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SCHRÖDINGER OPERATORS WITH $(\alpha\delta' + \beta\delta)$-LIKE POTENTIALS: NORM RESOLVENT CONVERGENCE AND SOLVABLE MODELS

YURIY GOLOVATY

Abstract. For real functions Φ and Ψ that are integrable and compactly supported, we prove the norm resolvent convergence, as $\varepsilon \to 0$, of a family S_ε of one-dimensional Schrödinger operators on the line of the form

$$S_\varepsilon = -\frac{d^2}{dx^2} + \alpha\varepsilon^{-2}\Phi(\varepsilon^{-1}x) + \beta\varepsilon^{-1}\Psi(\varepsilon^{-1}x).$$

The limit results are shape-dependent: without reference to the convergence of potentials in the sense of distributions the limit operator S_0 exists and strongly depends on the pair (Φ, Ψ). A class of nontrivial point interactions which are formally related the pseudo-Hamiltonian $-\frac{d^2}{dx^2} + \alpha\delta'(x) + \beta\delta(x)$ is singled out. The limit behavior, as $\varepsilon \to 0$, of the scattering data for such potentials is also described.

1. Introduction

The Schrödinger operators with singular potentials supported on a discrete set (such potentials are usually termed "point interactions") have attracted considerable attention both in the physical and mathematical literature from the early thirties of the last century. To understand the nature of quantum systems it appeared conceivable to analyze their general features about interactions with a range much smaller then the atomic size. Historically point interactions were introduced in quantum mechanics as limits of families of squeezed potentials. The quantum mechanical models that are based on the concept of zero range quantum interactions reveal an undoubted effectiveness whenever solvability together with non triviality is required. General references for this fascinating area are [1, 5], which provide extensive documentation of pertinent material.

In spite of all advantages of the solvable models, which are widely used in various applications to quantum physics, they give rise to many mathematical difficulties. One of the main difficulty of the analysis of zero-range interactions, compared to Schrödinger operators with short-range potentials, is that the Schrödinger operators with singular potentials are often only formal differential expressions, and for the corresponding differential equations no solution exists even in the sense of distributions. In 1961 Berezin and Faddeev [9] suggested how such formal Schrödinger operators can be constructed as mathematically well-defined objects, and for the first time a formal Hamiltonian was written as a self-adjoint operator derived by the theory of self-adjoint extensions of symmetric operators (see [22] for more details).

There exists a large body of results on this subject. It is impossible to refer to all relevant papers, and I confine myself to a brief overview of the one-dimensional case. In the last years there were obtained many results for the point interactions based on the theories of self-adjoint extensions of symmetric operators, singular quadratic forms, boundary triples and almost solvable extensions [4, 8, 10, 27, 28, 29, 30], and once more

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we refer to [1, 5] for a complete list of references therein. Another way to define a formal Schrödinger operator with potential $v \in \mathcal{D}'(\mathbb{R})$ is to approximate it by Schrödinger operators with more smooth ones v_ε obtained by a suitable regularization as well as to use the concept of quasi-derivatives [6, 17, 15, 23, 24, 34, 35, 36, 37]. In [16, 18, 31, 33] the solvable models for 1D Schrödinger operators were based on specific products of $\delta^{(k)}$ and discontinuous functions, where δ is the Dirac delta function. It is worth to note that the difficulty of dealing with the multiplication in $\mathcal{D}'$ may be also overcome by using the new algebras of generalized functions [7].

It is common knowledge that all nontrivial point interactions at a point x can be described by the coupling conditions

$$\begin{pmatrix}\psi(x+0)\\ \psi'(x+0)\end{pmatrix} = C\begin{pmatrix}\psi(x-0)\\ \psi'(x-0)\end{pmatrix}, \quad C = e^{i\varphi}\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}, \tag{1.1}$$

where $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $c_{kl} \in \mathbb{R}$ and $c_{11}c_{22}-c_{12}c_{21} = 1$. For the physically based classification of these interactions we refer the reader to the recent preprint [11], where in particular one singles out the four most important cases: δ-potentials, δ'-interactions, δ'-potentials and δ-magnetic potentials. For a quantum system described by the Schrödinger operator with a smooth enough potential localized in a neighborhood of x one can often assign a point interaction with some matrix in (1.1) and the zero-range model governs the quantum dynamics of the true interaction with admissible fidelity, first of all, for low-energy particles. However the connection between real short-range interactions and point ones is very complex and ambiguously determined. This is certainly the reason why there are a number of papers about this object occasionally with conflicting conclusions.

As for the δ-potential, any smooth approximation of $\beta\delta(x)$ leads to the same solvable model given by conditions (1.1) with the matrix

$$C = \begin{pmatrix}1 & 0\\ \beta & 1\end{pmatrix}. \tag{1.2}$$

Thus the result is shape-independent that is to say it is not determined by a way of regularization. In this case the limiting argument admits a straightforward interpretation. The nonzero off-diagonal element of C implicitly involves the integral of the approximating potential, and hence a slow particle on the line "feels" only the average value of a localized potential.

The situation changes if we turn now to the δ'-potential. The usual regularization of $\delta'(x)$ is a sequence $\varepsilon^{-2}v(\varepsilon^{-1}x)$ with a zero-mean function $v \in C_0^\infty(\mathbb{R})$. It was shown in [20] that for *almost all* functions v the best zero-range approximation to the Hamiltonian $H_\varepsilon = -\frac{d^2}{dx^2} + \varepsilon^{-2}v(\varepsilon^{-1}x)$ is the free Hamiltonian $-\frac{d^2}{dx^2}$ subject to the split boundary conditions $\psi(-0) = \psi(+0) = 0$. These conditions define a non-transparent interaction at the origin. However, there exist so-called *resonant potentials* v (see below for the precise definition) for which the limit behavior of quantum system can be characterized by the nontrivial point interaction with the coupling matrix

$$C = \begin{pmatrix}\theta & 0\\ 0 & \theta^{-1}\end{pmatrix}, \tag{1.3}$$

where $\theta = \theta(v)$ is a spectral characteristics of the potential v. Incidentally, it is of interest that for *any* shape v there exists a countable set of so-called resonant coupling constants α_k for which the spreading potentials $\alpha_k v$ are resonant. These results can be extended recently to potentials v of the Faddeev-Marchenko class [21] and generalized to the case of quantum graphs [32].

Therefore the results on δ'-potentials become shape-dependent: depending on v the Hamiltonians H_ε regularize different kinds of point interactions, nevertheless all of them involve the δ'-like potentials. Hence, it is impossible to assign just one self-adjoint operator to the pseudo-Hamiltonian $-\frac{d^2}{dx^2} + \alpha\delta'(x)$, which is a symbolic notation only for a wide variety of quantum systems with quite different qualitative and quantitative characteristics.

It has been known for a very long time that the δ'-potential defined through the regularization $\varepsilon^{-2}v(\varepsilon^{-1}x)$ is opaque acting as a perfect wall, see widely cited Šeba's paper [37] of 1986. However, such a conclusion is in contradiction with the analysis of H_ε with piece-wise constant potentials v performed recently by Zolotaryuk a.o. [13, 38, 39], where the resonances in the transmission probability for the scattering problem are established. In [19] a similar resonance phenomenon is also obtained in the asymptotics of eigenvalues for the Schrödinger operators perturbed by δ'-like potentials. The authors of [2, 12] faced the question on the convergence of H_ε in approximation of a smooth planar quantum waveguide with a quantum graph. Under the assumption that the mean value of v is different from zero, they also singled out the set of resonant potentials v producing a non-trivial limit of H_ε in the norm resolvent sense. After the revision in [20] of Šeba's proof, the situation concerning these controversial results was clarified thus. Curiously enough, P. Šeba was the first who, in 1985 [36], discovered the resonant potentials for a similar family of the Dirichlet Schrödinger operators on the half-line which produce in the limit the Robin boundary condition.

This paper can be viewed as a natural continuation of recent works [19, 20, 21] on the Schrödinger operators with δ'-like potentials to the case in which the potentials are a smooth enough regularization of the distribution $\alpha\delta'(x) + \beta\delta(x)$. Clearly it is to be expected that the limit results concerning such families of squeezed potentials will be also shape-dependent.

Notation. Throughout the paper, $W_2^l(\omega)$ stands for the Sobolev space of functions defined on a set $\omega \subset \mathbb{R}$ that belong to $L_2(\omega)$ together with their derivatives up to order l. The norm in $W_2^2(\omega)$ is given by $\|f\|_{W_2^2(\Omega)} = \left(\|f''\|^2_{L_2(\Omega)} + \|f\|^2_{L_2(\Omega)}\right)^{1/2}$, where $\|f\|_{L_2(\Omega)}$ is the usual L_2-norm. We shall write $\|f\|$ instead of $\|f\|_{L_2(\mathbb{R})}$.

2. Statement of problem and main result

Let us consider the formal Hamiltonian

$$H = -\frac{d^2}{dx^2} + \alpha\delta'(x) + \beta\delta(x), \quad x \in \mathbb{R},$$

where δ is the Dirac delta function. In accordance with the classic theory of distributions we have $(H\phi)(x) = -\phi''(x) + \alpha\phi(0)\delta'(x) + (\beta\phi(0) - \alpha\phi'(0))\delta(x)$ for any $\phi \in C^1(\mathbb{R})$. However, there exist no solutions in $\mathcal{D}'(\mathbb{R})$ to the equation $H\phi = \lambda\phi$ for a nonzero α, except for a trivial one. The reason for this at first sight surprising fact lies in the theory of distributions: $\mathcal{D}'(\mathbb{R})$ is not an algebra with respect to the "pointwise" multiplication.

Instead of H, we consider the family of Schrödinger operators

$$S_\varepsilon = -\frac{d^2}{dx^2} + \frac{\alpha}{\varepsilon^2}\Phi\left(\frac{x}{\varepsilon}\right) + \frac{\beta}{\varepsilon}\Psi\left(\frac{x}{\varepsilon}\right), \quad \operatorname{dom} S_\varepsilon = W_2^2(\mathbb{R}) \tag{2.1}$$

with integrable potentials Φ and Ψ of compact supports. Here ε is a small positive parameter, and the coupling constants α and β are assumed to be real. One of the questions of our primary interest in this paper is the behavior of S_ε as ε tends to zero. The motivation for this question stems from the fact that the potentials

$$V_\varepsilon = \alpha\varepsilon^{-2}\Phi(\varepsilon^{-1}\,\cdot\,) + \beta\varepsilon^{-1}\Psi(\varepsilon^{-1}\,\cdot\,)$$

approximate the pseudopotential $\alpha\delta' + \beta\delta$ in weak topology under some assumptions on Φ and Ψ. Indeed, if the following conditions hold

$$\int_{\mathbb{R}} \Phi\, ds = 0, \quad \int_{\mathbb{R}} s\Phi\, ds = -1 \quad \text{and} \quad \int_{\mathbb{R}} \Psi\, ds = 1, \tag{2.2}$$

then $\alpha\varepsilon^{-2}\Phi(\varepsilon^{-1}x)+\beta\varepsilon^{-1}\Psi(\varepsilon^{-1}x) \to \alpha\delta'(x)+\beta\delta(x)$ as $\varepsilon \to 0$ in the sense of distributions. In this case, we call Φ the shape of a δ'-like sequence and Ψ the shape of a δ-like one.

Notwithstanding the title of paper, all results presented here concern the potentials V_ε with arbitrary Φ and Ψ of compact support, and the $\alpha\delta' + \beta\delta$-like potentials are only a partial case in our considerations. Note that if the first condition in (2.2) is not fulfilled, then the potentials V_ε do not converge even in the distributional sense. However, surprisingly enough, without reference to the convergence of V_ε the limit of S_ε exists in the norm resolvent sense (i.e., in the sense of uniform convergence of resolvents).

Definition 2.1. ([26]) We say that the Schrödinger operator $-\frac{d^2}{ds^2} + q$ in $L_2(\mathbb{R})$ possess a *half-bound state* (or *zero-energy resonance*) provided there exists a solution u to the equation $-u'' + qu = 0$ in $\mathbb{R}$ that is bounded on the whole line, i.e. $u \in L^\infty(\mathbb{R})$. The potential q is then called *resonant*.

Such a solution u is unique up to a scalar factor and has nonzero limits $u(\pm\infty)$. Our main result reads as follows.

Main result. *Let Φ and Ψ be integrable and bounded real functions with compact support. Then the operator family S_ε given by (2.1) converges as $\varepsilon \to 0$ in the norm resolvent sense.*

If the potential $\alpha\Phi$ is resonant with a half-bound state u_α, and $u_\alpha^\pm = u_\alpha(\pm\infty)$, then the limit operator S_0 is a perturbation of the free Schrödinger operator defined by $S_0\phi = -\phi''$ on functions ϕ in $W_2^2(\mathbb{R} \setminus \{0\})$ obeying the boundary conditions at the origin

$$\phi(+0) - \theta_\alpha\phi(-0) = 0, \quad \phi'(+0) - \theta_\alpha^{-1}\phi'(-0) = \beta\,\varkappa_\alpha\phi(-0). \tag{2.3}$$

The parameters θ_α and $\varkappa_\alpha$ are specified by the potentials Φ and Ψ

$$\theta_\alpha = \frac{u_\alpha^+}{u_\alpha^-}, \quad \varkappa_\alpha = \frac{1}{u_\alpha^- u_\alpha^+}\int_{\mathbb{R}} \Psi u_\alpha^2\, dt.$$

Otherwise, in the non-resonant case, the limit S_0 is equal to the direct sum $S_- \oplus S_+$ of the Dirichlet half-line Schrödinger operators $S_\pm$.

The result is proved in Theorems 4.1 and 5.1 below. In addition, convergence of the scattering data for S_ε to the ones for S_0 is established in Theorem 6.1.

In the resonant case, the point interaction generated by the coupling matrix

$$C(\Phi,\Psi) = \begin{pmatrix} \theta_\alpha & 0 \\ \beta\varkappa_\alpha & \theta_\alpha^{-1} \end{pmatrix}$$

in (1.1) may be regarded as a first approximation to the real interaction governed by the Hamiltonian S_ε. The explicit relation between θ_α, $\varkappa_\alpha$ and the potentials Φ, Ψ makes it possible to carry out a quantitative analysis of the quantum system, for instance, to compute approximate values of the scattering data for given Φ and Ψ.

It is appropriate to mention here that in [3, 31] the pseudo-Hamiltonian $-\frac{d^2}{dx^2}+\alpha\delta'+\beta\delta$ was interpreted as a point interaction with the matrix

$$C = \begin{pmatrix} \frac{2+\alpha}{2-\alpha} & 0 \\ \frac{4\beta}{(2-\alpha)^2} & \frac{2-\alpha}{2+\alpha} \end{pmatrix}$$

and some split boundary conditions were associated with the singular values $\alpha = \pm 2$. The solvable model was derived from the assumption that the following product formulae

$$v(x)\delta(x) = \{v\}_0\,\delta(x), \quad v(x)\delta'(x) = \{v\}_0\,\delta'(x) - \{v'\}_0\,\delta(x)$$

hold, where $\{f\}_0 = \frac{1}{2}(f(-0)+f(+0))$ is the mean value of a discontinuous function f at $x=0$. The spectrum and scattering properties of this model were described in [16, 18].

3. Resonant sets and maps

Since the potential V_ε has compact support shrinking to the origin, there is no loss of generality in supposing that the supports both of Φ and Ψ are subsets of the interval $I=[-1,1]$. Denote by $\mathcal{P}$ the class of real integrable and bounded functions of compact support contained in I.

Definition 3.1. The *resonant set* Λ_Φ of potential $\Phi \in \mathcal{P}$ is the set of all real value α for which the operator $-\frac{d^2}{ds^2}+\alpha\Phi$ in $L_2(\mathbb{R})$ possesses a half-bound state.

Suppose that a potential $q \in \mathcal{P}$ is resonant, i.e. q possesses a half-bound state u. Then u, as a solution to the equation $-u''+qu=0$, is constant for $|s|>1$, because q is a zero function outside I. Moreover, the restriction of u to I is a nontrivial solution of the problem $-u''+qu=0$, $s\in(-1,1)$ and $u'(-1)=0$, $u'(1)=0$. Hence, the potential q is resonant if and only if zero is an eigenvalue of the operator $N=-\frac{d^2}{ds^2}+q$ in $L_2(I)$ subject to the Neumann boundary conditions at $s=\pm1$.

Consequently the resonant set Λ_Φ coincides with the set of eigenvalues of the problem

$$-u''+\alpha\Phi u=0,\quad s\in(-1,1),\quad u'(-1)=0,\quad u'(1)=0 \tag{3.1}$$

with respect to the spectral parameter α. In the case of a positive Φ it is clear that Λ_Φ is a countable subset of $\mathbb{R}_+$ without finite accumulation points, and all eigenvalues of (3.1) are simple. Otherwise, (3.1) is a problem with indefinite weight function [14].

Assume that Φ has only isolated turning points in I. This case was considered in [19]. Then problem (3.1) can be associated with an operator in an appropriate Krein space. Let $\mathcal{K}_\Phi$ be the weight L_2-space with the scalar product $(f,g)=\int_{-1}^1 f\overline{g}\,|\Phi|ds$ and the indefinite inner product $[f,g]=(Jf,g)$, where $Jf=\operatorname{sgn}\Phi\cdot f$. The operator J is called the fundamental symmetry. We can introduce in $\mathcal{K}_\Phi$ the operator

$$T=-\frac{1}{\Phi(s)}\frac{d^2}{ds^2},\quad \operatorname{dom}T=\left\{g\in\mathcal{K}_\Phi\colon g\in W_2^2(I),\ \Phi^{-1}g''\in\mathcal{K}_\Phi,\ g'(-1)=0,\ g'(1)=0\right\}$$

that is J-self-adjoint and J-nonnegative. The spectrum of T is real and discrete, and has the two accumulation points $-\infty$ and $+\infty$. All nonzero eigenvalues are simple, and $\alpha=0$ is semi-simple, generically. The reader can refers to [25] for the details of the theory. Obviously, $\Lambda_\Phi=-\sigma(T)$. Hence, the resonant set Λ_Φ is discrete and unbounded in both directions.

Now suppose that the support of Φ is a disconnected subset of I. For the sake of simplicity, assume that it has only one gap: $\operatorname{supp}\Phi=[-1,s_1]\cup[s_2,1]$ and $s_1<s_2$. Each solution u of (3.1) is then a linear function on $[s_1,s_2]$. Therefore $u'(s_2)=u'(s_1)$ and $u(s_2)-u(s_1)=lu'(s_1)$, where $l=s_2-s_1$ is the length of gap. Let us move the interval $[-1,s_1]$ up to $[s_2,1]$, identify the points s_1 and s_2, and thereafter rewrite problem (3.1) as

$$\begin{gathered}-v''+\alpha\Upsilon v=0,\quad x\in(l-1,s_2)\cup(s_2,1),\quad v'(l-1)=0,\quad v'(1)=0,\\ v'(s_2+0)=v'(s_2-0),\quad v(s_2+0)-v(s_2-0)=lv'(s_2).\end{gathered} \tag{3.2}$$

The new "glued" potential Υ coincides with Φ on $[s_2,1]$, $\Upsilon(s)=\Phi(s-l)$ for $s\in[l-1,s_2)$; hence Υ has a unique turning point $s=s_2$. The relation between solutions of problems (3.1) and (3.2) is obviously given by

$$u(s)=\begin{cases}v(s+l) & \text{for } s\in[-1,s_1),\\ v'(s_2)(s-s_1)+v(s_2-0) & \text{for } s\in[s_1,s_2],\\ v(s) & \text{for } s\in(s_2,1].\end{cases}$$

As in the previous case, we can now construct an J-self-adjoint and J-nonnegative operator in $\mathcal{K}_\Upsilon$ associated with problem (3.2) and derive the same properties of the resonant set Λ_Φ. The similar considerations can be applied to Φ with several gaps on its support.

In conclusion of this section, we introduce two characteristics of the potentials Φ and Ψ, which will turn out to be important for us later. Let us denote by u_α the half-bound state that corresponds to resonant potential $\alpha\Phi$. Clearly, $u_\alpha(\pm\infty) = u_\alpha(\pm 1)$. Let θ be the map of Λ_Φ to $\mathbb{R}$ such that

$$\theta(\alpha) = \frac{u_\alpha(+1)}{u_\alpha(-1)} \tag{3.3}$$

for all $\alpha \in \Lambda_\Phi$. Next, let the map $\varkappa$ is given by

$$\varkappa(\alpha) = \frac{1}{u_\alpha(-1)u_\alpha(+1)} \int_{\mathbb{R}} \Psi u_\alpha^2\, ds, \quad \alpha \in \Lambda_\Phi. \tag{3.4}$$

The value $\varkappa(\alpha)$ describes the interaction of potentials Φ and Ψ at the resonant α. Since the half-bound state is unique up to a scalar factor, both maps are well defined.

Definition 3.2. We call $\theta\colon \Lambda_\Phi \to \mathbb{R}$ the resonant map of Φ, and $\varkappa\colon \Lambda_\Phi \to \mathbb{R}$ the intercoupling map for a pair of potentials Φ and Ψ.

4. Convergence of S_ε in resonance case

In this section, we analyze the more difficult resonant case where α is a point of the resonant set Λ_Φ. Here and subsequently, $\theta_\alpha = \theta(\alpha)$ and $\varkappa_\alpha = \varkappa(\alpha)$, where θ and $\varkappa$ are the resonant and intercoupling maps for a given pair of Φ and Ψ. Let u_α be the eigenfunction of (3.1) corresponding to $\alpha \in \Lambda_\Phi$ such that $u_\alpha(-1) = 1$. From (3.3) and (3.4) it follows that $\theta_\alpha = u_\alpha(1)$ and

$$\varkappa_\alpha = \theta_\alpha^{-1} \int_{-1}^{1} \Psi u_\alpha^2\, dt.$$

Denote by $S(\mu,\nu)$ the free Schrödinger operator on the line acting via $S(\mu,\nu)\phi = -\phi''$ on the domain

$$\operatorname{dom} S(\mu,\nu) = \bigl\{\phi \in W_2^2(\mathbb{R}\setminus\{0\})\colon \phi(+0) = \mu\phi(-0),\ \phi'(+0) = \mu^{-1}\phi'(-0) + \nu\phi(-0)\bigr\}.$$

For each real ν and $\mu \neq 0$ the operator $S(\mu,\nu)$ is self-adjoint.

Theorem 4.1. *Assume that $\Phi, \Psi \in \mathcal{P}$ and α belongs to the resonant set Λ_Φ. Then the operator family S_ε defined by* (2.1) *converges to $S(\theta_\alpha, \beta\varkappa_\alpha)$ as $\varepsilon \to 0$ in the norm resolvent sense.*

We have divided the proof into a sequence of propositions. Fix an arbitrary $f \in L_2(\mathbb{R})$ and $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \neq 0$. The basic idea of the proof is to construct a fair approximation to the function $y_\varepsilon = (S_\varepsilon - \zeta)^{-1} f$, uniformly for f in bounded subsets of $L_2(\mathbb{R})$.

In the sequel, letters C_j, c_j and b_j denote various positive constants independent of ε and f, whose values might be different in different proofs, and $\|f\|$ stands for the $L_2(\mathbb{R})$-norm of a function f. For abbreviation, we let S_0 stand for $S(\theta_\alpha, \beta\varkappa_\alpha)$.

Set $y = (S_0 - \zeta)^{-1} f$. We show that y is a very satisfactory approximation to y_ε for $|x| > \varepsilon$. The problem of choosing a close approximation to y_ε on the support of potential V_ε is more subtle. Denote by v_ε the solution of the Cauchy problem

$$\begin{cases} -v_\varepsilon'' + \alpha\Phi(s)v_\varepsilon = \varepsilon f(\varepsilon s) - \beta y(-\varepsilon)\Psi(s)u_\alpha(s), & s \in (-1,1),\\ \;\; v_\varepsilon(-1) = 0, \quad v_\varepsilon'(-1) = y'(-\varepsilon). \end{cases} \tag{4.1}$$

Clearly, we have $v_\varepsilon \in W_2^2(-1,1)$ for any $\varepsilon > 0$. The next proposition establishes some asymptotic properties of v_ε as $\varepsilon \to 0$.

Proposition 4.2. *The following holds for any $f \in L_2(\mathbb{R})$ and $\varepsilon \in (0,1)$:*

$$\|v_\varepsilon\|_{W_2^2(-1,1)} \le C_1\|f\|, \quad |v_\varepsilon'(1) - y'(+0)| \le C_2\varepsilon^{1/2}\|f\|.$$

Proof. We first observe that $(S_0 - \zeta)^{-1}$ is a bounded operator from $L_2(\mathbb{R})$ to the domain of S_0 equipped with the graph norm. The latter space is a subspace of $W_2^2(\mathbb{R}\setminus\{0\})$, and therefore

$$\|y\|_{W_2^2(\mathbb{R}\setminus\{0\})} \le c_1\|f\|, \quad \|y\|_{C^1(\mathbb{R}\setminus\{0\})} \le c_2\|f\|, \tag{4.2}$$

since $W_2^2(\mathbb{R}\setminus\{0\}) \subset C^1(\mathbb{R}\setminus\{0\})$ by the Sobolev embedding theorem. Hence,

$$\|v_\varepsilon\|_{W_2^2(-1,1)} \le c_3\big(|y(-\varepsilon)| + |y'(-\varepsilon)| + \varepsilon\|f(\varepsilon\,\cdot)\|_{L_2(-1,1)}\big) \le C_1\|f\|,$$

because there exists a constant c_4 such that

$$\|f(\varepsilon\cdot)\|_{L_2(-1,1)} \le c_4\varepsilon^{-1/2}\|f\|. \tag{4.3}$$

Next, multiplying equation (4.1) by the eigenfunction u_α and integrating by parts yield

$$v_\varepsilon'(1) = \theta_\alpha^{-1}y'(-\varepsilon) + \beta\varkappa_\alpha y(-\varepsilon) - \varepsilon\theta_\alpha^{-1}\int_{-1}^1 f(\varepsilon s)\,u_\alpha(s)\,ds. \tag{4.4}$$

Recall that the function y satisfies the condition $y'(+0) = \theta_\alpha^{-1}y'(-0) + \beta\varkappa_\alpha y(-0)$. Subtracting this equality from (4.4) we finally obtain

$$\begin{aligned}\big|v_\varepsilon'(1) - y'(+0)\big| &\le |\theta_\alpha^{-1}|\,|y'(-\varepsilon) - y'(-0)| + |\beta|\,|\varkappa_\alpha|\,\big|y(-\varepsilon) - y(-0)\big| \\ &\quad + \varepsilon|\theta_\alpha^{-1}|\,\|f(\varepsilon\,\cdot)\|_{L_2(-1,1)}\|u_\alpha\|_{L_2(-1,1)} \le C_2\varepsilon^{1/2}\|f\|\end{aligned}$$

in view of (4.3) and the following estimates

$$\big|y^{(k)}(\pm\varepsilon) - y^{(k)}(\pm 0)\big| \le \left|\int_0^{\pm\varepsilon}|y^{(k+1)}(x)|\,dx\right| \le c_6\varepsilon^{1/2}\|y\|_{W_2^2(\mathbb{R}\setminus\{0\})} \le c_7\varepsilon^{1/2}\|f\|, \tag{4.5}$$

holding for $k = 0, 1$. □

Let us introduce the function w_ε such that $w_\varepsilon(x) = y(x)$ for $|x| > \varepsilon$ and $w_\varepsilon(x) = y(-\varepsilon)u_\alpha(x/\varepsilon) + \varepsilon v_\varepsilon(x/\varepsilon)$ for $|x| \le \varepsilon$. By construction, w_ε belongs to $W_2^2(\mathbb{R}\setminus\{\varepsilon\})$. Although w_ε is in general discontinuous at the point $x = \varepsilon$, its jump and the jump of its first derivative at this point are small. Indeed, $[w_\varepsilon]_{x=\varepsilon} = y(\varepsilon) - \theta_\alpha y(-\varepsilon) - \varepsilon v_\varepsilon(1)$ and $[w_\varepsilon']_{x=\varepsilon} = y'(\varepsilon) - v_\varepsilon'(1)$, where $[h]_{x=a} = h(a+0) - h(a-0)$ is a jump of a function h at $x = a$. Therefore, taking into account Proposition 4.2, estimates (4.5) and the equality $y(0+) = \theta_\alpha y(0-)$, we see that the jumps can be bounded as

$$\begin{aligned}\big|[w_\varepsilon]_{x=\varepsilon}\big| &\le |y(\varepsilon) - y(+0)| + |\theta_\alpha||y(-\varepsilon) - y(-0)| + \varepsilon\|v_\varepsilon\|_{W_2^2(-1,1)} \le b_1\varepsilon^{1/2}\|f\|, \\ \big|[w_\varepsilon']_{x=\varepsilon}\big| &\le |y'(\varepsilon) - y'(+0)| + |y'(+0) - v_\varepsilon'(1)| \le b_2\varepsilon^{1/2}\|f\|.\end{aligned} \tag{4.6}$$

Let us introduce functions η_0 and η_1 that are smooth outside the origin, have compact supports contained in $[0,1]$, and have the prescribed jumps $[\eta_0]_{x=0} = 1$, $[\eta_0']_{x=0} = 0$ and

FIGURE 1. Plots of the functions with the prescribed jumps at the origin

$[\eta_1]_{x=0} = 0$, $[\eta_1']_{x=0} = 1$ (see Fig. 1). Set $z_\varepsilon(x) = [w_\varepsilon]_{x=\varepsilon}\,\eta_0(x-\varepsilon) + [w_\varepsilon']_{x=\varepsilon}\,\eta_1(x-\varepsilon)$; then in view of (4.6)

$$\max_{x\geq\varepsilon} |z_\varepsilon^{(k)}(x)| \leq b_3\varepsilon^{1/2}\|f\| \tag{4.7}$$

for some b_3 and $k = 0, 1, 2$. Additionally, $z_\varepsilon = 0$ on $(-\infty, \varepsilon)$.

Clearly, the function

$$\tilde{y}_\varepsilon(x) = \begin{cases} y(x) - z_\varepsilon(x) & \text{if } |x| > \varepsilon, \\ y(-\varepsilon)u_\alpha(\frac{x}{\varepsilon}) + \varepsilon v_\varepsilon(\frac{x}{\varepsilon}) & \text{if } |x| \leq \varepsilon \end{cases} \tag{4.8}$$

is continuous on $\mathbb{R}$ along with its derivative and belongs to $\operatorname{dom} S_\varepsilon$.

Proposition 4.3. *Fix $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \neq 0$. Then the estimates*

$$\|y_\varepsilon - \tilde{y}_\varepsilon\| \leq C_1\varepsilon^{1/2}\|\|f\|, \quad \|\tilde{y}_\varepsilon - y\| \leq C_2\varepsilon^{1/2}\|f\| \tag{4.9}$$

hold for each $f \in L_2(\mathbb{R})$ and $\varepsilon > 0$, where $y_\varepsilon = (S_\varepsilon - \zeta)^{-1}f$ and $y = (S_0 - \zeta)^{-1}f$.

Proof. It is convenient now to rewrite the approximation $\tilde{y}_\varepsilon$ in the form

$$\tilde{y}_\varepsilon(x) = (1 - \chi_\varepsilon(x))y(x) + y(-\varepsilon)u_\alpha(\tfrac{x}{\varepsilon}) + \varepsilon v_\varepsilon(\tfrac{x}{\varepsilon}) - z_\varepsilon(x),$$

where χ_ε is the characteristic function of $[-\varepsilon, \varepsilon]$, and u_α and v_ε are extended by zero to the whole line. Recalling the definition of y, u_α and v_ε, we deduce

$$(S_\varepsilon - \zeta)\tilde{y}_\varepsilon(x) = \big(-\tfrac{d^2}{dx^2} - \zeta\big)(y(x) - z_\varepsilon(x)) = f(x) + z_\varepsilon''(x) + \zeta z_\varepsilon(x)$$

for $|x| > \varepsilon$, and

$$\begin{aligned}(S_\varepsilon - \zeta)\tilde{y}_\varepsilon(x) &= \Big(-\tfrac{d^2}{dx^2} + \alpha\varepsilon^{-2}\Phi(\varepsilon^{-1}x) + \beta\varepsilon^{-1}\Psi(\varepsilon^{-1}x) - \zeta\Big)\Big(y(-\varepsilon)u_\alpha(\tfrac{x}{\varepsilon}) + \varepsilon v_\varepsilon(\tfrac{x}{\varepsilon})\Big) \\ &= \varepsilon^{-2}y(-\varepsilon)\Big\{-u_\alpha'' + \alpha\Phi(\tfrac{x}{\varepsilon})u_\alpha\Big\} + \varepsilon^{-1}\Big\{-v_\varepsilon'' + \alpha\Phi(\tfrac{x}{\varepsilon})v_\varepsilon + \beta y(-\varepsilon)\Psi(\tfrac{x}{\varepsilon})u_\alpha\Big\} \\ &\quad + \beta\Psi(\tfrac{x}{\varepsilon})v_\varepsilon(\tfrac{x}{\varepsilon}) - \zeta\tilde{y}_\varepsilon(x) = f(x) + \big(\beta\Psi(\tfrac{x}{\varepsilon}) - \zeta\big)v_\varepsilon(\tfrac{x}{\varepsilon}) - \zeta y(-\varepsilon)u_\alpha(\tfrac{x}{\varepsilon})\end{aligned}$$

for $|x| \leq \varepsilon$. Therefore $(S_\varepsilon - \zeta)\tilde{y}_\varepsilon = f + r_\varepsilon$, where

$$r_\varepsilon(x) = \begin{cases} z_\varepsilon''(x) + \zeta z_\varepsilon(x) & \text{if } |x| > \varepsilon, \\ \big(\beta\Psi(\frac{x}{\varepsilon}) - \varepsilon\zeta\big)v_\varepsilon(\frac{x}{\varepsilon}) - \zeta y(-\varepsilon)u_\alpha(\frac{x}{\varepsilon}) & \text{if } |x| \leq \varepsilon. \end{cases}$$

Hence $\tilde{y}_\varepsilon - y_\varepsilon = (S_\varepsilon - \zeta)^{-1}r_\varepsilon$, and from this we conclude

$$\|y_\varepsilon - \tilde{y}_\varepsilon\| \leq \|(S_\varepsilon - \zeta)^{-1}\|\,\|r_\varepsilon\| \leq |\operatorname{Im}\zeta|^{-1}\|r_\varepsilon\|.$$

We can now employ Proposition 4.2 and estimates (4.2), (4.7) to derive the bound

$$\begin{aligned}\|r_\varepsilon\| &\leq c_1\|z_\varepsilon'' + \zeta z_\varepsilon\|_{L_2(\varepsilon, 1+\varepsilon)} + c_2\|v_\varepsilon(\varepsilon^{-1}\,\cdot\,)\|_{L_2(-\varepsilon,\varepsilon)} + c_3|y(-\varepsilon)|\,\|u_\alpha(\varepsilon^{-1}\,\cdot\,)\|_{L_2(-\varepsilon,\varepsilon)} \\ &\leq c_4\max_{x\geq\varepsilon}(|z_\varepsilon| + |z_\varepsilon''|) + c_5\varepsilon^{1/2}\big(\|v_\varepsilon\|_{L_2(-1,1)} + \|y\|_{C(\mathbb{R}\setminus\{0\})}\|u_\alpha\|_{L_2(-1,1)}\big) \\ &\leq c_6\varepsilon^{1/2}\|f\|.\end{aligned}$$

This proves the first inequality in (4.9). Similarly,

$$\begin{aligned}\|\tilde{y}_\varepsilon - y\| &= \|y(-\varepsilon)u_\alpha(\varepsilon^{-1}\,\cdot\,) + \varepsilon v_\varepsilon(\varepsilon^{-1}\,\cdot\,) - z_\varepsilon - \chi_\varepsilon y\| \leq c_7\varepsilon^{1/2}|y(-\varepsilon)|\|u_\alpha\|_{L_2(-1,1)} \\ &\quad + c_8\varepsilon^{3/2}\|v_\varepsilon\|_{L_2(-1,1)} + c_9\max_{x\geq\varepsilon}|z_\varepsilon| + c_{10}\|y\|_{C(\mathbb{R}\setminus\{0\})}\|\chi_\varepsilon\| \leq c_{11}\varepsilon^{1/2}\|f\|,\end{aligned}$$

and so finish the proof. ☐

Proof of Theorem 4.1. For each $f \in L_2(\mathbb{R})$ and $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \neq 0$ we can construct the approximation $\tilde{y}_\varepsilon$ to $y_\varepsilon = (S_\varepsilon - \zeta)^{-1}f$ given by (4.8). As above, set $y = (S_0 - \zeta)^{-1}f$. Applying Proposition 4.3, we discover

$$\|(S_\varepsilon - \zeta)^{-1}f - (S_0 - \zeta)^{-1}f\| = \|y_\varepsilon - y\| \leq \|\tilde{y}_\varepsilon - y_\varepsilon\| + \|\tilde{y}_\varepsilon - y\| \leq C\varepsilon^{1/2}\|f\|,$$

which establishes the norm resolvent convergence of S_ε to the operator $S(\theta_\alpha, \beta\varkappa_\alpha)$. ☐

5. Convergence of S_ε in non-resonance case

Now we study the non-resonant case when α does not belong to the resonant set Λ_Φ. Denote by S_0 the direct sum $S_- \oplus S_+$ of the unperturbed half-line Schrödinger operators $S_\pm = -d^2/dx^2$ on $\mathbb{R}_\pm$ subject to the Dirichlet boundary condition at $x = 0$. Hence $\operatorname{dom} S_0 = \{y \in W_2^2(\mathbb{R} \setminus \{0\})\colon y(-0) = y(+0) = 0\}$.

Theorem 5.1. *If $\alpha \notin \Lambda_\Phi$, then the operator family S_ε given by* (2.1) *converges to the direct sum $S_- \oplus S_+$, as $\varepsilon \to 0$, in the norm resolvent sense.*

Proof. Exactly the same considerations, as in the previous section, apply here, with one important difference: the function $y = (S_0 - \zeta)f$ is small in a neighborhood of the origin, since $y(0) = 0$; and we have to change the approximation to $y_\varepsilon = (S_\varepsilon - \zeta)^{-1} f$ on the support of V_ε. We set $\tilde{y}_\varepsilon(x) = y(x) - z_\varepsilon(x)$ for $|x| > \varepsilon$ and $\tilde{y}_\varepsilon(x) = \varepsilon v_\varepsilon(\frac{x}{\varepsilon})$ for $|x| \le \varepsilon$, where v_ε is a solution to the boundary value problem

$$-v_\varepsilon'' + \alpha\Phi(s)v_\varepsilon = \varepsilon f(\varepsilon s), \quad s \in (-1,1), \quad v_\varepsilon'(-1) = y'(-\varepsilon), \quad v_\varepsilon'(1) = y'(\varepsilon), \tag{5.1}$$

and $z_\varepsilon(x) = \big(\varepsilon v_\varepsilon(-1) - y(-\varepsilon)\big)\,\eta_0(-x-\varepsilon) + \big(y(\varepsilon) - \varepsilon v_\varepsilon(1)\big)\,\eta_0(x-\varepsilon)$. Note that the solution v_ε of (5.1) exists, because in the non-resonance case the number α is not an eigenvalue of the corresponding homogeneous problem. Moreover,

$$\|v_\varepsilon\|_{W_2^2(-1,1)} \le c_1\big(|y'(-\varepsilon)| + |y'(\varepsilon)| + \varepsilon\|f(\varepsilon\,\cdot)\|_{L_2(-1,1)}\big) \le c_2\|f\|,$$

due to (4.3) and the apparent estimates $\|y\|_{C^1(\mathbb{R}\setminus\{0\})} \le c_3\|y\|_{W_2^2(\mathbb{R}\setminus\{0\})} \le c_4\|f\|$.

Recalling that $y(0) = 0$, we deduce from (4.5) that $|y(\pm\varepsilon)| \le c_5\varepsilon\|f\|$. Therefore the corrector z_ε can be bounded as

$$\max_{|x|\ge\varepsilon} |z_\varepsilon^{(k)}(x)| \le c_6\varepsilon\|f\|$$

for some c_6 and $k = 0,1,2$, since $|y(\pm\varepsilon) - \varepsilon v_\varepsilon(\pm1)| \le |y(\pm\varepsilon)| + \varepsilon|v_\varepsilon(\pm1)| \le c_7\varepsilon\|f\|$.

Next, an easy computation shows that $(S_\varepsilon - \zeta)\tilde{y}_\varepsilon = f + r_\varepsilon$, where

$$r_\varepsilon(x) = \begin{cases} z_\varepsilon''(x) + \zeta z_\varepsilon(x) & \text{if } |x| > \varepsilon, \\ \big(\beta\Psi(\frac{x}{\varepsilon}) - \varepsilon\zeta\big)v_\varepsilon(\frac{x}{\varepsilon}) & \text{if } |x| \le \varepsilon. \end{cases}$$

As in Proposition 4.3, from the estimates above we can derive $\|r_\varepsilon\| \le c_8\varepsilon^{1/2}\|f\|$. Hence,

$$\begin{aligned}\|(S_\varepsilon - \zeta)^{-1}f - (S_0 - \zeta)^{-1}f\| &\le \|\tilde{y}_\varepsilon - y_\varepsilon\| + \|\tilde{y}_\varepsilon - y\| \\ &\le \|(S_\varepsilon - \zeta)^{-1}r_\varepsilon\| + \|\varepsilon v_\varepsilon(\varepsilon^{-1}\,\cdot) - z_\varepsilon - \chi_\varepsilon y\| \le C\varepsilon^{1/2}\|f\|,\end{aligned}$$

which establishes the norm resolvent convergence of S_ε to the operator $S_- \oplus S_+$. □

6. Scattering on $\alpha\delta' + \beta\delta$-like potentials

6.1. **Scattering problem for S_0.** First, let us discuss stationary scattering associated with the Hamiltonians $S(\theta_\alpha, \beta\varkappa_\alpha)$ and $-\frac{d^2}{dx^2}$, which corresponds to the resonant case. Consider the incoming monochromatic wave $e^{\mathrm{i}kx}$ with $k > 0$ coming from the left. Then the corresponding wave function has the form $\psi(x,k) = e^{\mathrm{i}kx} + R\,e^{-\mathrm{i}kx}$ for $x < 0$ and $\psi(x,k) = T\,e^{\mathrm{i}kx}$ for $x > 0$. Here R and T are respectively the reflection and transmission coefficients. The matching conditions (2.3) at the origin clearly yield

$$\begin{pmatrix} T \\ \mathrm{i}kT \end{pmatrix} = \begin{pmatrix} \theta_\alpha & 0 \\ \beta\varkappa_\alpha & \theta_\alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 + R \\ \mathrm{i}k(1 - R) \end{pmatrix}.$$

Then one obtains the reflection and transmission coefficients from the left, cf. [18, Eq 23]

$$R(k,\alpha) = \frac{\mathrm{i}k(\theta_\alpha^{-1} - \theta_\alpha) + \beta\varkappa_\alpha}{\mathrm{i}k(\theta_\alpha^{-1} + \theta_\alpha) - \beta\varkappa_\alpha}, \quad T(k,\alpha) = \frac{2\mathrm{i}k}{\mathrm{i}k(\theta_\alpha^{-1} + \theta_\alpha) - \beta\varkappa_\alpha}, \quad \alpha \in \Lambda_\Phi. \tag{6.1}$$

In the non-resonant case the scattering problem is trivial. The split condition $y(0) = 0$ leads to the equalities $R(k,\alpha) = -1$ and $T(k,\alpha) = 0$.

6.2. **Convergence of the scattering data.** Next we investigate the stationary scattering for $S_\varepsilon(\alpha,\beta,\Phi,\Psi)$ and $-\frac{d^2}{dx^2}$, and prove that the scattering data converge as $\varepsilon\to 0$ to the scattering data for S_0 obtained above. We look for the positive-energy solution to the equation $-\psi''+V_\varepsilon\psi=k^2\psi$ given in the form

$$\psi_\varepsilon(x,k,\alpha)=\begin{cases} e^{ikx}+R\,e^{-ikx} & \text{for } x<-\varepsilon,\\ A\,u_\varepsilon(\varepsilon^{-1}x,\alpha)+B\,v_\varepsilon(\varepsilon^{-1}x,\alpha) & \text{for } |x|<\varepsilon,\\ \qquad T\,e^{ikx} & \text{for } x>\varepsilon.\end{cases}$$

Here $u_\varepsilon=u_\varepsilon(s,\alpha)$ and $v_\varepsilon=v_\varepsilon(s,\alpha)$ are solutions of the equation

$$-w''+\alpha\Phi(s)w+\beta\varepsilon\Psi(s)w=\varepsilon^2k^2w,\quad s\in(-1,1) \tag{6.2}$$

subject to the initial conditions

$$u_\varepsilon(-1,\alpha)=1,\quad u_\varepsilon'(-1,\alpha)=0\quad\text{and}\quad v_\varepsilon(-1,\alpha)=0,\quad v_\varepsilon'(-1,\alpha)=1 \tag{6.3}$$

respectively. The coefficients R, A, B, and T can be found from the linear system

$$\begin{pmatrix} -e^{\mathrm{i}\varepsilon k} & 1 & 0 & 0\\ \mathrm{i}\varepsilon k e^{\mathrm{i}\varepsilon k} & 0 & 1 & 0\\ 0 & u_\varepsilon(1,\alpha) & v_\varepsilon(1,\alpha) & -e^{\mathrm{i}\varepsilon k}\\ 0 & u_\varepsilon'(1,\alpha) & v_\varepsilon'(1,\alpha) & -\mathrm{i}\varepsilon k e^{\mathrm{i}\varepsilon k}\end{pmatrix}\begin{pmatrix} R\\ A\\ B\\ T\end{pmatrix}=\begin{pmatrix} e^{-\mathrm{i}\varepsilon k}\\ \mathrm{i}\varepsilon k e^{-\mathrm{i}\varepsilon k}\\ 0\\ 0\end{pmatrix}$$

obtained by matching the solution and its first derivative at the points $x=\pm\varepsilon$. By Cramer's rule, we can derive

$$\begin{aligned} R_\varepsilon(k,\alpha)&=-e^{-2\mathrm{i}\varepsilon k}\,\frac{u_\varepsilon'(1,\alpha)-\mathrm{i}\varepsilon k\big(u_\varepsilon(1,\alpha)-v_\varepsilon'(1,\alpha)\big)+\varepsilon^2k^2v_\varepsilon(1,\alpha)}{u_\varepsilon'(1,\alpha)-\mathrm{i}\varepsilon k\big(u_\varepsilon(1,\alpha)+v_\varepsilon'(1,\alpha)\big)-\varepsilon^2k^2v_\varepsilon(1,\alpha)},\\ T_\varepsilon(k,\alpha)&=-e^{-2\mathrm{i}\varepsilon k}\,\frac{2\mathrm{i}\varepsilon k}{u_\varepsilon'(1,\alpha)-\mathrm{i}\varepsilon k\big(u_\varepsilon(1,\alpha)+v_\varepsilon'(1,\alpha)\big)-\varepsilon^2k^2v_\varepsilon(1,\alpha)}.\end{aligned} \tag{6.4}$$

Here we use the identity $u_\varepsilon(1,\alpha)v_\varepsilon'(1,\alpha)-u_\varepsilon'(1,\alpha)v_\varepsilon(1,\alpha)=1$ that follows from (6.3) and the constancy in ξ and ε of the Wronskian of u_ε and v_ε.

Theorem 6.1. *For each $k>0$ and $\alpha\in\mathbb{R}$ the scattering data $R_\varepsilon(k,\alpha)$ and $T_\varepsilon(k,\alpha)$ converge to $R(k,\alpha)$ and $T(k,\alpha)$ as $\varepsilon\to0$ respectively, where the limit values are given by* (6.1) *in the resonant case, and $R(k,\alpha)=-1$, $T(k,\alpha)=0$ otherwise.*

Proof. From the smooth dependence of a solution to the Cauchy problem on parameters, we see that u_ε and v_ε converge in $C^1(-1,1)$ to the solutions u and v respectively of the equation $-w''+\alpha\Phi w=0$ subject to the initial conditions

$$u(-1,\alpha)=1,\quad u'(-1,\alpha)=0\quad\text{and}\quad v(-1,\alpha)=0,\quad v'(-1,\alpha)=1.$$

The non-resonant case. Since α is not a eigenvalue of problem (3.1), we conclude that $u'(1,\alpha)$ is different from 0. From (6.4), it immediately follows that $R_\varepsilon(k,\alpha)=-1+O(\varepsilon)$ and $T_\varepsilon(k,\alpha)=O(\varepsilon)$ as $\varepsilon\to0$.

The resonant case. If α is resonant, then u is an eigenfunction of (3.1). Therefore $u'(1,\alpha)=0$ and $u(1,\alpha)=\theta_\alpha$. Throughout the proof, θ and $\varkappa$ denote the resonant and intercoupling maps for a pair (Φ,Ψ). Let us substitute functions u_ε and v_ε into (6.2) alternately. Multiplying the derived identities by u and integrating by parts yield

$$\begin{aligned}\theta_\alpha u_\varepsilon'(1,\alpha)&=\varepsilon\beta\int_{-1}^1\Psi u_\varepsilon u\,d\xi+\varepsilon^2k^2\int_{-1}^1 u_\varepsilon u\,d\xi,\\ \theta_\alpha v_\varepsilon'(1,\alpha)&=1+\varepsilon\beta\int_{-1}^1\Psi v_\varepsilon u\,d\xi+\varepsilon^2k^2\int_{-1}^1 v_\varepsilon u\,d\xi.\end{aligned} \tag{6.5}$$

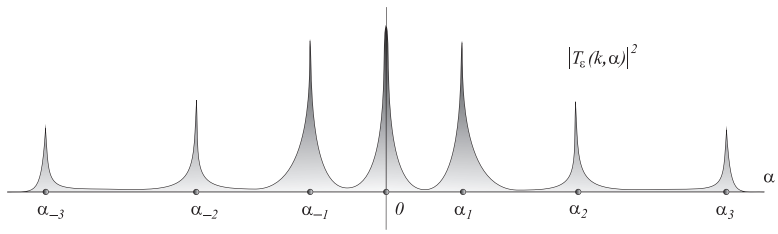

FIGURE 2. A plot of transmission probability $|T_\varepsilon(k,\alpha)|^2$ as function of α.

Therefore $u'_\varepsilon(1,\alpha) = \varepsilon\beta\varkappa_\alpha + o(\varepsilon)$ and $v'_\varepsilon(1,\alpha) = \theta_\alpha^{-1} + O(\varepsilon)$ as $\varepsilon \to 0$. Combining then these asymptotic formulae and (6.4), we have

$$R_\varepsilon(k,\alpha) = \frac{\mathrm{i}k(\theta_\alpha^{-1} - \theta_\alpha) + \beta\varkappa_\alpha}{\mathrm{i}k(\theta_\alpha^{-1} + \theta_\alpha) - \beta\varkappa_\alpha} + o(1), \quad T_\varepsilon(k,\alpha) = \frac{2\mathrm{i}k}{\mathrm{i}k(\theta_\alpha^{-1} + \theta_\alpha) - \beta\varkappa_\alpha} + o(1), \tag{6.6}$$

as $\varepsilon \to 0$, which completes the proof. □

6.3. **Resonances in the transmission probability.** It follows from the theorem above that the probability of transmission across the barrier $V_\varepsilon = \alpha\varepsilon^{-2}\Phi(\varepsilon^{-1}\cdot) + \beta\varepsilon^{-1}\Psi(\varepsilon^{-1}\cdot)$ is negligibly small for $\alpha \notin \Lambda_\Phi$. However, for the resonant coupling constants α this probability remains non-zero as $\varepsilon \to 0$, resulting in the existence of non-separated states. The resonances of transmission, as shown in Fig. 2, are sharp like one-point spikes, which spread for non-zero values as $\varepsilon \to 0$. The limit resonant values can be represented via the maps θ and $\varkappa$:

$$|T(k,\alpha)|^2 = \frac{4k^2}{k^2(\theta_\alpha^{-1} + \theta_\alpha)^2 + \beta^2\varkappa_\alpha^2}, \quad \alpha \in \Lambda_\Phi. \tag{6.7}$$

Several special cases are of interest.

6.3.1. *δ-potentials.* Set $\alpha = 0$, then $V_\varepsilon = \beta\varepsilon^{-1}\Psi(\varepsilon^{-1}\cdot)$. Assume also that $\int_{\mathbb{R}} \Psi\, dt = 1$. Consequently V_ε converges to $\beta\delta(x)$ in the sense of distributions. Note that the value $\alpha = 0$ belongs to Λ_Φ for each Φ. The corresponding eigenfunction u_0 of (3.1) is constant on $[-1,1]$, and both numbers θ_0 and $\varkappa_0$ are equal to 1 (see (3.3), (3.4)). Hence, the limit operator $S(1,\beta)$ describes the point interaction at the origin with the coupling matrix given by (1.2). In addition, from (6.7) we have $|T(k,0)|^2 = \frac{4k^2}{4k^2+\beta^2}$, which is common knowledge of the scattering by the $\beta\delta(x)$-potential.

6.3.2. *δ′-like potentials.* Now we set $\beta = 0$. Then $V_\varepsilon = \alpha\varepsilon^{-2}\Phi(\varepsilon^{-1}\cdot)$. If $\int_{\mathbb{R}} \Phi\, dt = 0$, then

$$V_\varepsilon(x) \to -\left(\int_{\mathbb{R}} s\Phi(s)\, ds\right)\delta'(x) \quad \text{in } \mathcal{D}'(\mathbb{R}),$$

and so in particular the limit can be zero. Otherwise, the family of potentials V_ε diverges in the sense of distributions. Without reference to the weak convergence of V_ε the transmission probability $|T_\varepsilon(k,\alpha)|^2$ for each $\alpha \in \Lambda_\Phi$ converges to the value

$$|T(k,\alpha)|^2 = \frac{4}{(\theta_\alpha^{-1} + \theta_\alpha)^2} \tag{6.8}$$

that does not depend on energy of particles (cf. [20, Sec. 5]). The limit operator $S(\alpha,0)$ corresponds to the point interaction at the origin with the coupling matrix given by (1.3), $\theta = \theta_\alpha$. As opposed to the case of the δ-potential, this result is shape-dependent via the resonant map θ.

Our analysis of the exactly solvable models with piecewise-constant potentials and the computer simulation of more complicated models may suggest that if $\varepsilon^{-2}\Phi(\varepsilon^{-1}\cdot) \to \delta'(x)$ as $\varepsilon \to 0$, then $|\theta_\alpha| \to +\infty$ as $\alpha \to +\infty$ and $|\theta_\alpha| \to 0$ as $\alpha \to -\infty$. Therefore the transmission probability for the δ'-like potentials given by (6.8) is very small for large $|\alpha|$. As we will see in the next special case, it is not true in general.

It is noteworthy that the operator $S(\alpha,0)$ can also appear as a solvable model for the Schrödinger operator with the potential V_ε when β is different from 0. Indeed, for some resonant values of α it is possible for potentials Φ and Ψ to be uncoupled, that is to say, $\varkappa_\alpha = 0$. If, for instance, a potential Ψ is sign-changing, then it is possible for the integral $\int_{-1}^{1}\Psi u_\alpha^2\,ds$ to be zero.

6.3.3. *Potentials with total transparency at resonances.* An interesting case occurs when for a pair of potentials Φ and Ψ the intercoupling map $\varkappa$ is identically zero and the resonant map θ is unimodular, $|\theta| = 1$. Then the marginal transmission probability across the potential V_ε is given by

$$|T(k,0)|^2 = \begin{cases} 1 & \text{if } \alpha \in \Lambda_\Phi, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, either the potential is asymptotically opaque for particles or else asymptotically totally transparent at resonances. For instance, this kind of case arises when Φ is an even function, whereas Ψ is an odd one. Then each eigenfunction u_α of (3.1) is either odd or even. In any case, $|\theta_\alpha| = 1$ and $\varkappa_\alpha = \theta_\alpha^{-1}\int_{-1}^{1}\Psi u_\alpha^2\,ds = 0$, since the square of u_α is an even function.

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Department of Mechanics and Mathematics, Ivan Franko National University of L'viv, 1 Universytets'ka, L'viv, 79000, Ukraine

E-mail address: yu_holovaty@franko.lviv.ua

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