

CLARK-OCONE TYPE FORMULAS ON SPACES OF TEST AND GENERALIZED FUNCTIONS OF MEIXNER WHITE NOISE ANALYSIS

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Dedicated to the memory of Professor A. G. Kostyuchenko

ABSTRACT. In the classical Gaussian analysis the Clark-Ocone formula can be written in the form

$$F = \mathbf{E}F + \int \mathbf{E}_t \partial_t F dW_t,$$

where the function (the random variable) F is square integrable with respect to the Gaussian measure and differentiable by Hida; \mathbf{E} denotes the expectation; \mathbf{E}_t denotes the conditional expectation with respect to the full σ -algebra that is generated by a Wiener process W up to the point of time t ; $\partial_t F$ is the Hida derivative of F ; $\int \circ(t) dW_t$ denotes the Itô stochastic integral with respect to the Wiener process. This formula has applications in the stochastic analysis and in the financial mathematics.

In this paper we generalize the Clark-Ocone formula to spaces of test and generalized functions of the so-called Meixner white noise analysis, in which instead of the Gaussian measure one uses the so-called generalized Meixner measure μ (depending on parameters, μ can be the Gaussian, Poissonian, Gamma measure etc.). In particular, we study properties of integrands in our (Clark-Ocone type) formulas.

INTRODUCTION

Denote by \mathcal{D} the Schwartz space of infinite-differentiable real-valued functions on $\mathbb{R}_+ := [0, +\infty)$ with compact supports; by \mathcal{D}' the distribution space that is dual of \mathcal{D} ; by $\langle \cdot, \cdot \rangle$ the pairing between elements of \mathcal{D}' and \mathcal{D} , this pairing is generated by the scalar product in the space of square integrable with respect to the Lebesgue measure functions on \mathbb{R}_+ ; by the subindex \mathbb{C} complexifications of spaces. The notation $\langle \cdot, \cdot \rangle$ will be preserved for tensor powers and complexifications of spaces.

Let μ be the standard Gaussian measure on $(\mathcal{D}', C(\mathcal{D}'))$ (here and below $C(\mathcal{D}')$ is the σ -algebra on \mathcal{D}' that is generated by cylindrical sets), i.e., a probability measure with the Laplace transform

$$l_\mu(\lambda) = \int_{\mathcal{D}'} e^{\langle x, \lambda \rangle} \mu(dx) = e^{\langle \lambda, \lambda \rangle / 2}, \quad \lambda \in \mathcal{D}_{\mathbb{C}}.$$

As is well known (e.g., [4, 27, 22]), any square integrable with respect to μ and differentiable by Hida complex-valued function F on \mathcal{D}' can be presented in the form

$$(0.1) \quad F = \mathbf{E}F + \int \mathbf{E}_t \partial_t F dW_t,$$

where \mathbf{E} denotes the expectation; \mathbf{E}_t denotes the conditional expectation with respect to the full σ -algebra $\sigma(W_s : s \leq t)$ that is generated by a Wiener process W up to the point of time t ; $\partial_t F$ is the Hida derivative of F ; $\int \circ(t) dW_t$ denotes the Itô stochastic integral with respect to W (usually for stochastic integrals on \mathbb{R}_+ we do not write limits

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of integration for simplification of notation). Formula (0.1) is called the *Clark-Ocone formula*. As is known (e.g., [5, 33]), formula (0.1) holds true (up to clear modifications) if instead of the Gaussian measure one considers the Poissonian one.

Clark-Ocone formulas and their generalizations (in this paper they will be called *Clark-Ocone type formulas*) have applications in the stochastic analysis and in the financial mathematics, see, e.g., [19, 1, 6, 29, 7, 28, 25, 10, 5, 33] and references therein. In order to satisfy demands of applications (for example, in some problems it is necessary to reconstruct an integrand by the result of integration, in another problems it is necessary to reconstruct a random variable by the family of conditional expectations of its stochastic derivative, etc.), different variants of such formulas on various spaces, with different stochastic derivatives and with stochastic integrals with respect to different random processes and measures were obtained, see, in particular, [20, 22, 1, 8, 2, 21, 6, 25, 33, 5]. For example, in [22, 21] a Clark-Ocone type formula that is connected with Lévy processes was obtained, this formula contains stochastic integrals with respect to a Wiener process and with respect to a compensated Poissonian random measure. In [6] another way of construction of Clark-Ocone type formulas that are connected with Lévy processes was offered, this way is based on the Nualart–Schoutens representation for a square integrable random variable [26, 31]; now the Clark-Ocone type formulas contain integrals with respect to special random processes. Moreover, these formulas were obtained in [6] not only for square integrable random variables, but also for generalized ones.

In the paper [13] the author obtained Clark-Ocone type formulas in the so-called Meixner white noise analysis. This analysis is connected with the generalized Meixner measure μ [30] (see also Subsection 1.1) that depending on parameters can be the Gaussian, Poissonian, Gamma measure etc., and with the corresponding Meixner random process M (the derivative of which is the Meixner white noise that is connected with μ). Note that under some assumptions (see Subsection 1.4) M is a Lévy process. Nevertheless, the constructions of [13] essentially differ from the constructions of [22, 21] and [6]: the author tried to preserve a "classical" form of Clark-Ocone type formulas and therefore exploited a Hida stochastic derivative and stochastic integrals with respect to M only. Of course, in the particular cases when μ is the Gaussian or Poissonian measure, the formulas from [13] reduce to the corresponding classical Clark-Ocone formulas. One of conditions on a random variable for which the Clark-Ocone type formulas in [13] were obtained is the *differentiability by Hida* in the classical sense. This condition arises naturally, but is very restrictive. Fortunately, some modification of the scheme of [13] allows to get rid of this contingency. In the short paper [14] the author described such a modification that is based on the use of the so-called *parameterized Kondratiev type spaces of generalized functions* ([12]). In the present paper we continue the researches that were started in [13, 14]: now our goal is to obtain and to study Clark-Ocone type formulas on parameterized Kondratiev type spaces of test and generalized functions of Meixner white noise analysis. In particular, we will show that the differentiability by Hida of a random variable does not play a significant role under construction of the above mentioned formulas.

The paper is organized in the following manner. In the first section we recall necessary definitions and results (the generalized Meixner measure, properties of the corresponding space of square integrable functions (L^2), the rigging of (L^2) by the parameterized Kondratiev type spaces of test and generalized functions, the extended (Skorohod) stochastic integral, the Hida stochastic derivative, properties of these operators). In the second section we deal with Clark-Ocone type formulas and related matters.

1. PRELIMINARIES

1.1. The generalized Meixner measure. Let us define the generalized Meixner measure (see [30] for more details and explanations). Let $\rho, \nu : \mathbb{R}_+ \rightarrow \mathbb{C}$ be smooth functions such that

$$(1.1) \quad \theta \stackrel{\text{def}}{=} \rho - \nu : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad \eta \stackrel{\text{def}}{=} \rho\nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

and, moreover, θ and η are bounded on \mathbb{R}_+ . Further, for each $t \in \mathbb{R}_+$ let $v_{\rho(t), \nu(t)}(ds)$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (here $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R}) that is defined by its Fourier transform

$$\int_{\mathbb{R}} e^{i\zeta s} v_{\rho(t), \nu(t)}(ds) = \exp \left\{ -i\zeta(\rho(t) + \nu(t)) + 2 \sum_{m=1}^{\infty} \frac{(\rho(t)\nu(t))^m}{m} \left[\sum_{n=2}^{\infty} \frac{(-i\zeta)^n}{n!} (\nu^{n-2}(t) + \nu^{n-3}(t)\rho(t) + \dots + \rho^{n-2}(t)) \right]^m \right\}.$$

Definition. A probability measure μ on the measurable space $(\mathcal{D}', C(\mathcal{D}'))$ with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle x, \xi \rangle} \mu(dx) = \exp \left\{ \int_{\mathbb{R}_+} dt \int_{\mathbb{R}} v_{\rho(t), \nu(t)}(ds) \frac{1}{s^2} (e^{is\xi(t)} - 1 - is\xi(t)) \right\}$$

is called the *generalized Meixner measure*.

Depending on parameters ρ and ν , μ can be, in particular, the Gaussian, Poissonian, Pascal, Meixner or Gamma measure.

It was proved in [30] that the generalized Meixner measure μ is the measure of a generalized random process [9] with independent values; and the Laplace transform $l_{\mu}(\cdot) = \int_{\mathcal{D}'} \exp\{\langle x, \cdot \rangle\} \mu(dx)$ of μ is a holomorphic at $0 \in \mathcal{D}_{\mathbb{C}}$ function.

1.2. The space of square integrable functions. Let $(L^2) := L^2(\mathcal{D}', \mu)$ be the space of complex-valued square integrable with respect to the generalized Meixner measure μ functions on \mathcal{D}' . We construct now a natural orthogonal basis in (L^2) . For $n \in \mathbb{N}$ denote by $\overline{\mathcal{P}}_n$ the closure in (L^2) of the set of all continuous polynomials on \mathcal{D}' of degree $\leq n$, $\overline{\mathcal{P}}_0 := \mathbb{C}$. Denote also $(L_n^2) := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $(L_0^2) := \mathbb{C}$. Since μ has a holomorphic at zero Laplace transform, the set of continuous polynomials on \mathcal{D}' is dense in (L^2) [32], therefore $(L^2) = \bigoplus_{n=0}^{\infty} (L_n^2)$.

Denote by $\widehat{\otimes}$ a symmetric tensor product. For each $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ ($\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$), we define $\langle x^{\widehat{\otimes} n}, f^{(n)} \rangle$, $x \in \mathcal{D}'$, as the orthogonal projection of $\langle x^{\widehat{\otimes} n}, f^{(n)} \rangle$ onto (L_n^2) . It follows from results of [30] that $\langle x^{\widehat{\otimes} n}, f^{(n)} \rangle := \langle P_n(x), f^{(n)} \rangle$, where $P_n(x) \in \mathcal{D}'^{\widehat{\otimes} n}$ are the kernels of (generalized Appell) polynomials with a generating function $\gamma(\lambda) \exp\{\langle x, \alpha(\lambda) \rangle\}$, $\lambda \in \mathcal{D}_{\mathbb{C}}$, i.e.,

$$\gamma(\lambda) \exp\{\langle x, \alpha(\lambda) \rangle\} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n(x), \lambda^{\widehat{\otimes} n} \rangle,$$

now

$$\begin{aligned} \alpha(\lambda) &= \lambda + \sum_{n=2}^{\infty} \frac{\lambda^n}{n} (\rho^{n-1} + \rho^{n-2}\nu + \dots + \nu^{n-1}), \\ \gamma(\lambda) &= \frac{1}{l_{\mu}(\alpha(\lambda))} \\ &= \exp \left\{ - \int_{\mathbb{R}_+} \left(\frac{\lambda^2(t)}{2} + \sum_{n=3}^{\infty} \frac{\lambda^n(t)}{n} (\rho^{n-2}(t) + \rho^{n-3}(t)\nu(t) + \dots + \nu^{n-2}(t)) \right) dt \right\}. \end{aligned}$$

Let us define (real, i.e., bilinear) scalar products $\langle \cdot, \cdot \rangle_{\text{ext}}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$

$$\langle f^{(n)}, g^{(n)} \rangle_{\text{ext}} := \frac{1}{n!} \int_{\mathcal{D}'} \langle P_n(x), f^{(n)} \rangle \langle P_n(x), g^{(n)} \rangle \mu(dx).$$

It follows from results of [30] that

$$\begin{aligned} (1.2) \quad \langle f^{(n)}, g^{(n)} \rangle_{\text{ext}} &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{l_1^{s_1} \dots l_k^{s_k} s_1! \dots s_k!} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} f^{(n)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1}, \dots, t_{s_1}}_{l_1}, \dots, \underbrace{t_{s_1 + \dots + s_k}, \dots, t_{s_1 + \dots + s_k}}_{l_k}) \\ &\times g^{(n)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1}, \dots, t_{s_1}}_{l_1}, \dots, \underbrace{t_{s_1 + \dots + s_k}, \dots, t_{s_1 + \dots + s_k}}_{l_k}) \eta^{l_1 - 1}(t_1) \dots \eta^{l_1 - 1}(t_{s_1}) \\ &\times \eta^{l_2 - 1}(t_{s_1 + 1}) \dots \eta^{l_2 - 1}(t_{s_1 + s_2}) \dots \eta^{l_k - 1}(t_{s_1 + \dots + s_{k-1} + 1}) \dots \eta^{l_k - 1}(t_{s_1 + \dots + s_k}) \\ &\times dt_1 \dots dt_{s_1 + \dots + s_k}. \end{aligned}$$

So, for example, for $n = 1$ $\langle f^{(1)}, g^{(1)} \rangle_{\text{ext}} = \langle f^{(1)}, g^{(1)} \rangle = \int_{\mathbb{R}_+} f^{(1)}(t) g^{(1)}(t) dt$, for $n = 2$ $\langle f^{(2)}, g^{(2)} \rangle_{\text{ext}} = \langle f^{(2)}, g^{(2)} \rangle + \int_{\mathbb{R}_+} f^{(2)}(t, t) g^{(2)}(t, t) \eta(t) dt$. If (see (1.2)) $\eta \equiv 0$ (the case of Gaussian or Poissonian μ) then $\langle f^{(n)}, g^{(n)} \rangle_{\text{ext}} = \langle f^{(n)}, g^{(n)} \rangle$, in the general case $\langle f^{(n)}, g^{(n)} \rangle_{\text{ext}} = \langle f^{(n)}, g^{(n)} \rangle + \dots$.

Let $|\cdot|_{\text{ext}}$ denote the norm that is generated by the scalar product $\langle \cdot, \cdot \rangle_{\text{ext}}$, i.e., for $n \in \mathbb{Z}_+$ $|f^{(n)}|_{\text{ext}} := \sqrt{\langle f^{(n)}, f^{(n)} \rangle_{\text{ext}}}$. Denote by $\mathcal{H}_{\text{ext}}^{(n)}$ the Hilbert space that is the completion in the classical sense of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to $|\cdot|_{\text{ext}}$ (in particular, $\mathcal{H}_{\text{ext}}^{(0)} = \mathbb{C}$).

Let $\mathcal{H} := L^2(\mathbb{R}_+)$ be the space of complex-valued square integrable with respect to the Lebesgue measure functions on \mathbb{R}_+ . It is clear that $\mathcal{H}_{\text{ext}}^{(1)} = \mathcal{H}$. For $n \in \mathbb{N} \setminus \{1\}$ one can identify $\mathcal{H}_{\text{ext}}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{\text{ext}}^{(n)}$ that consists of "vanishing on diagonals" elements (i.e., $f^{(n)}(t_1, \dots, t_n) = 0$ if there exist $i, j \in \{1, \dots, n\}$ such that $i \neq j$ but $t_i = t_j$). In this sense the space $\mathcal{H}_{\text{ext}}^{(n)}$ is an extension of $\mathcal{H}_{\text{ext}}^{\widehat{\otimes} n}$.

Note that, of course, the space $\mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N} \setminus \{1\}$, depends on the parametric function η , see (1.1) (for example, if $\eta \equiv 0$ then $\mathcal{H}_{\text{ext}}^{(n)} = \mathcal{H}_{\text{ext}}^{\widehat{\otimes} n}$), but we do not use η in the designation of this space for simplification of notation.

For $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{Z}_+$, we define a polynomial $\langle P_n, F^{(n)} \rangle \in (L^2)$ as

$$\langle P_n, F^{(n)} \rangle := (L^2)\text{-}\lim_{k \rightarrow \infty} \langle P_n, f_k^{(n)} \rangle,$$

where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \ni f_k^{(n)} \rightarrow F^{(n)}$ in $\mathcal{H}_{\text{ext}}^{(n)}$ as $k \rightarrow \infty$ (as is easy to verify, this definition is well-posed). The forthcoming statement easily follows from the construction of polynomials $\langle P_n, F^{(n)} \rangle$ (see also [30]).

Theorem. A function $F \in (L^2)$ if and only if there exists a sequence of kernels

$$(F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)})_{n=0}^{\infty}$$

such that F can be presented in the form

$$(1.3) \quad F = \sum_{n=0}^{\infty} \langle P_n, F^{(n)} \rangle,$$

where the series converges in (L^2) , i.e., the (L^2) -norm of F

$$(1.4) \quad \|F\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{\text{ext}}^2 < \infty.$$

Moreover, the system $\{\langle P_n, F^{(n)} \rangle, F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}, n \in \mathbb{Z}_+\}$ is an orthogonal basis in (L^2) in the sense that for $F, G \in (L^2)$ of form (1.3) the (real) scalar product in (L^2)

$$(F, G)_{(L^2)} = \sum_{n=0}^{\infty} n! \langle F^{(n)}, G^{(n)} \rangle_{\text{ext}}.$$

1.3. A rigging of (L^2) by parameterized Kondratiev type spaces of test and generalized functions. Let $\beta \in [0, 1]$, $q \in \mathbb{Z}$ in the case $\beta > 0$ and $q \in \mathbb{Z}_+$ in the case $\beta = 0$.

Definition. We define a parameterized Kondratiev type space of test functions $(L^2)_q^\beta \subseteq (L^2)$ as a Hilbert space of (classes of) functions $F : \mathcal{D}' \rightarrow \mathbb{C}$ of form (1.3), for which

$$(1.5) \quad \|F\|_{(L^2)_q^\beta}^2 \equiv \|F\|_{q,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |F^{(n)}|_{\text{ext}}^2 < \infty,$$

let also $(L^2)^\beta := \text{pr} \lim_q (L^2)_q^\beta$ – the projective limit of the spaces $(L^2)_q^\beta$ with the corresponding topology (e.g., [3]).

It is easy to see that the spaces $(L^2)_q^\beta, (L^2)^\beta$ are densely and continuously embedded into (L^2) , therefore one can consider a chain (a rigging of (L^2))

$$(L^2)^{-\beta} \supset (L^2)_{-q}^{-\beta} \supseteq (L^2) \supseteq (L^2)_q^\beta \supset (L^2)^\beta,$$

where $(L^2)_{-q}^{-\beta}, (L^2)^{-\beta} = \text{ind} \lim_q (L^2)_{-q}^{-\beta}$ (the inductive limit, e.g., [3]) are the spaces dual of $(L^2)_q^\beta, (L^2)^\beta$ with respect to (L^2) correspondingly.

Definition. The spaces $(L^2)_{-q}^{-\beta}, (L^2)^{-\beta}$ are called parameterized Kondratiev type spaces of generalized functions.

It is easy to see that $F \in (L^2)_{-q}^{-\beta}$ if and only if F can be presented in form (1.3) with

$$\|F\|_{(L^2)_{-q}^{-\beta}}^2 \equiv \|F\|_{-q,-\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F^{(n)}|_{\text{ext}}^2 < \infty.$$

Denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the (real, i.e., bilinear) pairing between generalized and test functions, this pairing is generated by the scalar product in (L^2) . It is easy to see that for a test function G and a generalized function F of form (1.3)

$$\langle\langle F, G \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}, G^{(n)} \rangle_{\text{ext}}.$$

In what follows, often it will be convenient to denote the spaces $(L^2)_q^\beta, (L^2)_{-q}^{-\beta}$ by the general symbol $(L^2)_q^\beta$, now $\beta \in [-1, 1]$, $q \in \mathbb{Z}$ (below we will accept this on default). Note that for $\beta = 0$ and $q \in \mathbb{N}$ $(L^2)_q^0$ is the space of test functions, $(L^2)_{-q}^0$ is the space of generalized functions, for $\beta = q = 0$ $(L^2)_0^0 = (L^2)$ (cf. (1.4) with (1.5)).

Remark. One can easily generalize the notion of spaces of test and generalized functions by using in the definitions of these spaces $q \in \mathbb{R}$, $\beta \in \mathbb{R}$, and K^{qn} ($K > 1$) instead of 2^{qn} (see (1.5)). But such a generalization is not essential in the framework of the Meixner white noise analysis.

1.4. The extended stochastic integral. By analogy with the Gaussian analysis, on the probability triplet $(\mathcal{D}', C(\mathcal{D}'), \mu)$ we define the Meixner random process M by setting for each $t \in \mathbb{R}_+$ $M_t := \langle P_1, 1_{[0,t)} \rangle \in (L^2)$ ($M_0 = 0$), here and below $1_B(y)$ is the indicator of the event $\{y \in B\}$.

Remark. If the parametric functions ρ and ν (see Subsection 1.1) are constants then M is a Lévy process; but, in general, it is not the case (M can be a not time-homogeneous process).

Using results of [30] one can show that M is a locally square integrable normal martingale (with respect to the generated by M flow of full σ -algebras) with orthogonal independent increments, therefore one can consider the Itô stochastic integral with respect to M .

Let us recall the construction of the extended (Skorohod) stochastic integral with respect to M (see [17] for details). Let $G \in (L^2)_q^\beta \otimes \mathcal{H}$. It follows from above-posed results that G can be presented in the form

$$(1.6) \quad G(\cdot) = \sum_{n=0}^{\infty} \langle P_n, G^{(n)} \rangle,$$

$G^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$, with

$$(1.7) \quad \|G\|_{(L^2)_q^\beta \otimes \mathcal{H}}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |G^{(n)}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}}^2 < \infty.$$

If in addition G is such that the kernels $G^{(n)}$ belong to $\mathcal{H}^{\widehat{\otimes} n} \otimes \mathcal{H} \subset \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$ (the embedding in the generalized sense described above) then one can show [17] that F can be presented in the form

$$(1.8) \quad G(\cdot) = \sum_{n=0}^{\infty} n! \int_0^\infty \int_0^\infty \dots \int_0^{t_n} G^{(n)}(t_1, \dots, t_n) dM_{t_1} \dots dM_{t_n},$$

i.e., as a series of repeated Itô stochastic integrals with respect to the Meixner process. In this case one can define the extended stochastic integral of G with respect to M as

$$(1.9) \quad \int G(t) \widehat{d}M_t := \sum_{n=0}^{\infty} (n+1)! \int_0^\infty \int_0^t \int_0^{t_n} \dots \int_0^{t_2} \widehat{G}^{(n)}(t_1, \dots, t_n, t) dM_{t_1} \dots dM_{t_n} dM_t \\ = \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{G}^{(n)} \rangle \in (L^2)_q^\beta$$

(cf. [11]), where $\widehat{G}^{(n)} \in \mathcal{H}^{\widehat{\otimes} n+1} \subset \mathcal{H}_{\text{ext}}^{(n+1)}$ are the projections of $G^{(n)}$ onto $\mathcal{H}^{\widehat{\otimes} n+1}$, if this series converges in $(L^2)_q^\beta$. Note that if in addition $\beta = q = 0$ and G is integrable by Itô then series (1.9) is the result of term by term integration of series (1.8). Moreover, in this case from $G \in (L^2) \otimes \mathcal{H}$ it follows that (1.9) converges in (L^2) .

For a general $G \in (L^2)_q^\beta \otimes \mathcal{H}$ the above mentioned definition cannot be accepted because it is impossible to project elements of $\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$ onto $\mathcal{H}_{\text{ext}}^{(n+1)}$, generally speaking. Nevertheless, the following natural generalization is possible. Let $G^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$. We select a representative (a function) $g^{(n)} \in G^{(n)}$ with the property $g_t^{(n)}(t_1, \dots, t_n) = 0$ if there exists $j \in \{1, \dots, n\}$ such that $t_j = t$. Let us define the element $\widehat{G}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ as the equivalence class in $\mathcal{H}_{\text{ext}}^{(n+1)}$ that is generated by the symmetrization of $g^{(n)}$ with respect to $n+1$ variables (note that for $n=0$ we have $\mathcal{H}_{\text{ext}}^{(0)} \otimes \mathcal{H} = \mathcal{H} \ni G^{(0)} = \widehat{G}^{(0)} \in \mathcal{H} = \mathcal{H}_{\text{ext}}^{(1)}$). It was proved in [17] that $\widehat{G}^{(n)}$ is well-defined and $|\widehat{G}^{(n)}|_{\text{ext}} \leq |G^{(n)}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}}$.

Definition. Let $G \in (L^2)_q^\beta \otimes \mathcal{H}$ and be such that

$$(1.10) \quad \sum_{n=0}^{\infty} ((n+1)!)^{1+\beta} 2^{q(n+1)} |\widehat{G}^{(n)}|_{\text{ext}}^2 < \infty,$$

where the elements $\widehat{G}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ are constructed as above by the kernels $G^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$ from decomposition (1.6) for G . We define the extended stochastic integral with respect to $M \int G(t) \widehat{d}M_t \in (L^2)_q^\beta$ by setting

$$\int G(t) \widehat{d}M_t := \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{G}^{(n)} \rangle.$$

In particular cases, when the generalized Meixner measure μ is the Gaussian or Poissonian one, the operator $\int \circ(t) \widehat{d}M_t$ is the classical extended Skorohod stochastic integral. Moreover, if $\beta = q = 0$ and $G \in (L^2) \otimes \mathcal{H}$ is integrable by Itô with respect to M (i.e., if G is adapted with respect to the generated by M flow of σ -algebras) then G is integrable in the extended sense and $\int G(t) \widehat{d}M_t = \int G(t) dM_t \in (L^2)$ ([17]), here and below $\int \circ(t) dM_t$ is the Itô stochastic integral.

Remark. The extended stochastic integral can be continued to a linear continuous operator acting from $(L^2)_q^\beta \otimes \mathcal{H}$ to $(L^2)_{q-1}^\beta$ (and from $(L^2)^\beta \otimes \mathcal{H}$ to $(L^2)^\beta$). Moreover, in the case $\beta = -1$ estimate (1.10) follows from the condition $G \in (L^2)_q^{-1} \otimes \mathcal{H}$, therefore $\int \circ(t) \widehat{d}M_t : (L^2)_q^{-1} \otimes \mathcal{H} \rightarrow (L^2)_q^{-1}$ is a linear continuous operator.

1.5. The Hida stochastic derivative. Finally, let us recall the notion of the Hida stochastic derivative in the Meixner white noise analysis (see [15, 16] for more details). First we note that, as it was proved in [17], any $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N}$, can be considered as an element $F^{(n)}(\cdot)$ of the space $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$, and $|F^{(n)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}} \leq |F^{(n)}|_{\text{ext}}$ (nevertheless, one can not understand the space $\mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N} \setminus \{1\}$, as a subspace of $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$: different elements of $\mathcal{H}_{\text{ext}}^{(n)}$ can coincide in $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$).

Definition. Let $F \in (L^2)_q^\beta$ and be such that

$$(1.11) \quad \sum_{n=1}^{\infty} (n!)^{1+\beta} n^{1-\beta} 2^{q(n-1)} |F^{(n)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}}^2 < \infty,$$

where $F^{(n)}(\cdot)$ are the kernels from decomposition (1.3) for F , in point as elements of $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$. We define the Hida stochastic derivative $\partial.F \in (L^2)_q^\beta \otimes \mathcal{H}$ by setting

$$\partial.F := \sum_{n=1}^{\infty} n \langle P_{n-1}, F^{(n)}(\cdot) \rangle.$$

It follows from results of [17, 12, 16] that the extended stochastic integral $\int \circ(t) \widehat{d}M_t : (L^2)_q^\beta \otimes \mathcal{H} \rightarrow (L^2)_q^\beta$ and the Hida stochastic derivative $\partial. : (L^2)_q^\beta \rightarrow (L^2)_q^\beta \otimes \mathcal{H}$ are adjoint one to another, and, in particular, are closed operators.

Remark. The Hida stochastic derivative can be continued to a linear continuous operator acting from $(L^2)_q^\beta$ to $(L^2)_{q-1}^\beta \otimes \mathcal{H}$ (and from $(L^2)^\beta$ to $(L^2)^\beta \otimes \mathcal{H}$). Moreover, in the case $\beta = 1$ estimate (1.11) follows from the condition $F \in (L^2)_q^1$, therefore $\partial. : (L^2)_q^1 \rightarrow (L^2)_q^1 \otimes \mathcal{H}$ is a linear continuous operator.

2. CLARK-OCONE TYPE FORMULAS AND RELATED MATTERS

2.1. A Clark-Ocone formula in the simplest particular case. For elements of spaces $(L^2)_q^\beta$, $\beta \in [-1, 0)$ and $q \in \mathbb{Z}$, or $\beta = 0$ and $q < 0$ (i.e., for generalized functions), let us extend the expectation \mathbf{E} and the conditional expectation \mathbf{E}_t as follows. Let $F \in (L^2)_q^\beta$, $t \in \mathbb{R}_+$. Set

$$(2.1) \quad \begin{aligned} \mathbf{E}F &:= \langle\langle F, 1 \rangle\rangle \in \mathbb{C}, \\ \mathbf{E}_t F &:= \langle P_0, F^{(0)} \rangle + \sum_{n=1}^{\infty} \langle P_n, F^{(n)} 1_{[0,t]^n} \rangle \in (L^2)_q^\beta, \end{aligned}$$

here $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (1.3) for F . If $F \in (L^2)$ then, as is easy to see, $\mathbf{E}F$ is the expectation of F , and as it follows from Theorem 4.2 in [18], $\mathbf{E}_t F$ is the conditional expectation of F with respect to the full σ -algebra $\sigma(M_s : s \leq t)$.

The next proposition is a simple generalization of the statement from Subsection 2.1 in [13].

Proposition 2.1. *Let $F \in (L^2)_q^\beta$ be such that all kernels $F^{(n)}$ from decomposition (1.3) belong to $\mathcal{H}^{\otimes n}$ (now we consider $\mathcal{H}^{\otimes n}$ as a subspace of $\mathcal{H}_{\text{ext}}^{(n)}$ in the generalized sense described in Subsection 1.2). Then the analog of classical Clark-Ocone formula (0.1) has a form*

$$(2.2) \quad F = \mathbf{E}F + \int \mathbf{E}_t \partial_t F dM_t.$$

2.2. A belonging of functions from $(L^2)_q^\beta$ to the range of values of the extended stochastic integral. Unfortunately, formula (2.2) is not valid for F with kernels $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, generally speaking (even for $F = \langle P_2, F^{(2)} \rangle$, see [13]). Moreover, in the general case not each $F \in (L^2)_q^\beta$ can be presented even as a result of extended stochastic integration. Therefore, in order to construct Clark-Ocone type formulas for general F , first we have to clarify this question.

Proposition 2.2. *Let $F \in (L^2)_q^\beta$. The following statements are equivalent:*

- (1) F can be presented in the form

$$(2.3) \quad F = \mathbf{E}F + \int G(t) \widehat{d}M_t,$$

where $G \in (L^2)_q^\beta \otimes \mathcal{H}$ in the case $\beta \geq 0$, and $G \in (L^2)_{q-1}^\beta \otimes \mathcal{H}$ in the case $\beta < 0$;

- (2) for each $n \in \mathbb{N} \setminus \{1\}$ the kernel $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ from decomposition (1.3) for F has a representative $f^{(n)}$ such that $f^{(n)}(t_1, \dots, t_n) = 0$ if for each $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\} \setminus \{i\}$ such that $t_i = t_j$.

Remark 2.1. If, for example, $\eta \equiv 0$ (see (1.1)) then for each $F \in (L^2)_q^\beta$ the condition of statement (2) is automatically fulfilled. In fact, it follows from (1.2) that considering properties of representatives of $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N} \setminus \{1\}$, one can ignore families of arguments $\{t_1, \dots, t_n\}$ for which there exist $i, j \in \{1, \dots, n\}$ such that $i \neq j$, $t_i = t_j$, $\eta(t_i) = 0$ (i.e., one can redefine these representatives on described families of arguments in compliance with necessity).

Proof. For $F = \langle P_n, F^{(n)} \rangle$, $n \in \mathbb{N} \setminus \{1\}$, $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, this statement is proved in [13] (in the cases $n = 0$ and $n = 1$ representation (2.3) is trivial). In the general case the implication "(1) \Rightarrow (2)" easily follows from the corresponding implication in the particular case; in order to prove the implication "(2) \Rightarrow (1)" it is sufficient to show that $G \in (L^2)_q^\beta \otimes \mathcal{H}$

in the case $\beta \geq 0$, and $G \in (L^2)_{q-1}^\beta \otimes \mathcal{H}$ in the case $\beta < 0$. In order to do this (and for the convenience of a reader) let us recall the construction of kernels from decomposition (1.6) for this G ([13]). Set $G^{(0)} := F^{(1)} \equiv F^{(1)}(\cdot)$. Further, let $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N} \setminus \{1\}$, be a kernel from decomposition (1.3) for F , $f^{(n)}$ be a representative of $F^{(n)}$ that is described in the condition of statement (2). Without loss of generality one can assume that $f^{(n)}$ is a symmetric function. We set

$$(2.4) \quad h_n(t_1, \dots, t_n) = \frac{1}{n} [1_{\{t_1 \neq t_n, t_2 \neq t_n, \dots, t_{n-1} \neq t_n\}} + 1_{\{t_n \neq t_{n-1}, t_1 \neq t_{n-1}, \dots, t_{n-2} \neq t_{n-1}\}} + \dots + 1_{\{t_2 \neq t_1, t_3 \neq t_1, \dots, t_n \neq t_1\}}]$$

(here and below 1_B is the indicator of the event B),

$$g_t^{(n-1)}(t_1, \dots, t_{n-1}) := \begin{cases} \frac{f^{(n)}(t_1, \dots, t_{n-1}, t)}{h_n(t_1, \dots, t_{n-1}, t)}, & \text{if } h_n(t_1, \dots, t_{n-1}, t) \neq 0 \\ 0, & \text{if } h_n(t_1, \dots, t_{n-1}, t) = 0 \end{cases}$$

(note that if $h_n(t_1, \dots, t_{n-1}, t) = 0$ then $f^{(n)}(t_1, \dots, t_{n-1}, t) = 0$ by the condition of statement (2)). It is proved in [13] that the function $g_t^{(n-1)}$ generates an element (an equivalence class) $G^{(n-1)} \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$, $\widehat{G}^{(n-1)} = F^{(n)}$, and $|G^{(n-1)}|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}} \leq \sqrt{n} |F^{(n)}|_{\text{ext}}$. Using this estimate, (1.7) and (1.5), we obtain for $\beta \geq 0$

$$\begin{aligned} \|G\|_{(L^2)_q^\beta \otimes \mathcal{H}}^2 &= \sum_{n=1}^{\infty} ((n-1)!)^{1+\beta} 2^{q(n-1)} |G^{(n-1)}|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}}^2 \\ &\leq 2^{-q} \sum_{n=1}^{\infty} (n!)^{1+\beta} 2^{qn} n^{-\beta} |F^{(n)}|_{\text{ext}}^2 \leq 2^{-q} \|F\|_{q,\beta}^2 < \infty, \end{aligned}$$

and for $\beta < 0$

$$\begin{aligned} \|G\|_{(L^2)_{q-1}^\beta \otimes \mathcal{H}}^2 &= \sum_{n=1}^{\infty} ((n-1)!)^{1+\beta} 2^{(q-1)(n-1)} |G^{(n-1)}|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}}^2 \\ &\leq 2^{1-q} \sum_{n=1}^{\infty} (n!)^{1+\beta} 2^{qn} [2^{-n} n^{-\beta}] |F^{(n)}|_{\text{ext}}^2 \leq 2^{1-q} \max_{n \in \mathbb{N}} [2^{-n} n^{-\beta}] \|F\|_{q,\beta}^2 < \infty, \end{aligned}$$

which required to be proved. □

Remark. Let $F \in (L^2)_q^\beta$ and be presentable in the form $F = \mathbf{E}F + \int \mathcal{G}(t) \widehat{d}M_t$, where $\mathcal{G}(\cdot) = \sum_{n=1}^{\infty} \langle P_{n-1}, \mathcal{G}^{(n-1)} \rangle$, $\mathcal{G}^{(n-1)} \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$, is a formal series, and $\int \mathcal{G}(t) \widehat{d}M_t$ is a formal stochastic integral, i.e., $\int \mathcal{G}(t) \widehat{d}M_t = \sum_{n=1}^{\infty} \langle P_n, \widehat{\mathcal{G}}^{(n-1)} \rangle$. Since now for each $n \in \mathbb{N}$ $F^{(n)} = \widehat{\mathcal{G}}^{(n-1)}$ in $\mathcal{H}_{\text{ext}}^{(n)}$ (see (1.3)), F satisfies the condition of statement (2) of Proposition 2.2 whence it follows that F can be presented in form (2.3) with an integrand $G \in (L^2)_q^\beta \otimes \mathcal{H}$ in the case $\beta \geq 0$ or $G \in (L^2)_{q-1}^\beta \otimes \mathcal{H}$ in the case $\beta < 0$ (note that $G \neq \mathcal{G}$, generally speaking). So, in what follows, in corresponding places we will write "F can be presented in form (2.3)" without the reminder that $G \in (L^2)_q^\beta \otimes \mathcal{H}$ or $G \in (L^2)_{q-1}^\beta \otimes \mathcal{H}$.

Finally we note that even a monomial $\langle P_3, F^{(3)} \rangle$, $F^{(3)} \in \mathcal{H}_{\text{ext}}^{(3)}$, can be presentable in form (2.3), but not in form (2.2) (see [13]).

2.3. Clark-Ocone type formulas. In this subsection we generalize the Clark-Ocone type formulas, proposed in [13], to the spaces $(L^2)_q^\beta$.

For $n \in \mathbb{N} \setminus \{1\}$ and $t_1, \dots, t_n \in \mathbb{R}_+$ set

$$\widehat{h}_n(t_1, \dots, t_n) := nh_n(t_1, \dots, t_n),$$

where the functions h_n are defined in (2.4); set also $h_1 \equiv 1$. Further, for $G^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{Z}_+$, set

$$\tilde{G}^{(n)}(\cdot_1, \dots, \cdot_n) := \begin{cases} \frac{G^{(n)}(\cdot_1, \dots, \cdot_n)}{h_{n+1}(\cdot_1, \dots, \cdot_n, \cdot)}, & \text{if } h_{n+1}(\cdot_1, \dots, \cdot_n, \cdot) \neq 0 \\ 0, & \text{if } h_{n+1}(\cdot_1, \dots, \cdot_n, \cdot) = 0 \end{cases}.$$

It is easy to see that $\tilde{G}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$ and

$$(2.5) \quad |\tilde{G}^{(n)}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}} \leq |G^{(n)}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}}.$$

For $G \in (L^2)_q^\beta \otimes \mathcal{H}$ we define

$$(AG)(\cdot) := \sum_{n=0}^{\infty} \langle P_n, \tilde{G}^{(n)} \rangle,$$

where the kernels $\tilde{G}^{(n)}$ are constructed by the kernels $G^{(n)}$ from decomposition (1.6) for G . It follows from estimate (2.5) that A is a linear *continuous* operator in $(L^2)_q^\beta \otimes \mathcal{H}$.

Proposition 2.3. *Let $F \in (L^2)_q^\beta$. Then for $\beta \geq 0$ $A\partial.F \in (L^2)_q^\beta \otimes \mathcal{H}$, and for $\beta < 0$ $A\partial.F \in (L^2)_{q-1}^\beta \otimes \mathcal{H}$, where we understand $\partial.$ as a linear continuous operator acting from $(L^2)_q^\beta$ to $(L^2)_{q-1}^\beta \otimes \mathcal{H}$.*

Proof. In the case $\beta < 0$ the result follows from properties of $\partial.$ and A . Let $\beta \geq 0$. Since $A\partial.F = \sum_{n=1}^{\infty} n \langle P_{n-1}, \tilde{F}^{(n)}(\cdot) \rangle$ (here $\tilde{F}^{(n)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$ are constructed by the kernels $F^{(n)}$ from decomposition (1.3) for F , in point as elements of $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}$), $h_n(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1+\dots+s_k}, \dots, t_{s_1+\dots+s_k}}_{l_k}, t) = 1_{\{l_k > 1\}} + (s_k + 1)1_{\{l_k = 1\}}$ for different arguments $t_1, \dots, t_{s_1+\dots+s_k}, t$ (here $k, l, s \in \mathbb{N}$, $l_1 > \dots > l_k$, $l_1 s_1 + \dots + l_k s_k = n - 1$) and therefore (see (1.2))

$$\begin{aligned} |n\tilde{F}^{(n)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}}^2 &= n \sum_{\substack{k, l, j, s_j \in \mathbb{N}: \\ j=1, \dots, k, l_1 > \dots > l_k > 1, \\ l_1 s_1 + \dots + l_k s_k + 1 = n}} \frac{n!}{l_1^{s_1} \dots l_k^{s_k} s_1! \dots s_k!} \\ &\times \int_{\mathbb{R}_+^{s_1+\dots+s_k+1}} |F^{(n)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1+\dots+s_k}, \dots, t_{s_1+\dots+s_k}}_{l_k}, t)|^2 \\ &\times \eta^{l_1-1}(t_1) \dots \eta^{l_k-1}(t_{s_1+\dots+s_k}) dt_1 \dots dt_{s_1+\dots+s_k} dt \\ &+ n \sum_{\substack{k, l, j, s_j \in \mathbb{N}: \\ j=1, \dots, k, l_1 > \dots > l_k = 1, \\ l_1 s_1 + \dots + l_{k-1} s_{k-1} + s_k + 1 = n}} \frac{n!}{l_1^{s_1} \dots l_{k-1}^{s_{k-1}} s_1! \dots (s_k + 1)!(s_k + 1)} \\ &\times \int_{\mathbb{R}_+^{s_1+\dots+s_k+1}} |F^{(n)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, t_{s_1+\dots+s_{k-1}+1}, \dots, t_{s_1+\dots+s_k}, t)|^2 \\ &\times \eta^{l_1-1}(t_1) \dots \eta^{l_{k-1}-1}(t_{s_1+\dots+s_{k-1}}) dt_1 \dots dt_{s_1+\dots+s_k} dt \leq n |F^{(n)}|_{\text{ext}}^2, \end{aligned}$$

we obtain (see (1.7) and (1.5))

$$\begin{aligned} \|A\partial.F\|_{(L^2)_q^\beta \otimes \mathcal{H}}^2 &= \sum_{n=1}^{\infty} ((n-1)!)^{1+\beta} 2^{q(n-1)} |n\tilde{F}^{(n)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}}^2 \\ &\leq 2^{-q} \sum_{n=1}^{\infty} (n!)^{1+\beta} 2^{qn} n^{-\beta} |F^{(n)}|_{\text{ext}}^2 \leq 2^{-q} \|F\|_{q, \beta}^2 < \infty, \end{aligned}$$

which required to be proved. \square

From the proof of Theorem 1 in [13] and this proposition we obtain the following result.

Theorem 2.1. *Let $F \in (L^2)_q^\beta$ and be presentable in form (2.3) (see Proposition 2.2). Then the representation*

$$(2.6) \quad F = \mathbf{E}F + \int A\partial_t F \widehat{d}M_t$$

is valid, where $\int A\partial_t F \widehat{d}M_t := \int (A\partial_t F)(t) \widehat{d}M_t$.

Corollary. *If $F \in (L^2)_q^\beta$ and can be presented in form (2.3) then an integrand G from (2.3) can be presented in the form*

$$G(\cdot) = A\partial_t F.$$

Formula (2.6) can be interpreted as a Clark-Ocone type formula in the Meixner white noise analysis, but this formula is not a closer analog of classical Clark-Ocone formula (0.1) (see [13] for details). Let us obtain a direct analog of formula (0.1). For $n \in \mathbb{N}$ and $t_1, \dots, t_n, t \in \mathbb{R}_+$ set

$$\chi_{n,t}(t_1, \dots, t_n) := \begin{cases} 0, & \text{if } \exists i \in \{1, \dots, n\} : t_i \geq t \text{ and } \forall j \in \{1, \dots, n\} \setminus \{i\} \ t_i \neq t_j \\ 1, & \text{in other cases} \end{cases},$$

i.e., $\chi_{n,t}(t_1, \dots, t_n) = 0$ if there exists t_i of multiplicity one such that $t_i \geq t$. For example, $\chi_{3,5}(6, 6, 4) = 1$ ($4 < 5$, 6 has the multiplicity two), $\chi_{3,5}(6, 6, 6) = \chi_{3,5}(4, 4, 4) = 1$ (no terms of multiplicity one), but $\chi_{3,5}(6, 4, 4) = 0$ ($6 > 5$, 6 has the multiplicity one). Set also $\chi_{0,\cdot} \equiv 1$. For $F \in (L^2)_q^\beta$ and $t \in \mathbb{R}_+$ define

$$(2.7) \quad \widetilde{\mathbf{E}}_t F := \sum_{n=0}^{\infty} \langle P_n, F^{(n)} \chi_{n,t} \rangle \in (L^2)_q^\beta,$$

where $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ are the kernels from decomposition (1.3) for F . As is easily seen, we have $|F^{(n)} \chi_{n,t}|_{\text{ext}} \leq |F^{(n)}|_{\text{ext}}$, therefore $\widetilde{\mathbf{E}}_t \circ$ is a linear continuous operator in $(L^2)_q^\beta$.

Remark. We use for the operator $\widetilde{\mathbf{E}}_t$ the notation that is similar to the designation of a conditional expectation because these operators are similar in a sense: cf. (2.7) and (2.1). Moreover, it is easy to see that in the Gaussian and Poissonian cases $\widetilde{\mathbf{E}}_t = \mathbf{E}_t$ because for $n \in \mathbb{N}$ $\chi_{n,t} = 1_{[0,t]^n}$ in $\mathcal{H}^{\widehat{\otimes} n}$ (i.e., these two functions belong to the same equivalence class in this space).

Further, for $G \in (L^2)_q^\beta \otimes \mathcal{H}$ we define

$$(\widetilde{\mathbf{E}}.G)(\cdot) := \sum_{n=0}^{\infty} \langle P_n, G^{(n)} \chi_{n,\cdot} \rangle \in (L^2)_q^\beta \otimes \mathcal{H},$$

where the kernels $G^{(n)}$ are from decomposition (1.6) for G . It is easy to see that $\widetilde{\mathbf{E}}$ is a linear continuous operator in $(L^2)_q^\beta \otimes \mathcal{H}$, and for fixed $t \in \mathbb{R}_+$

$$(\widetilde{\mathbf{E}}.G)(t) = \widetilde{\mathbf{E}}_t G(t)$$

(one can understand elements of $(L^2)_q^\beta \otimes \mathcal{H}$ as classes of functions acting from \mathbb{R}_+ to $(L^2)_q^\beta$, $G(t)$ and $(\widetilde{\mathbf{E}}.G)(t)$ are corresponding representatives of such classes at t).

Proposition 2.4. *Let $F \in (L^2)_q^\beta$. Then for $\beta \geq 0$ $\widetilde{\mathbf{E}}.\partial.F \in (L^2)_q^\beta \otimes \mathcal{H}$, and for $\beta < 0$ $\widetilde{\mathbf{E}}.\partial.F \in (L^2)_{q-1}^\beta \otimes \mathcal{H}$, where ∂ . as in Proposition 2.3.*

Proof. In the case $\beta < 0$ the result follows from properties of $\partial.$ and $\tilde{\mathbf{E}}.$ Let $\beta \geq 0$. Since $\tilde{\mathbf{E}}.\partial.F = \sum_{n=0}^{\infty} (n+1) \langle P_n, F^{(n+1)}(\cdot) \chi_{n,\cdot} \rangle$ (here $F^{(n+1)}(\cdot)$ are the kernels from decomposition (1.3) for F , in point as elements of $\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}$), in order to estimate the norm of $\tilde{\mathbf{E}}.\partial.F$ in $(L^2)_q^\beta \otimes \mathcal{H}$ we need the following technical result.

Lemma. For $n \in \mathbb{Z}_+$ and $F^{(n+1)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$

$$(n+1) |F^{(n+1)}(\cdot) \chi_{n,\cdot}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}}^2 \leq |F^{(n+1)}|_{\text{ext}}^2.$$

Proof. Using (1.2), the definition of $\chi_{n,\cdot}$, and the fact that for a symmetric integrable by Lebesgue function $f_m : \mathbb{R}_+^m \rightarrow \mathbb{R}$, $m \in \mathbb{N}$,

$$\begin{aligned} & \int_{\mathbb{R}_+^m} f_m(t_1, \dots, t_m) dt_1 \dots dt_m \\ &= \sum_{k=1}^m \int_{\mathbb{R}_+} dt_k \int_{[0, t_k]^{m-1}} f_m(t_1, \dots, t_m) dt_1 \dots dt_{k-1} dt_{k+1} \dots dt_m \\ &\equiv \sum_{k=1}^m \int_{\mathbb{R}_+} dt_1 \int_{[0, t_1]^{m-1}} f_m(t_k, t_2, \dots, t_{k-1}, t_1, t_{k+1}, \dots, t_m) dt_2 \dots dt_m \\ &= m \int_{\mathbb{R}_+} dt_1 \int_{[0, t_1]^{m-1}} f_m(t_1, \dots, t_m) dt_2 \dots dt_m, \end{aligned}$$

we obtain

$$\begin{aligned} (n+1) |F^{(n+1)}(\cdot) \chi_{n,\cdot}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}}^2 &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{(n+1)!}{l_1^{s_1} \dots l_k^{s_k} s_1! \dots s_k!} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} |F^{(n+1)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1 + \dots + s_k}, \dots, t_{s_1 + \dots + s_k}}_{l_k}, t)| \\ &\times \chi_{n,t}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1 + \dots + s_k}, \dots, t_{s_1 + \dots + s_k}}_{l_k})|^2 \\ &\times \eta^{l_1 - 1}(t_1) \dots \eta^{l_k - 1}(t_{s_1 + \dots + s_k}) dt_1 \dots dt_{s_1 + \dots + s_k} dt \\ &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k > 1, \\ l_1 s_1 + \dots + l_k s_k + 1 = n+1}} \frac{(n+1)!}{l_1^{s_1} \dots l_k^{s_k} s_1! \dots s_k!} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} |F^{(n+1)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1 + \dots + s_k}, \dots, t_{s_1 + \dots + s_k}}_{l_k}, t)|^2 \\ &\times \eta^{l_1 - 1}(t_1) \dots \eta^{l_k - 1}(t_{s_1 + \dots + s_k}) dt_1 \dots dt_{s_1 + \dots + s_k} dt \\ &+ \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k = 1, \\ l_1 s_1 + \dots + l_{k-1} s_{k-1} + s_k + 1 = n+1}} \frac{(n+1)!}{l_1^{s_1} \dots l_{k-1}^{s_{k-1}} s_1! \dots s_k!} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} |F^{(n)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, t_{s_1 + \dots + s_{k-1} + 1}, \dots, t_{s_1 + \dots + s_k}, t)| \\ &\times \chi_{n,t}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, t_{s_1 + \dots + s_{k-1} + 1}, \dots, t_{s_1 + \dots + s_k})|^2 \\ &\times \eta^{l_1 - 1}(t_1) \dots \eta^{l_{k-1} - 1}(t_{s_1 + \dots + s_{k-1}}) dt_1 \dots dt_{s_1 + \dots + s_k} dt \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k > 1, \\ l_1 s_1 + \dots + l_k s_k + 1 = n+1}} \frac{(n+1)!}{l_1^{s_1} \dots l_k^{s_k} s_1! \dots s_k!} \\
&\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} |F^{(n+1)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1 + \dots + s_k}, \dots, t_{s_1 + \dots + s_k}}_{l_k}, t)|^2 \\
&\times \eta^{l_1 - 1}(t_1) \dots \eta^{l_k - 1}(t_{s_1 + \dots + s_k}) dt_1 \dots dt_{s_1 + \dots + s_k} dt \\
&+ \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k = 1, \\ l_1 s_1 + \dots + l_{k-1} s_{k-1} + s_k + 1 = n+1}} \frac{(n+1)!}{l_1^{s_1} \dots l_{k-1}^{s_{k-1}} s_1! \dots s_k!} \\
&\times \int_{\mathbb{R}_+^{s_1 + \dots + s_{k-1} + 1}} dt dt_1 \dots dt_{s_1 + \dots + s_{k-1}} \eta^{l_1 - 1}(t_1) \dots \eta^{l_{k-1} - 1}(t_{s_1 + \dots + s_{k-1}}) \\
&\times \int_{[0, t]^{s_k}} |F^{(n)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, t_{s_1 + \dots + s_{k-1} + 1}, \dots, t_{s_1 + \dots + s_k}, t)|^2 \\
&\times dt_{s_1 + \dots + s_{k-1} + 1} \dots dt_{s_1 + \dots + s_k} \\
&= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k > 1, \\ l_1 s_1 + \dots + l_k s_k + 1 = n+1}} \frac{(n+1)!}{l_1^{s_1} \dots l_k^{s_k} s_1! \dots s_k!} \\
&\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} |F^{(n+1)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, \underbrace{t_{s_1 + \dots + s_k}, \dots, t_{s_1 + \dots + s_k}}_{l_k}, t)|^2 \\
&\times \eta^{l_1 - 1}(t_1) \dots \eta^{l_k - 1}(t_{s_1 + \dots + s_k}) dt_1 \dots dt_{s_1 + \dots + s_k} dt \\
&+ \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k = 1, \\ l_1 s_1 + \dots + l_{k-1} s_{k-1} + s_k + 1 = n+1}} \frac{(n+1)!}{l_1^{s_1} \dots l_{k-1}^{s_{k-1}} s_1! \dots (s_k + 1)!} \\
&\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} |F^{(n)}(\underbrace{t_1, \dots, t_1}_{l_1}, \dots, t_{s_1 + \dots + s_{k-1} + 1}, \dots, t_{s_1 + \dots + s_k}, t)|^2 \\
&\times \eta^{l_1 - 1}(t_1) \dots \eta^{l_{k-1} - 1}(t_{s_1 + \dots + s_{k-1}}) dt_1 \dots dt_{s_1 + \dots + s_k} dt \leq |F^{(n+1)}|_{\text{ext}}^2.
\end{aligned}$$

(Note that if $F^{(n+1)}$ satisfies the condition described in statement (2) of Proposition 2.2 then on the last step we have the equality, i.e., in this case $(n+1)|F^{(n+1)}(\cdot)\chi_{n,\cdot}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}}^2 = |F^{(n+1)}|_{\text{ext}}^2$.) \square

Using the result of this lemma, we obtain (see (1.7) and (1.5))

$$\begin{aligned}
\|\tilde{\mathbf{E}}.\partial.F\|_{(L^2)_q^\beta \otimes \mathcal{H}}^2 &= \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} (n+1)^2 |F^{(n+1)}(\cdot)\chi_{n,\cdot}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}}^2 \\
&\leq 2^{-q} \sum_{n=0}^{\infty} ((n+1)!)^{1+\beta} 2^{q(n+1)} (n+1)^{-\beta} |F^{(n+1)}|_{\text{ext}}^2 \leq 2^{-q} \|F\|_{q,\beta}^2 < \infty,
\end{aligned}$$

which required to be proved. \square

From the proof of Theorem 2 in [13] and this proposition we obtain the following result.

Theorem 2.2. Let $F \in (L^2)_q^\beta$ and be presentable in form (2.3) (see Proposition 2.2). Then the representation

$$(2.8) \quad F = \mathbf{E}F + \int \tilde{\mathbf{E}}_t \partial_t F \widehat{dM}_t$$

is valid.

Note that if the kernels $F^{(n)}$ from decomposition (1.3) for F can be considered as elements of $\mathcal{H}^{\widehat{\otimes} n}$ (see Subsection 1.2) then formula (2.8) reduces to (2.2).

Remark. In the case $\beta = q = 0$ (recall that $(L^2)_0^0 = (L^2)$) the statements of Theorems 2.1, 2.2 turn into the statements of Theorems 1, 2 from [13] correspondingly, but with one significant difference: now a function F , for which we construct the Clark-Ocone type formulas, can be not differentiable by Hida.

As we see, the use of the extended stochastic integral and of special operators in Clark-Ocone type formulas is stipulated by properties of the scalar products $\langle \cdot, \cdot \rangle_{\text{ext}}$. Nevertheless, in some particular cases one can use the Itô stochastic integral and the conditional expectation. Let us consider the question about this possibility in more details. From the proof of Theorem 3 in [13] and Proposition 2.4 one can easily obtain the following result.

Theorem 2.3. Let $F \in (L^2)_q^\beta$. Then the following statements are equivalent:

- (1) F can be presented in form (2.2) (now $\tilde{\mathbf{E}}.\partial.F \in (L^2)_q^\beta \otimes \mathcal{H}$ if $\beta \geq 0$, and $\tilde{\mathbf{E}}.\partial.F \in (L^2)_{q-1}^\beta \otimes \mathcal{H}$ if $\beta < 0$);
- (2) for each $n \in \mathbb{N} \setminus \{1\}$ the kernel $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ from decomposition (1.3) for F has a representative $f^{(n)} \in F^{(n)}$ such that $f^{(n)}(t_1, \dots, t_n) = 0$ if there exist $i, j \in \{1, \dots, n\}$, $i \neq j$, such that $\max\{t_1, \dots, t_n\} = t_i = t_j$ (i.e., if the multiplicity of maximal $t \in \{t_1, \dots, t_n\}$ is greater than one).

Remark. As is easy to see, if for some $F \in (L^2)_q^\beta$ the condition of statement (2) of this theorem is fulfilled (for example, it is so in the case $\eta \equiv 0$ (see (1.1))) then the condition of statement (2) of Proposition 2.2 is fulfilled too.

Note that if $F \in (L^2)_q^\beta$ can be presented in form (2.2) then $\tilde{\mathbf{E}}.\partial.F = \mathbf{E}.\partial.F$ (see the proof of Proposition 2.2 in [13]).

Remark. The results of Theorems 2.1–2.3 hold true if the random variable $F \in (L^2)^\beta$, $\beta \in [-1, 1]$, in this case the integrands belong to $(L^2)^\beta \otimes \mathcal{H}$.

Finally we note that the spaces similar to $\mathcal{H}_{\text{ext}}^{(n)}$ arise not only in the Meixner white noise analysis, but also in an analysis connected with Lévy processes, see [24, 23]. Therefore the results of this paper can be reformulated for the "Lévy analysis". A detailed study of this question and related topics will be given in another paper.

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