Methods of Functional Analysis and Topology
Vol. 18 (2012), no. 2, pp. 140–151

THE DIRICHLET PROBLEM FOR DIFFERENTIAL EQUATIONS IN A BANACH SPACE

M. L. GORBACHUK AND V. I. GORBACHUK

Dedicated to the blessed memory of A. G. Kostyuchenko

Abstract. In the paper, we consider an abstract differential equation of the form $\left(\frac{\partial^2}{\partial t^2} - B\right)^m y(t) = 0$, where B is a positive operator in a Banach space $\mathfrak{B}$. For solutions of this equation on $(0, \infty)$, it is established the analogue of the Phragmen-Lindelöf principle on the basis of which we show that the Dirichlet problem for the above equation is uniquely solvable in the class of vector-valued functions admitting an exponential estimate at infinity. The Dirichlet data may be both usual and generalized with respect to the operator $-B^{1/2}$.The formula for the solution is given, and some applications to partial differential equations are adduced.

1. Let $\mathfrak{B}$ be a Banach space with norm $\|\cdot\|$ over the field $\mathbb{C}$ of complex numbers, and let $E(\mathfrak{B})$ $(L(\mathfrak{B}))$ be the set of all densely defined closed (bounded) linear operators on $\mathfrak{B}$. In what follows $\{e^{tA}\}_{t\geq 0}$ denotes a C_0-semigroup of bounded linear operators on $\mathfrak{B}$ with infinitesimal generator A (for the theory of semigroups of linear operators on Banach and locally convex spaces we refer, for instance, to [1–4]). Recall only that a family $\mathbf{U} = \{U(t)\}_{t\geq 0}$ of operators $U(t) \in L(\mathfrak{B})$ forms a C_0-semigroup on $\mathfrak{B}$ if the following three properties are satisfied:
1) $U(0) = I$, the identity operator on $\mathfrak{B}$;
2) $U(t)U(s) = U(t+s)$ for all $t, s \geq 0$;
3) $\lim\limits_{t\to 0} \|U(t)x - x\| = 0$ for all $x \in \mathfrak{B}$.

The infinitesimal generator of $\mathbf{U}$, or briefly the generator, is the linear operator A with domain $\mathcal{D}(A)$ defined by

$$\mathcal{D}(A) = \left\{x \in \mathfrak{B} : \lim_{t\to 0} \frac{1}{t}(U(t)x - x) \text{ exists}\right\},$$

$$Ax = \lim_{t\to 0} \frac{1}{t}(U(t)x - x),\ x \in \mathcal{D}(A).$$

This operator is closed, its domain $\mathcal{D}(A)$ is dense in $\mathfrak{B}$ and $\mathbf{U}$-invariant, i.e. $U(t)x \in \mathcal{D}(A)$ for all $x \in \mathcal{D}(A),\ t \geq 0$, and $AU(t)x = U(t)Ax$. Moreover,

$$\frac{d}{dt}U(t)x = AU(t)x, \quad x \in \mathcal{D}(A).$$

Finally, assume that $\ker e^{tA} = \{0\}$ for any $t > 0$. Without loss of generality it may be also supposed $\{e^{tA}\}_{t\geq 0}$ to be a contraction semigroup.

We introduce the space $\mathfrak{B}_{-t}(A),\ t > 0$, as a completion of $\mathfrak{B}$ in the norm

$$\|x\|_{-t} = \left\|e^{tA}x\right\|.$$

2000 *Mathematics Subject Classification.* Primary 34G10.

Key words and phrases. Closed operator, entire vector, differential equation in a Banach space, C_0-semigroup and C_0-group of linear operators, analytic C_0-semigroup, entire vector-valued function, the Phragment-Lindelöf principle, the Dirichlet problem.

The norms $\|\cdot\|_{-t},\ t>0$, are compatible and comparable on $\mathfrak{B}$. So, for $t<t'$ we have a dense and continuous embedding

$$\mathfrak{B}\subseteq\mathfrak{B}_{-t}(A)\subseteq\mathfrak{B}_{-t'}(A). \tag{1}$$

We put $\mathfrak{B}_{-}(A)=\bigcap_{t>0}\mathfrak{B}_{-t}(A)$ and endow $\mathfrak{B}_{-}(A)$ with the projective limit topology of the Banach spaces $\mathfrak{B}_{-t}(A)$

$$\mathfrak{B}_{-}(A)=\operatorname{proj}\lim_{t\to 0}\mathfrak{B}_{-t}(A).$$

Observe that it suffices for obtaining $\mathfrak{B}_{-}(A)$ to restrict ourselves to the spaces $\mathfrak{B}_{-\frac{1}{n}}(A),\ n\in\mathbb{N}$. Thus, $\mathfrak{B}_{-}(A)$ is a complete countably normed space. (As for countably normed spaces and operators on them see [5]).

The operator e^{tA} admits a continuous extension $\widetilde{U}(t)$ onto $\mathfrak{B}_{-t}(A)$. Moreover, because of continuity of embedding (1),

$$\widetilde{U}(t')\restriction_{\mathfrak{B}_{-t}(A)}=\widetilde{U}(t)\quad\text{as}\quad 0<t<t'. \tag{2}$$

On the space $\mathfrak{B}_{-}(A)$ we define the operators $U(t)$ in such a way:

$$\forall x\in\mathfrak{B}_{-}(A):U(t)x=\widetilde{U}(t)x\quad\text{as}\quad t>0;\quad U(0)x=x.$$

In view of (2), this definition is correct.

The family $\mathbf{U}=\{U(t)\}_{t\ge 0}$ forms an equicontinuous C_0-semigroup on $\mathfrak{B}_{-}(A)$. As has been shown in [6 - 8], this semigroup possesses the following properties:

(i) $\forall t>0:U(t)\mathfrak{B}_{-}(A)\subseteq\mathfrak{B}$;

(ii) $\forall t>0,\forall x\in\mathfrak{B}:U(t)x=e^{tA}x$;

(iii) $\forall t>0,\forall x\in\mathfrak{B}_{-}(A):U(t+s)x=e^{tA}U(s)x=e^{sA}U(t)x$.

If the semigroup $\{e^{tA}\}_{t\ge 0}$ is differentiable on $(0,\infty)$, then (see [9]) the embedding of $\mathfrak{B}$ into $\mathfrak{B}_{-}(A)$ is strict, the generator $\widehat{A}$ of the semigroup $\{U(t)\}_{t\ge 0}$ is defined and continuous on the whole space $\mathfrak{B}_{-}(A)$, and $\widehat{A}$ is the closure of A in $\mathfrak{B}_{-}(A)$. So, the semigroup $\{e^{t\widehat{A}}\}_{t\ge 0}$ is infinitely differentiable in $\mathfrak{B}_{-}(A)$ on $[0,\infty)$. If $0\in\rho(A)$, ($\rho(\cdot)$ denotes the resolvent set of an operator), then the operator $\widehat{A}$ has a continuous inverse defined on the whole $\mathfrak{B}_{-}(A)$.

2. Consider now the equation

$$\left(\frac{d^2}{dt^2}-B\right)^m y(t)=0,\quad t\in(0,\infty),\quad m\in\mathbb{N}, \tag{3}$$

where B is a positive operator on $\mathfrak{B}$, that is, $B\in E(\mathfrak{B}),\ (-\infty,0]\subset\rho(B)$, and there exists a constant $M>0$ such that

$$\forall\lambda\ge 0:\|R_B(-\lambda)\|\le\frac{M}{1+\lambda}$$

($R_B(\lambda)=(B-\lambda I)^{-1}$ is the resolvent of the operator B).

As has been shown in [10, 11], for such an operator the fractional powers $B^\alpha,\ 0<\alpha<1$, can be determined, and the operator $A=-B^{1/2}$ is the generator of a bounded analytic semigroup $\{e^{tA}\}_{t\ge 0}$ on $\mathfrak{B}$ with negative type

$$\omega(A):=\lim_{t\to\infty}\frac{\ln\|e^{tA}\|}{t}=\inf_{t\in(0,\infty)}\frac{\ln\|e^{tA}\|}{t}=\sup\{\Re\lambda:\lambda\in\sigma(A)\}$$

($\sigma(\cdot)$ denotes the spectrum of an operator).

By a (classical) solution of equation (3) on $(0,\infty)$ we mean $2m$ times continuously differentiable on $(0,\infty)$ $\mathfrak{B}$-valued vector function $y(t)$ such that for any $t\in(0,\infty),\ y^{(2k)}(t)\in\mathcal{D}(B^{m-k})$ $(k=0,1,\dots,m)$, $B^{m-k}y^{(2k)}(t)$ are continuous in $\mathfrak{B}$ on $(0,\infty)$, and $y(t)$ satisfies (3). All the solutions of (3) on $(0,\infty)$ were described in [9]. Moreover, it was proved there that every solution $y(t)$ and its arbitrary order derivatives have boundary values at the point 0 in the space $\mathfrak{B}_-(A)$, i.e.

$$\lim_{t\to 0} y^{(k)}(t) = y_k \in \mathfrak{B}_-(A),$$

where the limit is taken in $\mathfrak{B}_-(A)$-topology.

The Dirichlet problem for equation (3) consists in finding solutions of this equation on $(0,\infty)$ satisfying the conditions

$$y^{(k)}(0) := \lim_{t\to 0} y^{(k)}(t) = y_k \in \mathfrak{B}_-(A), \quad k = 0,1,\dots,m-1. \tag{4}$$

In the case of $m = 1$, it was considered in [11–15].

Let us start from the homogeneous Dirichlet problem

$$y^{(k)}(0) = 0, \quad k = 0,1,\dots,m-1. \tag{5}$$

To solve this problem, we need the space of entire vectors of the operator A (see [16]), that is, the space

$$\mathfrak{G}_{(1)}(A) = \operatorname*{proj\,lim}_{0<\alpha\to 0} \mathfrak{G}_1^\alpha(A) = \bigcap_{\alpha>0} \mathfrak{G}_1^\alpha(A),$$

where

$$\mathfrak{G}_1^\alpha(A) = \left\{ x \in \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(A^n) \middle| \exists c = c(x) > 0, \forall k \in \mathbb{N}_0 = \{0\}\cup\mathbb{N} : \|A^k x\| \le c\alpha^k k^k \right\}$$

is a Banach space with norm

$$\|x\|_{\mathfrak{G}_1^\alpha(A)} = \sup_{k\in\mathbb{N}_0} \frac{\|A^k x\|}{\alpha^k k^k}.$$

The convergence in $\mathfrak{G}_{(1)}(A)$ means the convergence in each $\mathfrak{G}_1^\alpha(A)$, $\alpha > 0$. Note that $\mathfrak{G}_{(1)}(A)$ may be obtained if we confine ourselves only to $\alpha = \frac{1}{n}$, $n \in \mathbb{N}$. So, $\mathfrak{G}_{(1)}(A)$ is countably normed. Since the semigroup $\{e^{tA}\}_{t\ge 0}$ is analytic, the set $\mathfrak{G}_{(1)}(A)$ is dense in $\mathfrak{B}$,

$$\mathfrak{G}_{(1)}(A) = \bigcap_{t>0} \mathcal{R}(e^{tA})$$

($\mathcal{R}(\cdot)$ denotes the range of an operator), and the operator-valued function

$$\exp(zA) := \sum_{k=0}^{\infty} \frac{z^k}{k!} A^k$$

is entire in the space $\mathfrak{G}_{(1)}(A)$ (see [6, 17]). Moreover, the family $\{\exp(zA)\}_{z\in\mathbb{C}}$ forms an one-parameter C_0-group on $\mathfrak{G}_{(1)}(A)$, and

$$\forall x \in \mathfrak{G}_{(1)}(A) : \exp(tA)x = \begin{cases} e^{tA}x & \text{as} \quad t \ge 0, \\ \left(e^{-tA}\right)^{-1} x & \text{as} \quad t < 0. \end{cases}$$

Theorem 1. *The set of all solutions of the homogeneous Dirichlet problem* (3), (5) *represents an infinite-dimensional linear space, and every solution of this problem is entire vector-valued function in* $\mathfrak{G}_{(1)}(A)$.

Proof. In accordance with [9], a vector-valued function $y(t)$ is a solution of equation (3) on $(0,\infty)$ if and only if it may be written in the form

$$y(t) = y_1(t) + y_2(t), \tag{6}$$

where

$$y_1(t) = \sum_{k=0}^{m-1} t^k e^{t\widehat{A}} f_k, \quad f_k \in \mathfrak{B}_-(A),$$

$$y_2(t) = \sum_{k=0}^{m-1} t^k \exp(-tA) g_k, \quad g_k \in \mathfrak{G}_{(1)}(A),$$

with vectors f_k, g_k being uniquely determined by $y(t)$.

It is not difficult to count that

$$\left(\frac{d}{dt} - \widehat{A}\right)^k y_1(t) = \sum_{i=k}^{m-1} i(i-1)\dots(i-k+1)t^{i-k}e^{t\widehat{A}} f_i, \quad t > 0, \quad k = 0, 1, \dots, m-1. \tag{7}$$

On the other hand,

$$\left(\frac{d}{dt} - \widehat{A}\right)^k y_1(t) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i \widehat{A}^{k-i} y_1^{(i)}(t).$$

The continuity of $\widehat{A}$ in $\mathfrak{B}_-(A)$ implies

$$\lim_{t\to 0} \left(\frac{d}{dt} - \widehat{A}\right)^k y_1(t) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i \widehat{A}^{k-i} y_1^{(i)}(0),$$

whence, by virtue of (7),

$$k! f_k = \sum_{i=0}^{k} (-1)^{k-i} C_k^i \widehat{A}^{k-i} y_1^{(i)}(0), \quad k = 0, 1, \dots, m-1.$$

So,

$$\sum_{i=0}^{k} (-1)^{k-i} C_k^i \widehat{A}^{k-i} y^{(i)}(0) = k! f_k + \sum_{i=0}^{k} (-1)^{k-i} C_k^i A^{k-i} y_2^{(i)}(0), \quad k = 0, 1, \dots, m-1.$$

Hence, in the case of homogeneous Dirichlet problem,

$$f_k = -\frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} C_k^i A^{k-i} y_2^{(i)}(0), \quad k = 0, 1, \dots, m-1.$$

Then the inclusion

$$\sum_{i=0}^{k} (-1)^{k-i} C_k^i A^{k-i} y_2^{(i)}(0) \in \mathfrak{G}_{(1)}(A)$$

implies $f_k \in \mathfrak{G}_{(1)}(A)$ and this, in turn, makes possible to conclude that the vector-valued function $y(t)$ is entire in $\mathfrak{G}_{(1)}(A)$.

The infinite dimension of the set of all solutions of problem (3), (5) is verified at least due to the fact that vector-valued functions

$$y(t) = \frac{\sinh tA}{A} g = \frac{\exp(tA) - \exp(-tA)}{A} g,$$

where g goes through the whole space $\mathfrak{G}_{(1)}(A)$, are its solutions. □

As is seen from the proof of Theorem 1, in the case of $m = 1$, all the solutions of problem (3), (5) on $(0, \infty)$ are given by the formula

$$y(t) = \frac{\sinh tB^{1/2}}{B^{1/2}} \widetilde{g}_0, \quad \widetilde{g}_0 \in \mathfrak{G}_{(1)}(B^{1/2}). \tag{8}$$

Indeed, in this case, because of (6), a vector-valued function $y(t)$ is a solution of this problem if and only if

$$y(t) = e^{t\widehat{A}} f_0 + \exp(-tA) g_0, \quad f_0 \in \mathfrak{B}_-(A), \ g_0 \in \mathfrak{G}_{(1)}(A).$$

Since

$$y(0) = f_0 + g_0 = 0,$$

we arrive at (8) with $\widetilde{g}_0 = -2B^{1/2}g_0 \in \mathfrak{G}_{(1)}(B^{1/2})$.

If $m = 2$, then, in view of (6), every solution of (3), (5) has the form

$$y(t) = e^{t\widehat{A}}f_0 + te^{t\widehat{A}}f_1 + \exp(-tA)g_0 + t\exp(-tA)g_1,$$
$$f_k \in \mathfrak{B}_-(A), \quad g_k \in \mathfrak{G}_{(1)}(A), \quad k = 0, 1.$$

The condition $y(0) = y'(0) = 0$ yields

$$f_0 = -g_0 \in \mathfrak{G}_{(1)}(A), \quad f_1 = 2Ag_0 - g_1 \in \mathfrak{G}_{(1)}(A),$$

whence

$$y(t) = \left(\frac{\sinh tB^{1/2}}{B^{1/2}} - te^{tB^{1/2}}\right)\widetilde{g}_0 + \frac{t\sinh tB^{1/2}}{B^{1/2}}\widetilde{g}_1,$$

where $\widetilde{g}_0 = -2B^{1/2}g_0, \widetilde{g}_1 = -2g_1 \in \mathfrak{G}_{(1)}(B^{1/2})$.

It turns out to be that under certain conditions on behavior on infinity of solutions of (3), (5), this problem has at most the trivial one. To show this, we shall first prove that the analog of the Phragmen-Lindelöf principle for analytic functions is valid for solutions of equation (3) on $(0, \infty)$. Namely, the following assertion holds.

Theorem 2. *(Analog of the Phragmen-Lindelöf principle). Let $\omega = -\omega(A) > 0$ where $\omega(A)$ is the type of $\left\{e^{tA}\right\}_{t\geq 0}$, $A = -B^{1/2}$. If for a solution $y(t)$ of equation (3) on $(0, \infty)$ the relation*

$$\forall\omega' \in (0, \omega), \exists c_{\omega'} > 0 : \|y(t)\| \leq c_{\omega'}e^{\omega' t} \tag{9}$$

is fulfilled for large enough $t > 0$ ($c_{\omega'} = \mathrm{const}$), then

$$\forall\omega'' \in (0, \omega)\exists c_{\omega''} > 0 : \|y(t)\| \leq c_{\omega''}e^{-\omega'' t}, \quad t > 0. \tag{10}$$

for sufficiently large $t > 0$

Proof. Let $y(t)$ be a solution of equation (3) on $(0, \infty)$ satisfying (9). Then $y(t)$ may be represented in the form (6). It is not hard to show that for arbitrary fixed $\varepsilon > 0$ and $t_0 > 0$, there exists a constant $c_\varepsilon = c_\varepsilon(t_0) > 0$ such that

$$\forall t \geq t_0 : \|y_1(t)\| \leq c_\varepsilon e^{-(\omega-\varepsilon)t}. \tag{11}$$

Indeed, for any $f \in \mathfrak{B}_-(A)$ and $t \geq t_0$, the relation

$$\left\|e^{t\widehat{A}}f\right\| = \left\|e^{(t-t_0)A}e^{t_0\widehat{A}}f\right\| \leq \left\|e^{(t-t_0)A}\right\| \|\widetilde{f}\|$$

holds with $\widetilde{f} = e^{t_0\widehat{A}}f \in \mathfrak{B}$, whence, by definition of the type of a semigroup, for any $\delta > 0, t \geq t_0$,

$$\left\|e^{t\widehat{A}}f\right\| \leq c'_\delta e^{-(\omega-\delta)(t-t_0)}\|\widetilde{f}\| = c_\delta e^{-(\omega-\delta)t}, \tag{12}$$

where $c_\delta = c'_\delta e^{(\omega-\delta)t_0}\|\widetilde{f}\|$. This implies

$$\|y_1(t)\| = \left\|\sum_{k=0}^{m-1} t^k e^{t\widehat{A}}f_k\right\| \leq \sum_{k=0}^{m-1} t^k \left\|e^{t\widehat{A}}f_k\right\|$$

$$\leq \sum_{k=0}^{m-1} t^k c_{k\delta}e^{-(\omega-\delta)t} \leq \sum_{k=0}^{m-1} c_{k,\delta}e^{-(\omega-2\delta)t} = c_\varepsilon e^{-(\omega-\varepsilon)t}, \tag{13}$$

Here $t \geq t_0, \varepsilon = 2\delta > 0, c_\varepsilon = \sum_{k=0}^{m-1} c_{k\delta}$, $c_{k\delta} > 0$ is the constant in (12) when it is put $f = f_k$ there.

Now let $g \in \mathfrak{G}_{(1)}(A)$. Since for any $\varepsilon > 0$,

$$\|g\| = \left\|e^{tA}\exp(-tA)g\right\| \leq \left\|e^{tA}\right\| \|\exp(-tA)g\| \leq c'_\varepsilon e^{-(\omega-\varepsilon)t}\|\exp(-tA)g\|,$$

$$0 < c'_\varepsilon = \text{const},$$

we obtain

$$\forall t > 0 : \| \exp(-tA)g\| \geq c''_\varepsilon e^{(\omega-\varepsilon)t}\|g\|,$$

($c''_\varepsilon = c'^{-1}_\varepsilon$ does not depend on t). It follows from here that for any $\varepsilon > 0$,

$$\|y_2(t)\| = \| \exp(-tA)g(t)\| \geq c''_\varepsilon e^{(\omega-\varepsilon)t}\|g(t)\|,$$

where

$$g(t) = \sum_{k=0}^{m-1} t^k g_k \in \mathfrak{G}_{(1)}(A).$$

Suppose $y_2(t) \not\equiv 0$. This means that in representation (6) there exists at least one $g_k \neq 0$, $k = 0, 1, \ldots, m-1$; for the sake of simplicity we set $g_{m-1} \neq 0$. Then

$$\begin{aligned}\|y_2(t)\| &\geq c''_\varepsilon e^{(\omega-\varepsilon)t}\left(t^{m-1}\|g_{m-1}\| - \left\|\sum_{k=0}^{m-2} t^k g_k\right\|\right)\\ &= c''_\varepsilon e^{(\omega-\varepsilon)t} t^{m-1}\left(\|g_{m-1}\| - \left\|\sum_{k=0}^{m-2} t^{k-m+1} g_k\right\|\right).\end{aligned}$$

Taking into account that

$$\left\|\sum_{k=0}^{m-2} t^{k-m+1} g_k\right\| \to 0 \quad \text{as} \quad t \to \infty,$$

and for sufficiently large $t > 0$,

$$t^{m-1} > e^{-\varepsilon t},$$

we conclude that

$$\forall \varepsilon > 0 : \|y_2(t)\| \geq \widetilde{c}_\varepsilon e^{(\omega-2\varepsilon)t}$$

($t > 0$ is large enough); moreover, the constant $\widetilde{c}_\varepsilon$ does not depend on t.

On the other hand, due to (9), (13),

$$\begin{aligned}\forall \varepsilon \in (0,\omega), \exists \omega' \in (0,\omega) : \|y_2(t)\| &= \|y(t) - y_1(t)\| \leq \|y(t)\| + \|y_1(t)\|\\ &\leq c_{\omega'} e^{\omega' t} + c_\varepsilon e^{-(\omega-\varepsilon)t} \leq \widetilde{c}_{\omega'} e^{\omega' t}.\end{aligned}$$

Thus, we have a two-sided estimate

$$\widetilde{c}_\varepsilon e^{(\omega-2\varepsilon)t} \leq \|y_2(t)\| \leq \widetilde{c}_{\omega'} e^{\omega' t}.$$

Putting

$$\varphi(t) = \frac{\|y_2(t)\|}{\widetilde{c}_\varepsilon e^{(\omega-2\varepsilon)t}}, \quad \varepsilon = \frac{\omega-\omega'}{4},$$

we get for sufficiently large $t > 0$,

$$1 \leq \varphi(t) \leq c_{\varepsilon\omega'} e^{t\frac{\omega'-\omega}{2}},$$

where the constant $c_{\varepsilon\omega'} = \frac{\widetilde{c}_{\omega'}}{\widetilde{c}_\varepsilon} > 0$ does not depend on t, which is impossible as $\frac{\omega'-\omega}{2} < 0$. So, in representation (6) $y_2(t) \equiv 0$. Setting $\omega'' = \omega - \varepsilon$, we obtain

$$\forall \omega'' \in (0,\omega) : \|y(t)\| = \|y_1(t)\| \leq c_{\omega''} e^{-\omega'' t}.$$

□

Theorem 3. *In the class of vector-valued functions $y(t)$ satisfying the condition*

$$\forall \varepsilon \in (0,\omega), \exists c_\varepsilon > 0 : \|y(t)\| \leq c_\varepsilon e^{(\omega-\varepsilon)t} \tag{14}$$

for sufficiently large $t > 0$, the homogeneous Dirichlet problem (3), (5) has only a trivial solution.

Proof. When proving the previous theorem, we had established that if a solution $y(t)$ of equation (3) on $(0,\infty)$ satisfies (14), then in its representation (6) $y_2(t)\equiv 0$. Thus,

$$y(t)=\sum_{k=0}^{m-1} t^k e^{t\widehat{A}} f_k, \quad f_k \in \mathfrak{B}_-(A).$$

If, in addition, the condition (5) holds true, then the formula for $k!f_k$ from the proof of Theorem 1 shows that $f_k=0$ for all $k=0,1,\dots,m-1$, which is what had to be proved. □

3. Let $A=-B^{1/2}$. For any $x\in\mathcal{D}(A^n), n\in\mathbb{N}$, we put

$$\|x\|_{\mathfrak{H}^n(A)}=\|A^n x\|.$$

The space

$$\mathfrak{H}^n(A)=\left(\mathcal{D}(A^n),\ \|\cdot\|_{\mathfrak{H}^n(A)}\right)$$

is the Sobolev space of order n associated to $\left\{e^{tA}\right\}_{t\ge 0}$ (see [4]). Define the space $\mathfrak{H}^{-n}(A)$ as a completion of $\mathfrak{B}$ in the norm

$$\|x\|_{\mathfrak{H}^{-n}(A)}=\|A^{-n}x\|.$$

Obviously,

$$\mathfrak{H}^n(A)\subseteq\mathfrak{B}\subseteq\mathfrak{H}^{-n}(A)\subseteq\mathfrak{B}_-(A). \tag{15}$$

Moreover (see [4]), the embeddings (15) are dense and continuous, $A\mathfrak{H}^n(A)=\mathfrak{H}^{n-1}(A)$, $\left\{e^{tA}\restriction_{\mathfrak{H}^n(A)}\right\}_{t\ge 0}$ is a C_0-semigroup on $\mathfrak{H}^n(A)$, and the extensions of $e^{tA}, t\ge 0$, to $\mathfrak{H}^{-n}(A)$ form a C_0-semigroup on $\mathfrak{H}^{-n}(A)$.

4. Pass now to the inhomogeneous Dirichlet problem (3), (4).

Theorem 4. *There exists a unique solution of problem* (3), (4) *in the class of vector-valued functions $y(t)$ satisfying* (9)*. Moreover, if $y^{(i)}(0)=y_i\in\mathfrak{H}^{p-i}(A),\ p\in\mathbb{Z}$, then for the corresponding solution $y(t)$, the vector-valued functions $y^{(i)}(t),\ i=0,1,\dots,m-1$, are continuous on $[0,\infty)$ in the space $\mathfrak{H}^{p-i}(A)$*

Proof. As was shown in the proof of Theorem 3, every solution of (3) satisfying (9) may be represented in the form

$$y(t)=\sum_{k=0}^{m-1} t^k e^{t\widehat{A}} f_k, \quad f_k \in \mathfrak{B}_-(A), \tag{16}$$

where,

$$k!f_k=\sum_{i=0}^{k}(-1)^{k-i}C_k^i\widehat{A}^{k-i}y_i, \quad k=0,1,\dots,m-1. \tag{17}$$

Thus, f_k are uniquely determined by the vectors $y_i=y^{(i)}(0)$. Then the uniqueness of a solution of (3), (4) in the class of vector-valued functions satisfying (9) follows from Theorem 3.

Let X be an arbitrary Banach space, and $\{T(t)\}_{t\ge 0}$ be a bounded analytic C_0-semigroup on X with generator $A,\ 0\in\rho(A)$. Then the vector-valued function

$$y_k(t)=t^kA^kT(t)x, \quad k\in\mathbb{N}, \quad x\in X,$$

is continuous in X, and

$$\forall k\in\mathbb{N}: y_k(t)\to 0 \quad \text{in} \quad X \quad \text{as} \quad t\to 0.$$

Indeed, the bounded analyticity of $\{T(t)\}_{t\ge 0}$ implies (see e.g. [1–4]) the inequality

$$\|t^kA^kT(t)\|_X\le c_k, \quad 0<c_k=\text{const}.$$

If $x \in \mathcal{D}(A^k)$, then $x = A^{-k}g,\ g \in X$, and

$$\forall k \in \mathbb{N} : \|t^k A^k T(t)x\|_X = t^k \|T(t)g\|_X \to 0 \quad \text{as} \quad t \to 0.$$

Taking into account the density of $\mathcal{D}(A^k)$ in X, we conclude, on the basis of the Banach-Steinhaus theorem, that the vector-valued function $y_k(t) = t^k A^k T(t)x$ is continuous for any $x \in X$; moreover, for any $k \in \mathbb{N},\ y_k(t) \to 0$ as $t \to 0$.

Assume now $A = -B^{1/2}$ and $y^{(i)}(0) = y_i \in \mathfrak{H}^{p-i}(A)$ with some $p \in \mathbb{Z}$. Then, because of (17),

$$f_k = \widehat{A}^k h_k,$$

where

$$h_k = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}C_k^i \widehat{A}^{-i} y_i \in \mathfrak{H}^p(A), \quad k \in \mathbb{N}_0,$$

whence

$$y(t) = \sum_{k=0}^{m-1} t^k \widehat{A}^k e^{t\widehat{A}} h_k, \quad h_k \in \mathfrak{H}^p(A).$$

Setting in the above arguments $X = \mathfrak{H}^p(A),\ T(t) = e^{t\widehat{A}}$, we find that $y \in C([0,\infty), \mathfrak{H}^p(A))$, and $y(t) \to f_0 = y_0$ in the space $\mathfrak{H}^p(A)$ as $t \to 0$. Analogously,

$$y^{(i)}(t) = \widehat{A}^i \sum_{k=0}^{m-1} t^k \widehat{A}^k e^{t\widehat{A}} h_{ki} = \sum_{k=0}^{m-1} t^k \widehat{A}^k e^{t\widehat{A}} \widehat{A}^i h_{ki},$$

where $h_{ki} \in \mathfrak{H}^p(A)$ as a linear combination of h_k. So $\widehat{A}^i h_{ki} \in \mathfrak{H}^{p-i}(A)$. Putting $X = \mathfrak{H}^{p-i}(A)$ and $T(t) = e^{t\widehat{A}}$, we arrive at the inclusion $y^{(i)} \in C([0,\infty), \mathfrak{H}^{p-i}(A))$. □

5. Let $\mathfrak{B} = \mathfrak{H}$ be a Hilbert space with the scalar product $(\cdot,\cdot)$, and let A be a selfadjoint positive definite operator on $\mathfrak{H}$ whose spectrum $\sigma(A)$ is discrete. Denote by $\{\lambda_k\}_{k=1}^{\infty}$ the sequence of eigenvalues of A arranged in ascending order so that every eigenvalue is repeated according to its multiplicity. Let also $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis in $\mathfrak{H}$ composed from the eigenvectors of A. The operator $-A$ generates an analytic C_0-semigroup $\{e^{-tA}\}_{t\geq 0}$ acting on $\mathfrak{H}$ as

$$e^{-tA}f = \sum_{k=1}^{\infty} e^{-t\lambda_k}(f, e_k)e_k.$$

Put

$$\Phi_m(A) = \left\{ f \in \mathfrak{H} \middle| f = \sum_{k=1}^{m}(f, e_k)e_k \right\}.$$

Evidently, $\Phi_{m+1}(A) \supset \Phi_m(A)$. We set

$$\Phi(A) = \operatorname*{ind\,lim}_{m\to\infty} \Phi_m(A).$$

The set $\Phi(A)$ is A-invariant and dense in $\mathfrak{H}$. Since $\Phi_m(A)$ is a subspace of $\Phi_{m+1}(A)$, the above inductive limit is regular (see [18]). The convergence $f_n \to 0$ in $\Phi(A)$ as $n \to \infty$ means that

$$\exists m \in \mathbb{N},\ \forall n \in \mathbb{N} : f_n \in \Phi_m(A) \quad \text{and} \quad \|f_n\| \to 0, \quad n \to \infty.$$

The space $\Phi(A)$ is complete with respect to this convergence.

Denote by $\Phi'(A)$ the space of continuous antilinear functionals f on $\Phi(A)$ with the weak convergence:

$$\Phi'(A) \ni f_n \to f \text{ in } \Phi'(A),\ n \to \infty \Longleftrightarrow \forall \varphi \in \Phi(A) : (f_n, \varphi) \to (f, \varphi).$$

((f,φ) is the action of the functional $f \in \Phi'(A)$ onto $\varphi \in \Phi(A)$, which is the extension of the scalar product $(\cdot,\cdot)$ from to $\Phi'(A) \times \Phi(A)$ (see [19])). Associating with each $f \in \mathfrak{H}$ the functional $F_f \in \Phi'(A) : (F_f, \varphi) = (f, \varphi)$, we construct the chain

$$\Phi(A) \subset \mathfrak{H} \subset \Phi'(A)$$

of continuous embeddings. The elements of $\Phi'(A)$ are called generalized vectors connected with the operator A.

Denote by $\mathcal{S}$ the space of all number consequences $\{s_n\}_{n=1}^{\infty},\ s_n \in \mathbb{C}$, with convergence by coordinates, and associate with every $f \in \Phi'(A)$ the sequence

$$Jf = \{f_k\}_{k=1}^{\infty}, \quad f_k = (f, e_k).$$

The mapping $J : \Phi'(A) \mapsto \mathcal{S}$ is an isomorphism. Indeed, it is not difficult to see that if $f_1 \neq f_2,\ f_i \in \Phi'(A),\ i = 1, 2$, then there exists $k \in \mathbb{N}$ such that $f_{1k} \neq f_{2k}$. Otherwise, the equality $(f_1 - f_2, e_n) = 0$ for any $n \in \mathbb{N}$ implies $(f_1 - f_2, \varphi) = 0$ for an arbitrary $\varphi \in \Phi(A)$, that is, $f_1 = f_2$. Thus, J is an injection. Moreover, J is a bijection. Indeed, let f be the functional associated with a sequence $\{s_k\}_{k=1}^{\infty} \in \mathcal{S}$ in the way

$$\left(f, \sum_{k=1}^{n} \alpha_k e_k\right) = \sum_{k=1}^{n} \overline{\alpha_k} s_k.$$

This antilinear functional is continuous on $\Phi(A)$. In addition, the convergence in $\Phi'(A)$ is equivalent to the coordinate-wise one of the corresponding sequences from $\mathcal{S}$. So, J is one-to-one and mutually continuous map. Hence, $\Phi'(A)$ is complete under the weak convergence.

The isomorphism J maps $\Phi(A)$ onto the set of finite sequences from $\mathcal{S}$ and $\mathfrak{H}$ onto l_2. Moreover, the operation $\{f_k\}_{k=1}^{\infty} \mapsto \{\lambda_k f_k\}_{k=1}^{\infty}$ corresponds to the operator A. This operator may be extended to the continuous operator $\widehat{A} : \widehat{A}f = J^{-1}\{\lambda_k f_k\}_{k=1}^{\infty}$ on $\Phi'(A)$, and the following relations are valid:

$$\forall f \in \Phi'(A),\ \forall \varphi \in \Phi(A) : (\widehat{A}f, \varphi) = (f, A\varphi); \quad \left(e^{-t\widehat{A}} f, \varphi\right) = \left(f, e^{-tA}\varphi\right).$$

The series $\sum_{k=1}^{\infty} f_k e_k$, where $f_k = (f, e_k)$, is called the Fourier series of $f \in \Phi'(A)$. If $f \in \mathfrak{H}$, this series coincides with the usual expansion of f in orthonormal basis $\{e_k\}_{k=1}^{\infty}$.

Theorem 5. *Let $f \in \Phi'(A)$. Then its Fourier series converges to f in the $\Phi'(A)$-topology. Conversely, the sequence of partial sums of a series $\sum\limits_{k=1}^{\infty} s_k e_k,\ \{s_k\}_{k=1}^{\infty} \in \mathcal{S}$, converges in $\Phi'(A)$ to a certain $f \in \Phi'(A)$, and this series is the Fourier series of f.*

Proof. Let $f \in \Phi'(A)$ and $\sum\limits_{k=1}^{\infty} f_k e_k$ be its Fourier series. Under the map J, the partial sum $S_n(f) = \sum\limits_{k=1}^{n} f_k e_k$ is transformed into $s^n = \{f_1, \dots, f_n, 0, \dots, 0, \dots\} \in \mathcal{S}$. The sequence s^n converges to $Jf = \{f_k\}_{k=1}^{\infty}$. So, $S_n(f) = J^{-1}s^n \to f,\ n \to \infty$, in the space $\Phi'(A)$.

Take now an arbitrary series $\sum\limits_{k=1}^{\infty} s_k e_k,\ s_k \in \mathbb{C}$. The reasoning similar to the above one allows to conclude that this series converges to $f = J^{-1} \lim_{n\to\infty} JS_n$ in the space $\Phi'(A)$, where $S_n = \sum\limits_{k=1}^{n} s_k e_k$, and also

$$(f, e_k) = (J^{-1} \lim_{n\to\infty} JS_n, e_k) = \lim_{n\to\infty} (S_n, e_k) = s_k.$$

$\square$

Corollary 1. *The space $\Phi(A)$ is dense in $\Phi'(A)$.*

Theorem 5 shows that $\Phi'(A)$ may be considered as the space of formal series of the form $\sum_{k=1}^{\infty} f_k e_k$, and this, in turn, makes possible in the situation under consideration to describe various subspaces of usual and generalized vectors connected with the operator A in terms of behavior of their Fourier coefficients. For instance, for the spaces $\mathfrak{B}_-(A)$ and $\mathfrak{G}_{(1)}(A)$ the following theorem is valid.

Theorem 6. *Let $f = \sum_{k=1}^{\infty} f_k e_k \in \Phi'(A)$. Then*

$$f \in \mathfrak{B}_-(A) \Longleftrightarrow \forall t > 0,\ \exists c = c(t) > 0 : |f_k| \le c e^{t\lambda_k}; \tag{18}$$

$$f \in \mathfrak{G}_{(1)}(A) \Longleftrightarrow \forall t > 0,\ \exists c = c(t) > 0 : |f_k| \le c e^{-t\lambda_k}. \tag{19}$$

Proof. Let $f \in \mathfrak{B}_-(A)$, i.e. $f \in \mathfrak{B}_{-t}(A)$ for any $t > 0$ Then

$$\left\| e^{-t\widehat{A}} f \right\|^2 = \sum_{k=1}^{\infty} \left| \left(e^{-t\widehat{A}} f, e_k \right) \right|^2 = \sum_{k=1}^{\infty} \left| \left(f, e^{-tA} e_k \right) \right|^2 = \sum_{k=1}^{\infty} e^{-2t\lambda_k} |f_k|^2.$$

As the latter series is convergent, we have

$$\forall t > 0,\ \exists c = c(t) > 0 : e^{-t\lambda_k} |f_k| < c.$$

Likewise, it is proved that the inclusion $f \in \mathfrak{G}_{(1)}(A)$ implies the inequality in (19).

Assume now that for $f \in \Phi'(A)$, the inequality in (18) is fulfilled. It follows from here that the series $\sum_{k=1}^{\infty} e^{-2t\lambda_k} |f_k|^2$ converges for any $t > 0$, and its sum is equal to $\left\| e^{-t\widehat{A}} f \right\|^2$, whence $f \in \mathfrak{B}_{-t}(A)$ for any $t > 0$, that is, $f \in \mathfrak{B}_-(A)$.

Further, if the inequality in (19) is satisfied, then the series $\sum_{k=1}^{\infty} e^{2t\lambda_k} |f_k|^2$ converges for any $t > 0$. This means that $f \in \mathcal{D}(e^{tA})$ for any $t > 0$. The relation

$$\mathfrak{G}_{(1)}(A) = \bigcap_{t>0} \mathcal{R}(e^{-tA}) = \bigcap_{t>0} \mathcal{D}(e^{tA})$$

shows that $f \in \mathfrak{G}_{(1)}(A)$. $\square$

6. Examples. Put $\mathfrak{B} = L_2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^q$ with piecewise smooth boundary $\partial\Omega$, and denote by B' the operator generated on $L_2(\Omega)$ by the differential expression

$$(\mathcal{L}u)(x) = -\sum_{i=1}^{q} \sum_{k=1}^{q} \frac{\partial}{\partial x_i} \left(a_{ik}(x) \frac{\partial u(x)}{\partial x_k} \right) + c(x) u(x), \tag{20}$$

on

$$\mathcal{D}(B') = \left\{ u \in C^2(\overline{\Omega}) \middle| u \restriction_{\partial\Omega} = 0 \right\}. \tag{21}$$

It is assumed that $a_{ik}(x), c(x) \in C^\infty(\overline{\Omega}),\ c(x) \ge 0$. Suppose also the expression (20) to be of elliptic type in $\overline{\Omega}$. In this case all the eigenvalues $\mu_i(x),\ i = 1, \dots, q$, of the matrix $\|a_{ik}(x)\|_{i,k=1}^q,\ x \in \overline{\Omega}$, have the same sign; without loss of generality we may assume $\mu_i(x) > 0,\ x \in \overline{\Omega}$.

It is not hard to make sure that B' is a positive definite Hermitian operator with dense domain in $L_2(\Omega)$. So, B' admits closure to a positive definite selfadjoint operator B on $L_2(\Omega)$. We shall call B the operator generated by (20),(21). The spectrum of B is discrete, and for its eigenvalues $\lambda_1(B) < \lambda_2(B) < \cdots < \lambda_n(B) < \cdots$ the estimate

$$c_1 n^{2/q} \le \lambda_n(B) \le c_2 n^{2/q}, \quad 0 < c_i = \text{const}, \quad i = 1, 2. \tag{22}$$

is valid (see [20]). Denote by $e_n(x)$, $n \in \mathbb{N}$, the orthonormal basis in $L_2(\Omega)$, composed from the eigenfunctions of B. Set $A = -B^{1/2}$. Then $\lambda_n(A) = -\lambda_n^{1/2}(B)$, and the orthonormal basis consisting of the eigenvalues of A coincides with $e_n(x)$, $n \in \mathbb{N}$. In view of (22), the spaces $\mathfrak{B}_-(A)$ and $\mathfrak{G}_{(1)}(A)$ are described by the equivalence relations (18) and (19), in which

$$|f_k| \le ce^{tk} \quad \text{and} \quad |f_k| \le ce^{-tk}$$

respectively.

In the case where Ω is a q-dimensional cube $0 < x_k < a$, $k = 1, \ldots, q$, $a > 0$, and $\mathcal{L} = \sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2}$, the following formulas for the eigenvalues $\lambda_{n_1 \ldots n_q}$, $n_k \in \mathbb{N}$, and eigenfunctions $e_{n_1 \ldots n_q}(x)$ of the operator B hold:

$$\lambda_{n_1 \ldots n_q} = \frac{\pi^2}{a^2} \sum_{k=1}^{q} n_k^2; \quad e_{n_1 \ldots n_q}(x) = \left(\frac{2}{a}\right)^{q/2} \prod_{k=1}^{q} \sin \frac{\pi}{a} x_k.$$

Assume now that B in equation (3) is the operator generated by (20),(21). Taking into account (22) and Theorem 6, we arrive at the following assertion.

Theorem 7. *Each classical solution $u(t,x)$ on $(0,\infty) \times \Omega$ of the problem*

$$\left(\frac{\partial^2}{\partial t^2} + \sum_{i=1}^{q} \sum_{k=1}^{q} \frac{\partial}{\partial x_i}\left(a_{ik} \frac{\partial}{\partial x_k}\right) + c(x)\right)^m u(t,x) = 0, \tag{23}$$

$$u(t,x)\big|_{t>0, x \in \partial\Omega} = 0 \tag{24}$$

may be represented in the form

$$u(t,x) = \sum_{j=0}^{m} t^j \sum_{n=1}^{\infty} a_n e^{-t\lambda_n(A)} e_n(x) + \sum_{j=0}^{m} t^j \sum_{n=1}^{\infty} b_n e^{t\lambda_n(A)} e_n(x), \quad t > 0, \quad x \in \Omega,$$

where $a_n, b_n \in \mathbb{C}$ satisfy the conditions

$$\forall t > 0 : a_n = o(e^{tn}), \quad b_n = o(e^{-tn}) \quad \text{as} \quad n \to \infty.$$

Since for the type of the semigroup $\{e^{tA}\}_{t \ge 0}$ the equality $\omega(A) = -\lambda_1^{1/2}(B)$ takes place, we obtain, by Theorem 2, the next assertion.

Theorem 8. *Let $u(t,x)$ be a solution of problem* (23), (24). *If*

$$\forall \omega' < \lambda_1(B^{1/2}),\ \exists c_{\omega'} > 0 : \int_\Omega |u(t,x)|^2\, dx \le c_{\omega'} e^{2\omega' t} \tag{25}$$

($t > 0$ is sufficiently large), then

$$\forall \omega'' < \lambda_1(B^{1/2}),\ \exists c_{\omega''} > 0 : \int_\Omega |u(t,x)|^2\, dx \le c_{\omega''} e^{-2\omega'' t}.$$

In the special case, where $q = 1$, $\Omega = (0,1)$, and $\mathcal{L} = \frac{d^2}{dx^2}$, Theorem 8 contains in particular the well-known classical Phragmen-Lindelöf principle (see [21]): if a function harmonic in the strip $(0,\infty) \times (0,1)$ is equal to 0 on half-lines forming a part of the boundary of the strip mentioned above, and it is bounded in $(0,\infty) \times (0,1)$ for large $t > 0$, then this function is tending exponentially to 0 as $t \to \infty$.

In the class of functions satisfying (25), the Dirichlet problem, which consists in finding a solution $u(t,x)$ of (23),(24) that satisfies the condition

$$\lim_{t \to 0} \frac{\partial^k u(t,x)}{\partial t^k} = y_k, \quad k = 0, 1, \ldots, m-1,$$

(the limit is taken in the $\mathfrak{B}_-(A)$-topology) is uniquely solvable for arbitrary $y_k \in \mathfrak{B}_-(A)$. The solution is given by formulas (18), (19).

References

1. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc., Providence, RI, 1957.
2. K. Yosida, *Functional Analysis*, Springer, Berlin—Góttingen—Heidelberg, 1965.
3. V. V. Vasilyev, S. G. Krein, and S. I. Piskariov, *Semigroups of operators, cosine operator functions, and linear differential equations*, Itogi Nauki i Techniki, Ser. Math., Math. Anal. **28** (1990), 87–202. (Russian)
4. K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, Berlin—New York, 2000.
5. I. M. Gelfand and G. E. Shilov, *Generalized Functions,* Vol. 2*: Spaces of Test and Generalized Functions*, Fizmatgiz, Moscow, 1958. (Russian)
6. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary-Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984)
7. A. V. Knyazyuk, *Boundary values of infinitely differentiable semigroups*, Preprint no. 69, Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1985. (Russian)
8. V. I. Gorbachuk and A. V. Knyazyuk, *Boundary values of solutions of operator-differential equations*, Uspekhi Mat. Nauk **44** (1989), no. 3, 55–91. (Russian); English transl. Russian Math. Surveys **44** (1989), no. 3, 67–111.
9. M. L. Gorbachuk, V. I. Gorbachuk, *On solutions of elliptic type differential equations on semiaxis in a Banach space*, Mat. Met. i Fiz.-Mech. Polja **51** (2008), no. 2, 14–25. (Ukrainian)
10. H. Komatsu, *Fractional powers of operators*, Pacific J. Math. **19** (1966), no. 2, 285–346.
11. S. G. Krein, *Linear Differential Equations in a Banach Space*, Nauka, Moscow, 1967. (Russian); English transl. Transl. Math. Monographs, Vol. 29, Amer. Math. Soc., Providence, RI, 1971.
12. A. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Math. **10** (1960), no. 2, 419–439.
13. A. V. Knyazyuk, *The Dirichlet problem for second-order differential equations in a Banach space*, Ukrain. Mat. Zh. **37** (1985), no. 2, 256–260. (Russian)
14. V. M. Gorbachuk, *The Dirichlet problem for a differential-operator equation of second order on the half-line*, Applications of the Methods of Functional Analysis in Problems of Mathematical Physics, Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1987, pp. 89–97. (Russian)
15. V. M. Gorbachuk and M. L. Gorbachuk, *On the correct solvability of the Dirichlet problem for operator-differential equations in a Banach space*, Ukrain. Mat. Zh. **58** (2006), no. 11, 1462–1476. (Ukrainian); English transl. Ukrainian Math. J. **58** (2006), no. 11, 1656–1672.
16. R. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. **143** (1969), 55–76.
17. V. M. Gorbachuk, *On solutions of parabolic and elliptic differential equations on* $(-\infty, \infty)$ *in a Banach space*, Methods Funct. Anal. Topology **14** (2008), no. 2, 177–183.
18. L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Fizmatgiz, Moscow, 1959. (Russian); English transl. Pergamon Press, Oxford, 1964.
19. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova dumka, Kiev, 1965)
20. S. G. Mikhlin, *Linear Partial Differential Equations*, Vysshaya Shkola, Moscow, 1977. (Russian)
21. P. D. Lax, *A Phragmen-Lindelöf theorem in harmonic analysis and its application to some questions in the theory of elliptic equations*, Comm. Pure Appl. Math. **10** (1957), 361–389.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: imath@horbach.kiev.ua

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: imath@horbach.kiev.ua

Received 01/12/2011