ON COMMUTING SYMMETRIC OPERATORS

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Dedicated to the memory of A. G. Kostyuchenko

Abstract. In this paper we present some conditions for a pair of commuting symmetric operators with a joint invariant dense domain in a Hilbert space, to have a commuting self-adjoint extension in the same space. The remarkable Godič-Lucenko theorem allows to get a convenient description of all such extensions.

1. Introduction

The extension problems for commuting operators have been studied for a long time and have numerous applications (e.g. [1, Chapter VIII, Section 2], [2]–[8], etc.). We shall give some conditions for a pair of commuting symmetric operators with a joint invariant dense domain in a Hilbert space, to have a commuting self-adjoint extension in the same space. A convenient description of all such extensions can be obtained by the remarkable Godič-Lucenko theorem [9] (also see [10] for another proof and a generalization). Our conditions form an analog of Theorem 2.6 in [1, Chapter VIII] obtained in 1945 by Livšic, and independently later by Eskin. Notice that Theorem 2.6 regarded operators acting in the tensor products of Hilbert spaces and having a fixed structure. Our proof below is analogous to that of Theorem 2.6 except its part related to the Godič-Lucenko Theorem. Also, our result generalizes the result of Ismagilov in [3, Theorem 1]. A close result to Theorem 2.1 below, was given by Slinker in [4, Theorem 3.1].

Notations. We denote by \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{Z}_+ \) the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. Everywhere in this paper, all Hilbert spaces are assumed to be separable. By \( \langle \cdot , \cdot \rangle_H \) and \( \| \cdot \|_H \) we denote the scalar product and the norm in a Hilbert space \( H \), respectively. The indices may be omitted in obvious cases. For a set \( M \) in \( H \), by \( \overline{M} \) we mean the closure of \( M \) in the norm \( \| \cdot \|_H \). The identity operator in \( H \) is denoted by \( E_H \). For an arbitrary linear operator \( A \) in \( H \), the operators \( A^*, \overline{A}, A^{-1} \) mean its adjoint operator, its closure and its inverse (if they exist). By \( D(A) \) and \( R(A) \) we mean the domain and the range of the operator \( A \).

2. An extension of commuting symmetric operators in the same space

Theorem 2.1. Let \( A \) be a symmetric operator and \( B \) be an essentially self-adjoint operator with a common domain \( D = D(A) = D(B) \) in a Hilbert space \( H, \overline{D} = H \), and

\[ AD \subseteq D, \quad BD \subseteq D; \]

\[ AB = BA. \]

Suppose also that for some \( z_0 \in \mathbb{C} \setminus \mathbb{R} \), the operator \( B \) restricted to the domain \( (A - z_0 E_H)D \) is essentially self-adjoint in a Hilbert space \( (\overline{A} - z_0 E_H)D(\overline{A}) \).

If there exists a conjugation \( J \) in \( H \) such that \( JD \subseteq D \), and

\[ AJ = JA, \quad BJ = JB, \]

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then there exists a self-adjoint operator $\tilde{A} \supseteq A$, which commutes with $B$.

Proof. Consider the Cayley transformations of the operators $\overline{A}$ and $\overline{B}$

$$V_A := (\overline{A} - z_0 E_H)(\overline{A} - z_0 E_H)^{-1} = E + (z_0 - \overline{z_0})(\overline{A} - z_0 E_H)^{-1},$$

$$U_B := (\overline{B} + i E_H)(\overline{B} - i E_H)^{-1} = E + 2i(\overline{B} - i E_H)^{-1}.$$ 

Set

$$\mathbf{H}_1 = (\overline{A} - z_0 E_H)D(\overline{A}), \quad \mathbf{H}_2 = \mathbf{H} \ominus \mathbf{H}_1, \quad \mathbf{H}_3 = (\overline{A} - \overline{z_0} E_H)D(\overline{A}), \quad \mathbf{H}_4 = \mathbf{H} \ominus \mathbf{H}_3.$$ 

Observe that $V_A$ maps isometrically $\mathbf{H}_1$ onto $\mathbf{H}_3$, and $U_B$ is a unitary operator in $\mathbf{H}$. For arbitrary $h \in (\mathbf{B} - i E_H)(\mathbf{A} - z_0 E_H)D$, $h = (\mathbf{B} - i E_H)(\mathbf{A} - z_0 E_H)f$, $f \in D$, we may write

$$U_B V_A h = U_B(\overline{A} - \overline{z_0} E_H)(\overline{B} - i E_H) f = (\overline{B} + i E_H)(\overline{A} - \overline{z_0} E_H) f;$$

$$V_A U_B h = V_A(\overline{B} + i E_H)(\overline{A} - z_0 E_H) f = (\overline{A} - z_0 E_H)(\overline{B} + i E_H) f.$$ 

Therefore

$$U_B V_A h = V_A U_B h, \quad h \in (\mathbf{B} - i E_H)(\mathbf{A} - z_0 E_H)D.$$ 

Since $\mathbf{B}$ restricted to the domain $(\mathbf{A} - z_0 E_H)D$ is essentially self-adjoint, the sets $(\mathbf{B} \pm i E_H)(\mathbf{A} - z_0 E_H)D$ are dense in $\mathbf{H}_1$. By continuity, from (1) we derive

$$U_B V_A h = V_A U_B h, \quad h \in \mathbf{H}_1.$$ 

Observe that

$$U_B(\mathbf{B} - i E_H)(\mathbf{A} - z_0 E_H) f = (\mathbf{B} + i E_H)(\mathbf{A} - z_0 E_H) f, \quad f \in D.$$ 

Thus, $U_B$ maps a dense set in $\mathbf{H}_1$ on a dense set in $\mathbf{H}_1$. Therefore $U_B$ is a unitary operator in $\mathbf{H}_1$. Then $U_B$ is a unitary operator in $\mathbf{H}_2$, as well. By (2) we may write

$$U_B \mathbf{H}_3 = U_B V_A \mathbf{H}_1 = V_A U_B \mathbf{H}_1 = V_A \mathbf{H}_1 = \mathbf{H}_3.$$ 

Therefore $U_B$ is a unitary operator in subspaces $\mathbf{H}_3$ and $\mathbf{H}_4$, as well.

Let us construct an isometric operator $U_{2,4}$ which maps $\mathbf{H}_2$ onto $\mathbf{H}_1$, and commutes with $U_B$

$$U_B U_{2,4} x = U_{2,4} U_B x, \quad x \in \mathbf{H}_2.$$ 

The operator $U_B$ restricted to $\mathbf{H}_j$ we denote by $U_{B,j}$, $1 \leq j \leq 4$. Using the definition of the closure we get

$$J \overline{A} h = \overline{A} J h, \quad h \in D(\overline{A}),$$

$$J \overline{B} h = \overline{B} J h, \quad h \in D(\overline{B}).$$ 

Observe that

$$J U_B h = U_B^{-1} J h, \quad h \in D(U_B).$$ 

To check this relation we write for an arbitrary $x \in D$ and $g := (\mathbf{B} - i E_H)x$

$$(\mathbf{B} - i E_H) J x = J (\mathbf{B} + i E_H)x = J U_B g.$$ 

Multiply the latter equality by $U_B$

$$(\mathbf{B} + i E_H) J x = U_B J U_B g,$$ 

and we get

$$J g = U_B J U_B g, \quad g \in (\mathbf{B} - i E_H)D.$$ 

Moreover, we have

$$\overline{A}^* J x = J \overline{A}^* x, \quad x \in D(\overline{A}^*).$$ 

Indeed, for arbitrary $f_A \in D(\overline{A})$ and $g_{A^*} \in D(\overline{A}^*)$ we may write

$$\overline{(\mathbf{A} f_A, g_{A^*})} = (J \overline{A} f_A, g_{A^*}) = (\overline{J f_A}, \overline{g_{A^*}}) = (J f_A, \overline{\overline{A}^* g_{A^*}}) = (f_A, J \overline{A}^* g_{A^*}),$$
and (4) follows. Choose an arbitrary \( x \in H_2 \). We have \( \bar{A}^* x = \bar{a}_0 x \), and therefore \( \bar{A}^* Jx = J\bar{A}^* x = z_0 x \). Thus, we have \( JH_2 \subseteq H_4 \). In a similar manner we get \( JH_4 \subseteq H_2 \), and therefore

\[
JH_2 = H_1, \quad JH_4 = H_2.
\]

By the Godić-Lucenko Theorem we have a representation: \( U_{B,2} = KL \), where \( K \) and \( L \) are some conjugations in \( H_2 \). We set

\[
U_{2,4} := JK.
\]

From (5) it follows that \( U_{2,4} \) maps isometrically \( H_2 \) onto \( H_4 \). Notice that \( U_{2,4}^{-1} = KJ \). Then

\[
U_{2,4}U_{B,2}^{-1} = JKLKJx = JLKJx = JU_{B,2}^{-1}Jx = JU_{B,4}^{-1}Jx = U_{B,4}x, \quad x \in H_4.
\]

Therefore relation (3) is true. Set \( \tilde{U} = V_A \oplus U_{2,4} \). It is straightforward to check that \( \tilde{U} \) commutes with \( U_B \). Moreover, it is easy to check that all unitary operators \( U \supseteq V_A \) which commute with \( U_B \) have the following form:

\[
U = V_A \oplus W_{2,4},
\]

where \( W_{2,4} \) is an isometric operator which maps \( H_2 \) onto \( H_4 \), and commutes with \( U_B \). Finally, we set \( \tilde{A} \) to be the inverse Cayley transformation of \( \tilde{U} \). □

Remark 2.1. Notice that by (7),(6) we easily obtain that the following relation:

\[
W_{2,4} = U_{2,4}U_2,
\]

where \( U_2 \) is an arbitrary unitary operator in \( H_2 \) commuting with \( U_B \), provides all admissible operators for (7). Operators \( U_2 \) may be described in terms of the decomposable operators with respect to the spectral measure of the operator \( U_B \) restricted to \( H_2 \).

REFERENCES


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