STURM TYPE OSCILLATION THEOREMS FOR EQUATIONS WITH BLOCK-TRIANGULAR MATRIX COEFFICIENTS

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To the memory of Anatoliy G. Kostyuchenko, a prominent mathematician and a remarkable person

ABSTRACT. A relation is established between spectral and oscillation properties of the problem on a finite interval and a semi-axis for second order differential equations with block-triangular matrix coefficients.

0. INTRODUCTION

The Sturm type oscillation theory for scalar differential equations of the second order and finite systems with Hermitian coefficients on finite and infinite intervals was studied by many specialists. The previous works of the authors [14], [15], [17], [18], along with the new results contained therein, present an extended bibliography in this area for self-adjoint problems.

The asymptotic distribution of eigenvalues for self-adjoint singular differential operators has been considered in the well known monograph of A. G. Kostyuchenko and I. S. Sargsyan [8].

The present work establishes a relation between spectral and oscillation properties of the problem on a finite interval and a semi-axis for second order differential equations with block-triangular matrix coefficients.

Note that differential equations with triangular matrix potential in the context of inverse scattering problem were considered in [3], [19], [20].

Consider the differential equation with matrix coefficients

$$(1) \qquad l[y] = -\left(P(x)y'\right)' + \frac{i}{2}\left(\left(Q(x)y\right)' + Q(x)y'\right) + V(x)y = \lambda W(x)y,$$

where the coefficients $P(x)$, $Q(x)$, together with their derivatives, as well as the coefficients $V(x)$, $W(x)$ depend continuously on $x \in [0, \infty)$.

Suppose that the coefficients $P(x)$, $Q(x)$, $V(x)$ in (1) have a block-triangular form. This means, in particular, that the potential $V(x)$ is of the form

$$(2) \qquad V(x) = \begin{pmatrix} V_{11}(x) & V_{12}(x) & \ldots & V_{1r}(x) \\ 0 & V_{22}(x) & \ldots & V_{2r}(x) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & V_{rr}(x) \end{pmatrix}$$

and the weight coefficient $W(x)$ is given by a block-diagonal matrix of the form

$$W(x) = \begin{pmatrix} W_{11}(x) & 0 & \ldots & 0 \\ 0 & W_{22}(x) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & W_{rr}(x) \end{pmatrix}.$$

The diagonal blocks $P_{kk}(x)$, $V_{kk}(x)$, $Q_{kk}(x)$, $W_{kk}(x)$, $k = \overline{1,r}$, are Hermitian $m_k \times m_k$ matrices, $m_k \geq 1$ (in particular, with $m_k = 1$ these are scalar functions). Let us assume $\sum\limits_{k=1}^{r} m_k = m$.

Denote by H_m the Hilbert space of dimension $m < \infty$. A vector $h \in H_m$ is to be written below as $h = \operatorname{col}(h_1, h_2, \ldots, h_r)$, with $h_k \in H_{m_k}$, $k = \overline{1,r}$.

Suppose that for every k the diagonal blocks $P_{kk}(x)$ of the coefficient at the highest derivative of $P(x)$, together with the diagonal blocks $W_{kk}(x)$ of the weight coefficient $W(x)$, are all simultaneously positive or negative matrices at every $x \in [0, b)$, also with b included in the case when the latter is finite.

Remark 1. In the case when some of the diagonal blocks of the matrices $P(x)$ and $W(x)$ are negative, consider the equation

$$(3) \qquad m[y] = \widetilde{I} \cdot l[y] = -\left(\widetilde{P}(x)y'\right)' + \frac{i}{2}\left(\left(\widetilde{Q}(x)y\right)' + \widetilde{Q}(x)y'\right) + \widetilde{V}(x)y = \lambda \widetilde{W}(x)y,$$

where $\widetilde{I}$ is a diagonal matrix, whose diagonal blocks are as follows. If $P_{kk}(x) > 0$ and $W_{kk}(x) > 0$, the corresponding block of $\widetilde{I}$ is I_{m_k}, the $m_k \times m_k$ unit matrix; otherwise $(P_{kk}(x) < 0$ and $W_{kk}(x) < 0)$ the corresponding block is $(-I_{m_k})$. As for the rest of the coefficients in (3), they are assumed to satisfy the relations $\widetilde{P}(x) = \widetilde{I} \cdot P(x)$, $\widetilde{Q}(x) = \widetilde{I} \cdot Q(x)$, $\widetilde{V}(x) = \widetilde{I} \cdot V(x)$, $\widetilde{W}(x) = \widetilde{I} \cdot W(x)$. Under the listed assumptions, all the diagonal blocks $\widetilde{P}_{kk}(x)$ of $\widetilde{P}(x)$ and the diagonal blocks $\widetilde{W}_{kk}(x)$ of the weight coefficient $\widetilde{W}(x)$ in (3) are positive matrices. *In what follows we assume that either the latter condition is valid for the differential expression $l[y]$, or we replace it with the differential expression $m[y]$.*

In the case $m_k = 1$ for all $k = \overline{1,r}$, the coefficients of the differential equation are triangular matrices.

The geometric and algebraic multiplicities of eigenvalues are the same in the case of Hermitian matrices. But these multiplicities can be different for triangular matrices. Even in the case when the algebraic multiplicity of an eigenvalue is a constant function of a variable x, the geometric multiplicity can happen to be non-constant, as the Jordan structure of the matrix can vary. For example, consider the triangular matrix $V(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, whose eigenvalue is 1. Its algebraic multiplicity is 2 for all x, while the geometric multiplicity is 1 for $x \neq 0$ and is 2 for $x = 0$.

1. The problem on a finite interval

Suppose we are given the boundary conditions at the endpoints of the interval $(0, b)$, $b < \infty$, as follows:

$$(4) \qquad\qquad\qquad A \cdot y'(0) - B \cdot y(0) = 0,$$

$$(5) \qquad\qquad\qquad C \cdot y'(b) - D \cdot y(b) = 0,$$

where A and B, C and D are commuting block-triangular matrices of the same structure as that of coefficients of the differential equation. For example,

$$(6) \qquad A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1r} \\ 0 & A_{22} & \ldots & A_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_{rr} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} & \ldots & B_{1r} \\ 0 & B_{22} & \ldots & B_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & B_{rr} \end{pmatrix},$$

where A_{kk}, B_{kk} are Hermitian $m_k \times m_k$-matrices such that $m_k \geq 1$, $k = \overline{1,r}$, $\sum\limits_{k=1}^{r} m_k = m$, and

(7) $$\det \left(A^2 + B^2 \right) = \prod_{k=1}^{r} \det \left(A_{kk}^2 + B_{kk}^2 \right) \neq 0.$$

It follows from $AB = BA$ that

(8) $$A_{kk} \cdot B_{kk} = B_{kk} \cdot A_{kk}, \quad k = \overline{1,r}.$$

In a similar way, the matrices C and D should satisfy the following relations:

(9) $$C_{kk} \cdot D_{kk} = D_{kk} \cdot C_{kk}, \quad k = \overline{1,r},$$

(10) $$\det \left(C^2 + D^2 \right) = \prod_{k=1}^{r} \det \left(C_{kk}^2 + D_{kk}^2 \right) \neq 0.$$

Denote by $Y(x, \lambda)$ the matrix solution of the equation (1), which satisfy the initial conditions

$$Y(0, \lambda) = A, \quad Y'(0, \lambda) = B.$$

This solution has a block-triangular structure

(11) $$Y(x, \lambda) = \begin{pmatrix} Y_{11}(x, \lambda) & Y_{12}(x, \lambda) & \dots & Y_{1r}(x, \lambda) \\ 0 & Y_{22}(x, \lambda) & \dots & Y_{2r}(x, \lambda) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Y_{rr}(x, \lambda) \end{pmatrix},$$

where $Y_{kk}(x, \lambda)$ are $m_k \times m_k$-matrices, $m_k \geq 1$, $k = \overline{1,r}$.

$Y(x, \lambda)$ is a fundamental solution of the problem (1), (4) in the sense that

(1) For every $h \in H_m$ (recall our notation H_m for a Hilbert space of dimension m) the vector function $y = Y(x, \lambda)h$ is a solution of (1) which satisfy the boundary condition (4);

(2) every vector solution $y(x, \lambda)$ of the problem (1), (4) admits a representation in the form $y(x, \lambda) = Y(x, \lambda)h$;

(3) for some (hence for every) x one has $\det \left(Y^2(x, \lambda) + Y'^2(x, \lambda) \right) \neq 0$.

This means that, given $h \neq 0$, $Y(x, \lambda)h$ is a non-trivial solution of the problem (1), (4) (cf. the definition in [17], [18]).

The latter condition for the solution $Y(x, \lambda)$ of the problem (1), (4) is valid at $x = 0$ by virtue of (7).

Denote by L' the minimal with respect to b differential operator generated in the Hilbert space $L_2(H_m, (0, b), W(x)dx)$ by the differential expression

(12) $$l_W[y] = W^{-1}(x)l[y]$$

and the boundary condition (4).

Consider the system

(13) $$l_k[z_k] = - \left(P_{kk}(x)z_k' \right)' + \frac{i}{2} \left((Q_{kk}(x)z_k)' + Q_{kk}(x)z_k' \right) + V_{kk}(x)z_k = \lambda W_{kk}(x)z_k$$

$k = \overline{1,r}$, with the boundary conditions

(14) $$A_{kk} \cdot z_k'(0) - B_{kk} \cdot z_k(0) = 0,$$

where $z_k(x)$ is a vector function with values in H_{m_k} in the notation as in Introduction.

Under the assumptions (7), (8), the condition (14) is self-adjoint (see [17], [18]). Denote by L'_k the minimal with respect to b symmetric operator, generated by the differential expression

$$(15) \qquad l_{k,w}[z_k] = W_{kk}^{-1}(x) l_k[z_k],$$

together with the boundary conditions (14) and $z_k(b) = z'_k(b) = 0$.

Denote by L the differential operator generated by the differential expression (12) and the boundary conditions (4), (5). Let also L_k stand for the self-adjoint extension of the minimal with respect to b operator L'_k, determined by the boundary condition

$$(16) \qquad C_{kk} \cdot z'_k(b) - D_{kk} \cdot z_k(b) = 0,$$

where C_{kk}, D_{kk} are $m_k \times m_k$ Hermitian matrices, which satisfy (9), (10).

In the case when the matrix C in (5) has the form

$$(17) \qquad C = \begin{pmatrix} 0 & C_{12} & \dots & C_{1r} \\ 0 & 0 & \dots & C_{2r} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

the operator L will be denoted by L_0; otherwise, when the boundary condition (16) acquires the form

$$(18) \qquad z_k(b) = 0,$$

the operator L_k will be denoted by L_k^0.

Denote by $\sigma_k = \bigcup_s \{\lambda_{sk}\}_s$, $k = \overline{1, r}$, the set of eigenvalues for the self-adjoint operator L_k; let also $N_k(\lambda)$ be the number of eigenvalues $\lambda_{sk} < \lambda$ with k fixed, counted according to their multiplicities. The quantities λ_{sk}, $N_k(\lambda)$ for L_k^0 will be denoted by λ_{sk}^0, $N_k^0(\lambda)$, respectively.

Lemma 1. *The spectrum of L is discrete, real, and coincides to the union of spectra of self-adjoint operators L_k*

$$(19) \qquad \sigma(L) = \sigma_d(L) = \bigcup_{k=1}^{r} \sigma_k.$$

Proof. The eigenvalues of L coincide to the poles of the Green function $G(x, \xi, \lambda)$ for the operator $L - \lambda I$, i. e., to zeros of the determinant $\Delta(\lambda) := \det \Omega(b, \lambda)$, where

$$\Omega(b, \lambda) = C \cdot Y'(b, \lambda) - D \cdot Y(b, \lambda).$$

Since the matrices C, D, $Y(b, \lambda)$, $Y'(b, \lambda)$ have a block-triangular structure of the form (6), (11), one has $\Delta(\lambda) = \prod_{k=1}^{r} \Delta_k(\lambda)$, where $\Delta_k(\lambda) := \det \Omega_k(b, \lambda)$,

$$\Omega_k(b, \lambda) = C_{kk} \cdot Y'_{kk}(b, \lambda) - D_{kk} \cdot Y_{kk}(b, \lambda).$$

On the other hand, under the assumptions (9), (10) zeros of $\Delta_k(\lambda)$ are just eigenvalues of the self-adjoint operator L_k, hence real. It follows that the spectrum of L is real and is just the union of spectra of L_k. The Lemma is proved. $\square$

Remark 2. Given an eigenvalue λ_0 of L, whose geometric multiplicity is æ_g, then 0 is an eigenvalue of the matrix $\Omega(b, \lambda_0)$ with the same geometric multiplicity æ_g, and vice versa.

In fact, if λ_0 is an eigenvalue of L with geometric multiplicity æ_g, then there exist æ_g linear independent solutions $y_1(x, \lambda_0)$, $y_2(x, \lambda_0), \dots, y_{\text{æ}_g}(x, \lambda_0)$ of (1), which satisfy the boundary conditions (4), (5). Since $Y(x, \lambda)$ is a fundamental solution of the problem (1), (4), there exist æ_g linear independent vectors $h_1, h_2, \dots, h_{\text{æ}_g} \in H_m$ such that $y_k(x, \lambda_0) =$

$Y(x, \lambda_0)h_k$, $k = 1, 2, \ldots, æ_g$. One deduces from this fact that $\Omega(b, \lambda_0)h_k = 0$, $k = 1, 2, \ldots, æ_g$. Hence the geometric multiplicity $æ'_g$ of 0 as an eigenvalue of the matrix $\Omega(b, \lambda_0)$ is at least $æ_g$, that is, $æ'_g \geq æ_g$. The converse inequality can be proved in a similar way. Thus we have $æ'_g = æ_g$.

Let us enumerate the eigenvalues of L^0 in the ascending order

$$\lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_n^0 \leq \cdots,$$

and denote by $N_a^0(\lambda)$ the number of eigenvalues $\lambda_n^0 < \lambda$ of L^0 counted according to their algebraic multiplicities.

Given an arbitrary matrix T, we use below the conventional notation

(20) $$\operatorname{nul} T = \dim \operatorname{Ker} T, \quad \operatorname{Def} T = \dim \operatorname{Coker} T.$$

If T is Hermitian, one has $\operatorname{nul} T = \operatorname{Def} T$.

In the case of non-Hermitian T, together with the geometric multiplicity $æ_g$ of 0 as an eigenvalue of T, which is just $\operatorname{nul} T$, we will consider the algebraic multiplicity $æ_a$ of 0 as an eigenvalue of T. For T Hermitian, the geometric and algebraic multiplicities are the same, so we will omit indices g and a.

Denote by $\operatorname{nul}_a T$ the algebraic multiplicity of 0 as an eigenvalue of T.

It was proved in [14], [15], [17], [18] that, although the matrix solution $Y_{kk}(x, \lambda)$ of the differential equation (13) with Hermitian coefficients, in general can fail to be a Hermitian matrix, but one has

(21) $$\operatorname{nul} Y_{kk}(x, \lambda) = \operatorname{Def} Y_{kk}(x, \lambda).$$

For $m \geq 1$ we denote by $\operatorname{nul}_a Y(x, \lambda)$ the algebraic multiplicity of 0 as an eigenvalue of the matrix $Y(x, \lambda)$ with fixed x and λ. In particular, with $m = 1$ we have $\operatorname{nul}_a Y(x, \lambda) = 1$ if x is a root of the scalar equation $Y(x, \lambda) = 0$, and $\operatorname{nul}_a Y(x, \lambda) = 0$ otherwise.

Theorem 1. *Let an operator L^0 be generated by the differential expression* (12) *with block-triangular matrix coefficients and the boundary conditions* (4), (5) *with a matrix C of the form* (17). *Suppose that the blocks $P_{kk}(x)$ of the coefficient at the highest derivative of $P(x)$ and the blocks $W_{kk}(x)$ of the matrix weight coefficient $W(x)$ are simultaneously either Hermitian positive or Hermitian negative at every $x \in [0, b]$, and the blocks $V_{kk}(x)$ are Hermitian. Then for $\lambda \in \mathbb{R}$ one has*

(22) $$\sum_{x \in (0,b)} \operatorname{nul}_a Y(x, \lambda) = N_a^0(\lambda).$$

Here the sum is in those $x \in (0, b)$ where $\operatorname{nul}_a Y(x, \lambda) \neq 0$.

Corollary 1. *For $\lambda \in \mathbb{R}$ one has* $\sum_{k=1}^{r} \sum_{x \in (0,b)} \operatorname{nul} Y_{kk}(x, \lambda) = N_a^0(\lambda)$.

Proof of Theorem 1. By virtue of Lemma 1 and the definition of $N_k^0(\lambda)$ and $N_a^0(\lambda)$ we deduce that

(23) $$N_a^0(\lambda) = \sum_{k=1}^{r} N_k^0(\lambda).$$

Since the diagonal blocks $P_{kk}(x)$ of the coefficient at the highest derivative and the diagonal blocks $W_{kk}(x)$ of the weight coefficient $W(x)$ may be assumed positive at all $x \in [0, b]$, we can use the oscillation theorem[1] for the boundary problem with self-adjoint

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[1]This is the classical Sturm theorem for a scalar equation of the form $-z'' + V(x)z = \lambda z$ on a finite interval. This theorem for the equation of the form (13) with the first derivative has been proved in [15], [17], [18]. In the case of arbitrary even order differential equations with matrix and operator coefficients on either finite or infinite interval, the theorem has been proved in [14], [15], [17], [18].

positive coefficients $P_{kk}(x)$, $W_{kk}(x)$ and zero boundary condition (18) at the right endpoint, to establish that

$$(24) \qquad \sum_{x \in (0,b)} \operatorname{nul} Y_{kk}(x, \lambda) = N_k^0(\lambda),$$

where $\operatorname{nul} Y_{kk}(x, \lambda)$ is given by (20). Furthermore, (21) is valid.

Since $\operatorname{nul}_a Y(x, \lambda)$ stands for the algebraic multiplicity of 0 as an eigenvalue of the matrix $Y(x, \lambda)$ at fixed x and λ, we are in a position to apply (21) in order to deduce that

$$(25) \qquad \operatorname{nul}_a Y(x, \lambda) = \sum_{k=1}^{r} \operatorname{nul} Y_{kk}(x, \lambda).$$

Now we compare (23), (24), and (25) to derive (22). The Theorem is proved. $\qquad \square$

The Corollary 1 is due to this Theorem and (25).

Denote by $N_a(\lambda)$ the number of eigenvalues of L less than λ, counted according to their algebraic multiplicities. Similarly to (23), we obtain $N_a(\lambda) = \sum_{k=1}^{r} N_k(\lambda)$.

In the case of Hermitian diagonal coefficients of the differential equation (1), i.e., Hermitian coefficients in the equation (13), and Hermitian matrices A_{kk}, B_{kk}, C_{kk}, D_{kk} in the boundary conditions (14), (16), the oscillation theorem for equations with Hermitian matrix coefficients (see [14], [15], [17], [18]) implies

$$N_k(\lambda) - \min\{\operatorname{rg} C_{kk}, \, m_k - \mathrm{æ}_k(\lambda)\} \le \sum_{x \in (0;b)} \operatorname{nul} Y_{kk}(x, \lambda) = N_k^0(\lambda) \le N_k(\lambda),$$

where $\operatorname{nul} Y_{kk}(x, \lambda) = \operatorname{nul}_a Y_{kk}(x, \lambda)$, $\mathrm{æ}_k(\lambda)$ is the multiplicity of λ as an eigenvalue of the self-adjoint operator L_k. After summing in k and applying (25), we obtain the following Theorem.

Theorem 2. *For the problem* (1), (4), (5) *and* $\lambda \in \mathbb{R}$ *one has*

$$(26) \qquad N_a(\lambda) - \sum_{k=1}^{r} \min\{\operatorname{rg} C_{kk}, \, m_k - \mathrm{æ}_k(\lambda)\} \le \sum_{x \in (0;b)} \operatorname{nul}_a Y(x, \lambda) \le N_a(\lambda).$$

The extensions corresponding to the equation (1) with Hermitian coefficients $P(x)$, $V(x)$, $W(x)$ and generated by the boundary problem, are self-adjoint and such that algebraic and geometric multiplicities of eigenvalues coincide. Moreover, for the matrix solution $Y(x, \lambda)$ of the problem (1), (4) one has $\operatorname{nul} Y(x, \lambda) = \operatorname{Def} Y(x, \lambda)$. Thus one gets a problem of transferring Theorems 1, 2 for geometric multiplicities associated to the problems with block-triangular coefficients. However, this generalization fails, as one can see from Example 1 below.

A proof of the Theorem for equations with self-adjoint coefficients (see [14], [15], [17], [18]) is based on considering the behavior of eigenvalues for the self-adjoint operator L_ξ^0 corresponding to the problem (1), (4), $y(\xi) = 0$.

Let

$$\lambda_1^0(\xi) \le \lambda_2^0(\xi) \le \cdots \le \lambda_n^0(\xi) \le \cdots$$

be the enumerated discrete spectrum of L_ξ^0, with the geometric multiplicities $\mathrm{æ}_g\left(\lambda_n^0(\xi)\right)$ being taken into account. The functions $\lambda_n^0(\xi)$ are continuous, monotonously decreasing in ξ for $\xi \in (0; b]$ and $\lambda_n^0(\xi) \to +\infty$ as $\xi \to +0$ (see [14]–[18]). Some specific curves may coincide completely or partially, according to multiplicity of the corresponding eigenvalues. In the case of self-adjoint problem, geometric and algebraic multiplicities of eigenvalues coincide, hence the total geometric multiplicity along the curves $\lambda = \lambda_j^0(x)$ is constant.

However, in the case of a problem with block-triangular matrix coefficients, the picture becomes different. While the total algebraic multiplicity along the curves $\lambda = \lambda_j^0(x)$ is constant, the total geometric multiplicity along those curves may vary, as one can observe from the Example below. Hence the following Remark.

Remark 3. The claim of Theorem 1 for geometric multiplicities is not true without additional assumptions.

Example 1. Consider the differential equation with a triangular potential

$$(27) \qquad\qquad -y'' + V(x)y = \lambda y,$$

where

$$(28) \qquad\qquad V(x) = \begin{pmatrix} 0 & V_{12}(x) \\ 0 & 0 \end{pmatrix}, \quad V_{12}(x) = \begin{cases} 0, & 0 \le x \le \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

The solution $Y(x, \lambda)$ satisfying the initial conditions $Y(0, \lambda) = 0$, $Y'(0, \lambda) = I$, has the form

$$(29) \qquad\qquad Y(x, \lambda) = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} x & y_{12}(x, \lambda) \\ 0 & \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} x \end{pmatrix},$$

where

$$y_{12}(x, \lambda) = \begin{cases} 0, & 0 \le x \le \frac{\pi}{2}, \\ \frac{1}{2(\sqrt{\lambda})^3} \cos \sqrt{\lambda} \frac{\pi}{2} \sin \sqrt{\lambda}(x - \frac{\pi}{2}) - \frac{1}{2\lambda}(x - \frac{\pi}{2}) \cos \sqrt{\lambda} x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

If one considers the condition $y(\pi, \lambda) = 0$ at the right endpoint, the eigenvalues are to be found from the equation $\sin \sqrt{\lambda} \pi = 0$, i.e., $\sqrt{\lambda} = n$, $\lambda = n^2$.

The eigenvalues of L_x^0 are given by

$$\lambda_n^0(x) = \frac{\pi^2 n^2}{x^2}, \quad \lambda_n^0(x) \to +\infty \quad \text{as} \quad x \to +0.$$

The algebraic multiplicity is $\text{æ}_a(\lambda_n^0(x)) = 2$ at all $x \in (0; \pi]$, while the geometric multiplicity is

$$\text{æ}_g(\lambda_n^0(x)) = \begin{cases} 2, & 0 < x \le \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Remark 4. If for some $\mu < \infty$ the total geometric multiplicity is constant along the collection of all the curves $\lambda = \lambda_j^0(x)$ for the problem (1), (4), $y(b) = 0$ with block-triangular coefficients, then one has

$$(30) \qquad\qquad \sum_{x \in (0; b)} \text{nul}\, Y(x, \lambda) = N_g^0(\lambda), \quad \lambda \le \mu,$$

where $\text{nul}\, Y(x, \lambda)$ is given by (20) in such a way that (21) is valid for $Y(x, \lambda)$, $N_g^0(\lambda)$ is the number of eigenvalues $\lambda_n^0 < \lambda$ of L^0 generated by the problem (1), (4), $y(b) = 0$, counted according to their geometric multiplicities.

The sketch of a proof for Remark 4 is similar to that of the corresponding theorem for equations with Hermitian coefficients (see [14], [15], [17], [18]).

Example 2. Consider the differential equation (27) with a potential of the form (28), where

$$V_{12}(x) = \begin{cases} 1, & 0 \le x \le \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

In this case the solution $Y(x, \lambda)$ satisfying the initial conditions $Y(0, \lambda) = 0$, $Y'(0, \lambda) = I$ has also the form (29), but

$$y_{12}(x, \lambda) = \begin{cases} \frac{1}{2(\sqrt{\lambda})^3} \sin \sqrt{\lambda} x - \frac{x}{2\lambda} \cos \sqrt{\lambda} x, & 0 \le x \le \frac{\pi}{2}, \\ \frac{1}{2(\sqrt{\lambda})^3} \sin \sqrt{\lambda} \frac{\pi}{2} \cos(\frac{\pi}{2} - x) - \frac{\pi}{4\lambda} \cos \sqrt{\lambda} x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

Here again $\lambda_n^0(x) = \frac{\pi^2 n^2}{x^2}$, but with

$$\text{æ}_a(\lambda_n^0(x)) = 2, \quad \forall x \in (0; \pi],$$
$$\text{æ}_g(\lambda_n^0(x)) = 1, \quad \forall x \in (0; \pi].$$

Now let us look at the problem on $[0; \pi]$ under the conditions $y(\pi, \lambda) = 0$, $\lambda_n^0 = n^2$. With $\lambda = 0$ the solution $Y(x, \lambda)$ has the form

$$Y(x, 0) = \begin{pmatrix} x & y_{12}(x) \\ 0 & x \end{pmatrix}, \quad y_{12}(x) = \begin{cases} \frac{x^3}{6}, & 0 \le x \le \frac{\pi}{2}, \\ \frac{\pi^2}{8} x - \frac{\pi^3}{24}, & \frac{\pi}{2} \le x \le \pi, \end{cases}$$

and $\lambda = 0$ is not among the eigenvalues. Thus $\lambda_n^0 = n^2$, $n \in \mathbb{N}$.

For $n = 1$, $\lambda_1^0 = 1$ one has

$$Y(x, \lambda_1^0) = \begin{pmatrix} \sin x & y_{12}(x) \\ 0 & \sin x \end{pmatrix}, \quad y_{12}(x) = \begin{cases} \frac{1}{2} \sin x - \frac{x}{2} \cos x, & 0 \le x \le \frac{\pi}{2}, \\ \frac{1}{2} \sin x - \frac{\pi}{4} \cos x, & \frac{\pi}{2} < x \le \pi, \end{cases}$$

and $\text{nul}\, Y(x, \lambda_1^0) = 0$ for $x \in (0; \pi)$.

For $n = 2$, $\lambda_2^0 = 4$ one has

$$Y(x, \lambda_2^0) = \begin{pmatrix} \frac{1}{2} \sin 2x & y_{12}(x) \\ 0, & \frac{1}{2} \sin 2x \end{pmatrix}.$$

In this case $\text{nul}\, Y(\frac{\pi}{2}; \lambda_2^0) = 1 = N_g^0(\lambda_2^0)$. A similar computation can be done for other n. All this results in

$$\sum_{x \in (0; \pi)} \text{nul}\, Y(x, \lambda) = N_g^0(\lambda).$$

Note that the number of roots for the equation $\det Y(x, \lambda) = 0$ on $(0; \pi)$, counted according to their multiplicities, is $N_a^0(\lambda)$, i.e., (22) is also valid.

2. The problem on a semi-axis

Denote by L' the minimal with respect to $x = \infty$ (non-closed) differential operator generated on compactly supported functions in $L_2(H_m, (0, \infty), W(x)dx)$ (here H_m is an m-dimensional Hilbert space) by the differential expression (12) and the boundary condition (4) at 0. Let also L'_k, $k = \overline{1, r}$, be the minimal symmetric operator, generated in $L_2(H_{m_k}, (0, \infty), W_{kk}(x)dx)$ by the differential expression (15) and the boundary condition (14). Suppose that every symmetric operator L'_k is lower semi-bounded. (If either $P(x) = I_m$ or the grows of $P_k(x)$ as $x \to \infty$ is not too fast, then the minimal symmetric semi-bounded operators L'_k, $k = \overline{1, r}$, are essentially self-adjoint (see [2]), and their self-adjoint extensions are produced by closing the minimal operator).

In the general case, suppose that one has the self-adjoint boundary condition at the infinity for the symmetric operator L'_k as follows:

(31) $$U_k[y_k] = 0, \quad k = \overline{1, r},$$

where U_k is a linear map from $L_2(H_{m_k}, (0, \infty), W_{kk}(x)dx)$ to H_{m_k} such that $U_k[y_k] = U_k[z_k]$ if $y_k(x) = z_k(x)$ for x big enough.

Suppose one has some boundary condition at the infinity for L':

(32) $$U[y] = 0$$

such that one may set up
$$(U[y])_1 = U_1[y_1, y_2, \ldots, y_r],$$
$$(U[y])_2 = U_2[y_2, y_3, \ldots, y_r],$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$(U[y])_{r-1} = U_{r-1}[y_{r-1}, y_r],$$
$$(U[y])_r = U_r[y_r].$$

Denote by L any extension of L' determined by the boundary condition (32) and possessing the following properties:

(33)
$$U_1[y_1, 0, \ldots, 0] = U_1[y_1],$$
$$U_2[y_2, 0, \ldots, 0] = U_2[y_2],$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$U_{r-1}[y_{r-1}, 0] = U_{r-1}[y_{r-1}],$$

where the right hand sides correspond to (31).

If (31) determine the Friedrichs extension L_k^0 of the semi-bounded symmetric operator L_k', the corresponding extension of L' will be denoted by L^0. It was demonstrated in [12], [13] that the spectral function $\rho(\lambda)$ of the operator L_k^0 is derivable by a kind of passage to a limit as $b \to \infty$ from the spectral function $\rho_b(\lambda)$ of the problem (13), (14), (18) on $[0, b]$.

Denote by $\sigma_k = \bigcup_s \{\lambda_{sk}\}_s$, $k = \overline{1, r}$, the set of eigenvalues $\lambda_{sk} < \lambda_e(L_k)$ of the self-adjoint operator L_k for each fixed k; let $N_k(\lambda)$ be the number of eigenvalues $\lambda_{sk} < \lambda < \lambda_e(L_k)$ counted according to their multiplicities. Here $\lambda_e(L_k)$ stands for the the lower bound of the essential spectrum of the operator L_k. The quantities λ_{sk}, $N_k(\lambda)$ for L_k^0 will be denoted by λ_{sk}^0, $N_k^0(\lambda)$, respectively.

Lemma 2. *The discrete spectrum of L is real and is contained in the union of discrete spectra of L_k*

(34)
$$\sigma_d(L) \subseteq \bigcup_{k=1}^r \sigma_k.$$

Proof. Let $\lambda = \lambda_0$ be an eigenvalue of L, and
$$y(x, \lambda_0) = \mathrm{col}(y_1(x, \lambda_0), y_2(x, \lambda_0), \ldots, y_r(x, \lambda_0))$$
the corresponding vector eigenfunction. Here at least one coordinate of the vector $y(x, \lambda_0)$ is not identically zero. Let us show that λ_0 is an eigenvalue of the operator L_k for at least one k.

After writing down the system (1) and the boundary conditions (4) in a block-coordinate form, the latter relations acquire the form (13), (14) for $k = r$, $\lambda = \lambda_0$ with respect to the function $y_r(x, \lambda_0)$ instead of $z_k(x)$. If $y_r(x, \lambda_0) \not\equiv 0$ for $x \in (0, \infty)$, the function $y_r(x, \lambda_0)$ satisfy the differential equation (13), the boundary condition (14), and the condition at infinity (31) with $k = r$. Hence $y_r(x, \lambda_0)$ in this case is an eigenfunction, and λ_0 is an eigenvalue of the self-adjoint operator L_r; in particular, λ_0 is real. Now suppose that $y_k(x, \lambda_0) \equiv 0$ for all $1 \le s < k \le r$ while $y_s(x, \lambda_0) \not\equiv 0$. Then the last $r - s$ equations of the system (1) become the identities, the s-th equation in (1) becomes the same as the corresponding equation of the system (13), the s-th equality in the boundary condition (4) at 0 coincides with the corresponding condition in (14), and the boundary condition at infinity (31) with $k = s$ becomes $U_s[y_s, 0, \ldots, 0] = 0$. This means that λ_0 is an eigenvalue of the self-adjoint operator L_s, and $y_s(x, \lambda_0)$ is an eigenfunction of this operator, hence λ_0 is real (here λ_0 may also be an eigenvalue of L_k for some different k). Thus the discrete spectrum of L is real, which completes the proof of (34). $\square$

The Lemma is proved, but it will be of some interest to consider the possibility of converse inclusion. Let $\lambda_0 \in \sigma_1$. Then there exists $h_1 \in \mathrm{H}_{m_1}$, $h_1 \neq 0$, such that the vector function $y_{11}(x, \lambda_0) = Y_{11}(x, \lambda_0)h_1$ is a solution of the equation (13), satisfies the boundary condition (14) and the condition at infinity, which determine the self-adjoint operator L_1. Hence this vector function is an eigenfunction of the latter operator, and the vector function $y(x, \lambda_0) = Y(x, \lambda_0)h$, with $h = \mathrm{col}(h_1, 0, \ldots, 0)$, is an eigenfunction of L, whence $\lambda_0 \in \sigma(L)$.

Let $\lambda_0 \in \sigma_2$. Then there exists $h_2 \in \mathrm{H}_{m_2}$, $h_2 \neq 0$, such that the vector function $y_{22}(x, \lambda_0) = Y_{22}(x, \lambda_0)h_2$ satisfies the boundary condition at infinity and is an eigenfunction for L_2. In order to make sure that $y(x, \lambda_0) = Y(x, \lambda)h$, with

$$h = \mathrm{col}(0, h_2, 0, \ldots, 0), \quad y(x, \lambda_0) = \mathrm{col}(y_{12}(x, \lambda_0), y_{22}(x, \lambda_0), 0, \ldots, 0),$$

is an eigenfunction of L, it suffices to have that $y(x, \lambda_0) \in L_2(H_m, (0, \infty), W(x)dx)$. In a similar way, if $\lambda_0 \in \sigma_k$, $1 < k \leq r$, and $y_{kk}(x, \lambda_0) = Y_{kk}(x, \lambda_0)h_k$, $h_k \in \mathrm{H}_{m_k}$, is the corresponding eigenfunction for L_k, then in order to make sure that $\lambda_0 \in \sigma_d(L)$, it suffices to have

$$
\begin{aligned}
(35) \qquad y(x, \lambda_0) &= Y(x, \lambda_0)h \\
&= \mathrm{col}(y_{1k}(x, \lambda_0), \ldots, y_{kk}(x, \lambda_0), 0, \ldots, 0) \in L_2(H_m, (0, \infty), W(x)dx),
\end{aligned}
$$

where $h = \mathrm{col}(0, \ldots, 0, h_k, 0, \ldots, 0)$.

This condition is always valid for $b < \infty$ due to smoothness of coefficients. Hence for the problem on a finite interval one has

$$
(36) \qquad \sigma_d(L) = \bigcup_{k=1}^{r} \sigma_k
$$

(cf. (19)). However, for the problem on a semi-axis, (35) can fail. Sufficient conditions for the spectrum $\sigma_d(L)$ of L to coincide with the union of discrete spectra of L_k, $k = \overline{1, r}$, are given by the following

Example 3. (36) is valid for the differential equation of the form

$$
(37) \qquad -y'' + V(x)y = \lambda y
$$

with a block-triangular $m \times m$ matrix potential $V(x)$ (whose diagonal blocks are Hermitian), which has the first moment

$$
\int_0^\infty x \cdot |V(x)|dx < \infty.
$$

In this case, only the negative part of the spectrum is discrete.

A proof follows from the properties of spectrum for the scattering problem related to Hermitian systems on a semi-axis [1] and the properties of spectrum for the scattering problem with a triangular matrix potential [3].

Another sufficient condition for (37) in the case of triangular 2×2 matrix potential, with fast growth on the diagonal, is presented by the authors in the following Lemma. Its proof is to be expounded in another paper.

Lemma 3. *Suppose that the potential $V(x)$ in (37) has the form*

$$
V(x) = w(x)I + U(x), \quad U(x) = \begin{pmatrix} p(x) & q(x) \\ 0 & p(x) \end{pmatrix},
$$

where $w(x)$, $p(x)$, $q(x)$ are scalar functions, $w(x)$, $p(x)$ are real, $0 < w(x) \to \infty$ monotonously as $x \to \infty$ and faster than x^α with $\alpha > 2$, $w(x)$ has a monotonous absolutely continuous derivative, so that

$$
\int_0^\infty |U(t)| \cdot w^{-\frac{1}{2}}(t)dt < \infty,
$$

$$\int_0^\infty w'^2(t) \cdot w^{-\frac{5}{2}}(t)dt < \infty, \quad \int_0^\infty w''(t) \cdot w^{-\frac{3}{2}}(t)dt < \infty.$$

Then the discrete spectrum $\sigma_d(L)$ of L coincides to the union of discrete spectra of self-adjoint operators L_k, $k = \overline{1,r}$.

Condition Σ. In what follows we assume that the coefficients of the differential equation (1) for the problem on a semi-axis are such that the discrete spectrum of L coincides to the union of discrete spectra of self-adjoint operators L_k, $k = \overline{1,r}$, i.e., (36) is valid.

Recall that we denote by L^0 such extension of L' that the corresponding extensions of the operators L'_k are just the Friedrichs extensions L_k^0.

Let us enumerate the eigenvalues of L^0 in increasing order

$$\lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_n^0 \leq \cdots < \lambda_e\left(L^0\right).$$

Denote by $N_a^0(\lambda)$ the number of eigenvalues $\lambda_n^0 < \lambda < \lambda_e\left(L^0\right)$ of the operator L^0, counted according to their algebraic multiplicities.

Theorem 3. *Assume that Condition Σ is satisfied. Suppose the operator L^0 is generated by the differential expression (12) with matrix block-triangular coefficients, the boundary condition at 0 (4), and such boundary conditions at the infinity that one gets Friedrichs extensions for semi-bounded symmetric operators L'_k. Assume also that the diagonal blocks $P_{kk}(x)$ of the coefficient at the highest derivative of $P(x)$ and the diagonal blocks $W_{kk}(x)$ of the weight coefficient $W(x)$ are simultaneously either Hermitian positive or Hermitian negative at every $x \in [0;\infty)$, and the blocks $V_{kk}(x)$ are Hermitian. Then for $\lambda < \lambda_e\left(L^0\right)$ one has*

$$\sum_{x \in (0,\infty)} \mathrm{nul}_a Y(x,\lambda) = N_a^0(\lambda)$$

(the sum here is in those $x \in (0,\infty)$ for which $\mathrm{nul}_a Y(x,\lambda) \neq 0$).

The proof is analogous to that of Theorem 1 and uses the oscillation theorem for equations with self-adjoint positive coefficients $P_{kk}(x)$, $W_{kk}(x)$ on semi-axis (in the case of the scalar equation (13) with $Q_{kk}(x) \equiv 0$ the theorem has been proved in [5], [9]; in the case of differential equations of an arbitrary even order with matrix and operator coefficients on an infinite interval the theorem has been proved in [14], [15], [17], [18]).

Let L_k be an arbitrary self-adjoint extension of a semi-bounded symmetric operator L'_k, determined by a condition (31) at the infinity. A description of self-adjoint extensions for symmetric differential operators of arbitrary order (both even and odd) on an infinite interval (axis, semi-axis) in an absolutely indefinite case was obtained in [7] (see also [17], [18]). In the case of intermediate deficiency indices these questions have been investigated in [4], [6], [10], [11].

Denote by L an extension of an operator L' determined by the condition (32) so that (33) is satisfied.

Theorem 4. *Let the condition Σ be satisfied. With the operator L as above and $\lambda < \lambda_e(L)$ one has*

$$N_a(\lambda) - \sum_{k=1}^r p_k \leq \sum_{x \in (0,\infty)} \mathrm{nul}_a Y(x,\lambda) = N_a^0(\lambda) \leq N_a(\lambda),$$

where $p_k = \mathrm{Def}\left\{L_k\,\middle|\,D\left(L_k^0\right) \cap D\left(L_k\right)\right\}$. If λ is not an eigenvalue of L', then for $\lambda < \lambda_e(L)$ one has

$$N_a(\lambda) - \sum_{k=1}^r \min\{p_k, d_k - \text{æ}_k(\lambda)\} \leq \sum_{x \in (0,\infty)} \mathrm{nul}_a Y(x,\lambda) = N_a^0(\lambda) \leq N_a(\lambda),$$

where $d_k = \mathrm{Def}\, L'_k$, $æ_k(\lambda)$ *is the multiplicity of* λ *as an eigenvalue of the self-adjoint operator* L_k.

Proof. By virtue of the oscillation theorem (see [15], [17], [18]), for every equation of the split system (13) with Hermitian matrix coefficients and with $\lambda < \lambda_e(L_k)$ one has

$$N_k(\lambda) - p_k \leq \sum_{x \in (0;\infty)} \mathrm{nul}\, Y_{kk}(x, \lambda) = N_k^0(\lambda) \leq N_k(\lambda),$$

where $N_k(\lambda)$ is a counting function for eigenvalues less than λ for the self-adjoint extension L_k of the symmetric operator L'_k generated by the problem (13), (14), which is minimal with respect to the infinity,

$$p_k = \mathrm{Def}\left\{L_k \left| D\left(L_k^0\right) \cap D(L_k)\right.\right\}.$$

If λ is not an eigenvalue of L'_k, for $\lambda < \lambda_e(L_k)$ one has

$$N_k(\lambda) - \min\{p_k, d_k - æ_k(\lambda)\} \leq \sum_{x \in (0,\infty)} \mathrm{nul}\, Y_{kk}(x, \lambda) = N_k^0(\lambda) \leq N(\lambda),$$

where $d_k = \mathrm{Def}\, L'_k$.

$\square$

Now we sum up this in k to obtain the claim of Theorem 4.

Remark 5. *With a regular endpoint* $b < \infty$ *one has* $p_k = rg\, C_{kk}$, $d_k = m_k$.

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