

## MAPS OF 2-MANIFOLDS INTO THE PLANE

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ABSTRACT. We define a  $p$ -graph and describe how it changes under isotopy of projections for classification of maps of 2-manifolds into plane. The problem of graph implementation and maps classification are considered.

### INTRODUCTION

In 1995, H. Whitney described a class of structurally stable mappings between two-dimensional manifolds [5]. These mappings form an open everywhere dense sets in the space of all maps. Actually a local classification of such maps is received. In particular it was proved that for these mappings each point is a regular point, a fold point, or a cusp point. In this paper, stable mappings of two-dimensional manifold into plane are considered. The basic topological invariant of the map is a planar graph formed from fold and cusp points [3]. To describe the graph and the map, we introduced  $p$ -graphs. They show how a section of the map changes. For 3-manifolds it will be a change of the Reeb graph [1], [2].

The main aim is to find a new way of classification of maps from two-dimensional manifolds into the plane using  $p$ -graphs.

### 1. RULES OF PROJECTION CHANGES

Let us have a closed smooth 2-dimensional manifold  $M^2$  and a function  $f : M^2 \rightarrow \mathbb{R}^2$  on it (the projection of the manifold into the plane). This projection divides the plane on the fields. On the projection we will note a number of connected components that are projected into the corresponding field of the projection. For this projection we will construct a graph  $G$  [4] that shows how the projection changes during shifting along a line.

For simplicity we will depict the sequence of graphs  $G$  that are formed by movement along a line in the direction from left to right, as one graph that will show how the graph changes. To depict this change on one graph, we denote with different colors numbered edges that are added to the graph, and those that disappear in sequential order (Example 1). Further assume that simultaneously no adjacent edges can arise or disappear (this can be achieved by a small move of projection). Thus for any projection, we can find an appropriate graph  $G$ . Let us call it a  $p$ -graph. Consider how the graph changes after isomorphic movements of the projection.

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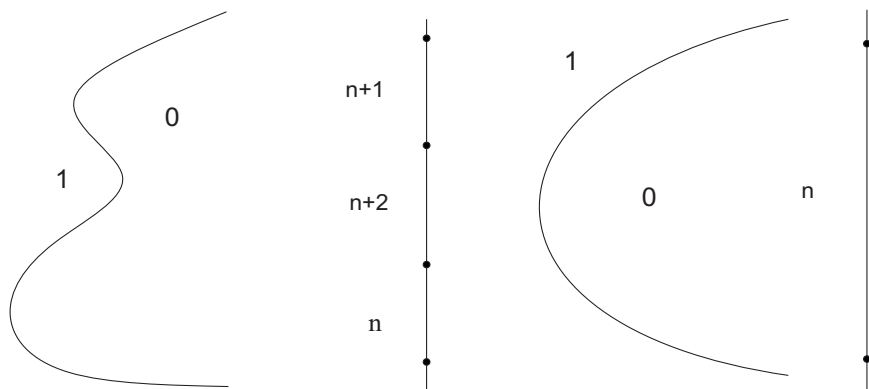


Fig. 1

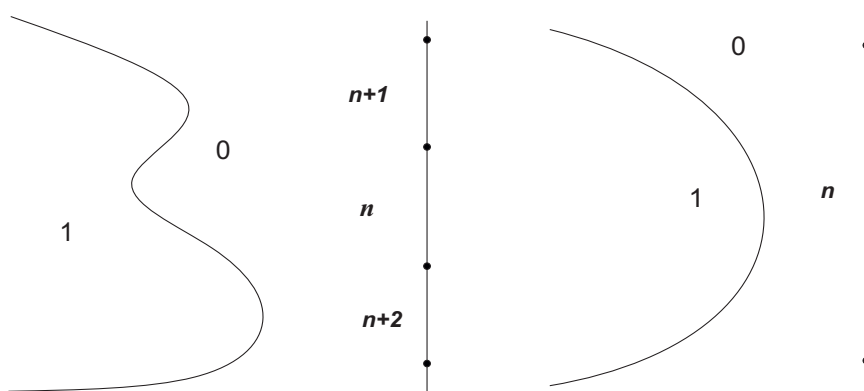


Fig. 2

Consider what happens with graph, when we “untwist” the projection.

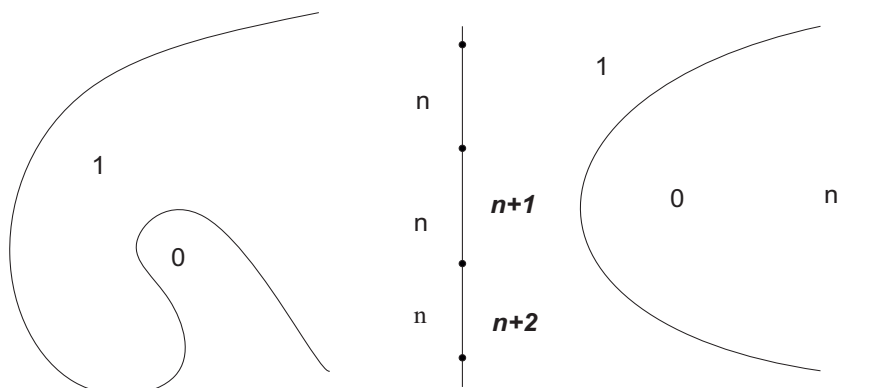


Fig. 3

Similar simplification can be made for parts of the projections that are symmetric about a line.

In the future, we will call these rules “Rule 1” (Fig. 1), “Rule 2” (Fig. 2), and “Rule 3” (Fig. 3).

In addition, there may be intersections and self-intersections of contours. We introduce the following additional rules that are allowed by projection isotopy and we find the corresponding changes on the graph.

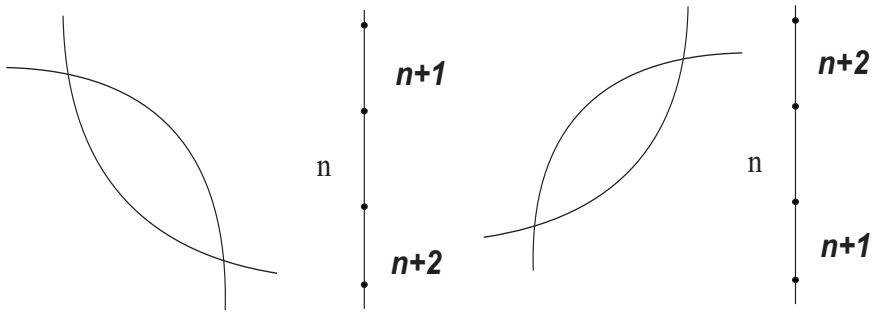


Fig. 4

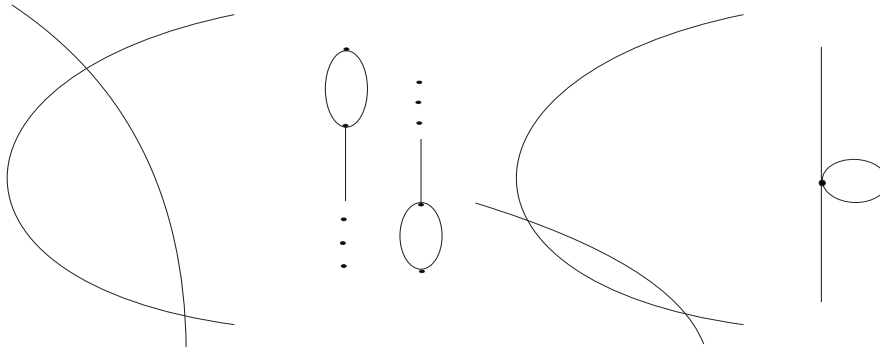


Fig. 5

Let us give more details on this rule, — if we change the order of appearance of two neighboring points, one of which corresponds to crossing the contour, another one is an intersection or self-intersection, then the  $p$ -graph will change from a loop to the following two consecutive appearances on the graph:

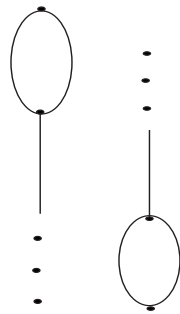


Fig. 6

These two rules will be called “Rule 4” and “Rule 5”, respectively.

## 2. IMPLEMENTATION THEOREM

**Theorem 1.** *For any  $p$ -graph there exists a unique corresponding projection (up to isotopy) on the plane.*

*Proof.* Consider a random  $p$ -graph. Split it into a sequence of graphs. Using this sequence we construct a table by the following rules.

Consider a rectangle. On the left side we note a number of points on the graph, the lower side we divide into the parts the number of which equal to the number of graphs. Then we move from left to right and depict the sequence of graphs, indicating on each segment the number of component connectivity.

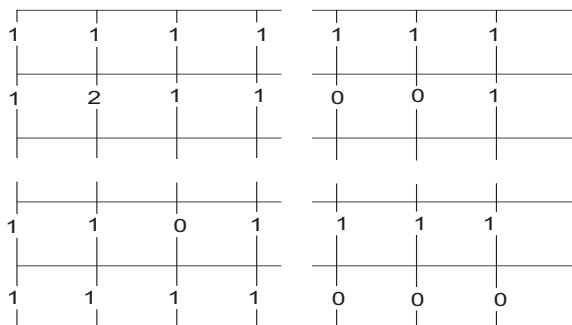


Fig. 7

For each of the segments, we continue it to the next right, marking the corresponding number of components that are projected, and combine the neighboring squares with the same number of components.

Separately, we consider the case where the graph has a loop. Then at some point there should be an additional area in the table:

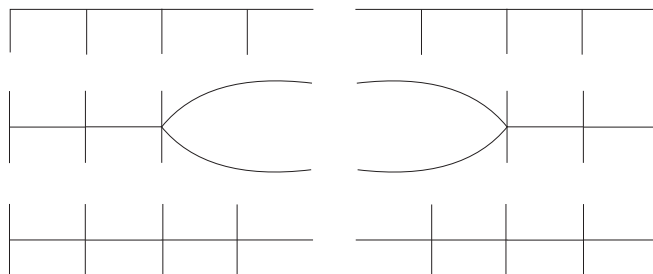


Fig. 8

If in the area of self-intersection, we have other changes in the projection, we must split a new area and go through a procedure similar to that we did for the table. It only remains to smooth the corners and get the look of the original projection.

The smoothing procedure is the following.

Consider the right angle, set in rectangular Cartesian coordinates by the points  $(-1,0)$

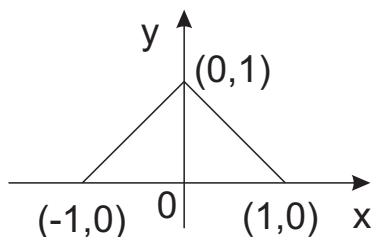


Fig. 9

Look for a function  $f(x)$  that would have the following properties:

1.  $f(x)$  is symmetric about the vertical axis.
2.  $f(-1) = 0, f(1) = 0$ .
3.  $f'(1) = -1, f'(-1) = 1$ .
4.  $f(x)$  is convex upward.
5. At the point 0, the function is arbitrarily close to 1.

Consider the function of the form

$$f(x) = ax^4 + bx^2 + c,$$

$$f'(x) = 4ax^3 + 2bx.$$

This function is symmetric about the vertical axis,

$$f(-1) = f(1) = a + b + c = 0,$$

$$f'(-1) = -4a - 2b = 1, f'(1) = 4a + 2b = -1.$$

So we have

$$a + b + c = 0,$$

$$4a + 2b = -1$$

in order to satisfy Condition 5, assign the constant value  $c := \frac{n-1}{n}$ , where  $n$  is any natural number.

Then we find that

$$a = \frac{n-2}{2n}, \quad b = \frac{-3n+4}{2n}.$$

The function, with these coefficients, satisfies all the necessary conditions. It remains only to apply the formula of rotation and stretching, and we get a function that approximates any right angle.

Separately, we should consider the case of smoothing areas of self-intersections. In this case, we can assume that we consider an arbitrarily narrow additional region.

Now we consider a right angle that meets the following two points: one we take from the middle of edge, the other lies in the middle of the arc, which limits the additional area. Then we smooth this angle. Due to the fact that the introduced region is narrow, we have no additional crossings with other areas.  $\square$

### 3. CLASSIFICATION OF PROJECTIONS

**Theorem 2.** *If the transition from one graph to another can go through the rules 1–5 or symmetric to them, then the corresponding projections are isotopic.*

*Proof.* Consider some projection. We simplify each of the separate curves that limit the field. We assign numbers to each curve. We mark the points of intersections with other curves on each curve.

For each of these closed curves, we choose some starting point and begin to move along the curve. Thus we find points at which the tangent is vertical, the points of intersections with other curves and the points of self-intersections. We consider that we have vertically directed tangents only in the points where the graph changes (this can be done by small projection movements). Then we get a sequence of tangents and the points.

Now consider how to apply our rules. Rule 1 is a replacement of three consecutive equally directed tangents to one similarly directed tangent. Rule 2 corresponds to the same replacement.

For Rule 3, we find a sequence of three tangents such that the third one is directed in the direction opposite to the first two and replace them with the third tangent. Rules 4 and 5 are changing the sequence of “special” points, and do not change their number and hence these rules are isotopic projection movements.

Using these rules, we can simplify every curve of our projection.

If we have intersection of two curves, then each of them can be simplified until we change the number of the intersections.

Thus, for each shift on the graph we can work out an appropriate projection isotopy that does not change the number of intersections and self-intersection.  $\square$

**Theorem 3.** *Suppose we have a projection on the plane of manifolds  $P_1$  and  $P_2$  and the corresponding  $p$ -graphs  $G_1$  and  $G_2$ . If the projection are isotopic, then from  $G_1$  to  $G_2$  we can go using the rules 1–5 and renumbering of vertices.*

*Proof.* We have shown that changing by the rules of graphs clearly fits the isotopic projection movements. However, we construct a  $p$ -graph, projecting the projection onto an arbitrary line. Changing the line is the same as turning on the projection by an angle. We have to show that rotation can be done using rules 1–5. We consider only some parts of the projection.

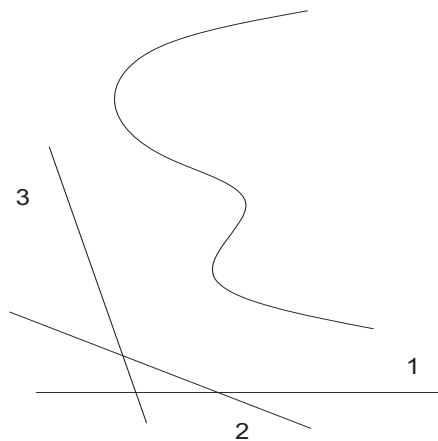


Fig. 10

Suppose we started by projecting onto the line 1. If we shift it a little, then the graph will not change, but if we continue to move straight, then at some time (line 3) the number of points on the graph changes. This projection isotopy, as in our example, corresponds to the rule 1. Consider first the case where there are no points of intersections and self-intersections. Then, considering three consecutive tangents we have the following situation:

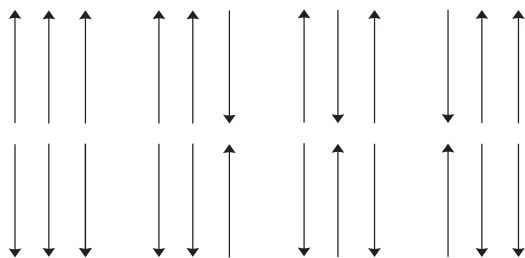


Fig. 11

Each replacement of such triples for one vector corresponds to one of the rules 1–3. Rules 4 and 5 make it possible to change the order of the intersections and adjacent to these points. □

#### 4. EXAMPLES

1. Constructing the  $p$ -graph. Let us have a projection of two-dimensional manifold into the plane. Let us construct the corresponding  $p$ -graph.

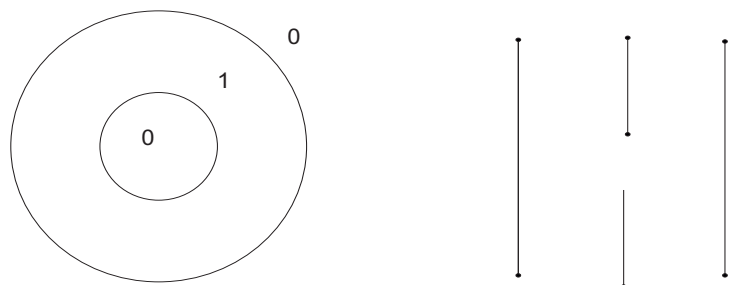


Fig. 12

By combining these graphs in one, we obtain

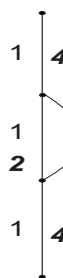


Fig. 13

Similarly, we construct  $p$ -graphs for the following projections:

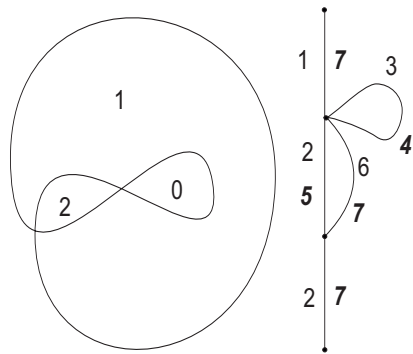


Fig. 14

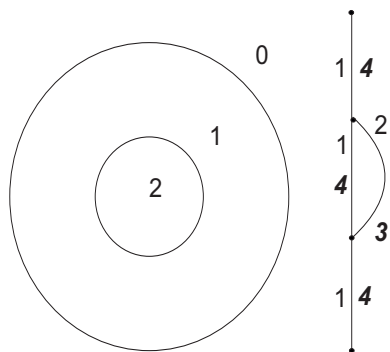


Fig. 15

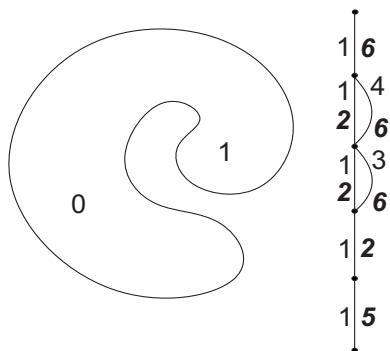


Fig. 16

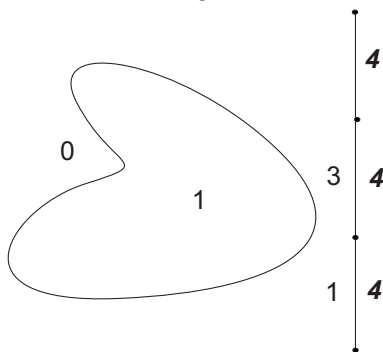


Fig. 17

2. Rules 1–3. Consider how using rules 1–3 we can simplify the graph. The first step distinguishes the fragmented graph that corresponds to the 3 after renumbering of edges. According to this principle we can replace it by one edge that occurs. Then we rename all the edges, keeping the order in which they arise and disappear. As a result we obtain a graph which can be simplified by using rule 1.

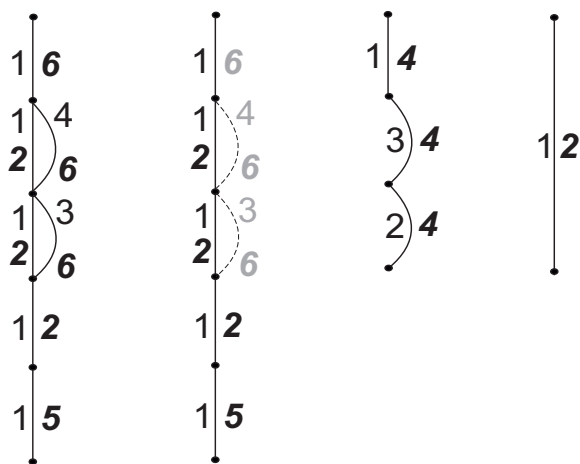


Fig. 18



3. Implementation of the graph

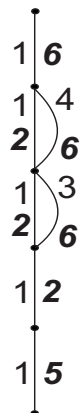


Fig. 19

Let us expand it into a sequence of graphs,

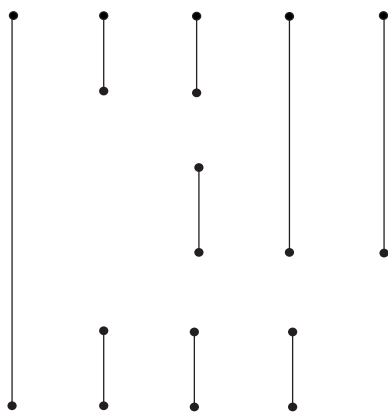


Fig. 20

Now let us construct the corresponding table,

1	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	0	0	0	0
1	1	1	1	0

Fig. 21

Now combine table cells having identical values obtaining

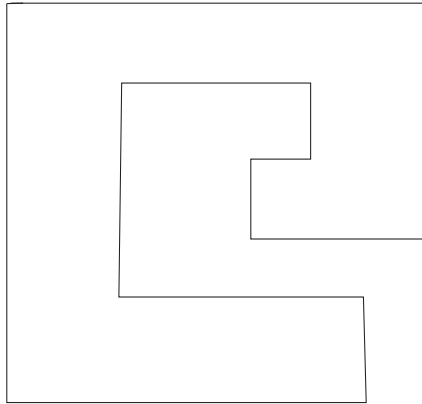


Fig. 22

After smoothing the corners we will get the same projection from which the original graph was constructed (Example 1).

## 5. CONCLUSIONS

In the present paper, maps from 2-manifolds into the plane are considered. For projections of the manifolds, we have built graphs that show how the graphs change for the family of Morse function. The theorem of realization and theorems of classification of map are proved, some examples are considered. The results obtained can be used in a classification of 3-manifolds into plane and can be applied in optics, engineering and applied sciences.

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