

SELF-ADJOINTNESS OF SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

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Dedicated to the memory of A. G. Kostyuchenko

ABSTRACT. We study one-dimensional Schrödinger operators S with real-valued distributional potentials q in $W_{2,\text{loc}}^{-1}(\mathbb{R})$ and prove an extension of the Povzner–Wienholtz theorem on self-adjointness of bounded below S thus providing additional information on its domain. The results are further specified for $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$.

1. INTRODUCTION AND MAIN RESULTS

In the Hilbert space $L_2(\mathbb{R})$, we consider a Schrödinger operator

$$S = -\frac{d^2}{dx^2} + q$$

with potential q that is a real-valued distribution from the space $W_{2,\text{loc}}^{-1}(\mathbb{R})$. Recall that $W_{2,\text{loc}}^{-1}(\mathbb{R})$ is the dual space to the space $W_{2,\text{comp}}^1(\mathbb{R})$ of functions in $W_2^1(\mathbb{R})$ with compact support and that every real-valued $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ can be represented as σ' for a real-valued function σ from $L_{2,\text{loc}}(\mathbb{R})$. The operator S can then be rigorously defined e.g. by the so-called regularization method that was used in [2] in the particular case $q(x) = 1/x$ and then developed for generic distributional potentials in $W_{2,\text{loc}}^{-1}(\mathbb{R})$ by Savchuk and Shkalikov [20, 21]; see also recent extensions to more general differential expressions in [9, 10]. Namely, the regularization method suggests to define S via

$$(1) \quad Sf = \ell(f) := -(f' - \sigma f)' - \sigma f'$$

on the natural maximal domain

$$(2) \quad \text{dom } S = \{f \in L_2(\mathbb{R}) \mid f, f' - \sigma f \in AC_{\text{loc}}(\mathbb{R}), \ell(f) \in L_2(\mathbb{R})\};$$

here $AC_{\text{loc}}(\mathbb{R})$ is the space of functions that are locally absolutely continuous. It is straightforward to see that $Sf = -f'' + qf$ in the sense of distributions, so that the above definition is independent of the particular choice of the primitive $\sigma \in L_{2,\text{loc}}(\mathbb{R})$.

One can also introduce the minimal operator S_0 , which is the closure of the restriction S'_0 of S onto the set of functions of compact support, i.e., onto

$$\text{dom } S'_0 = \{f \in L_{2,\text{comp}}(\mathbb{R}) \mid f, f' - \sigma f \in AC_{\text{loc}}(\mathbb{R}), \ell(f) \in L_2(\mathbb{R})\}.$$

The operator S'_0 (and hence S_0) is symmetric; moreover, in a standard manner [18] one proves that S is the adjoint of S_0 , so that S is the so-called maximal operator.

An important question preceding any further analysis of the operator S is whether it is self-adjoint. Recently, this question has attracted attention in the literature in the particular case where the distributional potential $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ contains the sum of Dirac delta-functions [1, 16, 13] or is periodic [18] (complex-valued periodic q are discussed

2010 *Mathematics Subject Classification.* Primary 34L05; Secondary 34L40, 47A05, 47B25.

Key words and phrases. Schrödinger operators, self-adjointness, singular potentials.

in [7]), or belongs to the space $W_{2,\text{unif}}^{-1}(\mathbb{R})$ [12]. We recall [12] that any $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ can be represented (not uniquely) in the form $q = \sigma' + \tau$, where σ and τ belong to $L_{2,\text{unif}}(\mathbb{R})$ and $L_{1,\text{unif}}(\mathbb{R})$, respectively, i.e.,

$$\begin{aligned} \|\sigma\|_{2,\text{unif}}^2 &:= \sup_{t \in \mathbb{R}} \int_t^{t+1} |\sigma(s)|^2 ds < \infty, \\ \|\tau\|_{1,\text{unif}} &:= \sup_{t \in \mathbb{R}} \int_t^{t+1} |\tau(s)| ds < \infty, \end{aligned}$$

and the derivative is understood in the sense of distributions. Given such a representation, the operator S is defined as

$$(3) \quad Sf = -(f' - \sigma f)' - \sigma f' + \tau f$$

on the domain (2); this definition is again independent of the particular choice of σ and τ above.

Theorem 3.5 of our paper [12] claims that for real-valued $q \in W_{2,\text{unif}}^{-1}$ the operator S as defined by (3) and (2) is self-adjoint and coincides with the operator T constructed by the form-sum method. However, as was pointed out in [18] and [8], the proof given in [12] is incomplete: namely, it establishes the inclusion $T \subset S$ but then derives the equality $S = T$ taking for granted that S is symmetric. However, since S_0 is symmetric, symmetry of S would immediately imply its self-adjointness, and only the claim that $S = T$ in Theorem 3.5 of [12] would remain non-trivial.

The fact that S is indeed self-adjoint is rigorously justified in the paper [18] for the particular case where $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ is periodic. The authors prove therein that S_0, S, T , and the Friedrichs extension of S_0 all coincide; however, the arguments heavily use periodicity of q and thus are not applicable for generic real-valued $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$.

Recently, Albeverio, Kostenko and Malamud [1] extended the Povzner–Wienholtz theorem stating that boundedness below of the minimal operator implies its self-adjointness (see [3] and the references therein) to the class of arbitrary distributional potentials in $W_{2,\text{loc}}^{-1}(\mathbb{R})$. The proof of Theorem I.1 in [1] is for the half-line and for the particular case where $q = q_0 + \sum_k \alpha_k \delta(\cdot - x_k)$, where $q_0 \in L_{1,\text{loc}}(\mathbb{R})$, α_k and x_k are real numbers, and δ is the Dirac delta-function; however, Remark III.2 explains that the same proof works in the more general situation of $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$. In particular, for $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ the minimal operator S_0 is shown in [12] to be bounded below; therefore, the operator $S_0 = S$ is then self-adjoint by the above extension of the Povzner–Wienholtz theorem. This fills out the gap in the proof of Theorem 3.5 of our paper [12].

The aim of this note is to give an alternative proof of the Povzner–Wienholtz theorem for distributional potentials $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$. Our approach has several merits; namely, it gives the representation of a positive operator S in the von Neumann form A^*A for some first order differential operator A and provides additional information on the domain of S . For regular q , possibility of such a representation is known to follow from disconjugacy of S on the whole line, i.e., from the Jacobi condition in the variational problem for the corresponding quadratic form of S , see [11, Ch. XI.10,11]. We also mention that the factorization of S as A^*A is of basic importance for the Darboux transformation method, also called Darboux–Crum, or single commutation method, see [4, 5, 6, 17].

Namely, assume that a real-valued distribution $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ is such that the minimal operator S_0 is bounded below. Adding a constant to q as necessary, we can make S_0 positive and shall assume this throughout the rest of the note. Then [14] the equation $y'' = qy$ has a (possibly not unique) solution that is positive over \mathbb{R} , and $r := y'/y \in L_{2,\text{loc}}(\mathbb{R})$ is a global distributional solution to the Riccati equation $r' + r^2 = q$. The function r is called the *Riccati representative* of q . Moreover, the differential expression ℓ

of (1) admits then a formal representation

$$\ell := -\frac{d^2}{dx^2} + q = -\left(\frac{d}{dx} + r\right)\left(\frac{d}{dx} - r\right).$$

This representation suggests that ℓ is also related to a differential operator A^*A , where A is the differential operator of first order given by

$$(4) \quad Af = f' - rf$$

on the maximal domain

$$(5) \quad \text{dom } A = \{f \in L_2(\mathbb{R}) \mid f' - rf \in L_2(\mathbb{R})\}.$$

The derivative f' for $f \in \text{dom } A$ is understood in the sense of distributions; observe, however, that $f' = rf + Af$ is locally integrable so that every $f \in \text{dom } A$ is locally absolutely continuous.

Our extension of the Povzner–Wienholtz theorem reads now as follows.

Theorem 1. *Assume that a real-valued distribution $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ is such that the minimal operator S_0 is positive and denote by $r \in L_{2,\text{loc}}(\mathbb{R})$ a Riccati representative of q . Then S_0 is self-adjoint; moreover, $S_0 = S = A^*A$, and for every $f \in \text{dom } S$ it holds that $f' - rf \in L_2(\mathbb{R})$.*

This theorem can further be specified if $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$. As we mentioned above, the operator S_0 is then automatically bounded below and thus self-adjoint; moreover, we can characterize its domain as follows.

Corollary 2. *Assume that a real-valued $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ is written as $q = \sigma' + \tau$ with some $\sigma \in L_{2,\text{unif}}(\mathbb{R})$ and $\tau \in L_{1,\text{unif}}(\mathbb{R})$. Then the corresponding maximal Schrödinger operator S is self-adjoint; moreover, $\text{dom } S \subset W_2^1(\mathbb{R})$ and $y' - \sigma y \in L_2(\mathbb{R})$ for every $y \in \text{dom } S$.*

We observe that Proposition 12 of [18] shows that if $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ is periodic, then the three statements:

- (a) S is self-adjoint;
- (b) $\text{dom } S \subset W_2^1(\mathbb{R})$;
- (c) for every $y \in \text{dom } S$, $y' - \sigma y \in L_2(\mathbb{R}) \cap AC_{\text{loc}}(\mathbb{R})$

are equivalent.

2. PROOFS

We start with the following simple observation.

Lemma 3. *The operator A defined in (4)–(5) is closed.*

Proof. Let $y_n \in \text{dom } A$ be such that $y_n \rightarrow y$ and $g_n := Ay_n \rightarrow g$ in $L_2(\mathbb{R})$ as $n \rightarrow \infty$. Since convergence in $L_{1,\text{loc}}(\mathbb{R})$ yields convergence in the space of distributions $\mathcal{D}'(\mathbb{R})$, we conclude that $y_n \rightarrow y$, $ry_n \rightarrow ry$, and $g_n \rightarrow g$ in $\mathcal{D}'(\mathbb{R})$. Therefore, $y'_n = ry_n + g_n \rightarrow ry + g$ in $\mathcal{D}'(\mathbb{R})$ as $n \rightarrow \infty$; on the other hand, $y'_n \rightarrow y'$ in $\mathcal{D}'(\mathbb{R})$ since differentiation is a continuous operation in $\mathcal{D}'(\mathbb{R})$. It follows that $y' = ry + g$, whence $y \in \text{dom } A$ and $Ay = g$ as required. \square

The von Neumann theorem [15, Thm. V.3.24] yields now the following result.

Corollary 4. *The operator $S_F := A^*A$ is self-adjoint on the domain*

$$\text{dom } S_F := \{f \in L_2(\mathbb{R}) \mid Af \in \text{dom } A^*\}.$$

Clearly, S_F is a self-adjoint extension of the minimal operator S_0 . It turns out that S_F is the Friedrichs extension of S_0 , see Chapter VI of Kato's classic book [15] for all relevant definitions.

Lemma 5. *The operator S_F is the Friedrichs extension of S_0 .*

Proof. We recall that the Friedrichs extension of S_0 is the self-adjoint operator associated with the closure \mathfrak{s}_0 of the quadratic form of S_0 (defined initially on $\text{dom } S_0$) via the first representation theorem [15, Thm. VI.2.1]. The quadratic form \mathfrak{s}_F of S_F is an extension of \mathfrak{s}_0 , and to prove that $\mathfrak{s}_0 = \mathfrak{s}_F$ it suffices to show that $\text{dom } S_0$ is a core for \mathfrak{s}_F .

It is straightforward to see that $\text{dom } \mathfrak{s}_F$ coincides with $\text{dom } A$ and that \mathfrak{s}_F -convergence is equivalent to the A -convergence. Therefore it suffices to show that $\text{dom } S_0$ is a core for A . By the von Neumann theorem [15, Thm. V.3.24] $\text{dom } A^*A$ is a core for A , and it suffices to show that $\text{dom } S_0$ is dense in $\text{dom } A^*A$ in the graph topology of A .

To this end let $f \in \text{dom } A^*A$ be arbitrary. Take $\chi \in C_0^\infty$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $(-1, 1)$, and set $\chi_n := \chi(\cdot/n)$ and $f_n := \chi_n f$. Then $f_n \rightarrow f$ and $Af_n = \chi_n(Af) + f\chi'_n \rightarrow Af$ in $L_2(\mathbb{R})$ as $n \rightarrow \infty$, i.e., f_n converge to f in the graph topology of A . Since $Af \in \text{dom } A^*$, we see that $Af_n = f'_n - rf_n$ is absolutely continuous. Recalling that $r' + r^2 = \sigma'$, we conclude that $r - \sigma$ is locally absolutely continuous, whence $f'_n - \sigma f_n$ is absolutely continuous as well. Thus f_n belong to the domain of S'_0 , which is henceforth dense in $\text{dom } A^*A$ in the graph topology of A , and the proof is complete. \square

Now we study the maximal operator S . The first observation is as follows.

Lemma 6. *For every $y \in \text{dom } S$, the quasi-derivative $y^{[1]} := y' - ry$ belongs to $L_2(\mathbb{R})$.*

Proof. Set $g := Sy$ and assume that $y^{[1]} = y' - ry$ is not in $L_2(\mathbb{R}^+)$. Integrating $\ell(y)\bar{y} = g\bar{y}$ by parts from 0 to x , we find that

$$\int_0^x g(t)\bar{y}(t) dt = \int_0^x |y^{[1]}(t)|^2 dt - y^{[1]}(x)\bar{y}(x) + y^{[1]}(0)\bar{y}(0).$$

It follows that

$$\frac{1}{T} \int_0^T \int_0^x |y^{[1]}(t)|^2 dt dx - \frac{1}{T} \int_0^T y^{[1]}(x)\bar{y}(x) dx = \frac{1}{T} \int_0^T \int_0^x g(t)\bar{y}(t) dt dx - y^{[1]}(0)\bar{y}(0)$$

remains bounded as $T \rightarrow \infty$; since $\int_0^x |y^{[1]}(t)|^2 dt$ grows to $+\infty$ as $x \rightarrow \infty$ by assumption, we conclude that

$$\frac{1}{T} \left| \int_0^T y^{[1]}(x)\bar{y}(x) dx \right| \rightarrow \infty$$

as $T \rightarrow \infty$ and, moreover, that

$$(6) \quad 2 \left| \int_0^T y^{[1]}(x)\bar{y}(x) dx \right| \geq \int_0^T \int_0^x |y^{[1]}(t)|^2 dt dx$$

for all T large enough. In view of the Cauchy–Bunyakovsky–Schwarz inequality

$$\left| \int_0^T y^{[1]}(x)\bar{y}(x) dx \right| \leq \|y\| \left(\int_0^T |y^{[1]}(x)|^2 dx \right)^{1/2},$$

(6) results in the inequality

$$\int_0^T |y^{[1]}(x)|^2 dx \geq \frac{1}{4\|y\|^2} \left(\int_0^T \int_0^x |y^{[1]}(t)|^2 dt dx \right)^2.$$

Set $I(T) := \int_0^T \int_0^x |y^{[1]}(t)|^2 dt dx$; then the above inequality can be written as

$$I'(T) \geq \frac{1}{4\|y\|^2} I^2(T),$$

and, upon integration, yields

$$(7) \quad \frac{1}{I(T_0)} - \frac{1}{I(T)} \geq \frac{T - T_0}{4\|y\|^2}$$

for every positive T and T_0 such that $T > T_0$ and $I(T_0) > 0$. However, the assumption that $y^{[1]} \notin L_2(\mathbb{R}^+)$ implies that $I(T) \rightarrow \infty$ as $T \rightarrow \infty$, which is in contradiction with (7). Therefore $y^{[1]} \in L_2(\mathbb{R}^+)$; the fact that $y^{[1]} \in L_2(\mathbb{R}^-)$ is proved analogously. \square

Remark 7. Similar arguments were used in [11, Lemma XI.7.1] and [14, Lemma 4.1] in the study of the Riccati equation.

Proof of Theorem 1. By Lemma 6, $\text{dom } S \subset \text{dom } A$. Further, $\text{dom } A = \text{dom } \mathfrak{s}_F$, where \mathfrak{s}_F is the quadratic form of S_F , the Friedrichs extension of S_0 . By the extremal property of the Friedrichs extension [15, Thm. VI.2.11] we conclude that every self-adjoint restriction of S , i.e., every self-adjoint extension of S_0 , coincides with S_F . This implies that the minimal operator S_0 is itself self-adjoint and that $S_0 = S_F = S$ as claimed. \square

It was proved in [12] that if $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$, then the operator S_0 is bounded below. Assuming that S_0 is already positive, we have as before $q = r' + r^2$ for some $r \in L_{2,\text{loc}}(\mathbb{R})$. It turns out that the function r in this representation has some special properties.

Lemma 8. *Assume that real-valued $q \in W_{2,\text{unif}}^{-1}(\mathbb{R})$ and $r \in L_{2,\text{loc}}(\mathbb{R})$ satisfy the equation $r' + r^2 = q$ in the sense of distributions. Then $r \in L_{2,\text{unif}}(\mathbb{R})$.*

Proof. We set

$$a_n := \int_n^{n+1} r^2(t) dt, \quad n \in \mathbb{Z},$$

and prove that $\sup_{n \in \mathbb{Z}} a_n$ is finite.

Denote by ϕ the function in $W_2^1(\mathbb{R})$ with support equal to $[-1, 2]$ and defined via

$$\phi(x) = \begin{cases} 1+x & x \in [-1, 0), \\ 1 & x \in [0, 1], \\ 2-x & x \in (1, 2]. \end{cases}$$

We also set $\phi_\xi := \phi(\cdot - \xi)$ and notice that $\|\phi_\xi\|_{L^\infty} = \|\phi'_\xi\|_{L^\infty} = 1$. Denoting by $\langle \cdot, \cdot \rangle$ the pairing between $W_{2,\text{loc}}^{-1}(\mathbb{R})$ and $W_{2,\text{comp}}^1(\mathbb{R})$, we find that

$$(8) \quad -\langle r, \phi'_\xi \rangle + \langle r^2, \phi_\xi \rangle = \langle q, \phi_\xi \rangle.$$

As $q = \sigma' + \tau$ with some $\sigma \in L_{2,\text{unif}}(\mathbb{R})$ and $\tau \in L_{1,\text{unif}}(\mathbb{R})$, the right-hand side of this equality admits the uniform estimate

$$(9) \quad |\langle q, \phi_\xi \rangle| \leq |\langle \sigma, \phi'_\xi \rangle| + |\langle \tau, \phi_\xi \rangle| \leq 3\|\sigma\|_{2,\text{unif}} + 3\|\tau\|_{1,\text{unif}} =: C;$$

we assume that $C > 0$ as otherwise $q \equiv r \equiv 0$ and there is nothing to prove. The inequalities

$$\langle r^2, \phi_n \rangle \geq a_n, \quad |\langle r, \phi'_n \rangle| \leq a_{n-1}^{1/2} + a_{n+1}^{1/2}$$

combined with (8) and (9) lead to the relation

$$(10) \quad a_n \leq a_{n-1}^{1/2} + a_{n+1}^{1/2} + C.$$

We shall prove below that

$$(11) \quad \liminf_{n \rightarrow -\infty} a_n \leq C/2, \quad \liminf_{n \rightarrow +\infty} a_n \leq C/2,$$

so that there exist sequences $(n_k^-)_{k \in \mathbb{N}}$ and $(n_k^+)_{k \in \mathbb{N}}$ tending respectively to $-\infty$ and $+\infty$ such that $a_{n_k^\pm} < C$ for all $k \in \mathbb{N}$. Given this, the proof is concluded as follows. We have

either $a_n \leq C$ for all $n \in \mathbb{Z}$, or otherwise $a_m > C$ for some $m \in \mathbb{Z}$. In the latter case, for every k so large that $m \in (n_k^-, n_k^+)$ the maximum

$$C_k := \max\{a_j \mid j = n_k^-, \dots, n_k^+\}$$

is assumed for some index m_k strictly between n_k^- and n_k^+ . Inequality (10) for $n = m_k$ then yields

$$C_k \leq 2C_k^{1/2} + C,$$

whence $C_k \leq 2C + 4$. Therefore in both cases $\sup_{n \in \mathbb{Z}} a_n$ is finite thus implying that $r \in L_{2, \text{unif}}(\mathbb{R})$ as claimed.

It remains to establish (11). To this end we take $a < b$ so that $b - a > 3$ and integrate (8) in ξ over (a, b) . As

$$\int_a^b \phi'_\xi(t) d\xi = \int_a^b \phi'(t - \xi) d\xi = \phi_a(t) - \phi_b(t),$$

the Fubini theorem yields

$$(12) \quad - \int_a^b \langle r, \phi'_\xi \rangle d\xi = \langle r, \phi_b \rangle - \langle r, \phi_a \rangle.$$

Similarly,

$$\int_a^b \langle r^2, \phi_\xi \rangle d\xi = \langle r^2, \psi \rangle$$

with

$$\psi(t) := \int_a^b \phi_\xi(t) d\xi.$$

Observing that $\text{supp } \psi = [a - 1, b + 2]$, that $\psi(t) = 2$ for $t \in [a + 2, b - 1]$ and that $\psi(t) \geq \frac{1}{2}\phi_a^2(t)$ for $t \in [a - 1, a + 2]$ and $\psi(t) \geq \frac{1}{2}\phi_b^2(t)$ for $t \in [b - 1, b + 2]$, we get

$$\langle r^2, \psi \rangle \geq 2 \int_{a+2}^{b-1} r^2(t) dt + \frac{1}{2} \langle r^2, \phi_a^2 \rangle + \frac{1}{2} \langle r^2, \phi_b^2 \rangle.$$

On the other hand, relations (8), (9), and (12) imply the inequality

$$\langle r^2, \psi \rangle \leq \left| \int_a^b \langle q, \phi_\xi \rangle d\xi \right| + \left| \int_a^b \langle r, \phi'_\xi \rangle d\xi \right| \leq C(b - a) + |\langle r, \phi_a \rangle| + |\langle r, \phi_b \rangle|.$$

Noticing that $|\langle r, \phi_\xi \rangle| \leq 2\langle r^2, \phi_\xi^2 \rangle^{1/2}$ by the Cauchy–Bunyakovsky–Schwarz inequality and that $2x - \frac{1}{2}x^2 \leq 2$ for $x \in \mathbb{R}$, we conclude that

$$\begin{aligned} 2 \int_{a+2}^{b-1} r^2(t) dt &\leq C(b - a) + 2\langle r^2, \phi_a^2 \rangle^{1/2} - \frac{1}{2}\langle r^2, \phi_a^2 \rangle + 2\langle r^2, \phi_b^2 \rangle^{1/2} - \frac{1}{2}\langle r^2, \phi_b^2 \rangle \\ &\leq C(b - a) + 4. \end{aligned}$$

This estimate yields (11) in a straightforward manner, and the proof is complete. \square

Proof of Corollary 2. We may again assume that the operator S is positive and denote by $r \in L_{2, \text{unif}}(\mathbb{R})$ the corresponding solution of the Riccati equation $r' + r^2 = q$ and by A the differential operator of (4)–(5). By Lemma 6, the domain of S is contained in $\text{dom } A$, so that it suffices to show that $\text{dom } A \subset W_2^1(\mathbb{R})$.

Take an arbitrary $y \in \text{dom } A$; thus y and $y' - ry = g$ are in $L_2(\mathbb{R})$. Set $\Delta_n := [n, n + 1]$, $g_n := \left(\int_{\Delta_n} |g(t)|^2 dt \right)^{1/2}$, and choose $\xi_n \in \Delta_n$ such that

$$|y(\xi_n)| \leq \left(\int_{\Delta_n} |y(t)|^2 dt \right)^{1/2} =: y_n.$$

For every $x \in \Delta_n$, we integrate the equality $y' = ry + g$ from ξ_n to x to get the estimates

$$|y(x)| \leq |y(\xi_n)| + \int_{\Delta_n} |r(t)y(t)| dt + \int_{\Delta_n} |g(t)| dt \leq y_n + y_n \|r\|_{2,\text{unif}} + g_n =: b_n$$

and

$$\int_{\Delta_n} |r(t)y(t)|^2 dt \leq b_n^2 \|r\|_{2,\text{unif}}^2.$$

Since the sequence (b_n) belongs to $\ell_2(\mathbb{Z})$, it follows that $ry \in L_2(\mathbb{R})$; thus $y' = ry + g \in L_2(\mathbb{R})$, and $y \in W_2^1(\mathbb{R})$.

Further, it was proved in [12] that $y \in W_2^1(\mathbb{R})$ and $\sigma \in L_{2,\text{unif}}(\mathbb{R})$ imply that $\sigma y \in L_2(\mathbb{R})$, whence the quasi-derivative $y' - \sigma y$ belongs to $L_2(\mathbb{R})$ as well. The proof is complete. \square

Acknowledgments. The authors thank Professors F. Gesztesy, A. Kostenko, M. Malamud, and V. Mikhailets for fruitful discussions and comments. R.H. acknowledges support from the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge for participation in the programme “Inverse Problems”, during which part of this work was done.

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Received 03/11/2012; Revised 21/03/2012