Methods of Functional Analysis and Topology
Vol. 18 (2012), no. 2, pp. 189–197

ON CHARACTERISTIC FUNCTIONS OF OPERATORS ON EQUILATERAL GRAPHS

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Dedicated to the memory of A. G. Kostyuchenko

Abstract. Known connection between discrete and continuous Laplacians in case of same symmetric potential on the edges of a quantum graph is used to construct characteristic functions of quantum graphs and to find some parameters of graphs using spectra of boundary value problems.

1. Introduction

Usually the term 'quantum graphs' means metric graphs considered as quasi-one-dimensional domains with differential operations defined on these domains [7], [8]. In quantum mechanics the Sturm-Liouville, the Dirac equation and in vibration theory the string equation is considered on the edges of a graph subject to matching and boundary conditions at the vertices. These are Dirichlet or Neumann or Robin conditions at pendant vertices and continuity conditions together with Kirchhoff's conditions at interior vertices. Such models are often used in problems of free-electron theory of conjugate molecules in chemistry and in the theory of quantum wires and thin wave-guides. The differential operations together with the matching and boundary conditions define an operator which is usually called continuous Laplacian. Since the literature on this topic is vast we refer just to some of the authors: [13]–[22], [24].

There are different definitions of the so called discrete (combinatorial) Laplacian (see [6], [4], [5]). We will use the one in [6] (see (2.3) below). This operator acting on a finite dimensional space is related to the adjacency matrix.

The question of connection between continuous and discrete Laplacian was pointed out in [1] and developed on rigorous level in [11] (see also [3], [2], [25], [26]). This connection exists in the case of free continuous Laplacian or under restrictive conditions on the potentials of the Sturm-Liouville equations on the edges of the graph. The edges must be of the same length, the potential must be the same on all the edges and to be symmetric with respect to the midpoint of the edge.

In this paper we use these relation between continuous and discrete Laplacians to describe some general features of the characteristic functions and spectra of quantum graphs. In [14] a general formula was obtained which allows to find the cyclomatic number of a graph using the spectrum of a boundary value problem defined on this graph. However, this formula is rather involved. Our results allow to determine the cyclomatic number in a very simple way for equilateral quantum graphs with a symmetric potential on the edges (see Remark 3.6). It is shown that symmetry of a graph leads to existence of multiple eigenvalues.

2010 *Mathematics Subject Classification.* Primary 34B45; Secondary 34B24, 34A55.

Key words and phrases. Sturm-Liouville equation, Dirichlet boundary condition, Neumann boundary condition, Kirchhoff's condition, eigenvalue, multiplicity, discrete Laplacian, continuous Laplacian, quantum graph, cyclically connected graph, tree.

2. Formulation of the problem

For a graph G we denote its vertices by v_i, $i = 1, 2, \ldots, p$, where p is the number of the vertices of G, its edges by e_j, $j = 1, 2, \ldots, g$, where g is the number of edges of G. For each i denote by $d(v_i)$ the degree of the vertex v_i and assume the lengths of edges to be the same.

Definition 2.1. Two vertices v and w of a connected graph G are said to be cyclically connected if a finite set of cycles $C_1, C_2, \ldots, C_k$ ($C_j \subset G$, $j = 1, 2, \ldots, k$) exists such that $v \in C_1$, $w \in C_k$ and each neighboring pair of cycles possesses at least one common vertex.

Definition 2.2. A graph is said to be cyclically connected if each pair of vertices in it is cyclically connected.

If we direct the edges of G to obtain an oriented graph then in addition to the degree $d(v_i)$ of a vertex v_i we introduce $d^+(v_i)$ the *indegree*, the number of edges directed towards the vertex and $d^-(v_i)$, the *outdegree*, the number of edges directed away from the vertex v_i.

The local coordinate on G identifies a directed edge e_j ($j = 1, 2, \ldots, g$) of G with the interval $[0, l]$ and the coordinate x increases in the direction of the edge. Thus, we assume the edges to have the same length.

To every cycle we ascribe any of the two possible directions. It is clear that the direction of an edge can be opposite to the direction of the cycle.

Definition 2.3. The matrix $M = \{M_{k,j}\}$, $k = 1, 2, \ldots, s$, $j = 1, 2, \ldots, g$, where s is the number of cycles, is said to be the matrix of cycles for an oriented graph G if

1) for an edge e_j which does not belong to the k-th cycle $M_{k,j} = 0$,

2) for an edge e_j which belongs to the k-th cycle and whose direction coincides with the direction of the cycle $M_{j,k} = 1$,

3) for an edge e_j which belongs to the k-th cycle and whose direction is opposite to the direction of the cycle $M_{j,k} = -1$.

Definition 2.4. A set of cycles in an oriented graph G is said to be linearly independent if the corresponding set of rows in the matrix of cycles is linearly independent. The rank μ of this matrix is said to be the cyclomatic number of the graph G.

It is known (see, e.g. [9], p. 545) that $\mu = g - p + 1$ where p is the number of vertices in G.

Let us equip each edge e_j with a real-valued function q_j which belongs to $\mathcal{L}_1[0, l]$ ($j = 1, 2, \ldots, g$). Now we introduce the operator $\mathcal{L}$ which we associate with the directed graph G equipped with the functions q_j. First we introduce the Sturm-Liouville operation p_j on each edge e_j. Let the domain $D(p_j)$ of the differential operation p_j be the set of functions f_j continuous on e_j (on $[0, l]$) which possess absolutely continuous derivatives f'_j and therefore f''_j exist a.e. on $[0, l]$. For $f_j \in D(p_j)$ we define the operation p_j by the equation $(p_j f)(x) = -\frac{d^2 f_j(x)}{dx^2} + q_j(x) f_j(x)$ a.e. on $[0, l]$ ($j = 1, 2, \ldots, g$). Let us consider vector-functions $F(x) = (f_1(x), f_2(x), \ldots, f_g(x))$ defined on $[0, l]$. We denote the set of vector-functions F such that $f_j \in \mathcal{L}_2[0, l]$, $j = 1, 2, \ldots, g$ by H. Defining multiplication by constant and addition in the usual way we equip H with the inner product

$$(F, B)_H = \sum_{j=1}^{g} \int_0^l f_j(x) \overline{b_j(x)} \, dx,$$

where $B = (b_1(x), b_2(x), \ldots, b_g(x)) \in H$. Thus, H is a Hilbert space. It is easy to see that this space is separable.

Let $D(P)$ be the set of vector-functions $Y(x) = (y_1(x), y_2(x), \dots, y_g(x))$, where $y_j \in D(p_j)$ $(j = 1, 2, \dots, g)$. For $Y \in D(P)$ we define the operation P by the equation

$$P(Y) = ((p_1 y_1)(x), (p_2 y_2)(x), \dots, (p_g y_g)(x)).$$

Let J be the set of numbers of the edges incident with pendant vertices, K be the set of numbers of interior vertices, W_i^- the set of numbers of edges outgoing away from the vertex v_i and W_i^+ the set numbers of edges incoming into the vertex v_i $(i = 1, 2, \dots, p)$.

Now we are ready to construct the operator $\mathcal{L}$. Its domain D is the set of vector-functions $F = F(x) = (f_1(x), f_2(x), \dots, f_g(x))$ such that

1) $F \in (H \cap D(P))$,
2) $P(F) \in H$,
3) if v_i is a pendant vertex and $W_i^- = \{j\}$ $(W_i^+ = \{j\})$ then

$$\left.\frac{df_j(x)}{dx}\right|_{x=0} = 0 \quad \left(\left.\frac{df_j(x)}{dx}\right|_{x=l} = 0 \right), \tag{2.1}$$

4) (continuity condition) for each $i \in K$ and each $j \in W_i^+$ and each $k \in W_i^-$: $f_j(l) = f_k(0)$,
5) (Kirchhoff condition) for each $i \in K$

$$\sum_{j \in W_i^+} \left.\frac{df_j(x)}{dx}\right|_{x=l} = \sum_{j \in W_i^-} \left.\frac{df_j(x)}{dx}\right|_{x=0} \qquad (i \in K). \tag{2.2}$$

By $\mathcal{L}$ we denote the operator acting in H according to

$$\mathcal{L}F = P(F)$$

with the domain D.

Remark 2.5. If W_i^+ or W_i^- is empty then the 0 must stand in the left- or the right-hand side of (2.2), correspondingly. Also condition 4) should look like $f_{j_1}(0) = f_{j_2}(0) = \dots = f_{j_{d^-(v_i)}}(0)$ or $f_{k_1}(l) = f_{k_2}(l) = \dots = f_{k_{d^+(v_i)}}(l)$, correspondingly.

Remark 2.6. To simplify notations we write $f_j'(0)$ and $f_j'(l)$ instead of $\left.\frac{df_j(x)}{dx}\right|_{x=0}$ and $\left.\frac{df_j(x)}{dx}\right|_{x=l}$ in the sequel. We admit the corresponding simplification for the second derivatives too.

It is easy to see that $\mathcal{L}$ is a self-adjoint operator in H (see, i.e. [13], [12], [18]). Since all the edges are of finite length and $q_j \in \mathcal{L}_1[0, l]$ $(j = 1, 2, \dots, g)$, the spectrum of $\mathcal{L}$ is discrete, i.e. it consists of normal (isolated Fredholm) eigenvalues which accumulate only at infinity. It should be mentioned that the self-adjoint matching condition which are considered in [13] are more general than (2.2).

Now let us define the discrete Laplacian [6].

We define the Laplacian for connected graphs without loops and multiple edges. To begin, we consider the matrix L, defined as follows:

$$L(u, v) = \begin{cases} d(v), \text{ if } u = v, \\ -1, \text{ if } u \text{ and } v \text{ are adjacent}, \\ 0, \text{ otherwise}. \end{cases}$$

Let $T = \operatorname{diag}\{d(v_1), d(v_2), \dots, d(v_p)\}$. The *Laplacian* of G is defined to be the matrix

$$\tilde{L}(u, v) = \begin{cases} 1, \text{ if } u = v, \\ -\frac{1}{\sqrt{d(v)d(u)}}, \text{ if } u \text{ and } v \text{ are adjacent}, \\ 0, \text{ otherwise}. \end{cases}$$

It is clear that for a connected graph having at least one edge

$$\tilde{L} = T^{-1/2} L T^{-1/2}.$$

The matrix $\tilde{L}$ can be considered as an operator on the space of functions $f : V(G) \to R$ which satisfies

$$\tilde{L}f(u) = \frac{1}{\sqrt{d(u)}} \sum_{v \sim u} \left(\frac{f(u)}{\sqrt{d(u)}} - \frac{f(v)}{\sqrt{d(v)}} \right).$$

If G is k-regular then

$$\tilde{L} = I - \frac{1}{k} A,$$

where A is the adjacency matrix. For a general connected graph

$$\tilde{L} = I - T^{-1/2} A T^{-1/2}. \tag{2.3}$$

3. Cyclically connected graphs

In this section the directions of edges are arbitrary.

We consider the following problem generated by the Sturm-Liouville equations on a cyclically connected graph

$$-y_j'' + q_j(x) y_j = \lambda y_j \tag{3.1}$$

with continuity conditions: if $i \in K$ then

$$\begin{gathered} y_{j_i}(l) = y_{h_i}(l) = y_{k_i}(0) = y_{r_i}(0) \\ \text{for all } j_i \in W_i^+, h_i \in W_i^+ \text{ and all } k_i \in W_i^-, r_i \in W_i^- \end{gathered} \tag{3.2}$$

and Kirchhoff's condition

$$\sum_{j \in W_i^+} y_j'(l) = \sum_{j \in W_i^-} y_j'(0), \quad i \in K. \tag{3.3}$$

Here and below we have 0 instead of the corresponding sum if $W_i^+ = \emptyset$ or $W_i^- = \emptyset$.

Problem (3.1)–(3.3) is the main one in this Section. Let us notice that in a cyclically connected graph all the vertices are interior.

The following theorem was proved in [21] for the general case of edges of different lengths.

Theorem 3.1. *The maximal multiplicity of an eigenvalue of problem (3.1)–(3.3) on a cyclically connected graph is $\mu + 1$.*

Remark 3.2. In the proof of this theorem in [21] it was shown that if $q_j(x) \equiv 0$ for all $j = 1, 2, \ldots, g$ then $\lambda = (\frac{2\pi}{l})^2$ is an eigenvalue of multiplicity $\mu + 1 = g - p + 2$.

Assumption. *In what follows we assume the potentials on the edges are the same, i.e. $q_1(x) \equiv q_2(x) \equiv \cdots \equiv q_g(x) \overset{\text{def}}{=} q(x)$ and symmetric with respect to the midpoint*

$$q(l - x) \overset{\text{a.e.}}{=} q(x).$$

Denote by $s(\lambda, x)$ and $c(\lambda, x)$ the solutions of (3.1) which satisfy the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = c'(\lambda, 0) = c(\lambda, 0) - 1 = 0$. From now on we consider symmetric potentials.

It is known (see e.g. [27], Proposition 2.1) that under this assumption

$$s'(\lambda, l) = c(\lambda, l) \tag{3.4}$$

and consequently

$$c'(\lambda, l) s(\lambda, l) = c^2(\lambda, l) - 1.$$

Following [11], [3], [25], [26] we introduce the solutions to (3.1):

$$f_j(\lambda, x) = \frac{f_j(l) - f_j(0) c(\lambda, l)}{s(\lambda, l)} s(\lambda, x) + f_j(0) c(\lambda, x), \tag{3.5}$$

where j is the index of an edge. Here it is assumed that λ is such that $s(\lambda, l) \neq 0$. It is clear that

$$f_j(\lambda, 0) = f_j(0), \quad f_j(\lambda, l) = f_j(l). \tag{3.6}$$

If $s(\lambda, l) \neq 0$, any solution on an edge e_j can be presented in the form of (3.5). Continuity conditions at a vertex v_i with incoming edges e_j $(j = j_1^+, j_2^+, \ldots, j_{d^+(v_i)})$ and outgoing edges e_j $(j = j_1^-, j_2^-, \ldots, j_{d^-(v_i)}^-)$ are

$$f_{j_1^+}(\lambda, l) = f_{j_2^+}(\lambda, l) = \cdots = f_{j^+_{d^+(v_i)}}(\lambda, l) = f_{j_1^-}(\lambda, 0) = f_{j_2^-}(\lambda, 0) = \cdots = f_{j^-_{d^-(v_i)}}(\lambda, 0)$$

or due to (3.6)

$$f_{j_1^+}(l) = f_{j_2^+}(l) = \cdots = f_{j^+_{d^+(v_i)}}(l) = f_{j_1^-}(0) = f_{j_2^-}(0) = \cdots = f_{j^-_{d^-(v_i)}}(0) \stackrel{\text{def}}{=} \Phi(v_i).$$

The Kirchhoff condition at v_i is

$$\sum_{j=j_1^+}^{j^+_{d^+(v_i)}} f_j'(\lambda, l) - \sum_{j=j_1^-}^{j^-_{d^-(v_i)}} f_j'(\lambda, 0) = 0$$

or

$$\sum_{j=j_1^+}^{j^+_{d^+(v_i)}} \frac{f_j(l)s'(\lambda, l) - f_j(0)}{s(\lambda, l)} - \sum_{j=j_1^-}^{j^-_{d^-(v_i)}} \frac{f_j(l) - f_j(0)c(\lambda, l)}{s(\lambda, l)} = 0. \tag{3.7}$$

Taking into account (3.4) and $s(\lambda, l) \neq 0$ we arrive at

$$c(\lambda, l)d(v_i)\Phi(v_i) - \sum_{v_j \sim v_i} \Phi(v_j) = 0, \tag{3.8}$$

where the sum is taken over all the vertices v_j adjacent with v_i. Equation (3.8) is the vector equation

$$zBF - AF = 0, \tag{3.9}$$

where we introduce $z = c(\lambda, l)$, $F = \{\Phi(v_1), \Phi(v_2), \ldots, \Phi(v_p)\}$, $B = \operatorname{diag}\{d(v_1), d(v_2), \ldots, d(v_p)\}$ and A is the adjacency matrix.

Since we exclude isolated vertices and, therefore $B > 0$, equation (3.9) can be written in the form

$$(zI - \tilde{A})F = 0, \tag{3.10}$$

where $\tilde{A} = B^{-1/2}AB^{-1/2}$. The spectrum of $\tilde{A}$ consists of p (with account of multiplicities) eigenvalues. Thus, if $s(\lambda_k, l) \neq 0$, then λ_k is an eigenvalue of problem (3.1)–(3.3) if and only if $c(\lambda_k, l) = \alpha_i$, where α_i $(i = 1, 2, \ldots, p)$ are the zeros of the polynomial $P_p(z) = det(zI - \tilde{A})$ of degree p, i.e. the eigenvalues of the matrix $\tilde{A}$, i.e. the eigenvalues of the adjacency matrix A. Let us notice that due to (2.3) α_k is an eigenvalue of $\tilde{A}$ if and only if $\tau_k = 1 - \alpha_k$ is an eigenvalue of the discrete Laplacian (see (2.3)). This means that the characteristic function is

$$\phi(\lambda) = s^{g-p}(\lambda, l)P_p(c(\lambda, l)), \tag{3.11}$$

where $P_p(z)$ is a polynomial of degree p.

Theorem 3.3.

$$\phi(\lambda) = s^{g-p}(\lambda, l)(c(\lambda, l) - 1)\tilde{P}_{p-1}(c(\lambda, l)),$$

where $\tilde{P}_{p-1}(z)$ is a polynomial of degree $p-1$ with $\tilde{P}_{p-1}(1) \neq 0$.

Proof. Let $q(x) \equiv 0$ then $s(\lambda, l) = \frac{\sin\sqrt{\lambda}l}{\sqrt{\lambda}}$ and $c(\lambda, l) = \cos\sqrt{\lambda}l$ equation (3.11) implies

$$\phi(\lambda) = b\left(\frac{\sin\sqrt{\lambda}l}{\sqrt{\lambda}}\right)^{g-p} \prod_{k=1}^{p} (\alpha_k - \cos\sqrt{\lambda}l),$$

where b is a nonzero constant.

According to Theorem 3.1 and Remark 3.2 $\phi(\lambda)$ must have a zero of multiplicity $g-p+2$ at $\lambda=(\frac{2\pi}{l})^2$. That means $\alpha_p=1$ and $\alpha_k\neq 1$ for $k\neq p$.

It is known [6] that the eigenvalues of the discrete Laplacian satisfy the conditions $\mathrm{Im}\tau_k=0$, $0\leq\tau_k\leq 2$ and, therefore, $\mathrm{Im}\alpha_k=0$ and $|\alpha_k|\leq 1$. These results follow also from the fact that the corresponding operator $\mathcal{L}$ is self-adjoint and, therefore, its eigenvalues are real and, consequently, the zeros of $\phi(\lambda)$ must be real and, therefore, $\mathrm{Im}\alpha_k=0$ and $|\alpha_k|\leq 1$ □

Theorem 3.4. *If in addition the graph is bipartite then*

$$\phi(\lambda)=s^{q-p}(\lambda,l)(c^2(\lambda,l)-1)c^m(\lambda,l)Q_{\frac{p-m}{2}-1}(c^2(\lambda,l)),$$

where $m\in N\cup\{0\}$, $m+p$ *is even number,* $Q_{\frac{p-m}{2}-1}(z)$ *is a polynomial of degree* $\frac{p-m}{2}-1$.

Proof. Equip the vertices with values 1 or -1 in such way that each edge is directed from a vertex valued with 1 to a vertex valued with -1. It is possible to do for any bipartite graph. Let the potential $q(x)\equiv 0$. Then $\lambda=(\frac{\pi}{l})^2$ is an eigenvalue with the eigenfunction constructed as follows. We direct the edges of the graph from the vertices valued 1 to the vertices valued -1. Then we consider a vector-function $\{\cos\frac{\pi x}{l},\cos\frac{\pi x}{l},\ldots,\cos\frac{\pi x}{l}\}^T$. It is easy to see that this is the eigenvector corresponding to the eigenvalue $\lambda=(\frac{\pi}{l})^2$. Since $\cos\sqrt{\lambda}l\Big|_{\sqrt{\lambda}=\pi/l}=-1$, we conclude that $\alpha_1=-1$.

Now we can use Lemma 1.8 (iii) from [6] to show that $\phi(\lambda)s^{p-g}(\lambda,l)$ is a polynomial in $c^2(\lambda,l)$ □

Corollary 3.5. *Let* G *be a cyclically connected not bipartite equilateral graph which is not a simple cycle with the same symmetric potential on the edges.*

$$p=g-m(\lambda_g), \tag{3.12}$$

where by $m(\lambda_k)$ *we denote the multiplicity of* λ_k.

Proof. Since the lowest nonnegative solution of the equation $c(\lambda,l)=\alpha_k$ for $\alpha_k\in(-1,1]$ is less than the lowest positive zero of $s(\lambda,l)$ we have

$$\lambda_1<\lambda_2\leq\lambda_3\leq\cdots\leq\lambda_p<\lambda_{p+1}=\lambda_{p+2}=\cdots=\lambda_g<\lambda_{g+1}\leq\cdots$$

□

Remark 3.6. A method of [14] allows to find cyclomatic number for an arbitrary graph but the corresponding formula is very complicated and involves all the eigenvalues. In our simple case we can find the cyclomatic number as $\mu=g-p+1$. The number of edges can be obtained from asymptotic formula (Weyl's law)

$$g=\frac{\pi}{l}\lim_{k\to\infty}\frac{k}{\lambda_k}$$

(see for example [14]) while p from (3.12).

Remark 3.7. Let G be a not bipartite cyclically connected graph which is not a simple cycle. Then the sequence $\{\lambda_{kg}\}_{-\infty,\ k\neq 0}^{\infty}$ is the set of zeros of $s(\lambda,l)$. Since the potential is symmetric with respect to the midpoint, this sequence is uniquely determines the potential. Moreover, $s(\lambda,l)=2s(\lambda,l/2)s'(\lambda,l/2)$ and $\{\lambda_{(2k-1)g}\}_{-\infty}^{\infty}$ is the set of zeros of $s'(\lambda,l/2)$ and $\{\lambda_{2kg}\}_{-\infty,\ k\neq 0}^{\infty}$ is the set of zeros of $s(\lambda,l/2)$. Thus, we can recover the potential on the interval $(0,l/2)$ using these two sequences according to the procedure in [23].

Examples

1. For the tetrahedron graph Theorem 3.3 gives

$$\phi(\lambda)=bs^2(\lambda,l)(c(\lambda,l)-1)P_3(c(\lambda,l)),$$

where b is a nonzero constant. A tetrahedron is invariant under the group of symmetry C_3 therefore it have three linearly independent eigenvectors corresponding to each eigenvalue λ_k which satisfies inequalities $s(\lambda_k, l) \neq 0$ and $c(\lambda_k, l) \neq 1$. That means $P_3(c(\lambda, l)) = (c(\lambda, l) - \alpha)^3$. Using Lemma 1.7 of [6] we obtain that the sum of zeros of $(z-1)P_3(z)$ is 0, i.e. $3\alpha + 1 = 0$. Therefore,

$$\phi(\lambda) = bs^2(\lambda, l)(c(\lambda, l) - 1)(c(\lambda, l) + 1/3)^3.$$

2. Let us consider an octahedron. In this case Theorem 3.3 gives

$$\phi(\lambda) = bs^6(\lambda, l)(c(\lambda, l) - 1)P_5(c(\lambda, l)).$$

Choose a square cross-section of the octahedron and equip its vertices with values 0. One of the two vertices which remain equip with 1 and the last vertex with -1. Let us construct a vector-function on the edges directed from the vertex of value 1 to the vertices of zero value we choose $y_j = \cos \frac{\pi x}{l}$, on the edges directed from the vertex of value -1 to the vertices of zero value we choose $y_j = -\cos \frac{\pi x}{l}$, on the rest of edges we choose $y_j \equiv 0$. The constructed vector is an eigenvector corresponding to the eigenvalue $\lambda = (\frac{\pi}{l})^2$. It is clear that this eigenvalue is of multiplicity 3 because there are three possibilities to choose the diagonal equipped with ± 1. This means that $P_5(c(\lambda, l)) = c^3(\lambda, l)P_2(\cos \lambda, l)$. Due to the symmetry of an octahedron we conclude that $P_2(z) = (z - \beta)^2$. Again using Lemma 1.7 of [6] we obtain $\beta = -\frac{1}{2}$ and

$$\phi(\lambda) = bs^6(\lambda, l)(c(\lambda, l) - 1)c^3(\lambda, l)(c(\lambda, l) + 1/2)^2.$$

3. For the cube graph using Theorem 3.4 and the symmetry of the problem we obtain

$$\phi(\lambda) = bs^4(\lambda, l)(c^2(\lambda, l) - 1)(c^2(\lambda, l) - 1/9)^3,$$

where b is a nonzero constant.

4. Connected graphs with Neumann conditions at pendant vertices

Now we consider a connected graph which can have pendant vertices. Let the boundary conditions at the pendant vertices be

$$y_j'(l) = 0, \quad j \in J.$$

In this case equation (3.11) remains true.

For a tree $g - p = -1$ and (3.11) takes on the form

$$\phi(\lambda) = s^{-1}(\lambda, l)P_p(c(\lambda, l)).$$

Since trees are bipartite, Theorem 3.4 implies

Theorem 4.1. *For a tree with the Neumann conditions at the pendant vertices*

$$\begin{aligned}\phi(\lambda) &= s^{-1}(\lambda, l)(c^2(\lambda, l) - 1)c^m(\lambda, l)Q_{\frac{p-m}{2}-1}(c^2(\lambda, l)) \\ &= c'(\lambda, l)c^m(\lambda, l)Q_{\frac{p-m}{2}-1}(c^2(\lambda, l)),\end{aligned}$$

where $m \in N \cup \{0\}$, $m + p$ *is even number,* $Q_{\frac{p-m}{2}-1}(z)$ *is a polynomial of degree* $\frac{p-m}{2} - 1$.

Example. For a star graph

$$\phi(\lambda) = b(s(\lambda, l))^{-1}(c^2(\lambda, l) - 1)c^{p-2}(\lambda, l) = c'(\lambda, l)c^{p-2}(\lambda, l).$$

5. Connected graphs with the Dirichlet conditions at pendant vertices

Now let us assume that the Dirichlet conditions are imposed at r of pendant vertices. Of course, in this case instead of conditions (2.1) one must impose $f_j(0) = 0$ ($f_j(l) = 0$) at these vertices in the definition of operator $\mathcal{L}$. It is clear that the operator remains selfadjoint.

Theorem 5.1. *If the Dirichlet conditions are imposed at* r *of pendant vertices then*

$$\phi(\lambda) = s^{g-p+r}(\lambda, l)P_{p-r}(c(\lambda, l)), \tag{5.1}$$

where $P_{p-r}(z)$ is a polynomial of degree $p-r$.

Proof. Let the Dirichlet boundary condition be imposed at a pendant vertex v_{j_1} instead of equation (3.9) with $j=j_1$ we have $f_{j_1}(0)=0$ or $f_{j_1}(l)=0$. In both cases instead of (3.10) we have

$$(zR-\tilde{A}_1)F=0,$$

where $R=\mathrm{diag}\{d(v_1),d(v_2),\ldots,d(v_{j_1-1}),0,d(v_{j_1+1}),\ldots,d(v_p)\}$. Thus, the characteristic polynomial, i.e. $\det(zR-\tilde{A}_1)$ is of degree $p-1$. If the Dirichlet conditions are imposed at r pendandant vertices then the degree of this polynomial is $p-r$ and, consequently, (5.1) is true. □

Theorem 5.2. *If the Dirichlet conditions are imposed at r of pendant vertices and the graph is bipartite then*

$$\phi(\lambda)=s^{g-p+r}(\lambda,l)c^{m}(\lambda,a)P_{\frac{p-r-m}{2}}(c^2(\lambda,l)),$$

where $P_{\frac{p-r-m}{2}}(z)$ is a polynomial of degree $\frac{p-r-m}{2}$.

Proof. To prove this theorem it is enough to use Theorem 5.1 and again Lemma 1.8 (iii) of [6]. □

Example. In [27] a rooted equilateral tree with the Dirichlet conditions at all the pendant vertices is considered. The root is of degree 2, all the other interior vertices of degree 3 and the combinatorial distance n from the root to each of the pendant vertex is the same. There the following result is obtained by direct calculations:

For each natural $n>1$

$$\phi_n(\lambda)=s^{m_n}(\lambda,l)c^{x_n}(\lambda,l)P_d(c^2(\lambda,l)),$$

where

$$m_n=2^n-1,$$

$$x_n=\begin{cases}\dfrac{1}{3}(2^n-1), & \text{if } n \text{ even},\\ \dfrac{2}{3}(2^{n-1}-1)+1, & \text{if } n \text{ odd}\end{cases}$$

and $P_d(z)$ is a polynomial of degree $d=\frac{1}{2}(2^{n+1}-2-m_n-x_n)$. Description of these polynomials can be found in [27].

This result meets Theorem 5.2 of the present paper.

Acknowledgments. The first author was partly supported by the Swiss National Science Foundation through grant IZ73Z0-128135.

References

1. S. Alexander, *Superconductivity of networks. A percolation approach to the effects of disorder*, Phys. Rev. B **27** (1983), 1541–1557.
2. J. von Below, J. A. Lubary, *The eigenvalues of the Laplacian on locally finite networks*, Results Math. **47** (2005), 199–225.
3. C. Cattaneo, *The spectrum of the continuous Laplacian on a graph*, Monatsh. Math. **124** (1997), no. 3, 215–235.
4. N. Biggs, *Algebraic Graph Theory*, 2nd ed., Cambridge University Press, Cambridge, 1993.
5. D. Cvetković, M. Doob, I. Gutman, Torgašev, *Resent Results in the Theory of Graph Spectra*, North Holland, Amsterdam, 1988.
6. Fan R. K. Chung, *Spectral Graph Theory*, Amer. Math. Soc., Providence, RI, 1997.
7. T. Kottos, U. Smilansky, *Quantum chaos on graphs*, Phys. Rev. Lett. **79** (1997), 4794–4797.

8. P. Kuchment, *Quantum graphs: an introduction and a brief survey*, Analysis on Graphs and its Applications, Proc. Symp. Pure Math., Amer. Math. Soc., 2008, pp. 291–314.
9. R. P. Grimaldi, *Discrete and Combinatorial Mathematics*, Pearson Edu. Ink., Boston, 2004.
10. P. Exner, *Weakly coupled states on branching graphs*, Letters Math. Phys. **38** (1996), 313–320.
11. P. Exner, *A duality between Schrödinger operators on graphs and certain Jacobi matrices*, Ann. Inst. H. Poincaré, Sec. A **66** (1997), 359–371.
12. R. Carlson, *Adjoint and self-adjoint differential operators on graphs*, Electronic J. Diff. Eqnuations, 1998, no. 6, 1–10. (electronic)
13. V. Kostrikin, R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A: Math. Gen. **32** (1999), 595–630.
14. P. Kurasov, *Schrödinger operators on graphs and geometry* I*: Essentially bounded potentials*, J. Funct. Anal. **254** (2008), 934–953.
15. V. Pivovarchik, *Inverse problem for the Sturm-Liouville equation on a simple graph*, SIAM J. Math. Anal. **32** (2000), 801–819.
16. B. Gutkin, U. Smilansky, *Can one hear the shape of a graph?*, J. Phys. A: Math. Gen. **34** (2001), 6061–6068.
17. M. Brown, R. Weikard, *A Borg-Levinson theorem for trees*, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci., vol. 461, no. 2062, 2005, pp. 3231–3243.
18. S. Currie, B. Watson, *Eigenvalue asymptotics for differential operators on graphs*, J. Comput. Appl. Math. **182** (2005), 13–31.
19. V. Pivovarchik, H. Woracek, *Sums of Nevanlinna functions and differential equations on star-shaped graphs*, Operators and Matrices **3** (2009), no. 4, 451–501.
20. A. Gomilko, V. Pivovarchik, *Inverse Sturm-Liouville problem on a figure-eight graph*, Ukrainian Math. J. **60** (2008), no. 9, 1360–1385.
21. I. Kac, V. Pivovarchik, *On multiplicity of quantum graph spectrum*, J. Phys. A: Math. Theor. **44** (2011), no. 10, 105301 (14 p.)
22. C. K. Law, V. Pivovarchik, *Characteristic functions of quantum graphs*, J. Phys. A: Math. Theor. **42** (2009), no. 3, 035302 (11 p.)
23. V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Naukova Dumka, Kiev, 1977. (Russian); English transl. Oper. Theory Adv. Appl. **22**, Birkhäuser Verlag, Basel, 1986.
24. M. Solomyak, *On the spectrum of the Laplacian on regular metric trees*, Waves Random Media **14** (2004), 155–171.
25. J. Friedman, J.-P. Tillich, *Wave equations for graphs and the edge-based Laplacian*, Pacific J. Math. **216** (2004), no. 2, 229–266.
26. K. Pankrashkin, *Spectra of Schrödinger operators on equilateral quantum graphs*, Letters Math. Phys. **77** (2006), no. 2, 139–154.
27. V. Pivovarchik, N. Rozhenko, *Inverse Sturm-Liouville problem on equilateral regular tree*, Applicable Analysis DOI:10.1080 / 00036811.2011.639765.

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Received 09/03/2012