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CONTROLLED FUSION FRAMES

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Abstract. We use two appropriate bounded invertible operators to define a controlled fusion frame with optimal fusion frame bounds to improve the numerical efficiency of iterative algorithms for inverting the fusion frame operator. We show that controlled fusion frames as a generalization of fusion frames give a generalized way to obtain numerical advantage in the sense of preconditioning to check the fusion frame condition. Also, we consider locally controlled frames for each locally space to obtain new globally controlled frames for our Hilbert space. We develop some well known results in fusion frames to the controlled fusion frames case.

1. Introduction

Fusion frames formally were introduced in [5], studied in [10] and generalized in [9] which is a weighted and distributed processing procedure that fuse together information in all subspaces of a Hilbert space H in a fusion frame system to obtain the global information in H.

Controlled frames for spherical wavelets were introduced in [3] to get a numerically more efficient approximation algorithm and the related theory for general frames was developed in [2]. For getting a numerical solution of a linear system of equations $Ax = b$, we can solve the system of equations $PAx = Pb$, where P is a suitable preconditioning matrix to get a better iterative algorithm, which was the main motivation for introducing controlled frames in [3]. We generalize this concept to the case of fusion frames. Although controlled fusion frames and fusion frames are mathematically equivalent, this different view-point of fusion frames, as stated in [2], gives opportunities for efficient implementations.

Furthermore, we give another reconstruction formula by using the controlled fusion frame operator for a pair of controlled Bessel fusion sequences. We construct new controlled fusion frames from a given one and we obtain some useful results about them.

Throughout this paper H is a separable Hilbert space, and $GL(H)$ denotes the set of all bounded linear operators which have bounded inverses. It is easy to see that if $S, T \in GL(H)$, then T^*, T^{-1} and ST are also in $GL(H)$. Let $GL^+(H)$ be the set of all positive operators in $GL(H)$.

A sequence $(f_i)_{i\in I}$ in H is called a *frame* for H, if there exist constants $0 < C \le D < \infty$ (*lower and upper frame bounds*) such that

$$C\|f\|^2 \le \sum_{i\in I} | < f, f_i > |^2 \le D\|f\|^2, \quad \forall f \in H.$$

If $C = D$, then $(f_i)_{i\in I}$ is called a *C-tight frame*, and if $C = D = 1$, it is called a *Parseval frame*. A *Bessel sequence* $(f_i)_{i\in I}$ is only required to fulfill the upper frame bound estimate but not necessarily the lower estimate.

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The *frame operator* $Sf = \sum_{i\in I} < f, f_i > f_i$ associated with a frame $(f_i)_{i\in I}$ is a bounded, invertible, and positive operator on H. This provides the reconstruction formulas

$$f = S^{-1}Sf = \sum_{i\in I} < f, f_i > S^{-1}f_i = \sum_{i\in I} < f, S^{-1}f_i > f_i.$$

Furthermore, $CId_H \leq S \leq DId_H$. See [7].

Let $T \in GL(H)$. A *frame controlled by the operator* T or T-*controlled frame* is a family of vectors $\{f_i\}_{i\in I}$ in H, such that there exist constants $0 < C_T \leq D_T < \infty$, verifying

$$C_T\|f\|^2 \leq \sum_{i\in I} < f, f_i >< Tf_i, f > \leq D_T\|f\|^2, \quad \forall f \in H.$$

The *controlled frame operator* S_T is defined by

$$S_Tf = \sum_{i\in I} < f, f_i > Tf_i.$$

Definition 1.1. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H. Let $\{\upsilon_i\}_{i\in I}$ be a family of weights, i.e., $\upsilon_i > 0$ for all $i \in I$. Then $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ is called a *fusion frame*, if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} \upsilon_i^2\|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \quad \forall f \in H,$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. We call C and D lower and upper fusion frame bounds, respectively. We call $\mathcal{W}$ a C-*tight fusion frame* if $C = D$ and a *Parseval fusion frame* if $C = D = 1$. If only the second inequality is required, we call $\mathcal{W}$ a *Bessel fusion sequence*. Moreover, we call it υ-*uniform* if $\upsilon = \upsilon_i$ for all $i \in I$. As we have in [5] the fusion frame operator $S_{\mathcal{W}}$ is defined by

$$S_{\mathcal{W}}(f) = \sum_{i\in I} \upsilon_i^2 \pi_{W_i} f, \quad \forall f \in H.$$

$S_{\mathcal{W}}$ is a positive and invertible operator and we have

$$CId_H \leq S_{\mathcal{W}} \leq DId_H,$$

hence

$$\frac{1}{D}Id_H \leq S_{\mathcal{W}}^{-1} \leq \frac{1}{C}Id_H.$$

Let $\{(W_i, \upsilon_i)\}_{i\in I}$ be a fusion frame for H, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for W_i for each $i \in I$. Then $\{(W_i, \upsilon_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is called a *fusion frame system* for H. C and D are called *associated fusion frame bounds*, if they are the fusion frame bounds for $\{(W_i, \upsilon_i)\}_{i\in I}$.

For convenience we state the following lemmas.

Lemma 1.2. [11]. *Let $\mathcal{A}$ be a C^*-algebra. If $a, b \in \mathcal{A}^+$, then*

(*i*) $a + b \in \mathcal{A}^+$*;*

(*ii*) $ab \in \mathcal{A}^+$ *if and only if* $ab = ba$.

Lemma 1.3. [5]. *Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H. Let $\{\upsilon_i\}_{i\in I}$ be a family of weights, and let $S_{\mathcal{W}}(f) = \sum_{i\in I} \upsilon_i^2\pi_{W_i} f$. Then the following conditions are equivalent:*

(*i*) $\{(W_i, \upsilon_i)\}_{i\in I}$ *is a (C, D)-fusion frame for H;*

(*ii*) *We have* $CId_H \leq S_{\mathcal{W}} \leq DId_H$.

Lemma 1.4. [2]. *Let $T : H \to H$ be a linear operator. Then the following statements are equivalent:*

(*i*) $CId_H \leq T \leq DId_H$*, for some* $0 < C \leq D < \infty$ *;*

(ii) T is positive and $C\|f\|^2 \leq \left\|T^{\frac{1}{2}}f\right\|^2 \leq D\|f\|^2$, for some $0 < C \leq D < \infty$;
(iii) $T \in GL^+(H)$.

The following well known result describes the relation between fusion frames, associated local frames and global frames for a Hilbert space.

Proposition 1.5. [4]. *For each $i \in I$, let W_i be a closed subspace of H, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for W_i with frame bounds A_i and B_i. Suppose that*

$$0 < A = \inf_{i\in I} A_i \leq \sup_{i\in I} B_i = B < \infty.$$

Then the following conditions are equivalent:
(i) $\{(W_i, \upsilon_i)\}_{i\in I}$ is a fusion frame for H;
(ii) $\{\upsilon_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for H.
In particular, if $\{(W_i, \upsilon_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is a fusion frame system for H with fusion frame bounds C and D, then $\{\upsilon_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for H with frame bounds AC and BD. Also if $\{\upsilon_i f_{ij}\}_{j\in J_i, i\in I}$ is a frame for H with frame bounds C and D, then $\{(W_i, \upsilon_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is a fusion frame system for H with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

2. Controlled fusion frames

In this section we introduce controlled fusion frames and we show that they are generalizations of fusion frames.

Definition 2.1. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H. Let $\{\upsilon_i\}_{i\in I}$ be a family of weights, and let $T, U \in GL(H)$. Then $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ is called a fusion frame controlled by T and U or (T, U)-*controlled fusion frame* if there exist two constants

$$0 < C_{TU} \leq D_{TU} < \infty$$

such that

$$C_{TU}\|f\|^2 \leq \sum_{i\in I} \upsilon_i^2 < \pi_{W_i} Uf, \pi_{W_i} Tf > \leq D_{TU}\|f\|^2, \quad \forall f \in H.$$

We call $\mathcal{W}$ a (T, U)-*controlled Parseval fusion frame* if $C_{TU} = D_{TU} = 1$. If only the second inequality is required, we call $\mathcal{W}$ a (T, U)-*controlled Bessel fusion sequence.* Moreover, we call it υ-*uniform* if $\upsilon = \upsilon_i$ for all $i \in I$.

If $\mathcal{W}$ is a (T, U)-controlled fusion frame and $T^*\pi_{W_i}U$ is a positive operator for each $i \in I$, then $T^*\pi_{W_i}U = U^*\pi_{W_i}T$ and we have

$$C_{TU}\|f\|^2 \leq \sum_{i\in I} \upsilon_i^2 \|(T^*\pi_{W_i}U)^{\frac{1}{2}} f\|^2 \leq D_{TU}\|f\|^2, \quad \forall f \in H.$$

We define *the controlled analysis operator* by

$$\theta_{TU} : H \to K, \quad \theta_{TU}(f) = (\upsilon_i(T^*\pi_{W_i}U)^{\frac{1}{2}} f)_{i\in I}, \quad \forall f \in H,$$

where

$$K = \left\{(\upsilon_i(T^*\pi_{W_i}U)^{\frac{1}{2}} f)_{i\in I} \mid f \in H\right\} \subseteq \Big(\bigoplus_{i\in I} H\Big)_{\ell^2}.$$

It is easy to see that K is closed and θ_{TU} is well defined. Moreover θ_{TU} is a bounded linear operator with adjoint (*the controlled synthesis operator*) θ_{TU}^* defined by

$$\theta_{TU}^* : K \to H, \quad \theta_{TU}^*((\upsilon_i(T^*\pi_{W_i}U)^{\frac{1}{2}} f)_{i\in I}) = \sum_{i\in I} \upsilon_i^2 T^*\pi_{W_i} Uf, \quad \forall f \in H.$$

Therefore, we define *the controlled fusion frame operator* S_{TU} on H by

$$S_{TU}f = \theta_{TU}^*\theta_{TU}(f) = \sum_{i\in I} \upsilon_i^2 T^*\pi_{W_i} Uf, \quad \forall f \in H.$$

It is easy to see that S_{TU} is well defined and

$$C_{TU} Id_H \leq S_{TU} \leq D_{TU} Id_H.$$

Hence S_{TU} is a bounded, invertible, self-adjoint and positive linear operator by Lemma 1.3. Therefore, we have $S_{TU} = S_{TU}^* = S_{UT}$.

A special case occurs when $U = Id_H$. In this case we call $\mathcal{W}$ a T-controlled fusion frame for H.

Similarly, we call $(f_i)_{i\in I}$ a (T,U)-*controlled frame* if there exists $0 < C_{TU} \leq D_{TU} < \infty$ such that

$$C_{TU}\|f\|^2 \leq \sum_{i\in I} < Uf, f_i >< f_i, Tf > \leq D_{TU}\|f\|^2, \quad \forall f \in H.$$

Also we call

$$S_{TU} f = \sum_{i\in I} < Uf, f_i > T^* f_i$$

the *controlled frame operator*. It is easy to see that if for each $i \in I$, the operator $f \mapsto \langle Uf, f_i\rangle T^* f_i$ is positive, then S_{TU} is well defined. It is a positive and invertible linear operator.

Controlled fusion frames are generalizations of fusion frames. Our next result proves this.

Proposition 2.2. (a) *Let $T, U \in GL(H)$ and $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ be a (T,U)-controlled fusion frame for H. Then $\mathcal{W}$ is a fusion frame for H. Furthermore $T^* S_{\mathcal{W}} U = U^* S_{\mathcal{W}} T$.*
(b) *Let $T \in GL(H)$. Then $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ is a fusion frame (resp. Bessel fusion sequence) if and only if it is a (T,T)-controlled fusion frame (resp. (T,T)-controlled Bessel fusion sequence) for H.*

Proof. (a) Let f be an arbitrary element of H. Since $\mathcal{W}$ is a (T,U)-controlled fusion frame for H, S_{TU} is a positive linear operator and since $(T^*)^{-1} \in GL(H)$, the map $\mathcal{S} : H \to H$ defined by

$$\mathcal{S}(f) = (T^*)^{-1} S_{TU} U^{-1}(f) = \sum_{i\in I} \upsilon_i^2 \pi_{W_i} f,$$

is well defined. It is easy to see that $\mathcal{S}$ is a bounded, positive linear operator on H, and also we have

$$\|\mathcal{S}^{-1}\| = \|U S_{TU}^{-1} T^*\| \leq \|U\|\|S_{TU}^{-1}\|\|T^*\| \leq \frac{1}{C_{TU}}\|U\|\|T\|.$$

Hence $\mathcal{S} \in GL^+(H)$. Then by Lemmas 1.3 and 1.4 $\{(W_i, \upsilon_i)\}_{i\in I}$ is a fusion frame and $\mathcal{S} = S_{\mathcal{W}}$ is its fusion frame operator. Furthermore,

$$T^* S_{\mathcal{W}} U = S_{TU} = S_{TU}^* = U^* S_{\mathcal{W}} T.$$

(b) Since $S_{TT} = T^* S_{\mathcal{W}} T$, then $C Id_H \leq S_{\mathcal{W}} \leq D Id_H$ is equivalent to

$$C T^* T \leq S_{TT} \leq D T^* T$$

for fusion frame bounds C, D. Furthermore,

$$\frac{1}{\|T^{-1}\|^2} Id_H \leq T^* T \leq \|T\|^2 Id_H$$

and we have the result. □

Remark. By the above proposition we have the following formulas:

$$f = S^{-1}(S(f)) = \sum_{i\in I} \upsilon_i^2 U S_{TU}^{-1} T^* \pi_{W_i}(f), \quad \forall f \in H,$$

and

$$f = S(S^{-1}(f)) = \sum_{i\in I} \upsilon_i^2 \pi_{W_i} U S_{TU}^{-1} T^*(f), \quad \forall f \in H.$$

Example 2.3. Let $T \in GL(H)$ and $\{f_i\}_{i\in I}$ be a T-controlled frame for H. If we take $W_i := \{\lambda f_i : \lambda \in \mathbb{C}\}$ for each $i \in I$, then $\pi_{W_i}(f) =< f, f_i > f_i$ for each $f \in H$. So $\{(W_i, 1) : i \in I\}$ is a (T, Id_H)-controlled fusion frame for H because there exist constants $0 < C_{TI} \le D_{TI} < \infty$ such that

$$C_{TI}\|f\|^2 \le \sum_{i\in I} < Tf, f_i >< f_i, f > \le D_{TI}\|f\|^2, \quad \forall f \in H.$$

But this is equivalent to

$$C_{TI}\|f\|^2 \le \sum_{i\in I} < \pi_{W_i} Tf, f > \le D_{TI}\|f\|^2, \quad \forall f \in H.$$

So we have the result.

Corollary 2.4. *Let $\mathcal{W}$ be a fusion frame for H and $T, U \in GL(H)$ be self-adjoint. Suppose that T ,U and $S_{\mathcal{W}}$ mutually commute and TU is positive. Then $\mathcal{W}$ is a (T, U)-controlled fusion frame for H.*

Proof. Since $S_{TU} = TS_{\mathcal{W}}U = TUS_{\mathcal{W}}$, we have the result. □

One of the main objects in frame and fusion frame theory is to solve the equation $Sf = h$ or $S_{\mathcal{W}} f = h$ and by using controlled frames and controlled fusion frames we choose an operator C as preconditioning matrix (see [2]) and we solve the equation

$$CSf = Ch \quad \text{or} \quad CS_{\mathcal{W}} f = Ch.$$

If we find an operator C such that

$$\|(CS_{\mathcal{W}})^{-1}\|\|CS_{\mathcal{W}}\| < \|S_{\mathcal{W}}^{-1}\|\|S_{\mathcal{W}}\|,$$

then the Neumann algorithm is more stable and we get a better approximation of the inverse operator, see [2].

Lemma 2.5. *Let $T \in GL^+(H)$ and let $\mathcal{W}$ be a T-controlled fusion frame for H. Then $\left(\frac{1}{\|(TS_{\mathcal{W}})^{-1}\|}, \|TS_{\mathcal{W}}\|\right)$, $\left(\frac{1}{\|T^{-1}\|}, \|T\|\right)$ and $\left(\frac{1}{\|(S_{\mathcal{W}})^{-1}\|}, \|S_{\mathcal{W}}\|\right)$, are the optimal bounds for the controlled fusion frame operator $TS_{\mathcal{W}}$, the operator T and the fusion frame operator $S_{\mathcal{W}}$, respectively. Moreover,*

$$\frac{Id_H}{\|T^{-1}\|\|S_{\mathcal{W}}^{-1}\|} \le TS_{\mathcal{W}} \le \|TS_{\mathcal{W}}\| Id_H \le \|T\|\|S_{\mathcal{W}}\| Id_H.$$

Proof. It is enough to observe that $TS_{\mathcal{W}} = S_{\mathcal{W}} T$. □

Our next result provides a sufficient condition on a family of closed subspaces of H to be a controlled fusion frame, in the presence of another controlled fusion frame. In fact it is a generalization of Proposition 2.4 in [1] and Proposition 4.1 in [6].

Proposition 2.6. *Let $T, U \in GL(H)$ and let $\{(W_i, \upsilon_i)\}_{i\in I}$ be a (T, U)-controlled fusion frame for H with lower and upper bounds C_{TU} and D_{TU}, respectively. Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of H. If there exists a number $0 < R < C_{TU}$ such that*

$$0 \le \sum_{i\in I} \upsilon_i^2 < T^*(\pi_{V_i} - \pi_{W_i})Uf, f > \le R\|f\|^2, \quad \forall f \in H,$$

then $\{(V_i, \upsilon_i)\}_{i\in I}$ is also a (T, U)-controlled fusion frame for H.

Proof. Let f be an arbitrary element of H. Since $\{(W_i, \upsilon_i)\}_{i\in I}$ is a (T,U)-controlled fusion frame for H, we have

$$C_{TU}\|f\|^2 \leq \sum_{i\in I}\upsilon_i^2 < T^*\pi_{W_i}Uf, f > \leq D_{TU}\|f\|^2.$$

Hence

$$\begin{aligned}\sum_{i\in I}\upsilon_i^2 < T^*\pi_{V_i}Uf, f > &= \sum_{i\in I}\upsilon_i^2 < T^*(\pi_{V_i}-\pi_{W_i})Uf, f > + \sum_{i\in I}\upsilon_i^2 < T^*\pi_{W_i}Uf, f > \\ &\leq R\|f\|^2 + D_{TU}\|f\|^2 = (R + D_{TU})\|f\|^2.\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{i\in I}\upsilon_i^2 < T^*\pi_{V_i}Uf, f > &= \sum_{i\in I}\upsilon_i^2 < T^*\pi_{W_i}Uf, f > + \sum_{i\in I}\upsilon_i^2 < T^*(\pi_{V_i}-\pi_{W_i})Uf, f > \\ &\geq \sum_{i\in I}\upsilon_i^2 < T^*\pi_{W_i}Uf, f > - \sum_{i\in I}\upsilon_i^2 < T^*(\pi_{V_i}-\pi_{W_i})Uf, f > \\ &\geq C_{TU}\|f\|^2 - R\|f\|^2 = (C_{TU} - R)\|f\|^2.\end{aligned}$$

So we have the result. □

Our next result is a generalization of Theorem 2.8 in [1].

Proposition 2.7. *Let $T, U \in GL(H)$ and let $\{(W_i, \upsilon_i)\}_{i\in I}$ be a (T,U)-controlled fusion frame for H. Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of H. Suppose that $\Phi : H \to H$ defined by*

$$\Phi(f) = \sum_{i\in I}\upsilon_i^2 T(\pi_{V_i} - \pi_{W_i})Uf, \quad \forall f \in H,$$

is a positive and compact operator. Then $\{(V_i, \upsilon_i)\}_{i\in I}$ is a (T,U)-controlled fusion frame for $\overline{span}\{V_i\}_{i\in I}$.

Proof. Let $\{(W_i, \upsilon_i)\}_{i\in I}$ be a (T,U)-controlled fusion frame for H. Then by Proposition 2.2, it is a fusion frame for H. On the other hand since Φ is a compact operator, $T^{-1}\Phi U^{-1}$ is also a compact operator. But

$$T^{-1}\Phi U^{-1}f = \sum_{i\in I}\upsilon_i^2(\pi_{V_i} - \pi_{W_i})f, \quad \forall f \in H.$$

Now by Theorem 2.8 in [1], we conclude that $\{(V_i, \upsilon_i)\}_{i\in I}$ is a fusion frame for $\overline{span}\{V_i\}_{i\in I}$. Let $S_{\mathcal{V}}$ be its associated fusion frame operator. Since $\Phi = TS_{\mathcal{V}}U - US_{\mathcal{W}}T$, then $TS_{\mathcal{V}}U = \Phi + US_{\mathcal{W}}T$. But Φ and $US_{\mathcal{W}}T (= S_{TU\mathcal{W}})$ are positive operators. Therefore, by Lemma 1.2 , $TS_{\mathcal{V}}U$ is a bounded positive operator. So $\{(V_i, \upsilon_i)\}_{i\in I}$ is a (T,U)-controlled fusion frame for $\overline{span}\{V_i\}_{i\in I}$. □

Example 2.8. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for $\mathbb{R}^3$ and $\mathcal{W} = \{(W_i, 1)\}_{i=1}^3$ be a 1-uniform fusion frame for it, in which $W_1 = \overline{span}\{e_1, e_2\}$,$W_2 = \overline{span}\{e_1, e_3\}$ and $W_3 = \overline{span}\{e_3\}$. It is easy to see that

$$\|f\|^2 \leq \sum_{i=1}^{3}\|\pi_{W_i}f\|^2 \leq 2\|f\|^2, \quad \forall f \in \mathbb{R}^3.$$

Let $T(x_1, x_2, x_3) = (5x_1, 4x_2, 5x_3)$ and $U(x_1, x_2, x_3) = (\frac{1}{6}x_1, \frac{1}{3}x_2, \frac{1}{6}x_3)$ be two operators on $\mathbb{R}^3$. It is easy to see that $T, U \in GL^+(\mathbb{R}^3)$, $TU = UT$, $TS_{\mathcal{W}} = S_{\mathcal{W}}T$ and $US_{\mathcal{W}} = S_{\mathcal{W}}U$. Now an easy computation shows that

$$\frac{4}{3}\|f\|^2 \leq \sum_{i=1}^{3} < \pi_{W_i}Tf, Uf > \leq \frac{5}{3}\|f\|^2, \quad \forall f \in \mathbb{R}^3.$$

So $\{(W_i, 1)\}_{i=1}^3$ is a (T,U)-controlled fusion frame for $\mathbb{R}^3$ as we expect from Corollary 2.4.

In the above example, we have $\frac{5}{4} = \frac{D_{TU}}{C_{TU}} < \frac{D}{C} = 2$. Therefore, $\|Id_H - S_{TU}\| \ll \|Id_H - S_{\mathcal{W}}\|$. Hence using a Neumann algorithm as described in [2], gives a good approximation of S_{TU}^{-1}. So an important work specially on infinite dimensional Hilbert spaces is to find appropriate operators T and U for which $\|Id_H - S_{TU}\| \ll \|Id_H - S_{\mathcal{W}}\|$.

In the following result we consider locally controlled frames for each W_i in a controlled fusion frame $\mathcal{W}$ to obtain new globally controlled frames for H.

Theorem 2.9. *Let $T, U \in GL(H)$ and let $\{W_i\}_{i\in I}$ be a family of closed subspaces of H and $\{v_i\}_{i\in I}$ be a family of weights. Suppose that $\{f_{ij} : j \in J_i\}$ is a frame for W_i with frame operator S_i, for each $i \in I$. Then $\{(W_i, v_i) : i \in I\}$ is a (T,U)-controlled fusion frame (Bessel) for H if and only if $\{v_i S_i^{-1/2} f_{ij} : i \in I, j \in J_i\}$ is a (T,U)-controlled frame (Bessel).*

Proof. Let $\{(W_i, v_i) : i \in I\}$ be a fusion frame, with bounds C and D. Since for each $i \in I$, $\{f_{ij} : j \in J_i\}$ is a frame for W_i with frame operator S_i, we have that $\{S_i^{-1/2} f_{ij} : j \in J_i\}$ is a Parseval frame for W_i and for every $f \in H$, we have

$$\pi_{W_i} Uf = \Sigma_{j\in J_i} \langle Uf, S_i^{-1/2} f_{ij}\rangle S_i^{-1/2} f_{ij}.$$

On the other hand by using (1) we conclude that

$$C||f||^2 \leq \Sigma_{i\in I} v_i^2 \langle \pi_{W_i} Uf, Tf\rangle \leq D||f||^2$$

is equivalent to

$$C||f||^2 \quad \leq \quad \Sigma_{i\in I}\Sigma_{j\in J_i} \langle Uf, v_i S_i^{-1/2} f_{ij}\rangle \langle v_i S_i^{-1/2} f_{ij}, Tf\rangle \leq D||f||^2,$$

which shows that $\{v_i S_i^{-1/2} f_{ij} : i \in I, j \in J_i\}$ is a (T,U)-controlled frame for H if and only if $\{(W_i, v_i) : i \in I\}$ is a (T,U)-controlled fusion frame for H. □

Our next result is a characterization theorem for (T,U)-controlled fusion frames.

Theorem 2.10. *Let $T, U \in GL(H)$ and let $\{W_i\}_{i\in I}$ be a family of closed subspaces of H and $\{v_i\}_{i\in I}$ be a family of weights. Suppose that $\{e_{ij} : j \in J_i\}$ is an orthonormal basis for W_i for each $i \in I$. Then $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ is a (T,U)-controlled fusion frame for H if and only if $\Psi = \{v_i U^* e_{ij} : i \in I, j \in J_i\}$ is a $T^*(U^*)^{-1}$-controlled frame for H with controlled frame operator $T^* S_{\mathcal{W}} U$.*

Proof. Let $\{e_{ij} : j \in J_i\}$ be an orthonormal basis for W_i for each $i \in I$. Then

$$\pi_{W_i}(Tf) = \sum_{j\in J_i} < \pi_{W_i} Tf, e_{ij} > e_{ij} = \sum_{j\in J_i} < Tf, e_{ij} > e_{ij}, \quad \forall f \in H.$$

Also,

$$\pi_{W_i}(Uf) = \sum_{j\in J_i} < \pi_{W_i} Uf, e_{ij} > e_{ij} = \sum_{j\in J_i} < Uf, e_{ij} > e_{ij}, \quad \forall f \in H.$$

Hence,

$$< \pi_{W_i}(Uf), \pi_{W_i} Tf >= \sum_{j\in J_i} < Uf, e_{ij} >< e_{ij}, Tf >= \sum_{j\in J_i} < f, U^* e_{ij} >< T^* e_{ij}, f > .$$

Now if we take $f_{ij} := v_i U^* e_{ij}$ and $\mathcal{C} := T^*(U^*)^{-1}$ then

$$C_{TU}\|f\|^2 \leq \sum_{i\in I} v_i^2 < \pi_{W_i} Uf, \pi_{W_i} Tf >\leq D_{TU}\|f\|^2$$

is equivalent to

$$C_{TU}\|f\|^2 \leq \sum_{i\in I}\sum_{j\in J_i} < f, f_{ij} >< \mathcal{C} f_{ij}, f >\leq D_{TU}\|f\|^2$$

and we have the first result. Furthermore, by Proposition 3.2 in [2], Ψ is a frame with the frame operator $T^*(U^*)^{-1}S_\Psi$. But $S_\Psi = U^* S_{\mathcal{W}} U$. So we have the result. □

If $UT^* = TU^*$, then $\mathcal{C} = T^*(U^*)^{-1}$ is self-adjoint and by Proposition 3.3 in [2] we have the following result.

Proposition 2.11. *Let $T, U \in GL(H)$ and $UT^* = TU^*$. Then $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ is a (T,U)-controlled fusion frame for H if and only if $\mathcal{W}$ is a fusion frame, $\mathcal{C} = T^*(U^*)^{-1}$ is positive and commutes with $T^* S_{\mathcal{W}} U$.*

Proof. The result follows by the above theorem and by Proposition 3.3 in [2]. □

By the above results, finding suitable operators T and U such that $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ forms a (T,U)-controlled fusion frame for H with optimal bounds, is equivalent to finding suitable operators T and U such that $\Psi = \{\upsilon_i U^* e_{ij} : i \in I, j \in J_i\}$ forms a $T^*(U^*)^{-1}$-controlled frame for H with optimal frame bounds. And for this, we can use the results obtained in [2].

3. Resolutions of the identity

In this section we want to find new resolutions of the identity. It is easy to see that the family $\{\upsilon_i^2 S_{TU}^{-1} T^* \pi_{W_i} U\}_{i\in I}$ of bounded operators is a resolution of the identity, where $T, U \in GL(H)$ and $\{(W_i, \upsilon_i)\}_{i\in I}$ is a (T,U)-controlled fusion frame for H. In fact we have

$$f = \sum_{i\in I} \upsilon_i^2 S_{TU}^{-1} T^* \pi_{W_i} U f = \sum_{i\in I} \upsilon_i^2 T^* \pi_{W_i} U S_{TU}^{-1} f.$$

Furthermore, as a corollary of Theorem 2.9 with its hypothesis we have the following reconstruction formula:

$$f = \sum_{i\in I} \sum_{j\in J_i} \upsilon_i^2 < Uf, S_i^{-1/2} f_{ij} > S_{TU}^{-1} T^* S_i^{-1/2} f_{ij}.$$

As mentioned before, by choosing suitable control operators we may have more suitable approximations for S_{TU}^{-1} and S_i^{-1} than $S_{\mathcal{W}}^{-1}$. Now we want to get a new resolution of the identity by using two controlled fusion frames.

Definition 3.1. Let $T, U \in GL(H)$ and let $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ and $\mathcal{V} = \{(V_i, \omega_i)\}_{i\in I}$ be (T,T)-controlled and (U,U)-controlled Bessel fusion sequences, respectively. We define a (T,U)-controlled fusion frame operator $S_{T\mathcal{W}\mathcal{V}U}$ for this pair of controlled fusion frames as follows:

$$S_{T\mathcal{W}\mathcal{V}U} f = \sum_{i\in I} \upsilon_i \omega_i T^* \pi_{W_i} \pi_{V_i} U f, \quad \forall f \in H.$$

As mentioned before, $\{(W_i, \upsilon_i)\}_{i\in I}$ and $\{(V_i, \omega_i)\}_{i\in I}$ are also two Bessel fusion sequences. So by [8] the fusion frame operator $S_{\mathcal{W}\mathcal{V}} f = \sum_{i\in I} \upsilon_i \omega_i \pi_{W_i} \pi_{V_i} f$ for this pair of Bessel fusion sequences is well defined and bounded. But $S_{T\mathcal{W}\mathcal{V}U} = T^* S_{\mathcal{W}\mathcal{V}} U$. Therefore, $S_{T\mathcal{W}\mathcal{V}U}$ is well defined and a bounded operator. Furthermore, $S_{T\mathcal{W}\mathcal{V}U}^* = S_{U\mathcal{V}\mathcal{W}T}$.

Theorem 3.2. *Let $T \in GL(H)$ and let $\mathcal{W} = \{(W_i, \upsilon_i)\}_{i\in I}$ be a (T,T)-controlled Bessel fusion sequence. Then $\mathcal{W}$ is a (T,T)-controlled fusion frame if and only if there exist an operator $U \in GL(H)$ and a (U,U)-controlled Bessel fusion sequence $\mathcal{V} = \{(V_i, \omega_i)\}_{i\in I}$ such that $S_{U\mathcal{V}\mathcal{W}T} \geq mId_H$ on H, for some $m > 0$.*

Proof. Let $\mathcal{W}$ be a (T,T)-controlled fusion frame with lower and upper fusion frame bounds C_T and D_T, respectively. Then we take $U = T$, $V_i = W_i$ and $\omega_i = \upsilon_i$, for all $i \in I$. Hence we have

$$\langle S_{T\mathcal{W}\mathcal{W}T} f, f\rangle = \langle \sum_{i\in I} \upsilon_i^2 T^* \pi_{W_i} T f, f\rangle = \sum_{i\in I} \upsilon_i^2 \langle \pi_{W_i} T f, \pi_{W_i} T f\rangle \geq C_T \|f\|^2$$

for all $f \in H$. Furthermore,

$$C_T\|f\|^2 \leq \left\|S^{\frac{1}{2}}_{T\mathcal{W}\mathcal{W}T}f\right\|^2 \leq D_T\|f\|^2.$$

So $S_{T\mathcal{W}\mathcal{W}T} = S_{TT} \in GL^+(H)$ by Lemma 1.4.

Conversely, suppose that there exist an operator $U \in GL(H)$ and a (U,U)-controlled Bessel fusion sequence $\mathcal{V} = \{(V_i, \omega_i)\}_{i\in I}$ with Bessel fusion bound D_U. Also let $0 < m$ be a constant such that

$$m\|f\|^2 \leq \langle S_{U\mathcal{V}\mathcal{W}T}f, f\rangle$$

for all $f \in H$. Then we have

$$\begin{aligned} m\|f\|^2 \leq \langle S_{U\mathcal{V}\mathcal{W}T}f, f\rangle &= \sum_{i\in I}\langle \upsilon_i\pi_{W_i}Tf, \omega_i\pi_{V_i}Uf\rangle \\ &\leq \Big(\sum_{i\in I}\upsilon_i^2\|\pi_{W_i}Tf\|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in I}\omega_i^2\|\pi_{V_i}Uf\|^2\Big)^{\frac{1}{2}} \\ &\leq \sqrt{D_U}\|f\|\Big(\sum_{i\in I}\upsilon_i^2\|\pi_{W_i}Tf\|^2\Big)^{\frac{1}{2}}, \end{aligned}$$

by the Cauchy-Schwartz inequality. Therefore,

$$\frac{m^2}{D_U}\|f\|^2 \leq \sum_{i\in I}\upsilon_i^2\|\pi_{W_i}Tf\|^2.$$

Hence we have the result. □

Our next result is an analog of Theorem 2.15 in [10].

Proposition 3.3. *Let $\mathcal{W}$ and $\mathcal{V}$ be controlled Bessel fusion sequences as mentioned in Definition 3.1. Suppose that there exists $0 < M < 1$ such that*

$$\|f - S_{U\mathcal{V}\mathcal{W}T}f\| \leq M\|f\|, \quad \forall f \in H.$$

Then $\mathcal{W}$ and $\mathcal{V}$ are (T,T)-controlled and (U,U)-controlled fusion frames, respectively. Furthermore, $S_{U\mathcal{V}\mathcal{W}T}$ is invertible.

Proof. Firstly $||Id_H - S_{U\mathcal{V}\mathcal{W}T}|| \leq M < 1$, therefore $S_{U\mathcal{V}\mathcal{W}T}$ is invertible, secondly let f be an arbitrary element of H. Then we have

$$\|S_{U\mathcal{V}\mathcal{W}T}f\| \geq \|f\| - \|f - S_{U\mathcal{V}\mathcal{W}T}f\| \geq (1-M)\|f\|.$$

Hence $S_{U\mathcal{V}\mathcal{W}T}$ is bounded below and we have

$$\begin{aligned} (1-M)\|f\| \leq \|S_{U\mathcal{V}\mathcal{W}T}f\| &= \sup_{g\in H, \|g\|=1}\Big|\Big\langle \sum_{i\in I}\upsilon_i\omega_i U^*\pi_{V_i}\pi_{W_i}Tf, g\Big\rangle\Big| \\ &\leq \sup_{g\in H, \|g\|=1}\Big(\sum_{i\in I}\upsilon_i^2\|\pi_{W_i}Tf\|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in I}\omega_i^2\|\pi_{V_i}Ug\|^2\Big)^{\frac{1}{2}} \\ &\leq \sqrt{D_U}\Big(\sum_{i\in I}\upsilon_i^2\|\pi_{W_i}Tf\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\frac{(1-M)^2}{D_U}\|f\|^2 \leq \sum_{i\in I}\upsilon_i^2\|\pi_{W_i}Tf\|^2,$$

where D_U is a controlled Bessel fusion bound for $\mathcal{V}$. Therefore, $\mathcal{W}$ is a (T,T)-controlled fusion frame. On the other hand we have

$$\|Id_H - S_{T\mathcal{W}\mathcal{V}U}\| = \|(Id_H - S_{U\mathcal{V}\mathcal{W}T})^*\| \leq M.$$

Hence similarly we can say that $\mathcal{V}$ is a (U,U)-controlled fusion frame. □

Let $S_{U\mathcal{V}\mathcal{W}T}$ and so $S_{T\mathcal{W}\mathcal{V}U}$ be invertible. Then the family

$$\{\upsilon_i\omega_i S^{-1}_{U\mathcal{V}\mathcal{W}T}U^*\pi_{V_i}\pi_{W_i}T\}_{i\in I}$$

is a resolution of the identity. Also we have the following reconstruction formulas:

$$f=\sum_{i\in I}\upsilon_i\omega_i S^{-1}_{U\mathcal{V}\mathcal{W}T}U^*\pi_{V_i}\pi_{W_i}Tf=\sum_{i\in I}\upsilon_i\omega_i U^*\pi_{V_i}\pi_{W_i}TS^{-1}_{U\mathcal{V}\mathcal{W}T}f$$

and

$$f=\sum_{i\in I}\upsilon_i\omega_i S^{-1}_{T\mathcal{W}\mathcal{V}U}T^*\pi_{W_i}\pi_{V_i}Uf=\sum_{i\in I}\upsilon_i\omega_i T^*\pi_{W_i}\pi_{V_i}US^{-1}_{T\mathcal{W}\mathcal{V}U}f.$$

Suppose that $\|Id_H-S_{U\mathcal{V}\mathcal{W}T}\|<1$. Then as we mentioned in Proposition 3.3, $S_{U\mathcal{V}\mathcal{W}T}$ is invertible and we have

$$S^{-1}_{U\mathcal{V}\mathcal{W}T}=\sum_{n=0}^{\infty}(Id_H-S_{U\mathcal{V}\mathcal{W}T})^n.$$

Furthermore,

$$\|S^{-1}_{U\mathcal{V}\mathcal{W}T}\|\le(1-\|Id_H-S_{U\mathcal{V}\mathcal{W}T}\|)^{-1}.$$

Therefore,

$$\{\upsilon_i\omega_i(Id_H-S_{U\mathcal{V}\mathcal{W}T})^nU^*\pi_{V_i}\pi_{W_i}T : i\in I, n\in\mathbb{Z}^+\}$$

is a new resolution of the identity.

References

1. M. S. Asgari, A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl. **308** (2005), 541–553.
2. P. Balazs, J.-P. Antoine, A. Gryboś, *Weighted and controlled frames: mutual relationship and first numerical properties*, International Journal of Wavelets, Multiresolution and Information Processing **8** (2010), no. 1, 109–132.
3. I. Bogdanova, P. Vandergheynst, J. P. Antoine, L. Jacques, and M. Morvidone, *Stereographic wavelet frames on the sphere*, Appl. Comput. Harmon. Anal. **16** (2005), 223–252.
4. P. G. Casazza, G. Kutyniok, *Frames of subspaces*, Wavelets, Frames, and Operator Theory (College Park, MD, 2003), Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 87–113.
5. P. G. Casazza, G. Kutyniok, S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal. **25** (2008), no. 1, 114–132.
6. O. Christensen, C. Heil, *Perturbations of Banach frames and atomic decomposition*, Math. Nachr. **185** (1997), 33–47.
7. O. Christensen, *Frames and Bases. An Introductory Course*, Birkhäuser, Boston, 2008.
8. P. Găvruţa, *On duality of fusion frames*, J. Math. Anal. Appl. **333** (2007), 871–879.
9. A. Khosravi, B. Khosravi, *Fusion frames and g-frames in Hilbert C^*-modules*, International Journal of Wavelets, Multiresolution and Information Processing **6** (2008), no. 3, 433–446.
10. A. Khosravi, K. Musazadeh, *Fusion frames and g-frames*, J. Math. Anal. Appl. **342** (2008), 1068-1083.
11. G. J. Murphy, *C^*-algebra and operator theory*, San Diego, California, Academic Press, 1990.

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