alpha)\theta} + R_{j;\alpha}(z),\tag{49}$$

where $R_{j;\alpha}(z)$ is some polynomial from $\mathcal{P}_{j;\alpha-1}$ if $\alpha > 0$ or from $\mathcal{P}_{j-1;2j-2}$ if $\alpha = 0$. Therefore $rR_{j;\alpha}(z)$ is some polynomial from $\mathcal{P}_{j+1;\alpha}$ (if $\alpha = 0$, then $rR_{j;\alpha}(z) \in \mathcal{P}_{j;2j-1}$ (see (33)). Multiplying (49) on r we conclude that

$$rP_{j;\alpha}(z) = k_{j;\alpha}r^{(j+1)-|\alpha-j|}e^{i(j-\alpha)\theta} + rR_{j;\alpha}(z),\tag{50}$$

where $rR_{j;\alpha}(z) \in \mathcal{P}_{j+1;\alpha}$ or to $\mathcal{P}_{j;2j-2}$.

On the other hand equality (49) for the case $P_{j+1;\alpha}(z)$ gives

$$\begin{aligned} P_{j+1;\alpha+1}(z) &= k_{j+1;\alpha+1}r^{(j+1)-|(\alpha+1)-(j+1)|}e^{i((j+1)-(\alpha+1))\theta} + R_{j+1;\alpha+1}(z) \\ &= k_{j+1;\alpha+1}r^{(j+1)-|(\alpha-j)|}e^{i(j-\alpha)\theta} + R_{j+1;\alpha+1}(z), \end{aligned}\tag{51}$$

where $R_{j+1;\alpha+1}(z) \in \mathcal{P}_{j+1;\alpha+1}$, since $\alpha+1 > 0$.

Finding

$$r^{(j+1)-|(\alpha-j)|}e^{i(j-\alpha)\theta} = r^{(j+1)-|((\alpha+1)-(j+1))|}e^{i((j+1)-(\alpha+1))\theta}$$

from (51) and substituting it into (50) gives

$$\begin{aligned} rP_{j;\alpha}(z) &= \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}}(P_{j+1;\alpha+1}(z) - R_{j+1;\alpha+1}(z)) + rR_{j;\alpha}(z) \\ &= \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}}P_{j+1;\alpha+1}(z) - \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}}R_{j+1;\alpha+1}(z) + rR_{j;\alpha}(z), \end{aligned}\tag{52}$$

where second two terms belong to $\mathcal{P}_{j+1;\alpha}$ and to $\mathcal{P}_{j+1;\alpha}$ or $\mathcal{P}_{j;2j-1}$ correspondingly and are in any case orthogonal to $P_{j+1;\alpha+1}(z)$.

Therefore substituting the expression (52) into (48) gives $a_{j+1,j;\alpha,\alpha} = \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}} > 0$. $\square$

In what follows we will use usual well known notations for the elements $a_{j,k}$ of the Jacobi matrix:

$$\begin{array}{lcrcl} a_n = a_{n+1,n} & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_{n+1}, \\ b_n = a_{n,n} & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_n, \\ c_n = a_{n,n+1} & : & \mathcal{H}_{n+1} & \longrightarrow & \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array}\tag{53}$$

Let U be a bounded linear operator defined on the space $\mathbf{l}_2$. It is possible to construct the operator matrix $(u_{j,k})_{j,k=0}^{\infty}$, where for each $j, k \in \mathbb{N}_0$ the element $u_{j,k}$ is an operator from $\mathcal{H}_k$ into $\mathcal{H}_j$, so that $\forall f, g \in \mathbf{l}_2$ we have like (30)

$$(Uf)_j = \sum_{k=0}^{\infty} u_{j,k} f_k, \quad j \in \mathbb{N}_0, \quad (Uf, g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} (u_{j,k} f_k, g_j)_{\mathcal{H}_j}. \tag{54}$$

To the proof of (54) we only need to write the usual matrix of the operator U on the space $l_2 \times l_2$ using the basis

$$(e_{0;0};\ e_{1;0}, e_{1;1}, e_{1;2};\ \ldots;\ e_{n;0}, e_{n;1}, \ldots, e_{n;2n};\ \ldots), \quad e_{0;0} = 1. \tag{55}$$

Then $u_{j,k}$ for each $j, k \in \mathbb{N}_0$ is the operator $\mathcal{H}_k \longrightarrow \mathcal{H}_j$ that has a matrix representation

$$u_{j,k;\alpha,\beta} = (Ue_{k;\beta}, e_{j;\alpha})_{\mathbf{l}_2}, \tag{56}$$

where $\alpha = 0, 1, \ldots, 2j$, $\beta = 0, 1, \ldots, 2k$. We will write: $u_{j,k} = (u_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{2j,2k}$ including cases

$$u_{0,0} = (u_{0,0;\alpha,\beta})_{\alpha,\beta=0}^{0,0} = u_{0,0;0,0}, \quad u_{0,1} = (u_{0,1;\alpha,\beta})_{\alpha,\beta=0}^{0,2}, \quad u_{1,0} = (u_{1,0;\alpha,\beta})_{\alpha,\beta=0}^{2,0}.$$

Note, that the same representation (54) is also valid for the general operator U on the space $\mathbf{l}_2$ defined on $\mathbf{l}_{\text{in}} \subset \mathbf{l}_2$, where $\mathbf{l}_{\text{fin}}$ denotes as usually the set of finite vectors from $\mathbf{l}_2$. In this case the first formula from (54) takes place for $f \in \mathbf{l}_{\text{fin}}$; in the second formula $f \in \mathbf{l}_{\text{fin}}$, $g \in \mathbf{l}_2$.

Let us consider the image $\hat{U} = IUI^{-1} : L^2 \longrightarrow L^2$ of the above bounded operator $U : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$ by the mapping (29). Its matrix in the basis (24),

$$(P_{0;0}(z);\ P_{1;0}(z), P_{1;1}(z), P_{1;2}(z);\ \ldots;\ P_{n;0}(z), P_{n;1}(z), \ldots, P_{n;2n}(z); \ldots),$$

is equal to the usual matrix of operator U understanding as the operator acting: $l_2 \times l_2 \longrightarrow l_2 \times l_2$ in the corresponding basis (55). Using (56) and the above mentioned procedure, we get the operator matrix $(u_{j,k})_{j,k=0}^{\infty}$ of $U : l_2 \times l_2 \longrightarrow l_2 \times l_2$. *By the definition this matrix is also the operator matrix of* $\hat{U} : L^2 \longrightarrow L^2$.

It is clear that $\hat{U}$ can be an arbitrary linear bounded operator in L^2.

Lemma 5. *Let* $\hat{U}$ *be the unitary operator of multiplication by* $e^{i\theta}$ *on the space* L^2

$$L^2 \ni \varphi(z) \longmapsto (\hat{U}\varphi)(z) = e^{i\theta}\varphi(z) \in L^2.$$

The operator matrix $(u_{j,k})_{j,k=0}^{\infty}$ *of* $\hat{U}$ *(i.e. of* $U = I^{-1}\hat{U}I$*) has a three-diagonal structure:* $u_{j,k} = 0$ *for* $|j-k| > 1$.

Proof. Using (56) for $e_{n;\gamma} = I^{-1}P_{n;\gamma}(z)$, $n \in \mathbb{N}_0$; $\gamma = 0, 1, \ldots, 2n$, we have $\forall j, k \in \mathbb{N}_0$

$$u_{j,k;\alpha,\beta} = (Ue_{k;\beta}, e_{j;\alpha})_{l_2} = \int_{\mathbb{C}} e^{i\theta} P_{k;\beta}(z) \overline{P_{j;\alpha}(z)}\, d\rho(z), \tag{57}$$

where $\alpha = 0, 1, \ldots, 2j$, $\beta = 0, 1, \ldots, 2k$. From (34), (35) we have $e^{i\theta}P_{k;\beta}(z) \in \mathcal{P}_{k+1;\beta}$ if $\beta = 0, 1, \ldots, k$ and $e^{i\theta}P_{k;\beta}(z) \in \mathcal{P}_{k+1;k}$ if $\beta = k+1, k+2, \ldots, 2k$. According to (27) the integral in (57) is equal to zero for $j > k+1$ in any case $\beta = 0, 1, \ldots, 2k$.

On another hand the integral in (57) has a form

$$u_{j,k;\alpha,\beta} = \overline{\int_{\mathbb{C}} e^{-i\theta} P_{j;\alpha}(z) \overline{P_{k;\beta}(z)}\, d\rho(z)}. \tag{58}$$

From (36), (37) we have now that $e^{-i\theta}P_{j;\alpha}(z) \in \mathcal{P}_{j+1;\alpha+2}$, if $\alpha = j, j+1, \ldots, 2j$ and $e^{-i\theta}P_{j;\alpha}(z) \in \mathcal{P}_{j;2j}$, if $\alpha = 0, 1, \ldots, j-1$. According to (27) the last integral is equal to zero for $k > j+1$ and for each $\alpha = 0, 1, \ldots, 2j$.

As result the integral in (58), i.e. coefficients $u_{j,k;\alpha,\beta}$, $j,k\in\mathbb{N}_0$, are equal to zero for $|j-k|>1$; $\alpha=0,1,\dots,2j$, $\beta=0,1,\dots,2k$. In the previous considerations it is necessary to take into account that $e_{0;0}=I^{-1}P_{0;0}(z)=1$. $\square$

In such a way the matrix $(u_{j,k})_{j,k=0}^{\infty}$ of the operator $\hat{U}$ has a three-diagonal block structure

$$\begin{bmatrix} u_{0,0} & u_{0,1} & 0 & 0 & 0\dots \\ u_{1,0} & u_{1,1} & u_{1,2} & 0 & 0\dots \\ 0 & u_{2,1} & u_{2,2} & u_{2,3} & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \vdots\ \ddots \end{bmatrix}. \tag{59}$$

A more careful analysis of expressions (57) gives a possibility to know about the zero and non zero elements of matrices $(u_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{2j,2k}$ in each case for $|j-k|\le 1$. We can also describe the permutation properties of matrix indexes j,k, and α,β.

Let us denote by $((u^*)_{j,k})_{j,k=0}^{\infty}$ the operator matrix of the operator $(\hat{U})^*$ which is adjoint to $\hat{U}$. Remark that $(\hat{U})^*$ is the operator of multiplication by $e^{-i\theta}$. Taking into account the expression (57) for $j,k\in\mathbb{N}_0$ we have

$$(u^*)_{j,k;\alpha,\beta}=\int_{\mathbb{C}} e^{-i\theta}P_{k;\beta}(z)\overline{P_{j;\alpha}(z)}\,d\rho(z)=\overline{\int_{\mathbb{C}} e^{i\theta}P_{j;\alpha}(z)\overline{P_{k;\beta}(z)}\,d\rho(z)}=\bar{u}_{k,j:\beta,\alpha}, \tag{60}$$

where $\alpha=0,1,\dots,2j$ and $\beta=0,1,\dots,2k$.

Lemma 6. *Let $(u_{j,k})_{j,k=0}^{\infty}$ be an operator matrix of multiplication by $e^{i\theta}$ in L^2, where $u_{j,k}:\mathcal{H}_k\longrightarrow\mathcal{H}_j$; $u_{j,k}=(u_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{2j,2k}$ are matrices of operators $u_{j,k}$ in the corresponding orthonormal basis. Then $\forall j\in\mathbb{N}$ we have*

$$u_{j,j+1;\alpha,\beta}=0,\qquad \alpha=0,1,\dots,j-1,\qquad \beta=0,1,\dots,2j+2; \tag{61}$$

$$u_{j,j+1;\alpha,\beta}=0,\qquad \alpha=j,j+1,\dots,2j-1,\qquad \beta=\alpha+3,\alpha+4,\dots,2j+2; \tag{62}$$

$$u_{j+1,j;\alpha,\beta}=0,\qquad \beta=0,1,\dots,j,\qquad \alpha=\beta+1,\beta+2,\dots,2j+2; \tag{63}$$

$$u_{j+1,j;\alpha,\beta}=0,\qquad \beta=j+1,j+1\dots,2j,\qquad \alpha=j+1,j+2,\dots,2j+2 \tag{64}$$

and $u_{1,0;1,0}=u_{1,0;2,0}=0$.

If we choose inside of each direction

$$\{e^{in\theta},r^1e^{i(n-1)\theta},r^2e^{i(n-2)\theta},\dots,r^n\dots,r^1e^{-i(n-1)\theta},e^{-in\theta}\},$$

in the Figure 1 another order (preserving the order of "lines with the corner" (see the comments after (23)) then the Lemma 6 is not true but it will still be possible to describe the zeros of matrices $(u_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{2j,2k}$. Such matrices $(u_{j,k})_{j,k=0}^{\infty}$ also have three-diagonal block structure and the zeros although in another places.

Proof. According to (57) and (36), (37) for $j\in\mathbb{N}$ we have for $u_{j,j+1;\alpha,\beta}$ with $\alpha=0,1,\dots,2j$, $\beta=0,1,\dots,2k$

$$u_{j,j+1;\alpha,\beta}=\int_{\mathbb{C}} e^{i\theta}P_{j+1,\beta}(z)\overline{P_{j;\alpha}(z)}\,d\rho(z)=\overline{\int_{\mathbb{C}} e^{-i\theta}P_{j,\alpha}(z)\overline{P_{j+1;\beta}(z)}\,d\rho(z)},$$

where $e^{-i\theta}P_{j;\alpha}(z)\in\mathcal{P}_{j+1;\alpha+2}$ for $\alpha=j,j+1,\dots,2j$. But according to (27) $\mathcal{P}_{j+1;\alpha+2}$ is orthogonal to $\mathcal{P}_{j+1;\beta}$ for $\beta>\alpha+2$ and, hence, the last integral is equal to zero. This gives the equalities in (62). And $e^{-i\theta}P_{j;\alpha}(z)\in\mathcal{P}_{j;2j}$ for $\alpha=0,1,\dots,j-1$. But according to (27) $\mathcal{P}_{j;2j}$ is orthogonal to $\mathcal{P}_{j+1;\beta}$ for $\beta=0,1,\dots,2j+2$ and, hence, the last integral is also equal to zero. This gives the equalities in (61).

Analogously from (57) and (34), (35) for $j \in \mathbb{N}_0$ we have $\alpha = 0, 1, \ldots, 2j+2$, $\beta = 0, 1, \ldots, 2j$

$$u_{j+1,j;\alpha,\beta} = \int_{\mathbb{C}} e^{i\theta} P_{j,\beta}(z) \overline{P_{j+1;\alpha}(z)}\, d\rho(z),$$

where $e^{i\theta} P_{j;\beta}(z) \in \mathcal{P}_{j+1;\beta}$ for $\beta = 0, 1, \ldots, j$. But according to (27) $\mathcal{P}_{j+1;\beta}$ is orthogonal to $\mathcal{P}_{j+1;\alpha}$ for $\beta < \alpha$ and, hence, the last integral is equal to zero for $\alpha = \beta+1, \beta+2, \ldots, 2j+2$. This gives the equalities in (63). And $e^{i\theta} P_{j;\alpha}(z) \in \mathcal{P}_{j+1;j}$ for $\beta = j+1, j+2, \ldots, 2j$. But according to (27) $\mathcal{P}_{j+1;j}$ is orthogonal to $\mathcal{P}_{j+1;\alpha}$ for $\alpha > j$ and, hence, the last integral is also equal to zero for $\alpha = j+1, j+2, \ldots, 2j+2$. This gives the equalities in (64) and $u_{1,0;1,0} = u_{1,0;2,0} = 0$.

Note that about $u_{0,1}$ we have no additional information. □

So, after these investigations we conclude that in (59) for $\forall j \in \mathbb{N}$ every under left corner of the matrices $u_{j+1,j}$ (starting from the second diagonal beginning from the left over corner) and $j+2$ last lines and every over right corner of the matrices $u_{j,j+1}$ (starting from the second diagonal beginning from the right under corner) and n first lines consist of zero elements. Taking into account (59) we can conclude that the unitary matrix of the multiplication operator by $e^{i\theta}$ is multi-diagonal usual scalar matrix i.e. in the usual basis of the space $l_2 \times l_2$.

Lemma 7. *The following elements of the matrix $(u_{j,k})_{j,k=0}^{\infty}$ from Lemma 6 are positive:*

$$\begin{aligned} &u_{0,1;0,2}, u_{1,0;0,0}; \\ &u_{j,j+1;\alpha,\alpha+2}, \qquad \alpha = j+1, j+2, \ldots, 2j; \\ &u_{j+1,j;\alpha,\alpha}, \qquad \alpha = 0, 1, \ldots j; \quad j \in \mathbb{N}. \end{aligned} \tag{65}$$

Proof. We start with a study of $u_{1,0;0,0}$. Using (57) and denoting by $P'_{1;0}(z) = e^{i\theta} - (e^{i\theta}, 1)_{L^2}$ $(1 = P_{0;0}(z))$ the non normalized vector $P_{1;0}(z)$ obtained due to the Schmidt orthogonalization procedure taking the vector $e^{i\theta}$, we get

$$\begin{aligned} u_{1,0;0,0} = \int_{\mathbb{C}} e^{i\theta} \overline{P_{1;0}(z)}\, d\rho(z) &= \|P'_{1;0}(z)\|_{L^2}^{-1} \int_{\mathbb{C}} e^{i\theta} \overline{(e^{i\theta} - (e^{i\theta}, 1)_{L^2})}\, d\rho(z) \\ &= \|P'_{1;0}(z)\|_{L^2}^{-1} (\|e^{i\theta}\|_{L^2}^2 - |(e^{-i\theta}, 1)_{L^2}|^2). \end{aligned} \tag{66}$$

The last difference is positive (see below, (68)), therefore $a_{1,0;0,0} > 0$.

Consider $a_{0,1;0,2}$. Denote as earlier by $P'_{1;2}(z)$ non normalized vector $P_{1;2}(z)$ obtained due to the Schmidt orthogonalization procedure taking the vector $e^{-i\theta}$. According to (23) and (24) we have

$$P'_{1;2}(z) = e^{-i\theta} - (e^{-i\theta}, P_{1;1}(z))_{L^2} P_{1;1}(z) - (e^{-i\theta}, P_{1;0}(z))_{L^2} P_{1;0}(z) - (e^{-i\theta}, 1)_{L^2}.$$

Therefore using (57) we get

$$\begin{aligned} u_{0,1;0,2} =& \int_{\mathbb{C}} e^{-i\theta} P_{1;2}(z)\, d\rho(z) = \|P'_{1;2}(z)\|_{L^2}^{-1} \int_{\mathbb{C}} e^{i\theta} P'_{1;2}(z)\, d\rho(z) \\ =& \|P'_{1;2}(z)\|_{L^2}^{-1} \int_{\mathbb{C}} e^{i\theta} (e^{-i\theta} - (e^{-i\theta}, P_{1;1}(z))_{L^2} P_{1;1}(z) \\ & - (e^{-i\theta}, P_{1;0}(z))_{L^2} P_{1;0}(z) - (e^{-i\theta}, 1)_{L^2})\, d\rho(z) \\ =& \|P'_{1;2}(z)\|_{L^2}^{-1} (\|e^{-i\theta}\|_{L^2}^2 - |(e^{-i\theta}, P_{1;1}(z))_{L^2}|^2 \\ & - |(e^{-i\theta}, P_{1;0}(z))_{L^2}|^2 - |(e^{-i\theta}, 1)_{L^2}|^2). \end{aligned} \tag{67}$$

Also using (68), we conclude that the last expression is positive and therefore $u_{0,1;0,2} > 0$.

Positiveness in (66) and (67) follows from the Parseval equality applied to the decomposition of the function $e^{-i\theta} \in L^2$ with respect to the orthonormal basis (24) on the space L^2:

$$|(e^{-i\theta}, 1)_{L^2}|^2 + |(e^{-i\theta}, P_{1;0}(z))_{L^2}|^2 + |(e^{-i\theta}, P_{1;1}(z))_{L^2}|^2 + \cdots = \|e^{-i\theta}\|^2_{L^2}. \tag{68}$$

Let us pass to the proof of the positiveness of $u_{j+1,j;\alpha,\alpha}$, where $j \in \mathbb{N}$, $\alpha = 0, 1, \ldots, j$. From (57) we have

$$u_{j+1,j;\alpha,\alpha} = \int_{\mathbb{C}} e^{i\theta} P_{j;\alpha}(z) \overline{P_{j+1;\alpha}(z)}\, d\rho(z). \tag{69}$$

According to (25) and (27)

$$P_{j;\alpha}(z) = k_{j;\alpha} r^{j-|\alpha-j|} e^{i(j-\alpha)\theta} + R_{j;\alpha}(z), \tag{70}$$

where $R_{j;\alpha}(z)$ is some polynomial from $\mathcal{P}_{j;\alpha-1}$ if $\alpha > 0$ or from $\mathcal{P}_{j-1;2(j-1)}$ if $\alpha = 0$. Therefore $e^{i\theta} R_{j;\alpha}(z)$ is some polynomial from $\mathcal{P}_{j+1;\alpha-1}$ for $\alpha = 1, 2, \ldots, j$, or from $\mathcal{P}_{j+1;j}$ for $\alpha = j+1, j+2, \ldots, 2j$, or from $\mathcal{P}_{j;2(j-1)}$ for $\alpha = 0$ (see (34), (35) and (27)). Multiplying (70) on $e^{i\theta}$ we conclude that

$$\begin{aligned} e^{i\theta} P_{j;\alpha}(z) &= k_{j;\alpha} r^{j-|\alpha-j)|} e^{i(j-\alpha+1)\theta} + e^{i\theta} R_{j;\alpha}(z) \\ &= k_{j;\alpha} r^{j+1-|\alpha-(j+1))|} e^{i(j+1-\alpha)\theta} + e^{i\theta} R_{j;\alpha}(z); \\ e^{i\theta} R_{j;\alpha}(z) &\in \begin{cases} \mathcal{P}_{j;2(j-1)}, & \alpha = 0; \\ \mathcal{P}_{j+1;\alpha-1}, & \alpha = 1, 2, \ldots, j; \\ \mathcal{P}_{j+1;j}, & \alpha = j+1, j+2, \ldots, 2j. \end{cases} \end{aligned} \tag{71}$$

On the other hand equality (70) for $P_{j+1;\alpha}(z)$ gives

$$\begin{aligned} P_{j+1;\alpha}(z) &= k_{j+1;\alpha} r^{j+1-|\alpha-(j+1)|} e^{i(j+1-\alpha)\theta} + R_{j+1;\alpha}(z); \\ R_{j+1;\alpha}(z) &\in \begin{cases} \mathcal{P}_{j;2j}, & \alpha = 0; \\ \mathcal{P}_{j+1;\alpha-1}, & \alpha = 1, 2, \ldots, 2j. \end{cases} \end{aligned} \tag{72}$$

Finding $r^{j+1-|\alpha-(j+1)|} e^{i(j+1-\alpha)\theta}$ from (72) and substituting it into (71) gives

$$\begin{aligned} e^{i\theta} P_{j;\alpha}(z) &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}} (P_{j+1;\alpha}(z) - R_{j+1;\alpha}(z)) + e^{i\theta} R_{j;\alpha}(z) \\ &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}} P_{j+1;\alpha}(z) - \frac{k_{j;\alpha}}{k_{j+1;\alpha}} R_{j+1;\alpha}(z) + e^{i\theta} R_{j;\alpha}(z), \end{aligned} \tag{73}$$

where the second term belong to $\mathcal{P}_{j+1;\alpha-1}$ ($\alpha \neq 0$) or to $\mathcal{P}_{j;2j}$ ($\alpha = 0$) and the third term belong to $\mathcal{P}_{j;2(j-1)}$ if $\alpha = 0$, $\mathcal{P}_{j+1;\alpha-1}$ if $\alpha = 1, 2, \ldots, j$, $\mathcal{P}_{j+1;j}$ if $\alpha = j+1, j+2, \ldots, 2j$ and in any case orthogonal to $P_{j+1;\alpha}(z)$ if $\alpha \leq j$. Therefore after the substitution the expression (73) into (69) we get that $a_{j+1,j;\alpha,\alpha+2} = \frac{k_{j;\alpha}}{k_{j+1;\alpha}} > 0$.

Consider at last the elements $u_{j,j+1;\alpha,\alpha+1}$ where $j \in \mathbb{N}$, $\alpha = j+1, j+2, \ldots, 2j$. From (57) we get

$$u_{j,j+1;\alpha,\alpha+2} = \int_{\mathbb{C}} e^{i\theta} P_{j+1,\alpha+2}(z) \overline{P_{j;\alpha}(z)}\, d\rho(z) = \overline{\int_{\mathbb{C}} e^{-i\theta} P_{j,\alpha}(z) \overline{P_{j+1;\alpha+2}(z)}\, d\rho(z)}. \tag{74}$$

For $P_{j;\alpha}(z)$ we have expression (70). Multiply it on $e^{-i\theta}$ we get similar to (71)

$$\begin{aligned} e^{-i\theta} P_{j;\alpha}(z) &= k_{j;\alpha} r^{(j-|(\alpha-1)|} e^{i(j-\alpha-1)\theta} + e^{-i\theta} R_{j;\alpha}(z) \\ &= k_{j;\alpha} r^{(j+1)-|(\alpha+2)-(j+1)|} e^{i(j+1-(\alpha+2))\theta} + e^{-i\theta} R_{j;\alpha}(z), \\ e^{-i\theta} R_{j;\alpha}(z) &\in \begin{cases} \mathcal{P}_{j-1;2(j-1)}, & \alpha = 0; \\ \mathcal{P}_{j;2j}, & \alpha = 1, 2, \ldots, j-1; \\ \mathcal{P}_{j+1;\alpha+2}, & \alpha = j, j+1, \ldots, 2j. \end{cases} \end{aligned} \tag{75}$$

Now the equality (70) for $\mathcal{P}_{j+1;\alpha+2}$ has a form

$$\begin{aligned} &P_{j+1;\alpha+2}(z) = k_{j+1;\alpha+2} r^{j+1-|(\alpha+2)-(j+1)|} e^{i((j+1)-(\alpha+2))\theta} + R_{j+1;\alpha+2}(z); \\ &R_{j+1;\alpha+2}(z) \in \left\{ \begin{array}{ll} \mathcal{P}_{j+1;\alpha+1}, & \alpha = 1, 2, \dots, 2j; \\ \mathcal{P}_{j;2j}, & \alpha = 0. \end{array} \right. \end{aligned} \tag{76}$$

Finding $r^{j+1-|(\alpha+2)-(j+1)|} e^{i((j+1)-(\alpha+2))\theta}$ from (76) and substituting it into (75) gives

$$\begin{aligned} e^{-i\theta} P_{j;\alpha}(z) &= \frac{k_{j;\alpha}}{k_{j+1;\alpha+2}} (P_{j+1;\alpha+2}(z) - R_{j+1;\alpha+2}(z)) + e^{-i\theta} R_{j;\alpha}(z) \\ &= \frac{k_{j;\alpha}}{k_{j+1;\alpha+2}} P_{j+1;\alpha+2}(z) - \frac{k_{j;\alpha}}{k_{j+1;\alpha+2}} R_{j+1;\alpha+2}(z) + e^{-i\theta} R_{j;\alpha}(z). \end{aligned} \tag{77}$$

As before the second term in (77) belongs to $\mathcal{P}_{j+1;\alpha+1}$, for $\alpha = 1, 2, \dots, 2j$ or to $\mathcal{P}_{j;2j}$ for $\alpha = 0$ and the third term belongs to $\mathcal{P}_{j+1;\alpha+1}$, for $\alpha = j+1, j+2, \dots, 2j$ or to $\mathcal{P}_{j;2j}$, for $\alpha = 1, 2, \dots, j-1$ and to $\mathcal{P}_{j-1;2(j-1)}$ if $\alpha = 0$ and are in any case orthogonal to $P_{j+1;\alpha+2}(z)$ for $\alpha = j+1, j+2, \dots, 2j$.

Therefore substituting of the expression (77) into (74) gives $u_{j,j+1;\alpha,\alpha+1} = \frac{k_{j;\alpha}}{k_{j+1;\alpha+2}} > 0$. ☐

In what follows we will use usual convenient notations for elements $u_{j,k}$ of the Jacobi matrix

$$\begin{array}{llcll} u_n = u_{n+1,n} & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_{n+1}, \\ w_n = u_{n,n} & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_n, \\ v_n = u_{n,n+1} & : & \mathcal{H}_{n+1} & \longrightarrow & \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{78}$$

All previous investigation are summarized in the following theorem.

Theorem 3. *The bounded Hermitian operator $\hat{A}$ of multiplication by r commuting with the unitary operator $\hat{U}$ of multiplication by $e^{i\theta}$ on the space L^2 in the orthonormal basis (24) of polynomials have the form of three-diagonal block Jacobi type symmetric matrix $J_A = (a_{j,k})_{j,k=0}^{\infty}$ and unitary matrix $J_U = (u_{j,k})_{j,k=0}^{\infty}$ that act on the space (28):*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0. \tag{79}$$

The norms of all operators $a_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$ and $u_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$ are uniformly bounded with respect to $j, k \in \mathbb{N}_0$.

In designations (53), the symmetric matrix has the block form

$$J_A = \left[\begin{array}{|cccccc} b_0 & c_0 & 0 & 0 & 0 & \dots \\ a_0 & b_1 & c_1 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & c_2 & 0 & \dots \\ 0 & 0 & a_2 & b_3 & c_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right] \tag{80}$$

and the scalar form

$$J_A=\left[\begin{array}{c|ccc|ccccc|ccccccc}
* & * & + & 0 & & & & & & & & & & & & \\ \hline
* & * & * & * & * & + & 0 & 0 & 0 & & & & & & & \\
+ & * & * & * & * & * & + & 0 & 0 & & & & 0 & & & \\
0 & * & * & * & * & * & * & + & 0 & & & & & & & \\ \hline
 & * & * & * & * & * & * & * & * & * & + & 0 & 0 & 0 & 0 & 0 \\
 & + & * & * & * & * & * & * & * & * & * & + & 0 & 0 & 0 & 0 \\
 & 0 & + & * & * & * & * & * & * & * & * & * & + & 0 & 0 & 0 \\
 & 0 & 0 & + & * & * & * & * & * & * & * & * & * & + & 0 & 0 \\
 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & + & 0 \\ \hline
 & & & & * & * & * & * & * & * & * & * & * & * & * & * \\
 & & & & + & * & * & * & * & * & * & * & * & * & * & * \\
 & & & & 0 & + & * & * & * & * & * & * & * & * & * & * \\
 & & 0 & & 0 & 0 & + & * & * & * & * & * & * & * & * & * \\
 & & & & 0 & 0 & 0 & + & * & * & * & * & * & * & * & * \\
 & & & & 0 & 0 & 0 & 0 & + & * & * & * & * & * & * & * \\
 & & & & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\ \hline
 & & & \ddots & & & & & \ddots & & & & & & & \ddots
\end{array}\right].$$

In (80) $\forall n\in\mathbb{N}_0$ b_n *is an* $((2n+1)\times(2n+1))$*-matrix:* $b_n=(b_{n;\alpha,\beta})_{\alpha,\beta=0}^{2n,2n}$, *(*$b_0=b_{0;0,0}$ *is a scalar);* a_n *is an* $((2n+3)\times(2n+1))$*-matrix:* $a_n=(a_{n;\alpha,\beta})_{\alpha,\beta=0}^{2n+2,2n}$*;* c_n *is an* $((2n+1)\times(2n+3))$*-matrix:* $c_n=(c_{n;\alpha,\beta})_{\alpha,\beta=0}^{2n,2n+2}$*. In these matrices* a_n *and* c_n *some elements are always equal to zero:* $\forall n\in\mathbb{N}$

$$\begin{aligned}
&c_{n;\alpha,\beta}=0, \quad \alpha=0,1,\dots,2n, \quad \beta=\alpha+2,\alpha+3,\dots,2n+2;\\
&a_{n;\alpha,\beta}=0, \quad \beta=0,1,\dots,2n, \quad \alpha=\beta+2,\beta+3,\dots,2n+2.
\end{aligned} \tag{81}$$

Some other their elements are positive, namely

$$c_{n;\alpha,\alpha+1}>0, \quad a_{n;\alpha+1,\alpha}>0, \quad \alpha=0,1,\dots,2n, \quad \forall n\in\mathbb{N}_0. \tag{82}$$

Thus, it is possible to say, that $\forall n\in\mathbb{N}_0$ *every under left corner of the matrices* a_n *(starting from the third diagonal) and every over right corner of the matrices* c_n *(starting from the third diagonal) consist of zero elements. All positive elements in (80) are denoted by "+").*

So, the matrix (80) in the scalar form has the indicated multi-diagonals structure.

The adjoint operator $(\hat{A})^*=\hat{A}$ *in basis (24) has the same form of analogously three-diagonal block Jacobi type matrix* J_A*.*

In designations (78), the unitary matrix has the block form

$$J_U=\left[\begin{array}{|c|c|c|c|c|c}
\hline
w_0 & v_0 & 0 & 0 & 0 & \dots \\ \hline
u_0 & w_1 & v_1 & 0 & 0 & \dots \\ \hline
0 & u_1 & w_2 & v_2 & 0 & \dots \\
0 & 0 & u_2 & w_3 & v_3 & \dots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right] \tag{83}$$

and the scalar form

$$J_U = \left[\begin{array}{c|ccc|ccccc|ccccccc}
* & * & * & + & & & & & & & & & & & & \\
\hline
+ & * & * & * & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & * & * & * & * & * & * & + & 0 & & & & 0 & & & \\
0 & * & * & * & * & * & * & * & + & & & & & & & \\
\hline
 & + & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & + & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & + & 0 & 0 \\
 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & + & 0 \\
 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & + \\
\hline
 & & & & + & * & * & * & * & * & * & * & * & * & * & * \\
 & & & & 0 & + & * & * & * & * & * & * & * & * & * & * \\
 & & & & 0 & 0 & + & * & * & * & * & * & * & * & * & * \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\
 & & & & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\
 & & & & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\
 & & & & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\
\hline
 & & & \ddots & & & & & \ddots & & & & & & & \ddots
\end{array}\right].$$

In (83) $\forall n \in \mathbb{N}_0$ w_n *is an* $((2n+1)\times(2n+1))$*-matrix:* $w_n = (b_{n;\alpha,\beta})_{\alpha,\beta=0}^{2n,2n}$, $(w_0 = w_{0;0,0}$ *is a scalar);* u_n *is an* $((2n+3)\times(2n+1))$*-matrix:* $u_n = (u_{n;\alpha,\beta})_{\alpha,\beta=0}^{2n+2,2n}$*;* v_n *is an* $((2n+1)\times(2n+3))$*-matrix:* $v_n = (c_{n;\alpha,\beta})_{\alpha,\beta=0}^{2n,2n+2}$. *In these matrices* u_n *and* v_n *some elements are always equal to zero:* $\forall n \in \mathbb{N}$

$$\begin{array}{llll}
 & v_{n;\alpha,\beta} = 0, & \alpha = 0,1,\ldots,n-1, & \beta = 0,1,\ldots,2n+2; \\
(84) & v_{n;\alpha,\beta} = 0, & \alpha = n,n+1,\ldots,2n-1, & \beta = \alpha+3,\alpha+4,\ldots 2n+2; \\
 & u_{n;\alpha,\beta} = 0, & \beta = 0,1,\ldots,n, & \alpha = \beta+1,\beta+2,\ldots,2n+2; \\
 & u_{n;\alpha,\beta} = 0, & \beta = n+1,n+2\ldots,2n, & \alpha = n+1,n+2,\ldots,2n+2
\end{array}$$

and $u_{0;1,0} = u_{0;2,0} = 0$.

Some other their elements are positive, namely

$$\begin{array}{lll}
 & v_{0;0,2}, u_{0;0,0}; & \\
(85) & v_{n;\alpha,\alpha+2}, & \alpha = n+1,n+2,\ldots,2n; \\
 & u_{n;\alpha,\alpha}, & \alpha = 0,1,\ldots,n; \quad n \in \mathbb{N}.
\end{array}$$

Thus, it is possible to say, that for $\forall n \in \mathbb{N}$ *every under left corner of the matrices* u_n *(starting from the second diagonal beginning in the left over corner) and* $n+2$ *last lines and every over right corner of the matrices* v_n *(starting from the second diagonal beginning in the right under corner) and* n *first lines consist of zero elements. All positive elements in (83) are denoted by "+").*

So, the matrix (83) in the scalar form has the indicated multi-diagonals structure.

The adjoint operator $(\hat{U})^*$ *in basis (24) has the form of analogously three-diagonal block Jacobi type matrix* J_{U^*}.

These matrices J_A, J_U *and* J_{U^*} *act as follows:* $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$

$$\begin{array}{ll}
 & (J_A f)_n = a_{n-1}f_{n-1} + b_n f_n + c_n f_{n+1}, \\
(86) & (J_U f)_n = u_{n-1}f_{n-1} + w_n f_n + v_n f_{n+1}, \\
 & (J_{U^*} f)_n = v_{n-1}^* f_{n-1} + w_n^* f_n + u_n^* f_{n+1}, \quad n \in \mathbb{N}_0, \quad f_{-1} = 0,
\end{array}$$

*(here by * is denoted the usual adjoint matrix).*

Let us conclude that the form of coefficients in the expression for J_{U^*} follows from (60) and (78).

4. THE DIRECT AND INVERSE SPECTRAL PROBLEMS FOR THE THREE-DIAGONAL BLOCK JACOBI TYPE OPERATORS

As it was mentioned above the main result of the previous section is, actually, the solution of the inverse problem for corresponding direct problem appearing in the title of this section.

We consider operators on the space $\mathbf{l}_2$ of the form (28). Additionally to the space $\mathbf{l}_2$ we consider its rigging

$$(\mathbf{l}_{\mathrm{fin}})' \supset \mathbf{l}_2(p^{-1}) \supset \mathbf{l}_2 \supset \mathbf{l}_2(p) \supset \mathbf{l}_{\mathrm{fin}}, \tag{87}$$

where $\mathbf{l}_2(p)$ is the weighted $\mathbf{l}_2$ space with a weight $p = (p_n)_{n=0}^{\infty}$, where $p_n \geq 1$, and $p^{-1} = (p_n^{-1})_{n=0}^{\infty}$. In our case $\mathbf{l}_2(p)$ is the Hilbert space of sequences $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$ for which we have

$$\|f\|^2_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} p_n, \quad (f,g)_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n} p_n. \tag{88}$$

The space $\mathbf{l}_2(p^{-1})$ is defined analogously; recall that $\mathbf{l}_{\mathrm{fin}}$ is the space of finite sequences and $(\mathbf{l}_{\mathrm{fin}})'$ is the space conjugate to $\mathbf{l}_{\mathrm{fin}}$. It is easy to show that the embedding $\mathbf{l}_2(p) \hookrightarrow \mathbf{l}_2$ is quasinuclear if $\sum_{n=0}^{\infty} n p_n^{-1} < \infty$ (see, for example, [3] Ch. 7; [5] Ch. 15).

Let A and U be a strongly commuting bounded selfadjoint a unitary operators connected with the chain (87). According to the projection spectral theorem (see [4] Ch. 3, Theorem 2.7; [3] Ch. 5; [5], Ch. 15;) such operator have representations

$$Af = \int_{\mathbb{C}} r \Phi(z)\, d\sigma(z) f, \quad Uf = \int_{\mathbb{C}} e^{i\theta} \Phi(z)\, d\sigma(z) f, \quad f \in \mathbf{l}_2, \tag{89}$$

where $\Phi(z) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is the generalized projection operator and $d\sigma(z)$ is spectral measure. The adjoint to U operator U^* has the same representation as in (89) where $e^{i\theta}\Phi(z)$ is replaced by $e^{-i\theta}\Phi(z)$. For every $f \in \mathbf{l}_{\mathrm{fin}}$ the projection $\Phi(z) f \in \mathbf{l}_2(p^{-1})$ is a generalized eigenvector of the operators U and U^* and A with corresponding eigenvalues $e^{i\theta}$, $e^{-i\theta}$ and r. For all $f, g \in \mathbf{l}_{\mathrm{fin}}$ we have Parseval equality

$$(f,g)_{\mathbf{l}_2} = \int_{\mathbb{C}} (\Phi(z) f, g)_{\mathbf{l}_2} d\sigma(z); \tag{90}$$

after closure by continuity the equality (90) takes place for $\forall f, g \in \mathbf{l}_2$.

Let us denote by π_n the operator of orthogonal projection in $\mathbf{l}_2$ on $\mathcal{H}_n$, $n \in \mathbb{N}_0$. Hence $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously on the spaces $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$ but possibly with the norm which is not equal to one.

Let us consider the operator matrix $(\Phi_{j,k}(z))_{j,k=0}^{\infty}$, where

$$\Phi_{j,k}(z) = \pi_j \Phi(z) \pi_k \ : \ \mathbf{l}_2 \longrightarrow \mathcal{H}_j, \ (\text{or } \mathcal{H}_k \longrightarrow \mathcal{H}_j). \tag{91}$$

The Parseval equality (90) can be written as follows: $\forall f, g \in \mathbf{l}_2$

$$\begin{aligned}(f,g)_{\mathbf{l}_2} &= \sum_{j,k=0}^{\infty} \int_{\mathbb{C}} (\Phi(z)\pi_k f, \pi_j g)_{\mathbf{l}_2} d\sigma(z) = \sum_{j,k=0}^{\infty} \int_{\mathbb{C}} (\pi_j \Phi(z) \pi_k f, g)_{\mathbf{l}_2} d\sigma(z)\\ &= \sum_{j,k=0}^{\infty} \int_{\mathbb{C}} (\Phi_{j,k}(z) f_k, g_j)_{\mathbf{l}_2} d\sigma(z).\end{aligned} \tag{92}$$

Let us now pass to a study of a more special bounded operators A and U that act on the space $\mathbf{l}_2$. Namely, let it be given by a matrices J_A and J_U that have a three-diagonal block structure of the form (80) and (83) correspondingly. So, these operators A and U are defined by the first and the second expressions in (86), the adjoint operator U^* is defined analogously by the third expression in (86). Recall that the norm of all elements a_n, b_n, c_n and, obviously, u_n, w_n and v_n are uniformly bounded with respect to $n \in \mathbb{N}_0$.

For the further investigations *we suppose that conditions (81), (82) and (84), (85) are fulfilled and additionally the operators A and U given by (80) and (83) are selfadjoint, unitary and commuting on* $\mathbf{l}_2$. The conditions, that would imply for operators A and U to be selfadjoint unitary and commuting will be investigate in the next Section.

At the next step we will rewrite the Parseval equality (92) in terms of generalized eigenvectors of operators A and U. At first we prove the following lemma.

Lemma 8. *Let $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n$, $z \in \mathbb{C}$, be a generalized eigenvector from $(\mathbf{l}_{\mathrm{fin}})'$ of commuting selfadjoint operator A and unitary operator U (U^*) with eigenvalue r and $e^{i\theta}$ ($e^{-i\theta}$) corresponding. Multiplying $\varphi(z)$ by the scalar constant (depending on $z = re^{i\theta}$) we can obtain, that $\varphi_0(z) = \varphi_0$ is independent of z. Thus $\varphi(z)$ is the solution from $(\mathbf{l}_{\mathrm{fin}})'$ of the system of three difference equations (see (86))*

$$\begin{gathered}
(J_A\varphi(z))_n = a_{n-1}\varphi_{n-1}(z) + b_n\varphi_n(z) + c_n\varphi_{n+1}(z) = r\varphi_n(z),\\
(J_U\varphi(z))_n = u_{n-1}\varphi_{n-1}(z) + w_n\varphi_n(z) + v_n\varphi_{n+1}(z) = e^{i\theta}\varphi_n(z),\\
(J_{U^*}\varphi(z))_n = v_{n-1}^*\varphi_{n-1}(z) + w_n^*\varphi_n(z) + u_n^*\varphi_{n+1}(z) = e^{-i\theta}\varphi_n(z),\\
n \in \mathbb{N}_0, \quad \varphi_{-1}(z) = 0,
\end{gathered} \tag{93}$$

with an initial condition $\varphi_0 \in \mathbb{C}$.

We assert that this solution is the following: $\forall n \in \mathbb{N}$

$$\varphi_n(z) = Q_n(z)\varphi_0 = (Q_{n;0}, Q_{n;1}, \dots, Q_{n;2n},)\varphi_0. \tag{94}$$

Here $Q_{n;\alpha}$, $\alpha = 0, 1, \dots, 2n$ are polynomials of r, $e^{i\theta}$ and $e^{-i\theta}$ and these polynomials have the form

$$Q_{n;\alpha}(z) = k_{n;\alpha}r^{n-|\alpha-n|}e^{-i(n-\alpha)\theta} + R_{n;\alpha}(z), \quad \alpha = 0, 1, \dots, 2n, \quad n \in \mathbb{N}_0, \tag{95}$$

where $k_{n;\alpha} > 0$ and $R_{n;\alpha}(z)$ is some linear combinations of r and $e^{i\theta}$ (law order see (23) and the Figure 1.).

Proof. For $n = 0$ the system (93) has the form (in convenient order):

$$J_{U^*}\varphi_0 = e^{-i\theta}\varphi_0, \quad J_A\varphi_0 = r\varphi_0, \quad J_U\varphi_0 = e^{i\theta}\varphi_0, \tag{96}$$

i.e.

$$\begin{aligned}
&\bar{w}_{0;0,0}\varphi_{0;0} + \bar{u}_{0;0,0}\varphi_{1;0} &&= e^{-i\theta}\varphi_{0;0},\\
&b_{0;0,0}\varphi_{0;0} + c_{0;0,0}\varphi_{1;0} + c_{0;0,1}\varphi_{1;1} &&= r\varphi_{0;0},\\
&w_{0;0,0}\varphi_{0;0} + v_{0;0,0}\varphi_{1;0} + v_{0;0,1}\varphi_{1;1} + v_{0;0,2}\varphi_{1;2} &&= e^{i\theta}\varphi_{0;0},
\end{aligned}$$

or

$$\begin{aligned}
&\bar{u}_{0;0,0}\varphi_{1;0} &&= (e^{-i\theta} - \bar{w}_{0;0,0})\varphi_{0;0},\\
&c_{0;0,0}\varphi_{1;0} + c_{0;0,1}\varphi_{1;1} &&= (r - b_{0;0,0})\varphi_{0;0},\\
&v_{0;0,0}\varphi_{1;0} + v_{0;0,1}\varphi_{1;1} + v_{0;0,2}\varphi_{1;2} &&= (e^{i\theta} - w_{0;0,0})\varphi_{0;0}.
\end{aligned} \tag{97}$$

Here and in what follows we denote

$$\varphi_0 = \varphi_{0;0} := Q_{0;0}; \quad \varphi_n(z) = (\varphi_{n;0}(z), \varphi_{n;1}(z), \dots, \varphi_{n;2n}(z)) \in \mathcal{H}_n, \quad \forall n \in \mathbb{N}.$$

Equalities in (96) we rewrite in the form

$$\begin{gathered}
\Delta_0\varphi_1(z) = \left((e^{-i\theta} - \bar{w}_{0;0,0})\varphi_0), (r - b_{0;0,0})\varphi_0, (e^{i\theta} - w_{0;0,0})\varphi_0\right),\\
\Delta_0 = \begin{pmatrix} \bar{u}_{0;0,0} & 0 & 0\\ c_{0;0,0} & c_{0;0,1} & 0\\ v_{0;0,0} & c_{0;0,1} & c_{0;0,2} \end{pmatrix},
\end{gathered} \tag{98}$$

where due to conditions (81), (82) and (84), (85) $\bar{u}_{0;0,0} > 0$, $c_{0;0,1} > 0$, $u_{0;0,2} > 0$. Hence $\Delta_0 > 0$. Therefore from (97) we obtain consequently

$$\begin{aligned}
\varphi_{1;0}(z) &= -\frac{\bar{w}_{0;0,0} - e^{-i\theta}}{\bar{u}_{0;0,0}}\varphi_0 =: Q_{1;0}(z)\varphi_0,\\
\varphi_{1;1}(z) &= -\left(\frac{b_{0;0,0} - r}{c_{0;0,1}} + \frac{c_{0;0,0}}{c_{0;0,1}}\frac{e^{-i\theta} - \bar{w}_{0;0,0}}{\bar{u}_{0;0,0}}\right)\varphi_0 =: Q_{1;1}(z)\varphi_0,\\
\varphi_{1;2}(z) &= -\left(\frac{w_{0;0,0} - e^{i\theta}}{v_{0;0,2}} - \frac{v_{0;0,0}}{v_{0;0,2}}\left[\frac{\bar{w}_{0;0,0} - e^{-i\theta}}{\bar{u}_{0;0,0}}\right]\right.\\
&\qquad \left. - \frac{v_{0;0,1}}{v_{0;0,2}}\left[\frac{b_{0;0,0} - r}{c_{0;0,1}} + \frac{c_{0;0,0}}{c_{0;0,1}}\frac{e^{-i\theta} - \bar{w}_{0;0,0}}{\bar{u}_{0;0,0}}\right]\right)\varphi_0 =: Q_{1;2}(z)\varphi_0.
\end{aligned}\tag{99}$$

In another words the solution $\varphi_n(z)$ by $n = 1$ of (93) has a form (94) with (95).

Suppose, by induction, that for $n \in \mathbb{N}$ the coordinates $\varphi_{n-1}(z)$ and $\varphi_n(z)$ of our generalized eigenvector $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$ have a form (94) with (95) and prove that $\varphi_{n+1}(z)$ is also of the form (94) with (95).

Our eigenvector $\varphi_{n+1}(z)$ satisfies the system (93) of equations. But this system is overdetermined: it consists of $3(2n+3)$ scalar equations from which it is necessary to find only $2n+3$ unknowns $\varphi_{n+1;0}, \varphi_{n+1;1}, \dots, \varphi_{n+1;2(n+1)}$ using as an initial data the previous $2n+1$ values $\varphi_{n;0}, \varphi_{n;1}, \dots, \varphi_{n;2n}$ of coordinates of the vector $\varphi_n(z)$. Analogously to (96) we have for $j = 0, 1, \dots, 2n$:

$$J_{U^*}\varphi_{n;j} = e^{-i\theta}\varphi_{n;j}, \quad J_A\varphi_{n;j} = r\varphi_{n;j}, \quad J_U\varphi_{n;j} = e^{i\theta}\varphi_{n;j}.$$

According to Theorem 3, especially to (81), (82) and (84), (85) the $((2n+3)\times(2n+1))$-matrices $\bar{u}_n$, c_n, v_n and act on $\varphi_{n+1} \in \mathcal{H}_n$ as follows:

$$\begin{aligned}
u_n^*\varphi_{n+1}(z) &= \left[\begin{array}{llllllll}
\bar{u}_{n;0,0} & 0 & \dots & 0 & 0 & \dots & 0\\
\bar{u}_{n;0,1} & \bar{u}_{n;1,1} & \dots & 0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\bar{u}_{n;0,n} & \bar{u}_{n;1,n} & \dots & \bar{u}_{n;n,n} & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\bar{u}_{n;0,2n-1} & \bar{u}_{n;1,2n-1} & \dots & \bar{u}_{n;n,2n-1} & 0 & \dots & 0\\
\bar{u}_{n;0,2n} & \bar{u}_{n;1,2n} & \dots & \bar{u}_{n;n,2n} & 0 & \dots & 0
\end{array}\right]\varphi_{n+1}(z),\\
c_n\varphi_{n+1}(z) &= \left[\begin{array}{llllll}
c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0\\
c_{n;1,0} & c_{n;1,1} & c_{n;1,2} & \dots & 0 & 0\\
c_{n;2,0} & c_{n;2,1} & c_{n;2,2} & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
c_{n;2n-1,0} & c_{n;2n-1,1} & c_{n;2n-1,2} & \dots & 0 & 0\\
c_{n;2n,0} & c_{n;2n,1} & c_{n;2n,2} & \dots & c_{n;2n,2n+1} & 0
\end{array}\right]\varphi_{n+1}(z),\\
v_n\varphi_{n+1}(z) &= \left[\begin{array}{llll}
0 & \dots 0 & \dots 0 & 0\\
\vdots & \ddots \vdots & \ddots \vdots & \vdots\\
0 & \dots 0 & \dots 0 & 0\\
v_{n;n,0} & \dots v_{n;n,n+2} & \dots 0 & 0\\
\vdots & \ddots \vdots & \ddots \vdots & \vdots\\
v_{n;2n-1,0} & \dots v_{n;2n-1,n+2} & \dots v_{n;2n-1,2n+1} & 0\\
v_{n;2n,0} & \dots v_{n;2n,n+2} & \dots v_{n;2n,2n+1} & v_{n;2n,2n+2}
\end{array}\right]\varphi_{n+1}(z),
\end{aligned}\tag{100}$$

where $\varphi_{n+1}(z) = (\varphi_{n+1;0}(z), \varphi_{n+1;1}(z), \dots, \varphi_{n+1;2n+2}(z))$.

Construct the similar to (98) the following combination using the matrices from (100): the $((2n+2)\times(2n+2))$-matrix

$$\Delta_n\varphi_{n+1}(z)=\begin{bmatrix} \bar{u}_{n;0,0} & 0 & 0 & \dots & 0 & 0 \\ c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0 \\ c_{n;1,0} & c_{n;1,1} & c_{n;1,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n;2n,0} & c_{n;2n,1} & c_{n;2n,2} & \dots & c_{n;2n,2n+1} & 0 \\ v_{n;2n,0} & v_{n;2n,1} & v_{n;2n,2} & \dots & v_{n;2n,2n+1} & v_{n;2n,2n+2} \end{bmatrix}\varphi_{n+1}(z), \tag{101}$$

where $\varphi_{n+1}(z)=(\varphi_{n+1;0}(z),\varphi_{n+1;1}(z),\dots,\varphi_{n+1;2n+2}(z))$.

The matrix (101) is invertable, because its elements are positive on the main diagonal (see (81), (82) and (84), (85)). Rewrite the equalities (93) as follows:

$$\begin{aligned} u_n^*\varphi_{n+1}(z) &= e^{-i\theta}\varphi_n(z)-v_{n-1}^*\varphi_{n-1}(z)-w_n^*\varphi_n(z),\\ c_n\varphi_{n+1}(z) &= r\varphi_n(z)-a_{n-1}\varphi_{n-1}(z)-b_n\varphi_n(z),\\ v_n\varphi_{n+1}(z) &= e^{i\theta}\varphi_n(z)-u_{n-1}\varphi_{n-1}(z)-w_n\varphi_n(z), \quad n\in\mathbb{N}, \end{aligned}$$

i.e.

$$\begin{aligned} u_n^*\varphi_{n+1}(z) &= (e^{-i\theta}-w_n^*)\varphi_n(z)-v_{n-1}^*\varphi_{n-1}(z),\\ c_n\varphi_{n+1}(z) &= (r-b_n)\varphi_n(z)-a_{n-1}\varphi_{n-1}(z),\\ v_n\varphi_{n+1}(z) &= (e^{i\theta}-w_n)\varphi_n(z)-u_{n-1}\varphi_{n-1}(z), \quad n\in\mathbb{N}, \end{aligned} \tag{102}$$

We see, that scalar equations (102)) have the form

$$\bar{u}_{n;0,0}\varphi_{n+1,0}(z)=\left\{(e^{-i\theta}1-w_n^*)Q_n(z)-v_{n-1}^*Q_{n-1}(z)\right\}_{n;j},$$

where $''1''$ is the identity $(2n+2)\times(2n+2)$ matrix. Hence for $j=0$ we obtain

$$Q_{n+1,0}(z):=\varphi_{n+1,0}(z)=\frac{1}{\bar{u}_{n;0,0}}\left\{(e^{-i\theta}1-w_n^*)Q_n(z)-v_{n-1}^*Q_{n-1}(z)\right\}_{n;0}.$$

Hence $Q_{n+1,0}(z)=k_{n+1,0}e^{-i(n+1)\theta}+\cdots$

For the case $j=1$ we have

$$c_{n;0,0}\varphi_{n+1,0}(z)+c_{n;0,1}\varphi_{n+1,1}(z)=\{(r1-b_n)Q_n(z)-a_{n-1}Q_{n-1}(z)\}_{n;1},$$

$$\begin{aligned} Q_{n+1,1}(z) &: = \varphi_{n+1,1}(z)\\ &= \frac{1}{\bar{c}_{n;0,1}}\{-c_{n;0,0}Q_{n+1,0}(z)+(r1-b_n)Q_n(z)-a_{n-1}Q_{n-1}(z)\}_{n;1}. \end{aligned}$$

Hence $Q_{n+1,1}(z)=k_{n+1,1}re^{-in\theta}+\cdots$

For $j=2,3,\dots,2n+1$ we have

$$\begin{aligned} c_{n;j,0}\varphi_{n+1,0}(z)+c_{n;j,1}\varphi_{n+1,1}(z)+\cdots+c_{n;j,j+1}\varphi_{n+1,j}(z)\\ =\{(r1-b_n)Q_n(z)-a_{n-1}Q_{n-1}(z)\}_{n;j}, \end{aligned}$$

$$\begin{aligned} Q_{n+1,j}(z):=\varphi_{n+1,j}(z)=\frac{1}{c_{n;j,j+1}}\{&-c_{n;j,0}\varphi_{n+1,0}(z)-c_{n;j,1}\varphi_{n+1,1}(z)\\ &-\cdots-c_{n;j,j}\varphi_{n+1,j-1}(z)+(r1-b_n)Q_n(z)-a_{n-1}Q_{n-1}(z)\}_{n;j}. \end{aligned}$$

Hence $Q_{n+1,j}(z)=k_{n+1,j}r^{n+1-|j-(n+1)|}e^{i(n+1-j)\theta}+\cdots$

And for the last equation from (102) (i.e $j=2n+2$) we have

$$\begin{aligned} v_{n;2n,0}\varphi_{n+1,0}(z)+v_{n;2n,1}\varphi_{n+1,1}(z)+\cdots+v_{n;2n,2n+2}\varphi_{n+1,2n+2}(z)\\ =\left\{(e^{i\theta}1-w_n)Q_n(z)-u_{n-1}Q_{n-1}(z)\right\}_{n;2n+2}, \end{aligned}$$

$$
\begin{aligned}
Q_{n+1,2n+2}(z) := \varphi_{n+1,2n+2}(z) = \frac{1}{v_{n;2n,2n+2}} \{ & -v_{n;j,0}\varphi_{n+1,0}(z) \\
& - v_{n;j,1}\varphi_{n+1,1}(z) - \cdots - v_{n;2n,2n+1}\varphi_{n+1,2n+1}(z) \\
& +(e^{i\theta}1 - w_n)Q_n(z) - u_{n-1}Q_{n-1}(z)\}_{n;2n+2}.
\end{aligned}
$$

Hence $Q_{n+1,2n+2}(z) = k_{n+1,2n+2} r^{n+1-|(2n+2)-(n+1)|} e^{i(n+1)-(2n+2))\theta} + \cdots$

It is necessary to take into account that the next diagonal elements $\bar{u}_{n;0,0}, c_{n;0,1}, c_{n;1,2}, \ldots, c_{n;2n,2n+1}, v_{n;2n,2n+2}$ of the matrix Δ_n are positive due to (82) and (85). This completes the induction and finishes the proof. $\square$

Remark 2. Note, that we did not assert, that the solution of the overdetermined system (93) exist for an arbitrary initial data $\varphi_0 \in \mathbb{C}$: we prove only, that the generalized eigenvector from $(\mathbf{l}_{\text{fin}})'$ of operators A and U is a solution of (93) and has the form (94) and (95).

In what follows, it will be convenient to look at $Q_n(z)$ with fixed z as a linear operator that acts from $\mathcal{H}_0$ into $\mathcal{H}_n$, i.e., $\mathcal{H}_0 \ni \varphi_0 \longmapsto Q_n(z)\varphi_0 \in \mathcal{H}_n$. We also regard $Q_n(z)$ as an operator valued polynomial of $z = re^{i\theta} \in \mathbb{C}$, i.e. r and $e^{i\theta}$; hence, for the adjoint operator we have $Q_n^*(z) = (Q_n(z))^* : \mathcal{H}_n \longrightarrow \mathcal{H}_0$. Using these polynomials $Q_n(z)$ we construct the following representation for $\Phi_{j,k}(z)$.

Lemma 9. *The operator $\Phi_{j,k}(z)$, $\forall z \in \mathbb{C}$ has the following representation*

$$\Phi_{j,k}(z) = Q_j(z)\Phi_{0,0}(z)Q_k^*(z) : \mathcal{H}_k \longrightarrow \mathcal{H}_j, \quad j, k \in \mathbb{N}_0, \tag{103}$$

where $\Phi_{0,0}(z) \geq 0$ is a scalar.

Proof. For a fixed $k \in \mathbb{N}_0$ the vector $\varphi = \varphi(z) = (\varphi_j(z))_{j=0}^{\infty}$, where

$$\varphi_j(z) = \Phi_{j,k}(z) = \pi_j\Phi(z)\pi_k \in \mathcal{H}_j, \quad z \in \mathbb{C}, \tag{104}$$

is generalized solution, in $(\mathbf{l}_{\text{fin}})'$ of the system of equations

$$J_{U^*}\varphi = e^{-i\theta}\varphi, \quad J_A\varphi = r\varphi, \quad J_U\varphi = e^{i\theta}\varphi, \tag{105}$$

since $\Phi(z)$ is a projector on to generalized eigenvectors for operator A and U (U^*) with correspondence generalized eigenvalues r and $e^{i\theta}$, $(e^{-i\theta})$. Therefore $\forall g \in \mathbf{l}_{\text{fin}}$ we have

$$(\varphi, J_A g)_{\mathbf{l}_2} = r(\varphi, g)_{\mathbf{l}_2}, \quad (\varphi, J_{U^*} g)_{\mathbf{l}_2} = e^{i\theta}(\varphi, g)_{\mathbf{l}_2}, \quad (\varphi, J_U g)_{\mathbf{l}_2} = e^{-i\theta}(\varphi, g)_{\mathbf{l}_2}.$$

Hence, it follows that $\varphi = \varphi(z) \in \mathbf{l}_2(p^{-1})$ exists as an usual solution of equation (105) with initial condition $\varphi_0 = \pi_0\Phi(z)\pi_k \in \mathcal{H}_0$.

Using Lemma 8 and due to (94) we obtain

$$\Phi_{j,k}(z) = Q_j(z)(\Phi_{0,k}(z)), \quad j \in \mathbb{N}_0. \tag{106}$$

The operator $\Phi(z) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is formally selfadjoint on $\mathbf{l}_2$, being the derivative of the resolution of identity of operators A and U (U^*) on $\mathbf{l}_2$ with respect to the spectral measure. Hence, according to (103) we get

$$(\Phi_{j,k}(z))^* = (\pi_j\Phi(z)\pi_k)^* = \pi_k\Phi(z)\pi_j = \Phi_{k,j}(z), \quad j, k \in \mathbb{N}_0. \tag{107}$$

For a fixed $j \in \mathbb{N}_0$ from (107) and previous conversation, it follows that the vector

$$\psi = \psi(z) = (\psi_k(z))_{k=0}^{\infty}, \quad \psi_k(z) = \Phi_{k,j}(z) = (\Phi_{j,k}(z))^*$$

is an usual solution of the equations (105) with initial condition $\psi_0 = \Phi_{0,j}(z) = (\Phi_{j,0}(z))^*$.

Again using Lemma 8 we obtain the representation of the type (106),

$$\Phi_{k,j}(z) = Q_k(z)(\Phi_{0,j}(z)), \quad k \in \mathbb{N}_0. \tag{108}$$

Taking into account (107) and (108) we get

$$\Phi_{0,k}(z) = (\Phi_{k,0}(z))^* = (Q_k(z)\Phi_{0,0}(z))^* = \Phi_{0,0}(z)(Q_k(z))^*, \quad k \in \mathbb{N}_0 \tag{109}$$

(here we used $\Phi_{0,0}(z) \geq 0$, this inequality follows from (90) and (91)). Substituting (109) into (106) we obtain (103). □

Now it is possible to rewrite the Parseval equality (92) in a more concrete form. To this end, we substitute the expression (103) for $\Phi_{j,k}(z)$ into (92) and get:

$$\begin{aligned}(f,g)_{\mathbf{l}_2} &= \sum_{j,k=0}^{\infty} \int_{\mathbb{C}} (\Phi_{j,k}(z)f_k, g_j)_{\mathbf{l}_2} d\sigma(z)\\ &= \sum_{j,k=0}^{\infty} \int_{\mathbb{C}} (Q_j(z)\Phi_{0,0}(z)Q_k^*(z)f_k, g_j)_{\mathbf{l}_2} d\sigma(z)\\ &= \sum_{j,k=0}^{\infty} \int_{\mathbb{C}} (Q_k^*(z)f_k, Q_j^*(z)g_j)_{\mathbf{l}_2} d\rho(z)\\ &= \int_{\mathbb{C}} \Big(\sum_{k=0}^{\infty} Q_k^*(z)f_k\Big)\overline{\Big(\sum_{j=0}^{\infty} Q_j^*(z)g_j\Big)} d\rho(z),\\ d\rho(z) &= \Phi_{0,0}(z)\, d\sigma(z) \quad \forall f,g \in \mathbf{l}_{\text{fin}}.\end{aligned} \tag{110}$$

Introduce the Fourier transform $\widehat{\ }$ induced by the bounded self adjoint operator A commuting with unitary operator U on the space $\mathbf{l}_2$, $\forall f \in \mathbf{l}_{\text{fin}}$

$$\mathbf{l}_2 \supset \mathbf{l}_{\text{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) = \sum_{n=0}^{\infty} Q_n^*(z)f_n \in L^2(\mathbb{C}, d\rho(z)). \tag{111}$$

Hence, (110) gives the Parseval equality in a final form, $\forall f,g \in \mathbf{l}_{\text{fin}}$

$$(f,g)_{\mathbf{l}_2} = \int_{\mathbb{C}} \hat{f}(z)\overline{\hat{g}(z)}\, d\rho(z). \tag{112}$$

Extending (112) by continuity, it becomes valid $\forall f,g \in \mathbf{l}_2$.

Orthogonality of the polynomials $Q_n^*(z)$ follows from (111) and (112). Namely, it is sufficient only to take $f = (0,\ldots,0,f_k,0,\ldots)$, $f_k \in \mathcal{H}_k$, $g = (0,\ldots,0,g_j,0,\ldots)$, $g_j \in \mathcal{H}_j$ in (111) and (112) . Then $\forall k,j \in \mathbb{N}_0$

$$\int_{\mathbb{C}} (Q_k^*(z)f_k)\overline{(Q_j^*(z)g_j)}\, d\rho(z) = \delta_{j,k}(f_j,g_j)_{\mathcal{H}_j}. \tag{113}$$

Using representation (94) for these polynomials we can rewrite the equality (113) in a more classical scalar form. To do this, we remark that $Q_0^*(z) = \bar{Q}_0(z)$ and for $n \in \mathbb{N}$ according to (94)

$$Q_n(z) = (Q_{n;0}(z), Q_{n;1}(z), \ldots, Q_{n;2n}(z)) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n.$$

Hence, for the adjoint operator $Q_n^*(z) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ we have $\forall x \in \mathcal{H}_0$, and $y = (y_0, y_1, \ldots, y_{2n}) \in \mathcal{H}_n$

$$\begin{aligned}(Q_n(z)x,y)_{\mathcal{H}_n} &= ((Q_{n;0}(z)x, Q_{n;1}(z)x, \ldots, Q_{n;2n}(z)x),(y_0,y_1,\ldots,y_{2n}))_{\mathcal{H}_n}\\ &= Q_{n;0}(z)x\bar{y}_0 + Q_{n;1}(z)x\bar{y}_1 + \cdots + Q_{n;2n}(z)x\bar{y}_{2n}\\ &= x\overline{(\overline{Q_{n;0}(z)}y_0 + \overline{Q_{n;1}(z)}y_1 + \cdots + \overline{Q_{n;2n}(z)}y_{2n})} = (x, Q_n^*(z)y)_{\mathcal{H}_0},\end{aligned}$$

that is, $Q_n^*(z)y = \overline{Q_{n;0}(z)}y_0 + \overline{Q_{n;1}(z)}y_1 + \cdots + \overline{Q_{n;2n}(z)}y_{2n}$.

Due to the last equality for $n \in \mathbb{N}$ and $f_n = (f_{n,0}, f_{n,1}, \ldots, f_{n,2n}) \in \mathcal{H}_n$, $z \in \mathbb{C}$, we obtain

$$Q_n^*(z)f_n = \overline{Q_{n;0}(z)}f_{n;0} + \overline{Q_{n;1}(z)}f_{n;1} + \cdots + \overline{Q_{n;2n}(z)}f_{n;2n}, \quad Q_0^*(z) = 1. \tag{114}$$

Therefore (113) has the form: $\forall f_{k;0}, f_{k;1}, \dots, f_{k;2k},\ g_{j;0}, g_{j;1}, \dots, g_{j;2j} \in \mathbb{C},\ j, k \in \mathbb{N}_0$,

$$\int_{\mathbb{C}} \Big(\sum_{\alpha=0}^{2k} \overline{Q_{k;\alpha}(z)} f_{k;\alpha} \Big) \overline{\Big(\sum_{\beta=0}^{2j} \overline{Q_{j;\beta}(z)} f_{j;\beta} \Big)}\, d\rho(z) = \delta_{j,k} \sum_{\alpha=0}^{2j} f_{j;\alpha} \bar{g}_{j;\alpha}.$$

This equality is equivalent to the following orthogonality relation in the usual classical form: $\forall j, k \in \mathbb{N}_0$, $\forall \alpha = 0, 1, \dots, 2j, \beta = 0, 1, \dots, 2k$

$$\int_{\mathbb{C}} \overline{Q^*_{k;\beta}(z)} Q_{j;\alpha} d\rho(z) = \delta_{j,k} \delta_{\alpha,\beta} \quad (Q_{0;0} = Q_0(z)). \tag{115}$$

Due to (114) the Fourier transform (111) can be rewritten as

$$\hat{f}(z) = \sum_{n=0}^{\infty} \sum_{\alpha=0}^{2n} \overline{Q_{n;\alpha}(z)} f_{n;\alpha}, \quad z \in \mathbb{C}, \quad \forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2. \tag{116}$$

Using the stated above results of this section, we can formulate the following spectral theorem for our commuting bounded selfadjoint operator A and the unitary operator U.

Theorem 4. *Consider the space (28)*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0, \tag{117}$$

and commuting linear operators A and U that are defined on finite vectors $\mathbf{l}_{\mathrm{fin}}$ by a block three-diagonal Jacobi type matrix J_A of the form (80) and J_U of the form (83) with the help of the expression in (86). We suppose that all its coefficients a_n, b_n, c_n, $n \in \mathbb{N}_0$, are uniformly bounded (u_n, w_n, c_n, $n \in \mathbb{N}_0$, are also uniformly bounded), some elements of these matrices are equal to zero or positive according to (81), (82) and (84), (85) and the closure of A by continuity is bounded selfadjoint operator and closure of U by continuity is unitary operator on this space.

The eigenfunction expansion of operators A and U has the following form. According to Lemma 8 we represent, using $\varphi_0 \in \mathbb{C}$, the solution $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n$, of equations (93) (which exists thanks to projection spectral theorem) for $z = re^{i\theta} \in \mathbb{C}$

$$\varphi_n(z) = Q_n(z)\varphi_0 = (Q_{n;0}(z), Q_{n;1}(z), \dots, Q_{n;2n}(z))\varphi_0,$$

where $Q_{n;\alpha}(z)$, $\alpha = 0, 1, \dots, 2n$ are polynomials of r, $e^{i\theta}$ and $e^{i\theta}$. Then the Fourier transform has the form

$$\begin{aligned} \mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) &= \sum_{n=0}^{\infty} Q^*_n(z) f_n \\ &= \sum_{n=0}^{\infty} \sum_{\alpha=0}^{2n} \overline{Q_{n;\alpha}(z)} f_{n;\alpha} \in L^2(\mathbb{C}, d\rho(z)). \end{aligned} \tag{118}$$

Here $Q^*_n(z) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ is adjoint to the operator $Q_n(z) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$, $d\rho(z)$ is the probability spectral measure of A and U.

The Parseval equality has the following form: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$\begin{aligned} (f, g)_{\mathbf{l}_2} &= \int_{\mathbb{C}} \hat{f}(z) \overline{\hat{g}(z)}\, d\rho(z), & (J_A f, g)_{\mathbf{l}_2} &= \int_{\mathbb{C}} r \hat{f}(z) \overline{\hat{g}(z)}\, d\rho(z), \\ (J_U f, g)_{\mathbf{l}_2} &= \int_{\mathbb{C}} e^{i\theta} \hat{f}(z) \overline{\hat{g}(z)}\, d\rho(z), & (J_{U^*} f, g)_{\mathbf{l}_2} &= \int_{\mathbb{C}} e^{-i\theta} \hat{f}(z) \overline{\hat{g}(z)}\, d\rho(z). \end{aligned} \tag{119}$$

Identities (118) and (119) are extended by continuity to $\forall f, g \in \mathbf{l}_2$ making the operator (118) unitary, which maps $\mathbf{l}_2$ onto the whole $L^2(\mathbb{C}, d\rho(z))$.

The polynomials $\overline{Q_{n;\alpha}(z)}$, $n \in \mathbb{N}$, $\alpha = 0, 1, \dots, 2n$, and $Q_{0;0}(z) = 1$, form an orthonormal system in $L^2(\mathbb{C}, d\rho(z))$ in the sense of (115) and it is total on this space.

Proof. It is only necessary to show that the orthogonal polynomials $\overline{Q_{n;\alpha}(z)}$, / $n \in \mathbb{N}$, $\alpha = 0, 1, \ldots, 2n$, and $Q_{0;0}(z) = 1$ form a total set in the space $L^2(\mathbb{C}, d\rho(z))$. For this reason we remark at first that due to the compactness of the support of the measure $d\rho(z)$ on $\mathbb{C}$, the elements $r^t e^{ij\theta}$, $t \in \mathbb{N}_0$, $j \in \mathbb{Z}$ form a total set in $L^2(\mathbb{C}, d\rho(z))$.

Let us suppose the contrary, i.e. that our system of polynomials is not total. Then there exist non zero function $h(z) \in L^2(\mathbb{C}, d\rho(z))$ that is orthogonal to all these polynomials and hence, according to (95) to all $r^t e^{ij\theta}$, $t \in \mathbb{N}_0$, $j \in \mathbb{Z}$. Hence $h(z) = 0$. □

The last theorem solves the direct problem for the bounded selfadjoint operator A commuting with unitary operator U that are generated on the space $\mathbf{l}_2$ by the matrix J_A of the form (80) and J_U of the form (83).

The inverse problem consists of the construction from a given measure $d\rho(z)$ on $\mathbb{C}$ with compact support the bounded hermitian matrix J_A of the form (80) commuting with unitary matrix J_U of the form (83) and that has its spectral measure equal to $d\rho(z)$. *This construction is conducted according to Theorem 3, with the use of the Schmidt orthogonalization procedure for the system (23). For the matrix J_A of the form (80) and J_U of the form (83), that are constructed from $d\rho(z)$, the spectral measure of corresponding bounded selfadjoint operator A and unitary operator U is coinciding with the start measure.*

Proof. The claim holds true, since the system of orthogonal polynomials, connected with A, $\overline{Q_{n,\alpha}(z)}$ $\alpha = 0, 1, \ldots, 2n$, $n \in \mathbb{N}_0$, are orthonormal in $L^2(\mathbb{C}, d\rho(z))$ and, according to Lemma 8, are constructed by $r^t e^{ij\theta}$, $t \in \mathbb{N}_0$, $j \in \mathbb{Z}$, in the same way as the system (24) is constructed by $r^t e^{ij\theta}$, $t \in \mathbb{N}_0$, $j \in \mathbb{Z}$. Hence, $\forall n \in \mathbb{N}$

$$Q_0(z) = 1 = P_0(z), \quad \overline{Q_{n,\alpha}(z)} = P_{n;\alpha}(z), \quad \alpha = 0, 1, \ldots, 2n, \tag{120}$$

Since both system of polynomials form a total set in $L^2(\mathbb{C}, d\rho(z))$, then (120) shows that the spectral measures constructed from operators and the given one coincides. □

Let us remark that expressions (40) and (57) (as it was known in the classical theory of Jacobi matrices) reconstruct the initial matrices (80) and (83) from the spectral measure $d\rho(z)$ of the operators constructed from J_A and J_U on $\mathbf{l}_2$.

5. JACOBI TYPE BLOCK MATRICES DESCRIBED IN THEOREM 3

We will find the condition which guarantee that the matrix J_U of type (13) is a unitary operator and commute with the matrix J_A of type (9) that is bounded hermitian operator. The formal adjoint matrix J_{U^*} has the form

$$J_{U^*} = \begin{bmatrix} w_0^* & u_0^* & 0 & 0 & \cdots \\ v_0^* & w_1^* & u_1^* & 0 & \cdots \\ 0 & v_1^* & w_2^* & u_2^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{array}{rcccl} v_n^* & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_{n+1}, \\ w_n^* & : & \mathcal{H}_n & \longrightarrow & \mathcal{H}_n, \\ u_n^* & : & \mathcal{H}_{n+1} & \longrightarrow & \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{121}$$

Multiplying matrices (13) and (121) we get $J_U J_{U^*}$. The expression for $J_{U^*} J_U$ is obtained analogous with the change u_n, w_n and v_n on v_n^*, w_n^* and u_n^* respectively and vice versa. Comparing these expressions for $J_U J_{U^*}$ and $J_{U^*} J_U$ we conclude that the equality $J_U J_{U^*} = J_{U^*} J_U$ is equivalent to fulfillment of the following system of equalities (we take into account that w_0 is a scalar, $w_0^* = \bar{w}_0$) $n \in \mathbb{N}_0$:

$$\begin{aligned} & v_0 v_0^* = u_0^* u_0; \\ & v_n u_{n+1}^* = u_n^* v_{n+1}, \\ & w_n u_n^* + v_n w_{n+1}^* = w_n^* v_n + u_n^* w_{n+1} \equiv 0, \\ & u_n u_n^* + w_{n+1} w_{n+1}^* + v_{n+1} v_{n+1}^* = v_n^* v_n + w_{n+1}^* w_{n+1} + u_{n+1}^* u_{n+1}. \end{aligned} \tag{122}$$

Taking initial matrices u_0, w_0, v_0 and finding from (122) step by step u_1, w_1, v_1; u_2, w_2, v_2; ... etc. (in non-uniquely manner) we can construct some normal matrix J_U.

Multiplying matrices (9) and (13) we get $J_A J_U$. The expression for $J_U J_A$ is obtained analogously with the change of u_n, w_n, v_n and a_n, b_n, c_n.

Comparing these expressions for $J_A J_U$ and $J_U J_A$ we conclude that the equality $J_U J_A = J_A J_U$ is equivalent to fulfillment of the following system of equalities (we take into account that w_0 and b_0 are a scalar and $b_0 = \bar{b}_0$) $n \in \mathbb{N}_0$:

$$\begin{aligned}
&c_0 u_0 = v_0 a_0;\\
&c_n v_{n+1} = v_n c_{n+1},\\
&a_{n+1} u_n = u_{n+1} a_n,\\
&b_n v_n + c_n w_{n+1} = w_n c_n + v_1 b_{n+1},\\
&a_n w_n + b_{n+1} u_n = u_n b_n + w_{n+1} a_n,\\
&a_n v_n + b_{n+1} w_{n+1} + c_{n+1} u_{n+1} = u_n c_n + w_{n+1} b_{n+1} + v_{n+1} a_{n+1}.
\end{aligned} \tag{123}$$

Taking initial matrices a_0, b_0, c_0 and the matrix J_U finding step by step a_1, b_1, c_1; a_2, b_2, c_2; … from (123) etc. (in non-uniquely manner) we can construct some matrix J_A, commuting with J_U.

But for such matrices Theorem 4 in general is not valid, because it is necessary to find these matrices in such way that u_n and v_n must be of the form (14), (15), (16) and such matrix must commute on l_{fin} with the matrix U_A of the form (10), (11), (12). Only in this case according to Lemma 8 and (109) the functions (23) are linearly independent and Theorem 4 is applicable.

Remark 3. The describing and convenient parameterizations of matrices u_n, w_n and v_n, $n \in \mathbb{N}_0$ which are a solution of equations (122), (123) and u_n, v_n, a_n, c_n have the form (14), (15), (16) and (10), (11), (12) respectively is an enough complicated open problem. With this connections we propose only some example.

Example. Let us put for the matrix (9) with conditions (10), (11), (12) following forms:

$$a_{n;j+1,j} = c_{n;j,j+1} = \alpha_{n-|n-j|} > 0, \quad n \in \mathbb{N}_0; \tag{124}$$
other elements are equal zero.

$$u_{n;j,j} = v_{n;j,j+2} = 1, \quad 0 = 1, 2, \dots, n, \quad n \in \mathbb{N}_0; \tag{125}$$
other elements are equal zero.

Hence the matrix J_A generated by (124) commutes with the matrix J_U generated by (125) on the finite vectors. (It is not difficult to verify.) If additionally $\sum_{n=1}^{\infty} |\alpha_i\| < \infty$ in (124), then A is obviously bounded operator and satisfies the condition of this article.

We have another non trivial examples of matrices J_U and J_A such that satisfies the conditions described in mains theorems, bat we are out of the admissible for authors scope of the article.

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