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SELF-ADJOINTNESS OF SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

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Dedicated to the memory of A. G. Kostyuchenko

Abstract. We study one-dimensional Schrödinger operators S with real-valued distributional potentials q in $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ and prove an extension of the Povzner–Wienholtz theorem on self-adjointness of bounded below S thus providing additional information on its domain. The results are further specified for $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$.

1. Introduction and main results

In the Hilbert space $L_2(\mathbb{R})$, we consider a Schrödinger operator

$$S = -\frac{d^2}{dx^2} + q$$

with potential q that is a real-valued distribution from the space $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$. Recall that $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ is the dual space to the space $W^{1}_{2,\mathrm{comp}}(\mathbb{R})$ of functions in $W^1_2(\mathbb{R})$ with compact support and that every real-valued $q \in W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ can be represented as σ' for a real-valued function σ from $L_{2,\mathrm{loc}}(\mathbb{R})$. The operator S can then be rigorously defined e.g. by the so-called regularization method that was used in [2] in the particular case $q(x) = 1/x$ and then developed for generic distributional potentials in $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ by Savchuk and Shkalikov [20, 21]; see also recent extensions to more general differential expressions in [9, 10]. Namely, the regularization method suggests to define S via

$$Sf = \ell(f) := -(f' - \sigma f)' - \sigma f' \tag{1}$$

on the natural maximal domain

$$\operatorname{dom} S = \{f \in L_2(\mathbb{R}) \mid f,\ f' - \sigma f \in AC_{\mathrm{loc}}(\mathbb{R}),\ \ell(f) \in L_2(\mathbb{R})\}; \tag{2}$$

here $AC_{\mathrm{loc}}(\mathbb{R})$ is the space of functions that are locally absolutely continuous. It is straightforward to see that $Sf = -f'' + qf$ in the sense of distributions, so that the above definition is independent of the particular choice of the primitive $\sigma \in L_{2,\mathrm{loc}}(\mathbb{R})$.

One can also introduce the minimal operator S_0, which is the closure of the restriction S_0' of S onto the set of functions of compact support, i.e., onto

$$\operatorname{dom} S_0' = \{f \in L_{2,\mathrm{comp}}(\mathbb{R}) \mid f, f' - \sigma f \in AC_{\mathrm{loc}}(\mathbb{R}),\ \ell(f) \in L_2(\mathbb{R})\}.$$

The operator S_0' (and hence S_0) is symmetric; moreover, in a standard manner [18] one proves that S is the adjoint of S_0, so that S is the so-called maximal operator.

An important question preceding any further analysis of the operator S is whether it is self-adjoint. Recently, this question has attracted attention in the literature in the particular case where the distributional potential $q \in W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ contains the sum of Dirac delta-functions [1, 16, 13] or is periodic [18] (complex-valued periodic q are discussed

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in [7]), or belongs to the space $W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$ [12]. We recall [12] that any $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$ can be represented (not uniquely) in the form $q = \sigma' + \tau$, where σ and τ belong to $L_{2,\mathrm{unif}}(\mathbb{R})$ and $L_{1,\mathrm{unif}}(\mathbb{R})$, respectively, i.e.,

$$\|\sigma\|^2_{2,\mathrm{unif}} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |\sigma(s)|^2 ds < \infty,$$
$$\|\tau\|_{1,\mathrm{unif}} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |\tau(s)|\, ds < \infty,$$

and the derivative is understood in the sense of distributions. Given such a representation, the operator S is defined as

$$Sf = -(f' - \sigma f)' - \sigma f' + \tau f \tag{3}$$

on the domain (2); this definition is again independent of the particular choice of σ and τ above.

Theorem 3.5 of our paper [12] claims that for real-valued $q \in W^{-1}_{2,\mathrm{unif}}$ the operator S as defined by (3) and (2) is self-adjoint and coincides with the operator T constructed by the form-sum method. However, as was pointed out in [18] and [8], the proof given in [12] is incomplete: namely, it establishes the inclusion $T \subset S$ but then derives the equality $S = T$ taking for granted that S is symmetric. However, since S_0 is symmetric, symmetry of S would immediately imply its self-adjointness, and only the claim that $S = T$ in Theorem 3.5 of [12] would remain non-trivial.

The fact that S is indeed self-adjoint is rigorously justified in the paper [18] for the particular case where $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$ is periodic. The authors prove therein that S_0, S, T, and the Friedrichs extension of S_0 all coincide; however, the arguments heavily use periodicity of q and thus are not applicable for generic real-valued $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$.

Recently, Albeverio, Kostenko and Malamud [1] extended the Povzner–Wienholtz theorem stating that boundedness below of the minimal operator implies its self-adjointness (see [3] and the references therein) to the class of arbitrary distributional potentials in $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$. The proof of Theorem I.1 in [1] is for the half-line and for the particular case where $q = q_0 + \sum_k \alpha_k \delta(\cdot - x_k)$, where $q_0 \in L_{1,\mathrm{loc}}(\mathbb{R})$, α_k and x_k are real numbers, and δ is the Dirac delta-function; however, Remark III.2 explains that the same proof works in the more general situation of $q \in W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$. In particular, for $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$ the minimal operator S_0 is shown in [12] to be bounded below; therefore, the operator $S_0 = S$ is then self-adjoint by the above extension of the Povzner–Wienholtz theorem. This fills out the gap in the proof of Theorem 3.5 of our paper [12].

The aim of this note is to give an alternative proof of the Povzner–Wienholtz theorem for distributional potentials $q \in W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$. Our approach has several merits; namely, it gives the representation of a positive operator S in the von Neumann form A^*A for some first order differential operator A and provides additional information on the domain of S. For regular q, possibility of such a representation is known to follow from disconjugacy of S on the whole line, i.e., from the Jacobi condition in the variational problem for the corresponding quadratic form of S, see [11, Ch. XI.10,11]. We also mention that the factorization of S as A^*A is of basic importance for the Darboux transformation method, also called Darboux–Crum, or single commutation method, see [4, 5, 6, 17].

Namely, assume that a real-valued distribution $q \in W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ is such that the minimal operator S_0 is bounded below. Adding a constant to q as necessary, we can make S_0 positive and shall assume this throughout the rest of the note. Then [14] the equation $y'' = qy$ has a (possibly not unique) solution that is positive over $\mathbb{R}$, and $r := y'/y \in L_{2,\mathrm{loc}}(\mathbb{R})$ is a global distributional solution to the Riccati equation $r' + r^2 = q$. The function r is called the *Riccati representative* of q. Moreover, the differential expression ℓ

of (1) admits then a formal representation

$$\ell := -\frac{d^2}{dx^2} + q = -\Big(\frac{d}{dx} + r\Big)\Big(\frac{d}{dx} - r\Big).$$

This representation suggests that ℓ is also related to a differential operator A^*A, where A is the differential operator of first order given by

$$Af = f' - rf \tag{4}$$

on the maximal domain

$$\operatorname{dom} A = \{f \in L_2(\mathbb{R}) \mid f' - rf \in L_2(\mathbb{R})\}. \tag{5}$$

The derivative f' for $f \in \operatorname{dom} A$ is understood in the sense of distributions; observe, however, that $f' = rf + Af$ is locally integrable so that every $f \in \operatorname{dom} A$ is locally absolutely continuous.

Our extension of the Povzner–Wienholtz theorem reads now as follows.

Theorem 1. *Assume that a real-valued distribution $q \in W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ is such that the minimal operator S_0 is positive and denote by $r \in L_{2,\mathrm{loc}}(\mathbb{R})$ a Riccati representative of q. Then S_0 is self-adjoint; moreover, $S_0 = S = A^*A$, and for every $f \in \operatorname{dom} S$ it holds that $f' - rf \in L_2(\mathbb{R})$.*

This theorem can further be specified if $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$. As we mentioned above, the operator S_0 is then automatically bounded below and thus self-adjoint; moreover, we can characterize its domain as follows.

Corollary 2. *Assume that a real-valued $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$ is written as $q = \sigma' + \tau$ with some $\sigma \in L_{2,\mathrm{unif}}(\mathbb{R})$ and $\tau \in L_{1,\mathrm{unif}}(\mathbb{R})$. Then the corresponding maximal Schrödinger operator S is self-adjoint; moreover,* $\operatorname{dom} S \subset W^1_2(\mathbb{R})$ *and $y' - \sigma y \in L_2(\mathbb{R})$ for every $y \in \operatorname{dom} S$.*

We observe that Proposition 12 of [18] shows that if $q \in W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ is periodic, then the three statements:

(a) S is self-adjoint;
(b) $\operatorname{dom} S \subset W^1_2(\mathbb{R})$;
(c) for every $y \in \operatorname{dom} S$, $y' - \sigma y \in L_2(\mathbb{R}) \cap AC_{\mathrm{loc}}(\mathbb{R})$

are equivalent.

2. Proofs

We start with the following simple observation.

Lemma 3. *The operator A defined in* (4)–(5) *is closed.*

Proof. Let $y_n \in \operatorname{dom} A$ be such that $y_n \to y$ and $g_n := Ay_n \to g$ in $L_2(\mathbb{R})$ as $n \to \infty$. Since convergence in $L_{1,\mathrm{loc}}(\mathbb{R})$ yields convergence in the space of distributions $\mathcal{D}'(\mathbb{R})$, we conclude that $y_n \to y$, $ry_n \to ry$, and $g_n \to g$ in $\mathcal{D}'(\mathbb{R})$. Therefore, $y_n' = ry_n + g_n \to ry + g$ in $\mathcal{D}'(\mathbb{R})$ as $n \to \infty$; on the other hand, $y_n' \to y'$ in $\mathcal{D}'(\mathbb{R})$ since differentiation is a continuous operation in $\mathcal{D}'(\mathbb{R})$. It follows that $y' = ry + g$, whence $y \in \operatorname{dom} A$ and $Ay = g$ as required. ☐

The von Neumann theorem [15, Thm. V.3.24] yields now the following result.

Corollary 4. *The operator $S_F := A^*A$ is self-adjoint on the domain*

$$\operatorname{dom} S_F := \{f \in L_2(\mathbb{R}) \mid Af \in \operatorname{dom} A^*\}.$$

Clearly, S_F is a self-adjoint extension of the minimal operator S_0. It turns out that S_F is the Friedrichs extension of S_0, see Chapter VI of Kato's classic book [15] for all relevant definitions.

Lemma 5. *The operator S_F is the Friedrichs extension of S_0.*

Proof. We recall that the Friedrichs extension of S_0 is the self-adjoint operator associated with the closure $\mathfrak{s}_0$ of the quadratic form of S_0 (defined initially on $\operatorname{dom} S_0$) via the first representation theorem [15, Thm. VI.2.1]. The quadratic form $\mathfrak{s}_F$ of S_F is an extension of $\mathfrak{s}_0$, and to prove that $\mathfrak{s}_0 = \mathfrak{s}_F$ it suffices to show that $\operatorname{dom} S_0$ is a core for $\mathfrak{s}_F$.

It is straightforward to see that $\operatorname{dom} \mathfrak{s}_F$ coincides with $\operatorname{dom} A$ and that $\mathfrak{s}_F$-convergence is equivalent to the A-convergence. Therefore it suffices to show that $\operatorname{dom} S_0$ is a core for A. By the von Neumann theorem [15, Thm. V.3.24] $\operatorname{dom} A^*A$ is a core for A, and it suffices to show that $\operatorname{dom} S_0$ is dense in $\operatorname{dom} A^*A$ in the graph topology of A.

To this end let $f \in \operatorname{dom} A^*A$ be arbitrary. Take $\chi \in C_0^\infty$ such that $0 \le \chi \le 1$ and $\chi \equiv 1$ on $(-1,1)$, and set $\chi_n := \chi(\cdot/n)$ and $f_n := \chi_n f$. Then $f_n \to f$ and $Af_n = \chi_n(Af) + f\chi_n' \to Af$ in $L_2(\mathbb{R})$ as $n \to \infty$, i.e., f_n converge to f in the graph topology of A. Since $Af \in \operatorname{dom} A^*$, we see that $Af_n = f_n' - rf_n$ is absolutely continuous. Recalling that $r' + r^2 = \sigma'$, we conclude that $r - \sigma$ is locally absolutely continuous, whence $f_n' - \sigma f_n$ is absolutely continuous as well. Thus f_n belong to the domain of S_0', which is henceforth dense in $\operatorname{dom} A^*A$ in the graph topology of A, and the proof is complete. □

Now we study the maximal operator S. The first observation is as follows.

Lemma 6. *For every $y \in \operatorname{dom} S$, the quasi-derivative $y^{[1]} := y' - ry$ belongs to $L_2(\mathbb{R})$.*

Proof. Set $g := Sy$ and assume that $y^{[1]} = y' - ry$ is not in $L_2(\mathbb{R}^+)$. Integrating $\ell(y)\overline{y} = g\overline{y}$ by parts from 0 to x, we find that

$$\int_0^x g(t)\overline{y}(t)\,dt = \int_0^x |y^{[1]}(t)|^2\,dt - y^{[1]}(x)\overline{y}(x) + y^{[1]}(0)\overline{y}(0).$$

It follows that

$$\frac{1}{T}\int_0^T\int_0^x |y^{[1]}(t)|^2\,dt\,dx - \frac{1}{T}\int_0^T y^{[1]}(x)\overline{y}(x)\,dx = \frac{1}{T}\int_0^T\int_0^x g(t)\overline{y}(t)\,dt\,dt - y^{[1]}(0)\overline{y}(0)$$

remains bounded as $T \to \infty$; since $\int_0^x |y^{[1]}(t)|^2\,dt$ grows to $+\infty$ as $x \to \infty$ by assumption, we conclude that

$$\frac{1}{T}\Big|\int_0^T y^{[1]}(x)\overline{y}(x)\,dx\Big| \to \infty$$

as $T \to \infty$ and, moreover, that

$$2\Big|\int_0^T y^{[1]}(x)\overline{y}(x)\,dx\Big| \ge \int_0^T\int_0^x |y^{[1]}(t)|^2\,dt\,dx \tag{6}$$

for all T large enough. In view of the Cauchy–Bunyakovsky–Schwarz inequality

$$\Big|\int_0^T y^{[1]}(x)\overline{y}(x)\,dx\Big| \le \|y\|\Big(\int_0^T |y^{[1]}(x)|^2\,dx\Big)^{1/2},$$

(6) results in the inequality

$$\int_0^T |y^{[1]}(x)|^2\,dx \ge \frac{1}{4\|y\|^2}\Big(\int_0^T\int_0^x |y^{[1]}(t)|^2\,dt\,dx\Big)^2.$$

Set $I(T) := \int_0^T\int_0^x |y^{[1]}(t)|^2\,dt\,dx$; then the above inequality can be written as

$$I'(T) \ge \frac{1}{4\|y\|^2} I^2(T),$$

and, upon integration, yields

$$\frac{1}{I(T_0)} - \frac{1}{I(T)} \geq \frac{T-T_0}{4\|y\|^2} \tag{7}$$

for every positive T and T_0 such that $T > T_0$ and $I(T_0) > 0$. However, the assumption that $y^{[1]} \notin L_2(\mathbb{R}^+)$ implies that $I(T) \to \infty$ as $T \to \infty$, which is in contradiction with (7). Therefore $y^{[1]} \in L_2(\mathbb{R}^+)$; the fact that $y^{[1]} \in L_2(\mathbb{R}^-)$ is proved analogously. $\square$

Remark 7. Similar arguments were used in [11, Lemma XI.7.1] and [14, Lemma 4.1] in the study of the Riccati equation.

Proof of Theorem 1. By Lemma 6, $\operatorname{dom} S \subset \operatorname{dom} A$. Further, $\operatorname{dom} A = \operatorname{dom} \mathfrak{s}_F$, where $\mathfrak{s}_F$ is the quadratic form of S_F, the Friedrichs extension of S_0. By the extremal property of the Friedrichs extension [15, Thm. VI.2.11] we conclude that every self-adjoint restriction of S, i.e., every self-adjoint extension of S_0, coincides with S_F. This implies that the minimal operator S_0 is itself self-adjoint and that $S_0 = S_F = S$ as claimed. $\square$

It was proved in [12] that if $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$, then the operator S_0 is bounded below. Assuming that S_0 is already positive, we have as before $q = r' + r^2$ for some $r \in L_{2,\mathrm{loc}}(\mathbb{R})$. It turns out that the function r in this representation has some special properties.

Lemma 8. *Assume that real-valued $q \in W^{-1}_{2,\mathrm{unif}}(\mathbb{R})$ and $r \in L_{2,\mathrm{loc}}(\mathbb{R})$ satisfy the equation $r' + r^2 = q$ in the sense of distributions. Then $r \in L_{2,\mathrm{unif}}(\mathbb{R})$.*

Proof. We set

$$a_n := \int_n^{n+1} r^2(t)\,dt, \quad n \in \mathbb{Z},$$

and prove that $\sup_{n\in\mathbb{Z}} a_n$ is finite.

Denote by ϕ the function in $W^1_2(\mathbb{R})$ with support equal to $[-1,2]$ and defined via

$$\phi(x) = \begin{cases} 1+x & x \in [-1,0), \\ 1 & x \in [0,1], \\ 2-x & x \in (1,2]. \end{cases}$$

We also set $\phi_\xi := \phi(\,\cdot - \xi)$ and notice that $\|\phi_\xi\|_{L_\infty} = \|\phi'_\xi\|_{L_\infty} = 1$. Denoting by $\langle\,\cdot\,,\,\cdot\,\rangle$ the pairing between $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ and $W^1_{2,\mathrm{comp}}(\mathbb{R})$, we find that

$$-\langle r, \phi'_\xi\rangle + \langle r^2, \phi_\xi\rangle = \langle q, \phi_\xi\rangle. \tag{8}$$

As $q = \sigma' + \tau$ with some $\sigma \in L_{2,\mathrm{unif}}(\mathbb{R})$ and $\tau \in L_{1,\mathrm{unif}}(\mathbb{R})$, the right-hand side of this equality admits the uniform estimate

$$|\langle q, \phi_\xi\rangle| \leq |\langle \sigma, \phi'_\xi\rangle| + |\langle \tau, \phi_\xi\rangle| \leq 3\|\sigma\|_{2,\mathrm{unif}} + 3\|\tau\|_{1,\mathrm{unif}} =: C; \tag{9}$$

we assume that $C > 0$ as otherwise $q \equiv r \equiv 0$ and there is nothing to prove. The inequalities

$$\langle r^2, \phi_n\rangle \geq a_n, \quad |\langle r, \phi'_n\rangle| \leq a^{1/2}_{n-1} + a^{1/2}_{n+1}$$

combined with (8) and (9) lead to the relation

$$a_n \leq a^{1/2}_{n-1} + a^{1/2}_{n+1} + C. \tag{10}$$

We shall prove below that

$$\liminf_{n\to-\infty} a_n \leq C/2, \quad \liminf_{n\to+\infty} a_n \leq C/2, \tag{11}$$

so that there exist sequences $(n_k^-)_{k\in\mathbb{N}}$ and $(n_k^+)_{k\in\mathbb{N}}$ tending respectively to $-\infty$ and $+\infty$ such that $a_{n_k^\pm} < C$ for all $k \in \mathbb{N}$. Given this, the proof is concluded as follows. We have either $a_n \le C$ for all $n \in \mathbb{Z}$, or otherwise $a_m > C$ for some $m \in \mathbb{Z}$. In the latter case, for every k so large that $m \in (n_k^-, n_k^+)$ the maximum

$$C_k := \max\{a_j \mid j = n_k^-, \dots, n_k^+\}$$

is assumed for some index m_k strictly between n_k^- and n_k^+. Inequality (10) for $n = m_k$ then yields

$$C_k \le 2C_k^{1/2} + C,$$

whence $C_k \le 2C + 4$. Therefore in both cases $\sup_{n\in\mathbb{Z}} a_n$ is finite thus implying that $r \in L_{2,\mathrm{unif}}(\mathbb{R})$ as claimed.

It remains to establish (11). To this end we take $a < b$ so that $b - a > 3$ and integrate (8) in ξ over (a, b). As

$$\int_a^b \phi_\xi'(t)\,d\xi = \int_a^b \phi'(t-\xi)\,d\xi = \phi_a(t) - \phi_b(t),$$

the Fubini theorem yields

$$-\int_a^b \langle r, \phi_\xi'\rangle\,d\xi = \langle r, \phi_b\rangle - \langle r, \phi_a\rangle. \tag{12}$$

Similarly,

$$\int_a^b \langle r^2, \phi_\xi\rangle\,d\xi = \langle r^2, \psi\rangle$$

with

$$\psi(t) := \int_a^b \phi_\xi(t)\,d\xi.$$

Observing that $\operatorname{supp}\psi = [a-1, b+2]$, that $\psi(t) = 2$ for $t \in [a+2, b-1]$ and that $\psi(t) \ge \frac12\phi_a^2(t)$ for $t \in [a-1, a+2]$ and $\psi(t) \ge \frac12\phi_b^2(t)$ for $t \in [b-1, b+2]$, we get

$$\langle r^2, \psi\rangle \ge 2\int_{a+2}^{b-1} r^2(t)\,dt + \tfrac12\langle r^2, \phi_a^2\rangle + \tfrac12\langle r^2, \phi_b^2\rangle.$$

On the other hand, relations (8), (9), and (12) imply the inequality

$$\langle r^2, \psi\rangle \le \Big|\int_a^b \langle q, \phi_\xi\rangle\,d\xi\Big| + \Big|\int_a^b \langle r, \phi_\xi'\rangle\,d\xi\Big| \le C(b-a) + |\langle r, \phi_a\rangle| + |\langle r, \phi_b\rangle|.$$

Noticing that $|\langle r, \phi_\xi\rangle| \le 2\langle r^2, \phi_\xi^2\rangle^{1/2}$ by the Cauchy–Bunyakovsky–Schwarz inequality and that $2x - \frac12 x^2 \le 2$ for $x \in \mathbb{R}$, we conclude that

$$\begin{aligned} 2\int_{a+2}^{b-1} r^2(t)\,dt &\le C(b-a) + 2\langle r^2, \phi_a^2\rangle^{1/2} - \tfrac12\langle r^2, \phi_a^2\rangle + 2\langle r^2, \phi_b^2\rangle^{1/2} - \tfrac12\langle r^2, \phi_b^2\rangle \\ &\le C(b-a) + 4. \end{aligned}$$

This estimate yields (11) in a straightforward manner, and the proof is complete. □

Proof of Corollary 2. We may again assume that the operator S is positive and denote by $r \in L_{2,\mathrm{unif}}(\mathbb{R})$ the corresponding solution of the Riccati equation $r' + r^2 = q$ and by A the differential operator of (4)–(5). By Lemma 6, the domain of S is contained in $\operatorname{dom} A$, so that it suffices to show that $\operatorname{dom} A \subset W_2^1(\mathbb{R})$.

Take an arbitrary $y \in \operatorname{dom} A$; thus y and $y' - ry = g$ are in $L_2(\mathbb{R})$. Set $\Delta_n := [n, n+1)$, $g_n := \big(\int_{\Delta_n} |g(t)|^2\,dt\big)^{1/2}$, and choose $\xi_n \in \Delta_n$ such that

$$|y(\xi_n)| \le \Big(\int_{\Delta_n} |y(t)|^2\,dt\Big)^{1/2} =: y_n.$$

For every $x \in \Delta_n$, we integrate the equality $y' = ry + g$ from ξ_n to x to get the estimates

$$|y(x)| \le |y(\xi_n)| + \int_{\Delta_n} |r(t)y(t)|\,dt + \int_{\Delta_n} |g(t)|\,dt \le y_n + y_n \|r\|_{2,\mathrm{unif}} + g_n =: b_n$$

and

$$\int_{\Delta_n} |r(t)y(t)|^2\,dt \le b_n^2 \|r\|_{2,\mathrm{unif}}^2.$$

Since the sequence (b_n) belongs to $\ell_2(\mathbb{Z})$, it follows that $ry \in L_2(\mathbb{R})$; thus $y' = ry + g \in L_2(\mathbb{R})$, and $y \in W_2^1(\mathbb{R})$.

Further, it was proved in [12] that $y \in W_2^1(\mathbb{R})$ and $\sigma \in L_{2,\mathrm{unif}}(\mathbb{R})$ imply that $\sigma y \in L_2(\mathbb{R})$, whence the quasi-derivative $y' - \sigma y$ belongs to $L_2(\mathbb{R})$ as well. The proof is complete. $\square$

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