Methods of Functional Analysis and Topology
Vol. 18 (2012), no. 3, pp. 266–287

ABSTRACT INTERPOLATION PROBLEM IN GENERALIZED NEVANLINNA CLASSES

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Abstract. The abstract interpolation problem (AIP) in the Schur class was posed by V. Katznelson, A. Kheifets, and P. Yuditskii in 1987. In the present paper we consider an analog of AIP for the generalized Nevanlinna class $N_\kappa(\mathcal{L})$ in the non-degenerate case. We associate with the data set of the AIP a symmetric linear relation $\widehat{A}$ acting in a Pontryagin space. The description of all solutions of the AIP is reduced to the problem of description of all $\mathcal{L}$-resolvents of this symmetric linear relation $\widehat{A}$. The latter set is parametrized by application of the indefinite version of Kreĭn's representation theory for symmetric linear relations in Pontryagin spaces developed by M. G. Kreĭn and H. Langer in [22] and a formula for the $\mathcal{L}$-resolvent matrix obtained by V. Derkach and M. Malamud in [11].

1. Introduction

The abstract interpolation problem (AIP) was posed by V. Katznelson, A. Kheifets, and P. Yuditskii in [15] as an extension of the V. P. Potapov's approach to interpolation problems [19]. A description of the set of all solutions of the AIP was reduced in [15] to the description of all scattering matrices of unitary extensions of a given partial isometry V ([4]). In several papers ([5], [24], [17], [16]) it was shown that many problems of analysis, such as the bitangential interpolation problem, moment problem, lifting problem, and others can be included into the general scheme of the AIP.

A parallel version of the AIP for the Nevanlinna class $N(\mathcal{L})$ was considered by V. A. Derkach in [10]. The class $N[\mathcal{L}]$ consists of all operator valued functions $m(\lambda)$ which are holomorphic in the upper halfplane $\mathbb{C}_+$ and take values in the set $[\mathcal{L}]$ of bounded linear operators in a Hilbert space $\mathcal{L}$, such that the kernel

$$\mathsf{N}_\omega^m(\lambda) = \frac{m(\lambda) - m(\omega)^*}{\lambda - \bar{\omega}} \tag{1.1}$$

is nonnegative on $\mathbb{C}_+$. In the present paper we consider an analog of AIP for the generalized Nevanlinna class $N_\kappa(\mathcal{L})$ ([21]) in the non-degenerate case.

The kernel $\mathsf{N}_\omega^m(\lambda)$ is said to have κ negative squares and it is written $sq_-\mathsf{N} = \kappa$ if for any choice set of points $\omega_1, \dots, \omega_n$ in Ω and vectors $u_1, \dots, u_n$ the quadratic form

$$\left[\left(\mathsf{N}_{\omega_j}(\omega_i) u_j,\, u_i \right)_{\mathcal{L}} \right]_{i,j=1}^n$$

has at most κ negative eigenvalues, and for some choice of n, ω_j, u_j it has exactly κ negative squares ([1]). Remind that the class $N_\kappa(\mathcal{L})$ consists of operator valued functions $m(\lambda)$ meromorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$ such that $m(\overline{\lambda}) = m(\lambda)^*$ and the kernel (1.1) has κ negative squares on the domain of holomorphy $\mathfrak{h}_m$ of the function m.

2000 *Mathematics Subject Classification.* Primary 47A57; Secondary 39B42, 47A06, 47B25, 47B50.

Key words and phrases. Interpolation problem, symmetric relation, selfadjoint extension, reproducing kernel, Pontryagin space, boundary triplet, resolvent matrix.

The author thanks V. A. Derkach and M. M. Malamud for many suggestions and observations.

Let $\mathcal{H}(m)$ be the Pontryagin space of vector valued functions meromorphic in $\mathbb{C}\setminus\mathbb{R}$ with the reproducing kernel $\mathsf{N}^m_\omega(\lambda)$ (see [7], [3]). This space is characterized by the properties:

(1) $\mathsf{N}^m_\omega(\cdot)u \in \mathcal{H}(m)$ for all $\omega\in\mathbb{C}\setminus\mathbb{R}$ and $u\in\mathcal{L}$;

(2) for every $f\in\mathcal{H}(m)$ the following identity holds

$$\langle f(\cdot), \mathsf{N}^m_\omega(\cdot)u\rangle_{\mathcal{H}(m)} = u^* f(\omega), \quad \omega\in\mathbb{C}\setminus\mathbb{R}, \quad u\in\mathcal{L}. \tag{1.2}$$

Let $\mathcal{X}$ be a complex linear space, let $\mathcal{L}$ be a Hilbert space, let B_1, B_2 be linear operators in $\mathcal{X}$, let C_1, C_2 be linear operators from $\mathcal{X}$ to $\mathcal{L}$ and let K be a nonnegative sesquilinear form on $\mathcal{X}$ which has ν negative squares. Consider the following indefinite analog of the AIP.

Problem $AIP_\kappa(B_1,B_2,C_1,C_2,K)$. Let the data set (B_1,B_2,C_1,C_2,K) satisfies the assumptions

(A1) $K(B_2h,B_1g)-K(B_1h,B_2g)=(C_1h,C_2g)_{\mathcal{L}}-(C_2h,C_1g)_{\mathcal{L}} \quad \forall h,g\in\mathcal{X}$;

(A2) $\ker K=\{0\}$, where $\ker K=\{h\in\mathcal{X}: \ K(h,u)=0 \ \ \forall u\in\mathcal{X}\}$.

Find a function $m(\lambda)$ from the class $N_\kappa(\mathcal{L})$ such that for some linear mapping $F:\mathcal{X}\to\mathcal{H}(m)$ the following conditions hold for all $h\in\mathcal{X}$:

(C1) $(FB_2h)(\lambda)-\lambda(FB_1h)(\lambda)=\begin{bmatrix} I_{\mathcal{L}} & -m(\lambda)\end{bmatrix}\begin{bmatrix} C_1h \\ C_2h\end{bmatrix}$;

(C2) $\langle Fh,\, Fh\rangle_{\mathcal{H}(m)}\le K(h,h)$.

Note that the condition $\nu\le\kappa$ is necessary for the solvability AIP_κ (see Remark 3.1).

Define the Pontryagin space $\mathcal{H}$ as the completion of $\mathcal{X}$ endowed with the scalar product

$$\langle h,g\rangle_{\mathcal{H}} = K(h,g), \quad h,g\in\mathcal{X}. \tag{1.3}$$

We identify the linear operators $B_1,\ B_2:\mathcal{X}\to\mathcal{X}$ with the linear operators $B_1,\ B_2: \mathcal{X}\to\mathcal{H}$.

In the present paper we will use the notion of a linear relation in a Pontryagin space $\mathfrak{H}$. Recall, that a subspace T of $\mathfrak{H}^2$ is called the linear relation in $\mathfrak{H}$. For a linear relation T in $\mathfrak{H}$ the symbols $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ stand for the domain, kernel, range, and the multivalued part, respectively, (see [1]). The adjoint T^* is the closed linear relation in $\mathfrak{H}$ defined by

$$T^*=\{\,\{h,k\}\in\mathfrak{H}^2:\ \langle k,f\rangle_{\mathfrak{H}}=\langle h,g\rangle_{\mathfrak{H}},\ \{f,g\}\in T\,\}.$$

Recall that a linear relation T in $\mathfrak{H}$ is called symmetric (selfadjoint), if $T\subset T^*$ ($T=T^*$, respectively).

It follows from (A1) that the linear relation

$$\widehat{A}=\left\{\left\{\begin{bmatrix} B_1h \\ C_1h\end{bmatrix},\begin{bmatrix} B_2h \\ C_2h\end{bmatrix}\right\}: h\in\mathcal{X}\right\}$$

is symmetric in $\mathcal{H}\oplus\mathcal{L}$.

Let us impose some additional assumptions on the data set

(A3) $B_2=I_{\mathcal{X}}$ and the operators $B_1:\mathcal{X}\subseteq\mathcal{H}\to\mathcal{H}$, $C_1,C_2:\mathcal{X}\subseteq\mathcal{H}\to\mathcal{L}$ are bounded.

(A4) for some choice of $\lambda_j\in\mathbb{C}_+$ $(j=1,\dots,\kappa)$ the following condition holds:

$$\ker\begin{bmatrix} C_2^* & (1-\lambda_1B_1^*)^{-1}C_2^* & (1-\lambda_2B_1^*)^{-1}C_2^* & \cdots & (1-\lambda_\kappa B_1^*)^{-1}C_2^*\end{bmatrix}=\{0\}.$$

The main result of the paper is the following description of all solutions of the AIP_κ.

Theorem 1.1. *Let the data set (B_1,B_2,C_1,C_2,K) satisfy the assumptions (A1)–(A4). Then the Problem $AIP_\kappa(B_1,B_2,C_1,C_2,K)$ is solvable and the set of its solutions $m(\lambda)$ is parameterized as follows*

$$m(\lambda)=P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}\restriction_{\mathcal{L}}\left(I_{\mathcal{L}}+\lambda P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}\restriction_{\mathcal{L}}\right)^{-1},$$

where $\widetilde{A}$ *ranges over the set of all selfadjoint* $\mathcal{L}$*-regular extensions of* $\widehat{A}$ *with the exit in a Pontryagin space* $\widetilde{\mathcal{H}} \oplus \mathcal{L} \supset \mathcal{H} \oplus \mathcal{L}$*. The corresponding linear mapping* $F : \mathcal{X} \to \mathcal{H}(m)$ *is given by*

$$(Fh)(\lambda) = P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}h, \quad h \in \mathcal{X}.$$

Due to Theorem 1.1 the description of all solutions m of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ is reduced to the problem of description of all $\mathcal{L}$-regular $\mathcal{L}$-resolvents of the linear relation $\widehat{A}$. The latter description goes back to M. G. Kreĭn in [20] (see also [23], [11]).

The pair $\{p, q\}$ of the $[\mathcal{L}]$-valued function is called the Nevanlinna pair if the functions $p(\cdot)$, $q(\cdot)$ are holomorphic on $\mathbb{C}_\pm$ and such that

(i) $\mathsf{N}^{pq}_\omega(\lambda) = \dfrac{q(\bar\lambda)^* p(\bar\omega) - p(\bar\lambda)^* q(\bar\omega)}{\lambda - \bar\omega}$ is nonnegative on $\mathbb{C}_+$;
(ii) $q(\bar\lambda)^* p(\lambda) - p(\bar\lambda)^* q(\lambda) = 0$, $\lambda \in \mathbb{C}_\pm$;
(iii) $0 \in \rho(p(\lambda) - \lambda q(\lambda))$, $\lambda \in \mathbb{C}_\pm$.

The set of Nevanlinna pairs of $[\mathcal{L}]$-valued functions is denoting by $\mathbf{N}(\mathcal{L})$. Two Nevanlinna pairs $\{p, q\}$ and $\{p_1, q_1\}$ are said to be equivalent if $p_1(\lambda) = p(\lambda)\chi(\lambda)$ and $q_1(\lambda) = q(\lambda)\chi(\lambda)$ for some operator function $\chi(\cdot) \in [\mathcal{L}]$, which is holomorphic and invertible on $\mathbb{C}_\pm$. The set of all equivalence classes of Nevanlinna pairs in $\mathcal{L}$ will be denoted by $\widetilde{\mathbf{N}}(\mathcal{L})$.

Define the operator valued function Θ by the formula

$$\Theta(\lambda) = \begin{bmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{bmatrix} = I_{\mathcal{L}\oplus\mathcal{L}} - \lambda \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I_{\mathcal{H}} - \lambda B_1)^{-1} \begin{bmatrix} -C_2^* & C_1^* \end{bmatrix}. \tag{1.4}$$

Theorem 1.2. *Let the* AIP_κ *data set* (B_1, B_2, C_1, C_2, K) *satisfy the assumptions (A1)–(A4)* $\kappa = \nu$*, and let* $\Theta(\lambda)$ *be defined by* (1.4)*. Then the formula*

$$m(\lambda) = (\theta_{11}(\lambda)q(\lambda) + \theta_{12}(\lambda)p(\lambda))(\theta_{21}(\lambda)q(\lambda) + \theta_{22}(\lambda)p(\lambda))^{-1} \tag{1.5}$$

establishes a one-to-one correspondence between the set of all solutions $m(\lambda)$ *of the* $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ *and the set of all equivalence classes of Nevanlinna pairs* $\{p, q\} \in \widetilde{\mathbf{N}}(\mathcal{L})$ *such that the function* m *defined by the formula* (1.5) *belongs to the class* $N_\kappa(\mathcal{L})$*.*

The assertions of Theorem 1.1 and Theorem 1.2 remain valid under less restrictive assumptions. Namely, (A1)–(A2) for Theorem 1.1, and (A1)–(A3) for Theorem 1.2. However, in this case the problem AIP_κ should be considered in the class of the multi-valued N_κ-functions i.e. the N_κ-pairs (see Definition 2.1).

The paper is organized as follows. In Section 2 we recall the definition of the class of N_κ-pairs $\widetilde{\mathbf{N}}_\kappa(\mathcal{L})$. To each selfadjoint linear relation $\widetilde{A}$ and a scale spaces $\mathcal{L}$ we associate an N_κ-pair $\{\varphi, \psi\}$, which is normalized by the condition $\varphi(\lambda) - \lambda\psi(\lambda) = I_{\mathcal{L}}$, by the formula

$$\psi(\lambda) := P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{L}}, \quad \varphi(\lambda) := I_{\mathcal{L}} + \lambda P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{L}}, \quad \lambda \in \rho(\widetilde{A}). \tag{1.6}$$

Conversely, given an N_κ-pair $\{\varphi, \psi\}$ we construct a functional model for a selfadjoint linear relation $\widetilde{A} = A(\varphi, \psi)$ such that the pair $\{\varphi, \psi\}$ is related to $A(\varphi, \psi)$ via (1.6). The main results of this part were presented by the author in [26]. In Section 3 we formulate the AIP in the classes of N_κ-pairs. In this section we prove a structure theorem for the solutions of the AIP_κ. In Section 4 we give explicit formulas describing all solutions of the problem in terms of the N_κ-pair. In this section we study the uniqueness of the construction of the map F via a solution of the AIP_κ. In Section 5 we prove the main results of this paper Theorem 1.1 and Theorem 1.2.

Applications of these results to an indefinite moment problem will be presented in another paper.

2. Functional model of a selfadjoint linear relation

2.1. Generalized Nevanlinna pairs. Let $\mathcal{L}$ be a Hilbert space.

Definition 2.1. A pair $\{\Phi, \Psi\}$ of $[\mathcal{L}]$-valued functions $\Phi(\cdot)$, $\Psi(\cdot)$ meromorphic on $\mathbb{C}\backslash\mathbb{R}$ with a common domain of holomorphy $\mathfrak{h}_{\Phi\Psi}$ is said to be an N_κ-pair (generalized Nevanlinna pair) if

(i) the kernel
$$\mathsf{N}_\omega^{\Phi\Psi}(\lambda) = \frac{\Psi(\bar{\lambda})^*\Phi(\bar{\omega}) - \Phi(\bar{\lambda})^*\Psi(\bar{\omega})}{\lambda - \bar{\omega}}$$
has κ negative squares on $\mathfrak{h}_{\Phi\Psi}$;

(ii) $\Psi(\bar{\lambda})^*\Phi(\lambda) - \Phi(\bar{\lambda})^*\Psi(\lambda) = 0$ for all $\lambda \in \mathfrak{h}_{\Phi\Psi}$;

(iii) for all $\lambda \in \mathfrak{h}_{\Phi\Psi} \cap \mathbb{C}_+$ there exists $\mu \in \mathbb{C}_+$ such that
$$0 \in \rho\big(\Phi(\lambda) - \mu\Psi(\lambda)\big) \text{ and } 0 \in \rho\big(\Phi(\overline{\lambda}) - \overline{\mu}\Psi(\overline{\lambda})\big).$$

Note that for $\kappa = 0$ we get the definition of the Nevanlinna pair.

Two N_κ-pairs $\{\Phi, \Psi\}$ and $\{\Phi_1, \Psi_1\}$ are said to be equivalent, if $\Phi_1(\lambda) = \Phi(\lambda)\chi(\lambda)$ and $\Psi_1(\lambda) = \Psi(\lambda)\chi(\lambda)$ for some $[\mathcal{L}]$-valued function $\chi(\cdot)$, which is holomorphic and invertible on $\mathfrak{h}_{\Phi\Psi}$. The set of all equivalence classes of N_κ-pairs in $\mathcal{L}$ will be denoted by $\widetilde{\mathbf{N}}_\kappa(\mathcal{L})$. We will write, for short, $\{\Phi, \Psi\} \in \widetilde{\mathbf{N}}_\kappa(\mathcal{L})$ for the generalized Nevanlinna pair $\{\Phi, \Psi\}$.

If $\Phi(\lambda) \equiv I_\mathcal{L}$, where $I_\mathcal{L}$ is the identity operator in the space $\mathcal{L}$ then the Definition 2.1 means that $\Psi(\lambda)$ is an $N_\kappa(\mathcal{L})$-function in the sense of [21]. In this case the condition (iii) is satisfied automatically. Clearly, if $\{\Phi, \Psi\}$ is an N_κ-pair such that $0 \in \rho\big(\Phi(\lambda)\big)$ $\lambda \in \mathfrak{h}_{\Phi\Psi}$, then it is equivalent to the pair $\big\{I_\mathcal{L}, \Psi(\lambda)\Phi(\lambda)^{-1}\big\}$, where $\Psi\Phi^{-1} \in N_\kappa(\mathcal{L})$.

Definition 2.2. An N_κ-pair $\{\varphi, \psi\}$ is said to be a normalized N_κ-pair if

(iii′) $\varphi(\lambda) - \lambda\psi(\lambda) \equiv I_\mathcal{L}$ for all $\lambda \in \mathfrak{h}_{\varphi\psi}$.

Clearly, every N_κ-pair $\{\Phi, \Psi\}$ such that $0 \in \rho\big(\Phi(\lambda) - \lambda\Psi(\lambda)\big)$ for $\lambda \in \mathfrak{h}_{\Phi\Psi}$ is equivalent to a unique normalized N_κ-pair $\{\varphi, \psi\}$ given by

$$\varphi(\lambda) = \Phi(\lambda)\big(\Phi(\lambda) - \lambda\Psi(\lambda)\big)^{-1}, \quad \psi(\lambda) = \Psi(\lambda)\big(\Phi(\lambda) - \lambda\Psi(\lambda)\big)^{-1}. \tag{2.1}$$

2.2. N_κ-pair corresponding to a selfadjoint linear relation and a scale. Let $\mathfrak{H}$ – be a vector space with a sesquilinear form $[\cdot,\cdot]_\mathfrak{H} : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$. Two elements u and v of $\mathfrak{H}$ are said to be orthogonal if $[u, v]_\mathfrak{H} = 0$. Similarly, two subspaces of $\mathfrak{H}$ are said to be orthogonal if every element of the first space is orthogonal to every element of the second. The linear space $\big(\mathfrak{H}, [\cdot,\cdot]\big)$ is called a Pontryagin space if there exists a direct orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, where $\mathfrak{H}_+$ is a Hilbert space with the form $[\cdot,\cdot]_\mathfrak{H}$ and $\mathfrak{H}_-$ is a Hilbert space of finite dimension with the form $-[\cdot,\cdot]_\mathfrak{H}$. The space $\mathfrak{H}$ is called Pontryagin space with the negative index κ (Π_κ-space) if the dimension of $\mathfrak{H}_-$ is $\kappa < \infty$ ([1]).

Let $\mathcal{H}$ be a Pontryagin space and $\mathcal{L}$ be a Hilbert space.

Definition 2.3. A linear relation $\widetilde{A} = \widetilde{A}^*$ in $\mathcal{H} \oplus \mathcal{L}$ is said to be $\mathcal{L}$-minimal if

$$\mathcal{H}_0 := \overline{\operatorname{span}}\{P_\mathcal{H}(\widetilde{A} - \lambda)^{-1}\mathcal{L} : \lambda \in \rho(\widetilde{A})\} = \mathcal{H}, \tag{2.2}$$

where $P_\mathcal{H}$ is the orthogonal projection onto the Pontryagin space $\mathcal{H}$.

Definition 2.4. A linear relation $\widetilde{A} = \widetilde{A}^*$ in $\mathcal{H} \oplus \mathcal{L}$ is said to be $\mathcal{L}$-regular if $\mathcal{H} \ominus \mathcal{H}_0$ is a Hilbert space. Here $\mathcal{H}_0$ is defined by (2.2).

In the case of an isometric operator in a Pontryagin space the notion of $\mathcal{L}$-regularity was given in [9].

Let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H}\oplus\mathcal{L}$ and let $P_{\mathcal{L}}$ be the orthogonal projection onto the scale space $\mathcal{L}$. Define the operator valued functions

$$\varphi(\lambda) := I_{\mathcal{L}} + \lambda P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}\restriction_{\mathcal{L}}, \quad \psi(\lambda) := P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}\restriction_{\mathcal{L}} \quad (\lambda\in\rho(\widetilde{A})). \tag{2.3}$$

Clearly,

$$\varphi(\lambda)^* = \varphi(\overline{\lambda}), \quad \psi(\lambda)^* = \psi(\overline{\lambda}) \quad (\lambda\in\rho(\widetilde{A})). \tag{2.4}$$

Proposition 2.5. ([26]) *Let $\mathcal{H}$ be a Π_κ-space, let $\mathcal{L}$ be a Hilbert space and let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H}\oplus\mathcal{L}$. The pair of operator valued functions $\{\varphi,\psi\}$ associated with $\widetilde{A}$ via* (2.3) *is a normalized $N_{\kappa'}$-pair with $0\le\kappa'\le\kappa$. If, additionally, the linear relation $\widetilde{A}$ is $\mathcal{L}$-minimal then $\kappa'=\kappa$.*

Definition 2.6. The pair of operator valued functions $\{\varphi,\psi\}$ determined by (2.3) will be called the N_κ-pair corresponding to the selfadjoint linear relation $\widetilde{A}$ and the scale $\mathcal{L}$.

Note that for vector valued functions $\varphi(\lambda)$ and $\psi(\lambda)$ defined by (2.3) we have $\mathfrak{h}_{\varphi\psi} = \mathfrak{h}_\varphi = \mathfrak{h}_\psi$.

2.3. **Functional model of a selfadjoint linear relation.** Consider the Pontryagin space $\mathcal{H}(\varphi,\psi)$ with the reproducing kernel $\mathsf{N}_\omega^{\varphi\psi}(\lambda)$. It follows from (1.2) that the evaluation operator

$$E(\lambda): f\mapsto f(\lambda) \quad (f\in\mathcal{H}(\varphi,\psi))$$

is a bounded operator from $\mathcal{H}(\varphi,\psi)$ to $\mathcal{L}$. Note also that the set of functions $\{\mathsf{N}_\omega^{\varphi\psi}(\cdot)u : \omega\in\mathfrak{h}_{\varphi\psi},\ u\in\mathcal{L}\}$ is total in $\mathcal{H}(\varphi,\psi)$ ([1]).

In [26] author constructed A functional model of a selfadjoint linear relation $\widetilde{A}$ connected with a given N_κ-pair $\{\varphi,\psi\}$. Recall the main results of this work.

Theorem 2.7. ([26]) *Let $\mathcal{L}$ be a Hilbert space and let $\{\varphi,\psi\}$ be a normalized pair from $\widetilde{\mathbf{N}}_\kappa(\mathcal{L})$. Then the linear relation*

$$A(\varphi,\psi) = \left\{\left\{\begin{bmatrix} f\\ u\end{bmatrix},\begin{bmatrix} f'\\ u'\end{bmatrix}\right\} : \begin{array}{c} f,f'\in\mathcal{H}(\varphi,\psi);\ u,u'\in\mathcal{L};\\ f'(\lambda)-\lambda f(\lambda)=\varphi(\lambda)u-\psi(\lambda)u' \quad \lambda\in\mathfrak{h}_{\varphi\psi}\end{array}\right\} \tag{2.5}$$

is a selfadjoint linear relation in $\mathcal{H}(\varphi,\psi)\oplus\mathcal{L}$ and the pair $\{\varphi,\psi\}$ is the N_κ-pair corresponding to $A(\varphi,\psi)$ and $\mathcal{L}$.

Remark 2.8. The linear relation $A(\varphi,\psi)$ given by (2.5) is $\mathcal{L}$-minimal.

Remark 2.9. Let the function $m(\lambda)$ belong to the class $N_\kappa(\mathcal{L})$ and let the operator $I-\lambda m(\lambda)$ be invertible for all $\lambda\in\mathbb{C}_\pm\cap\mathfrak{h}_m$. Then the pair $\{I_{\mathcal{L}},\, m(\lambda)\}$ is equivalent to the normalized N_κ-pair

$$\{\varphi,\psi\} = \left\{\big(I_{\mathcal{L}}-\lambda m(\lambda)\big)^{-1},\ m(\lambda)\big(I_{\mathcal{L}}-\lambda m(\lambda)\big)^{-1}\right\},$$

and the corresponding functional model can be rewritten as

$$A(\varphi,\psi) = \left\{\left\{\begin{bmatrix} g\\ u\end{bmatrix},\begin{bmatrix} g'\\ u'\end{bmatrix}\right\} : \begin{array}{c} g,g'\in\mathcal{H}(m);\ u,u'\in\mathcal{L};\\ g'(\lambda)-\lambda g(\lambda)=u-m(\lambda)u' \quad \lambda\in\mathbb{C}_\pm\cap\mathfrak{h}_m\end{array}\right\}. \tag{2.6}$$

The functions f,f' from the functional model (2.5) are connected with the functions g,g' from the functional model (2.6) by the relationships

$$g(\lambda)=(I-\lambda m(\lambda))f(\lambda) \quad\text{and}\quad g'(\lambda)=(I-\lambda m(\lambda))f'(\lambda).$$

Proposition 2.10. ([26]) *For every normalized N_κ-pair $\{\varphi,\psi\}$ and $h\in\mathcal{H}(\varphi,\psi)$ the following identity holds:*

$$P_{\mathcal{L}}(A(\varphi,\psi)-\lambda)^{-1}\begin{bmatrix} h\\ 0\end{bmatrix} = h(\lambda) \quad (\lambda\in\mathfrak{h}_{\varphi\psi}). \tag{2.7}$$

The operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}(\varphi, \psi)$ given by the formula

$$h \mapsto (\mathcal{F}h)(\lambda) = P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}h \quad (h \in \mathcal{H}) \tag{2.8}$$

is called the *generalized Fourier transform* associated with $\widetilde{A}$ and the scale $\mathcal{L}$.

Theorem 2.11. ([26]) *Let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H} \oplus \mathcal{L}$ and let $\{\varphi, \psi\}$ be the corresponding N_κ-pair and let $\mathcal{H}_0$ be defined by* (2.2). *Then the generalized Fourier transform $\mathcal{F}$ maps isometrically the subspace $\mathcal{H}_0$ onto $\mathcal{H}(\varphi, \psi)$ and $\mathcal{F}$ is identically equal to* 0 *on $\mathcal{H} \ominus \mathcal{H}_0$.*

Corollary 2.12. *Let the linear relation $\widetilde{A}$ be $\mathcal{L}$-minimal. Then it is unitary equivalent to the linear relation $A(\varphi, \psi)$ via the formula*

$$A(\varphi, \psi) = \left\{ \left\{ \begin{bmatrix} \mathcal{F}f \\ u \end{bmatrix}, \begin{bmatrix} \mathcal{F}f' \\ u' \end{bmatrix} \right\} : \left\{ \begin{bmatrix} f \\ u \end{bmatrix}, \begin{bmatrix} f' \\ u' \end{bmatrix} \right\} \in \widetilde{A} \right\}. \tag{2.9}$$

The operator $\mathcal{F} \oplus I_{\mathcal{L}}$ establishes this unitary equivalence.

Proposition 2.13. ([26]) *For every $\left\{ \begin{bmatrix} f \\ u \end{bmatrix}, \begin{bmatrix} f' \\ u' \end{bmatrix} \right\} \in \widetilde{A}$ the following identity holds:*

$$E(\lambda)\mathcal{F}(f' - \lambda f) = \begin{bmatrix} \varphi(\lambda) & -\psi(\lambda) \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}. \tag{2.10}$$

3. Solubility of the abstract interpolation problem

3.1. **Formulation of the problem.** Let $\mathcal{X}$ be a complex linear space, let $\mathcal{L}$ be a Hilbert space, let B_1, B_2 be linear operators in $\mathcal{X}$, let C_1, C_2 be linear operators from $\mathcal{X}$ to $\mathcal{L}$. Let K be a sesquilinear form on $\mathcal{X}$ which has ν negative squares. Consider the following continuous analog of the abstract interpolation problem.

Problem $AIP_\kappa(B_1, B_2, C_1, C_2, K)$. Let the data set (B_1, B_2, C_1, C_2, K) satisfies the assumptions (A1)-(A2). Find a N_κ-pair $\{\varphi, \psi\} \in \widetilde{N}(\mathcal{L})$ such that for some linear mapping $F : \mathcal{X} \to \mathcal{H}(\varphi, \psi)$ the following conditions hold:

(C1) $(FB_2h)(\lambda) - \lambda(FB_1h)(\lambda) = \begin{bmatrix} \varphi(\lambda) & -\psi(\lambda) \end{bmatrix} \begin{bmatrix} C_1h \\ C_2h \end{bmatrix}$, for all $h \in \mathcal{X}$;

(C2) $\langle Fh, Fh \rangle_{\mathcal{H}(\varphi,\psi)} \leq K(h, h)$, for all $h \in \mathcal{X}$.

The AIP with $\kappa = \nu = 0$ had been considered in [10].

Remark 3.1. For the solvability of the problem it is necessary that $\kappa \geq \nu$. Indeed, let us decompose the space $\mathcal{X}$ into the direct sum of the negative subspace $\mathcal{X}_-$ and the non-negative subspace $\mathcal{X}_+$ according to the form $K(h, h)$

$$\mathcal{X} = \mathcal{X}_- \dot{+} \mathcal{X}_+. \tag{3.1}$$

Also $\dim \operatorname{clos}(\mathcal{X}_-) = \nu < \infty$ hence the set $\mathcal{X}_-$ is the close subspace of the space $\mathcal{X}$ it follows from (C2) that $\langle Fh, Fh \rangle^2_{\mathcal{H}(\varphi,\psi)} < 0$ for any $h (\neq 0) \in \mathcal{X}_-$. Thus the linear mapping F is not equal to 0 at the subspace $\mathcal{X}_-$ (except at the zero-vector). Hence the restriction of the mapping F to the finite-dimensional subspace $\mathcal{X}_-$ is invertible and thus it saves the dimension

$$\dim F(\mathcal{X}_-) = \nu.$$

The set $F(\mathcal{X}_-)$ is a part of the negative subspaces of $\mathcal{H}(\varphi, \psi)$. Therefore the dimension of the negative subspace of the space $\mathcal{H}(\varphi, \psi)$ is not less than ν.

Let the space $\mathcal{H}$ be the completion of the space $\mathcal{X}$ with respect to the scalar product

$$\langle h, g \rangle_{\mathcal{H}} = K(h, g), \quad h, g \in \mathcal{X}. \tag{3.2}$$

Clearly the space $\mathcal{H}$ is a Pontryagin space with the negative index ν. We identify linear operators $B_1,\ B_2 : \mathcal{X} \to \mathcal{X}$ with linear operators $B_1,\ B_2 : \mathcal{X} \to \mathcal{H}$.

3.2. **Structure of the solutions.**

Proposition 3.2. *Let the data set (B_1, B_2, C_1, C_2, K) satisfies the assumption (A1). Then the linear relation*

$$\widehat{A} = \left\{ \left\{ \begin{bmatrix} B_1 h \\ C_1 h \end{bmatrix}, \begin{bmatrix} B_2 h \\ C_2 h \end{bmatrix} \right\} : h \in \mathcal{X} \right\} \tag{3.3}$$

is symmetric in $\mathcal{H} \oplus \mathcal{L}$.

Proof. The statement is implied by (A1) since

$$\begin{aligned} &\left\langle \begin{bmatrix} B_2 h \\ C_2 h \end{bmatrix}, \begin{bmatrix} B_1 h \\ C_1 h \end{bmatrix} \right\rangle_{\mathcal{H}\oplus\mathcal{L}} - \left\langle \begin{bmatrix} B_1 h \\ C_1 h \end{bmatrix}, \begin{bmatrix} B_2 h \\ C_2 h \end{bmatrix} \right\rangle_{\mathcal{H}\oplus\mathcal{L}} \\ &= K(B_2 h, B_1 h) + (C_2 h, C_1 h)_{\mathcal{L}} - K(B_1 h, B_2 h) - (C_1 h, C_2 h)_{\mathcal{L}} = 0. \end{aligned}$$

□

Theorem 3.3. *Let the data set (B_1, B_2, C_1, C_2, K) satisfies the assumptions (A1)–(A2). Then the Problem $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ is solvable and the set of its normalized solutions is parametrized by the formula*

$$\begin{bmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{bmatrix} = \begin{bmatrix} I_{\mathcal{L}} & 0 \\ \lambda & I_{\mathcal{L}} \end{bmatrix} \begin{bmatrix} P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1} \upharpoonright_{\mathcal{L}} \\ I_{\mathcal{L}} \end{bmatrix}, \tag{3.4}$$

where $\widetilde{A}$ ranges over the set of all selfadjoint $\mathcal{L}$-regular extensions of $\widehat{A}$ with the exit in a Pontryagin space $\widetilde{\mathcal{H}} \oplus \mathcal{L} \supset \mathcal{H} \oplus \mathcal{L}$. The corresponding linear mapping $F : \mathcal{X} \to \mathcal{H}(\varphi, \psi)$ is given by

$$(Fh)(\lambda) = P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1} h, \quad h \in \mathcal{X}. \tag{3.5}$$

Proof. Sufficiency. Let $\widetilde{A}$ be a selfadjoint extension of $\widehat{A}$ and let $\{\varphi, \psi\}$ be the normalized N_κ-pair corresponding to $\widetilde{A}$ and the scale $\mathcal{L}$ and let $\mathcal{F} : \widetilde{\mathcal{H}} \to \mathcal{H}(\varphi, \psi)$ be the corresponding generalized Fourier transform. Then the formula (3.4) is implied by (2.3).

Let $\mathcal{F} : \widetilde{\mathcal{H}} \to \mathcal{H}(\varphi, \psi)$ be the generalized Fourier transform associated with $\widetilde{A}$ and the scale $\mathcal{L}$. In view of (2.8) the linear mapping $F : \mathcal{X} \to \mathcal{H}(\varphi, \psi)$ given by (3.5) is connected to $\mathcal{F}$ via the formula

$$Fh = \mathcal{F}h \quad (h \in \mathcal{X}). \tag{3.6}$$

Since $\mathcal{F}$ satisfies the identity (2.10) and

$$\left\{ \begin{bmatrix} B_1 h \\ C_1 h \end{bmatrix}, \begin{bmatrix} B_2 h \\ C_2 h \end{bmatrix} \right\} \in \widehat{A} \subset \widetilde{A}$$

one obtains from (2.10)

$$\begin{aligned} (FB_2 h)(\lambda) - \lambda(FB_1 h)(\lambda) &= (\mathcal{F}B_2 h)(\lambda) - \lambda(\mathcal{F}B_1 h)(\lambda) \\ &= \begin{bmatrix} \varphi(\lambda) & -\psi(\lambda) \end{bmatrix} \begin{bmatrix} C_1 h \\ C_2 h \end{bmatrix} \quad \forall h \in \mathcal{X}. \end{aligned} \tag{3.7}$$

Thus we have shown that the condition (C1) holds.

Since $\widetilde{A}$ is the $\mathcal{L}$-regular extensions of $\widehat{A}$ then one obtains from Theorem 2.11

$$\langle \mathcal{F}h, \mathcal{F}h \rangle_{\mathcal{H}(\varphi,\psi)} \le \langle h, h \rangle_{\mathcal{H}} \quad (h \in \mathcal{H}). \tag{3.8}$$

Indeed, the generalized Fourier transform $\mathcal{F}$ maps isometrically the Pontryagin space $\mathcal{H}_0$ on $\mathcal{H}(\varphi, \psi)$ and is identically equal to 0 on a Hilbert space $\mathcal{H} \ominus \mathcal{H}_0$.

Next, it follows from (3.6) and (3.2) that for all $h \in \mathcal{X}$

$$\langle Fh, Fh \rangle_{\mathcal{H}\langle\varphi,\psi\rangle} = \langle \mathcal{F}h, \mathcal{F}h \rangle_{\mathcal{H}(\varphi,\psi)} \le \langle h, h \rangle_{\mathcal{H}} = K(h, h),$$

since $\widetilde{A}$ is $\mathcal{L}$-regular. This proves (C2).

Hence N_κ-pair $\{\varphi,\psi\}$ is a solution of the AIP_κ.

Necessity. Let a normalized N_κ-pair $\{\varphi,\psi\}$ be a solution of the AIP_κ and let the mapping $F:\mathcal{X}\to\mathcal{H}(\varphi,\psi)$ satisfies (C1)–(C2). We will construct a selfadjoint $\mathcal{L}$-regular exit space extension $\widetilde{A}$ of $\widehat{A}$ such that (3.4) and (3.5) hold.

Step 1. Isometric embedding of $\mathcal{H}$. Consider the mapping $\widehat{F}:\mathcal{H}\to\mathcal{H}(\varphi,\psi)$ as an extension of $F:\mathcal{X}\to\mathcal{H}(\varphi,\psi)$ to the space $\mathcal{H}\supset\mathcal{X}$.

The mapping $\widehat{F}$ is contractive due to (C2)

$$\left\langle(\widehat{F}h)(\lambda),(\widehat{F}h)(\lambda)\right\rangle_{\mathcal{H}(\varphi,\psi)}=\left\langle(Fh)(\lambda),(Fh)(\lambda)\right\rangle_{\mathcal{H}(\varphi,\psi)}\le K(h,h)=\langle h,h\rangle_{\mathcal{H}}.$$

Thus the operator $I_{\mathcal{H}}-\widehat{F}^*\widehat{F}:\mathcal{H}\to\mathcal{H}$ is nonnegative.

Let $D=D^*(\ge 0)$ be the defect operator of the contraction $\widehat{F}$ defined by

$$D^2=I-\widehat{F}^*\widehat{F}:\mathcal{H}\to\mathcal{H} \tag{3.9}$$

and let $\mathcal{D}=\overline{\operatorname{ran}}\,D$ be the defect subspace of $\widehat{F}$ in $\mathcal{H}$. The space $\mathcal{D}$ is Hilbert space since the operator D is nonnegative. Consider the column extension $\widetilde{F}$ of the operator $\widehat{F}$ to the isometric mapping from $\mathcal{H}$ to $\mathcal{D}\oplus\mathcal{H}(\varphi,\psi)$ by the formula

$$\widetilde{F}h=\begin{bmatrix} Dh\\ \widehat{F}h\end{bmatrix},\quad h\in\mathcal{H}. \tag{3.10}$$

Step 2. Construction of a selfadjoint linear relation $\widetilde{A}$. Let $A_{\mathcal{D}}$ be a linear relation in $\mathcal{D}$ defined by

$$A_{\mathcal{D}}=\left\{\{DB_1h,DB_2h\}:h\in\mathcal{X}\right\}.$$

We will prove that $A_{\mathcal{D}}$ is a symmetric linear relation in $\mathcal{D}$. We must show that

$$\langle DB_2h,DB_1h\rangle_{\mathcal{H}}-\langle DB_1h,DB_2h\rangle_{\mathcal{H}}=0. \tag{3.11}$$

It follows from (3.9) that

$$\begin{aligned}\langle DB_2h,&DB_1h\rangle_{\mathcal{H}}-\langle DB_1h,DB_2h\rangle_{\mathcal{H}}\\ &=\langle(I-\widehat{F}^*\widehat{F})B_2h,B_1h\rangle_{\mathcal{H}}-\langle(I-\widehat{F}^*\widehat{F})B_1h,B_2h\rangle_{\mathcal{H}}\\ &=\langle B_2h,B_1h\rangle_{\mathcal{H}}-\langle B_1h,B_2h\rangle_{\mathcal{H}}\\ &-\langle\widehat{F}B_2h,\widehat{F}B_1h\rangle_{\mathcal{H}(\varphi,\psi)}+\langle\widehat{F}B_1h,\widehat{F}B_2h\rangle_{\mathcal{H}(\varphi,\psi)}\\ &=K(B_2h,B_1h)-K(B_1h,B_2h)\\ &-\langle\widehat{F}B_2h,\widehat{F}B_1h\rangle_{\mathcal{H}(\varphi,\psi)}+\langle\widehat{F}B_1h,\widehat{F}B_2h\rangle_{\mathcal{H}(\varphi,\psi)}.\end{aligned} \tag{3.12}$$

As follows from (C1)

$$\begin{aligned}(\widehat{F}B_2h)(\lambda)-\lambda(\widehat{F}B_1h)(\lambda)&=(FB_2h)(\lambda)-\lambda(FB_1h)(\lambda)\\ &=\begin{bmatrix}\varphi(\lambda) & -\psi(\lambda)\end{bmatrix}\begin{bmatrix}C_1h\\ C_2h\end{bmatrix}\quad\forall h\in\mathcal{X}.\end{aligned} \tag{3.13}$$

The Theorem 2.7 implies that

$$\left\{\begin{bmatrix}\widehat{F}B_1h\\ C_1h\end{bmatrix},\begin{bmatrix}\widehat{F}B_2h\\ C_2h\end{bmatrix}\right\}\in A(\varphi,\psi).$$

Since $A(\varphi,\psi)$ is a selfadjoint linear relation then

$$\langle\widehat{F}B_2h,\widehat{F}B_1h\rangle_{\mathcal{H}(\varphi,\psi)}+(C_2h,C_1h)_{\mathcal{L}}-\langle\widehat{F}B_1h,\widehat{F}B_2h\rangle_{\mathcal{H}(\varphi,\psi)}-(C_1h,C_2h)_{\mathcal{L}}=0.$$

Therefore the right hand part of (3.12) can be rewritten as

$$K(B_2h,B_1h)-K(B_1h,B_2h)-(C_1h,C_2h)_{\mathcal{L}}+(C_2h,C_1h)_{\mathcal{L}},$$

which is vanishing due to (A1). Thus the equality (3.11) has showed and hence the relation $A_{\mathcal{D}}$ is a symmetric linear relation in $\mathcal{D}$.

Let $\widetilde{A}_{\mathcal{D}}$ be a selfadjoint extension of $A_{\mathcal{D}}$ in a Hilbert space $\widetilde{\mathcal{D}} \supset \mathcal{D}$ and let

$$\widetilde{A} = \widetilde{A}_{\mathcal{D}} \oplus A(\varphi, \psi). \tag{3.14}$$

Note that $\widetilde{A}$ is a selfadjoint linear relation as a direct sum of selfadjoint linear relations.

Step 3. Linear relation $\widetilde{A}$ satisfies (3.4) *and* (3.5). Under the identification of the vector $h \in \mathcal{H}$ with $\widetilde{F}h$ the symmetric linear relation $\widehat{A}$ in $\mathcal{H} \oplus \mathcal{L}$ can be identified with the symmetric linear relation

$$\begin{aligned} A_1 &= (\widetilde{F} \oplus I_{\mathcal{L}})\widehat{A}(\widetilde{F} \oplus I_{\mathcal{L}})^{-1} \\ &= \left\{ \left\{ \begin{bmatrix} DB_1h \\ \widehat{F}B_1h \\ C_1h \end{bmatrix}, \begin{bmatrix} DB_2h \\ \widehat{F}B_2h \\ C_2h \end{bmatrix} \right\} : h \in \mathcal{X} \right\} \end{aligned}$$

in $\widetilde{\mathcal{H}} := \widetilde{\mathcal{D}} \oplus \mathcal{H}(\varphi, \psi) \oplus \mathcal{L}$. Moreover, it follows from the results of Step 2 of this theorem that A_1 is contained in the selfadjoint linear relation $\widetilde{A} = \widetilde{A}_{\mathcal{D}} \oplus A(\varphi, \psi)$. Indeed, $\{DB_1h, DB_2h\} \in A_{\mathcal{D}} \subset \widetilde{A}_{\mathcal{D}}$ and

$$\left\{ \begin{bmatrix} \widehat{F}B_1h \\ C_1h \end{bmatrix}, \begin{bmatrix} \widehat{F}B_2h \\ C_2h \end{bmatrix} \right\} \in A(\varphi, \psi).$$

The formula (3.4) is implied by the analogous formula for $A(\varphi, \psi)$

$$\begin{bmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{bmatrix} = \begin{bmatrix} I_{\mathcal{L}} & 0 \\ \lambda I_{\mathcal{L}} & I_{\mathcal{L}} \end{bmatrix} \begin{bmatrix} P_{\mathcal{L}}(A(\varphi, \psi) - \lambda)^{-1}\restriction_{\mathcal{L}} \\ I_{\mathcal{L}} \end{bmatrix}$$

since

$$P_{\mathcal{L}}(A(\varphi, \psi) - \lambda)^{-1}\restriction_{\mathcal{L}} = P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}\restriction_{\mathcal{L}}.$$

It follows from the definition of the linear relation $\widetilde{A}$ and the mapping $\widehat{F}$ that for all $h \in \mathcal{X}$

$$\begin{aligned} P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1} \begin{bmatrix} \widetilde{F}h \\ 0 \end{bmatrix} &= P_{\mathcal{L}}(A(\varphi, \psi) - \lambda)^{-1} \begin{bmatrix} \widehat{F}h \\ 0 \end{bmatrix} \\ &= P_{\mathcal{L}}(A(\varphi, \psi) - \lambda)^{-1} \begin{bmatrix} Fh \\ 0 \end{bmatrix}. \end{aligned} \tag{3.15}$$

Next, substituting (2.7) in the equality (3.15) we get

$$P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1} \begin{bmatrix} \widetilde{F}h \\ 0 \end{bmatrix} = (Fh)(\lambda).$$

In view of the identification $\widetilde{F}h$ and h this completes the proof of the formula (3.5).

Step 4. The $\mathcal{L}$-regularity of the extensions $\widetilde{A}$. Consider the space

$$\widetilde{\mathcal{H}}_0 = \overline{\text{span}} \Big\{ P_{\widetilde{\mathcal{D}} \oplus \mathcal{H}(\varphi, \psi)}(\widetilde{A} - \lambda)^{-1}\mathcal{L} : \quad \lambda \in \rho(\widetilde{A}) \Big\}.$$

Since $A(\varphi, \psi)$ in $\mathcal{L}$-minimal and

$$P_{\widetilde{\mathcal{D}} \oplus \mathcal{H}(\varphi, \psi)}(\widetilde{A} - \lambda)^{-1}\restriction_{\mathcal{L}} = P_{\mathcal{H}(\varphi, \psi)}(A(\varphi, \psi) - \lambda)^{-1}\restriction_{\mathcal{L}} ,$$

then $\widetilde{\mathcal{H}}_0 = \mathcal{H}(\varphi, \psi)$. Thus the subspace

$$\Big(\widetilde{\mathcal{D}} \oplus \mathcal{H}(\varphi, \psi) \Big) \ominus \widetilde{\mathcal{H}}_0$$

coincides with the Hilbert space $\widetilde{\mathcal{D}}$. □

4. Description of solutions of abstract interpolation problem

In this section we will assume that the number of negative squares of the kernel $\mathsf{N}_{\omega}^{\varphi\psi}(\lambda)$ is equal to the number of negative squares of the form K. In other words we will study AIP_κ with $\kappa = \nu$.

Let $\widetilde{A}$ be a selfadjoint $\mathcal{L}$-regular extensions of A with the exit in a Pontryagin space $\widetilde{\mathcal{H}} \oplus \mathcal{L}$.

Definition 4.1. The compression $\mathbf{R}_\lambda = P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathcal{L}$ of the resolvent of $\widetilde{A}$ on the space $\mathcal{L}$ is said to be the $\mathcal{L}$-resolvent of the relation $\widetilde{A}$.

The $\mathcal{L}$-resolvent $\mathbf{R}_\lambda$ is said to be $\mathcal{L}$-regular if $\widetilde{A}$ is the $\mathcal{L}$-regular lineal relation.

In view of Theorem 3.3 a description of the set of solutions of the AIP_κ is reduced to the description of $\mathcal{L}$-regular $\mathcal{L}$-resolvents of the linear relation $\widehat{A}$.

4.1. **Symmetric linear relation** A. Let us impose some additional assumptions on the data set (B_1, B_2, C_1, C_2, K):

(A3) $B_2 = I_{\mathcal{X}}$ and the operators $B_1 : \mathcal{X} \subseteq \mathcal{H} \to \mathcal{H}$, $C_1, C_2 : \mathcal{X} \subseteq \mathcal{H} \to \mathcal{L}$ are bounded.

The symmetric linear relation $\widehat{A}$ can be rewritten as

$$\widehat{A} = \left\{ \left\{ \begin{bmatrix} B_1 h \\ C_1 h \end{bmatrix}, \begin{bmatrix} h \\ C_2 h \end{bmatrix} \right\} : h \in \mathcal{X} \right\}. \tag{4.1}$$

In view of (A3) the closures of $\widehat{A}$ take the form

$$A := \operatorname{clos} \widehat{A} = \left\{ \left\{ \begin{bmatrix} B_1 h \\ C_1 h \end{bmatrix}, \begin{bmatrix} h \\ C_2 h \end{bmatrix} \right\} : h \in \mathcal{H} \right\}. \tag{4.2}$$

A point $\lambda \in \mathbb{C}$ is said to be a regular type point for a closed symmetric linear relation A if $\operatorname{ran}(A - \lambda)$ is closed in $\mathcal{H} \oplus \mathcal{L}$. Let $\widehat{\rho}(A)$ be the set of all regular type points for linear relation A and let $\rho_s(A, \mathcal{L}) := \rho(A, \mathcal{L}) \cap \overline{\rho(A, \mathcal{L})}$. It is well known ([13]) that for symmetric linear relation A in Π_κ-space $\widehat{\rho}(A)$ coincides with $\mathbb{C}_+ \cup \mathbb{C}_-$ (with a possible exception of at most κ pairs of complex numbers symmetric with respect to the real axis) and the defect subspaces

$$\mathfrak{N}_\lambda(A) := (\mathcal{H} \oplus \mathcal{L}) \ominus \operatorname{ran}(A - \bar{\lambda})$$

have the same dimensions $n_+(A)$ and $n_-(A)$ for $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$, respectively, which are called the defect numbers of the symmetric linear relation A. In the following proposition we show, that the symmetric linear relation A in (4.2) has equal defect numbers $n_+(A) = n_-(A) = \dim \mathcal{L}$ and, moreover, $0 \in \widehat{\rho}(A)$.

Proposition 4.2. *Let the data set (B_1, B_2, C_1, C_2, K) satisfy the assumptions (A1)–(A3). Then*

1) *the adjoint linear relation A^* takes the form*

$$A^* = \left\{ \widehat{g} = \left\{ \begin{bmatrix} g \\ v \end{bmatrix}, \begin{bmatrix} g' \\ v' \end{bmatrix} \right\} : \begin{array}{c} v, v' \in \mathcal{L}, g' \in \mathcal{H}; \\ g = B_1^* g' + C_1^* v' - C_2^* v \end{array} \right\}; \tag{4.3}$$

2) *the set $\widehat{\rho}(A)$ of regular type points for symmetric linear relation A contains the resolvent set of the linear relation B_1^{-1}*

$$\rho(B_1^{-1}) = \left\{ \lambda \in (\mathbb{C} \setminus \{0\}) : 1/\lambda \in \rho(B_1) \right\} \cup \{0\},$$

and the defect subspace $\mathfrak{N}_\lambda(A)$ for $\lambda \in \rho(B_1^{-1})$ consists of vectors

$$\begin{bmatrix} -F(\overline{\lambda})^* u \\ u \end{bmatrix}, \quad u \in \mathcal{L}, \tag{4.4}$$

where the function $F(\lambda):\mathcal{H}\to\mathcal{L}$ is defined by

$$F(\lambda)=(C_2-\lambda C_1)(I_{\mathcal{H}}-\lambda B_1)^{-1}. \tag{4.5}$$

Proof. 1) Let

$$\widehat{g}=\left\{\begin{bmatrix} g \\ v \end{bmatrix},\begin{bmatrix} g' \\ v' \end{bmatrix}\right\}\in A^* \quad (g,g'\in\mathcal{H};v,v'\in\mathcal{L}).$$

Then it follows from (4.2) that

$$\langle g',B_1h\rangle_{\mathcal{H}}-\langle g,h\rangle_{\mathcal{H}}+(v',C_1h)_{\mathcal{L}}-(v,C_2h)_{\mathcal{L}}=0$$

for all $h\in\mathcal{H}$. From the nondegeneracy of the Pontryagin space $\mathcal{H}$ it follows that

$$B_1^*g'-g+C_1^*v'-C_2^*v=0$$

or in the equivalent form

$$g=B_1^*g'+C_1^*v'-C_2^*v. \tag{4.6}$$

2) It follows from (4.2) that

$$A-\lambda=\left\{\left\{\begin{bmatrix} B_1h \\ C_1h \end{bmatrix},\begin{bmatrix} (I_{\mathcal{H}}-\lambda B_1)h \\ (C_2-\lambda C_1)h \end{bmatrix}\right\}:h\in\mathcal{H}\right\}, \tag{4.7}$$

and hence

$$\operatorname{ran}(A-\lambda)=\left\{\begin{bmatrix} h \\ F(\lambda)h \end{bmatrix}:\ h\in\mathcal{H}\right\},$$

where $F(\lambda)$ is given by (4.5). Therefore the set $\operatorname{ran}(A-\lambda)$ in closed for all $\lambda\in\rho(B_1^{-1})$.

If $\lambda\in\rho(B_1^{-1})$ and $\widehat{g}\in\mathfrak{N}_\lambda(A)=\ker(A^*-\lambda)$ then $g'=\lambda g$, $v'=\lambda v$. Substituting these equalities in (4.6) one obtains

$$(I_{\mathcal{H}}-\lambda B_1^*)g=-(C_2^*-\lambda C_1^*)v.$$

This proves the second statement since $g=-F(\bar{\lambda})^*v$. □

4.2. Kreĭn's representation theory for symmetric linear relation A.

Recall some facts from M. G. Kreĭn's representation theory following ([20], [12]).

Definition 4.3. Let A a symmetric linear relation in $\mathcal{H}\oplus\mathcal{L}$. It is said that λ is $\mathcal{L}$-regular point for A (in short $\lambda\in\rho(A,\mathcal{L})$) if λ s a regular type point for A and

$$\mathcal{H}\oplus\mathcal{L}=\operatorname{ran}(A-\lambda)\dot{+}\mathcal{L}. \tag{4.8}$$

It is also put $\rho_s(A,\mathcal{L}):=\rho(A,\mathcal{L})\cap\overline{\rho(A,\mathcal{L})}$.

For every $\lambda\in\rho(A,\mathcal{L})$ the operator valued function $\mathcal{P}(\lambda):\mathcal{H}\to\mathcal{L}$ is defined as a skew projection onto $\mathcal{L}$ in the decomposition (4.8) and $\mathcal{Q}(\lambda):\mathcal{H}\to\mathcal{L}$ is given by

$$\mathcal{Q}(\lambda)=P_{\mathcal{L}}(A-\lambda)^{-1}(I-\mathcal{P}(\lambda)),\quad \lambda\in\rho(A,\mathcal{L}). \tag{4.9}$$

Let the matrix $J\in\mathcal{L}\oplus\mathcal{L}$ be given by

$$J=\begin{bmatrix} 0 & -iI_{\mathcal{L}} \\ iI_{\mathcal{L}} & 0 \end{bmatrix}.$$

In the next theorem we will describe $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ through the data set of the problem AIP_κ.

Theorem 4.4. *Let B_1, B_2, C_1, C_2, K satisfy the assumptions* (A1)–(A3). *We define the operator valued function $F(\lambda)$ by the formula* (4.5). *Then*

1) *$\rho(A,\mathcal{L}) = \rho(B_1^{-1})$ and for $\lambda \in \rho(A,\mathcal{L})$ the operator valued functions $\mathcal{P}(\lambda)$, $\mathcal{Q}(\lambda)$ are given by*

$$\mathcal{P}(\lambda)\begin{bmatrix} f \\ u \end{bmatrix} = u - F(\lambda)f, \quad f \in \mathcal{H}, \quad u \in \mathcal{L}, \tag{4.10}$$

$$\mathcal{Q}(\lambda)\begin{bmatrix} f \\ u \end{bmatrix} = C_1(I_{\mathcal{H}} - \lambda B_1)^{-1}f, \quad f \in \mathcal{H}. \tag{4.11}$$

2) *The adjoint operators to $\mathcal{P}(\lambda), \mathcal{Q}(\lambda) : \begin{bmatrix} \mathcal{H} \\ \mathcal{L} \end{bmatrix} \to \mathcal{L}$ take the form*

$$\mathcal{P}(\lambda)^* u = \begin{bmatrix} -F(\lambda)^* u \\ u \end{bmatrix} \quad u \in \mathcal{L}, \quad \lambda \in \rho(A,\mathcal{L}), \tag{4.12}$$

$$\mathcal{Q}(\lambda)^* u = \begin{bmatrix} (I_{\mathcal{H}} - \bar{\lambda} B_1^*)^{-1} C_1^* u \\ 0 \end{bmatrix} \quad u \in \mathcal{L}, \quad \lambda \in \rho(A,\mathcal{L}). \tag{4.13}$$

Proof. 1) Assume that $\lambda \in \rho(A,\mathcal{L})$ and the decomposition (4.8) holds. Then for $f \in \mathcal{H}$, $u \in \mathcal{L}$ there are unique $h \in \mathcal{H}$ and $v \in \mathcal{L}$ such that

$$(I_{\mathcal{H}} - \lambda B_1)h = f, \quad (C_2 - \lambda C_1)h + v = u. \tag{4.14}$$

This implies, in particular, that $\lambda \in \rho(B_1^{-1})$.

Conversely, if $\lambda \in \rho(B_1^{-1})$, then the equations (4.14) have unique solutions $h \in \mathcal{H}$ and $v \in \mathcal{L}$. Hence $\lambda \in \rho(A,\mathcal{L})$. In view of (4.14) these solutions take the form

$$h = (I_{\mathcal{H}} - \lambda B_1)^{-1} f, \quad v = \mathcal{P}(\lambda)\begin{bmatrix} f \\ u \end{bmatrix} = u - F(\lambda)f. \tag{4.15}$$

It follows from (4.9), (4.15) and (4.7) that

$$\begin{aligned}
\mathcal{Q}(\lambda)\begin{bmatrix} f \\ u \end{bmatrix} &= P_{\mathcal{L}}(A-\lambda)^{-1}(I - \mathcal{P}(\lambda))\begin{bmatrix} f \\ u \end{bmatrix} \\
&= P_{\mathcal{L}}(A-\lambda)^{-1}\begin{bmatrix} f \\ F(\lambda)f \end{bmatrix} \\
&= P_{\mathcal{L}}\begin{bmatrix} B_1(I_{\mathcal{H}} - \lambda B_1)^{-1} f \\ C_1(C_2 - \lambda C_1)^{-1} F(\lambda) f \end{bmatrix} \\
&= C_1(I_{\mathcal{H}} - \lambda B_1)^{-1} f.
\end{aligned}$$

2) The formulas (4.12), (4.13) are implied by (4.10), (4.11) and the equalities

$$\left\langle \mathcal{P}(\lambda)^* v, \begin{bmatrix} f \\ u \end{bmatrix} \right\rangle_{\mathcal{H}\oplus\mathcal{L}} = (v, u - F(\lambda)f)_{\mathcal{L}} = \left\langle \begin{bmatrix} -F(\lambda)^* v \\ v \end{bmatrix}, \begin{bmatrix} f \\ u \end{bmatrix} \right\rangle_{\mathcal{H}\oplus\mathcal{L}},$$

$$\begin{aligned}
\left\langle \mathcal{Q}(\lambda)^* v, \begin{bmatrix} f \\ u \end{bmatrix} \right\rangle_{\mathcal{H}\oplus\mathcal{L}} &= (v, C_1(I_{\mathcal{H}} - \lambda B_1)^{-1} f)_{\mathcal{L}} \\
&= \left\langle \begin{bmatrix} (I_{\mathcal{H}} - \lambda B_1^*)^{-1} C_1^* v \\ 0 \end{bmatrix}, \begin{bmatrix} f \\ u \end{bmatrix} \right\rangle_{\mathcal{H}\oplus\mathcal{L}}.
\end{aligned}$$

□

4.3. **The boundary triplet for A^*.** Let us recall the definition of a boundary triplet which will be used later in calculation of $\mathcal{L}$-resolvent matrix of the operator A. For a Hilbert space densely determined symmetric operator A it was introduced in [18] (see also [14]). To the case of non-densely defined symmetric operator it was extended in [25] (see also [12]). For a Pontryagin space symmetric operator it was extended in [8].

Definition 4.5. A triplet $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ where $\Gamma_i : A^* \to \mathcal{L}$, $i = 1, 2$, is said to be a boundary triplet for A^* if for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in A^*$ the abstract Green's formula holds

$$\langle f', g\rangle_{\mathcal{H}\oplus\mathcal{L}} - \langle f, g'\rangle_{\mathcal{H}\oplus\mathcal{L}} = (\Gamma_1\widehat{f}, \Gamma_2\widehat{g})_{\mathcal{L}} - (\Gamma_2\widehat{f}, \Gamma_1\widehat{g})_{\mathcal{L}} \tag{4.16}$$

and the mapping $\Gamma := \begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix} : A^* \to \begin{bmatrix}\mathcal{L}\\ \mathcal{L}\end{bmatrix}$ is surjective.

As was proved in [8] the set of all selfadjoint extensions $\widetilde{A}$ of A can be parametrized by the set of selfadjoint linear relations τ in $\mathcal{L}$ via the formula

$$f \in \widetilde{A} \Leftrightarrow \Gamma f \in \tau.$$

Let the operator-valued functions $\widehat{\mathcal{P}}(\lambda)^*$ and $\widehat{\mathcal{Q}}(\lambda)^*$ be given by

$$\widehat{\mathcal{P}}(\lambda)^* u = \{\mathcal{P}(\lambda)^* u, \bar{\lambda}\mathcal{P}(\lambda)^* u\}, \quad u \in \mathcal{L}, \tag{4.17}$$

$$\widehat{\mathcal{Q}}(\lambda)^* u = \{\mathcal{Q}(\lambda)^* u, u + \bar{\lambda}\mathcal{Q}(\lambda)^* u\}, \quad u \in \mathcal{L}, \tag{4.18}$$

where $\mathcal{P}(\lambda)^*$, $\mathcal{Q}(\lambda)^* : \mathcal{L} \to \mathcal{H}$ are adjoint operators to $\mathcal{P}(\lambda)$, $\mathcal{Q}(\lambda) : \mathcal{H} \to \mathcal{L}$.

Theorem 4.6. ([8], [11], [12]) *Let $\lambda \in \rho_s(A, \mathcal{L})$ then the linear relation A^* can be decomposed in the following direct sum:*

$$A^* = A \dotplus \widehat{\mathcal{P}}(\lambda)^*\mathcal{L} \dotplus \widehat{\mathcal{Q}}(\lambda)^*\mathcal{L}. \tag{4.19}$$

Proposition 4.7. *Let the data set (B_1, B_2, C_1, C_2, K) satisfy the assumptions (A1)-(A3). Then a boundary triplet $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ for A^* can be defined by*

$$\Gamma_1\widehat{g} = v - C_1 g', \quad \Gamma_2\widehat{g} = -v' + C_2 g'. \tag{4.20}$$

Proof. For two vectors

$$\widehat{f} = \left\{\begin{bmatrix} f\\ u\end{bmatrix}, \begin{bmatrix} f'\\ u'\end{bmatrix}\right\}, \quad \widehat{g} = \left\{\begin{bmatrix} g\\ v\end{bmatrix}, \begin{bmatrix} g'\\ v'\end{bmatrix}\right\} \in A^*$$

one obtains

$$\begin{aligned}\langle f', g\rangle_{\mathcal{H}} - \langle f, g'\rangle_{\mathcal{H}} + (u', v)_{\mathcal{L}} - (u, v')_{\mathcal{L}} &= (u', v)_{\mathcal{L}} - (u, v')_{\mathcal{L}}\\ &+ \langle f', B_1^* g' + C_1^* v' - C_2^* v\rangle_{\mathcal{H}} - \langle B_1^* f' + C_1^* u' - C_2^* u, g'\rangle_{\mathcal{H}}.\end{aligned} \tag{4.21}$$

It follows from (1) that the right hand part of (4.21) takes the form

$$\begin{aligned}&\langle B_1 f', g'\rangle_{\mathcal{H}} - \langle f', B_1 g'\rangle_{\mathcal{H}} + (C_1 f', v')_{\mathcal{L}} - (C_2 f', v)_{\mathcal{L}}\\ &\quad - (u', C_1 g')_{\mathcal{L}} + (u, C_2 g')_{\mathcal{L}} + (u', v)_{\mathcal{L}} - (u, v')_{\mathcal{L}}\\ &\quad = (C_2 f', C_1 g')_{\mathcal{L}} - (C_1 f', C_2 g')_{\mathcal{L}} + (C_1 f', v')_{\mathcal{L}} - (C_2 f', v)_{\mathcal{L}}\\ &\quad - (u', C_1 g')_{\mathcal{L}} + (u, C_2 g')_{\mathcal{L}} + (u', v)_{\mathcal{L}} - (u, v')_{\mathcal{L}}\\ &\quad = (C_2 f' - u', C_1 g' - v)_{\mathcal{L}} - (C_1 f' - u, C_2 g' - v')_{\mathcal{L}}.\end{aligned}$$

Hence the abstract Green's formula (4.16) holds for the maps Γ_1 and Γ_2 defined by the formulas (4.20).

Since $0 \in \rho(A, \mathcal{L})$ one can rewrite the formula (4.19) in the form

$$A^* = A \dotplus \widehat{\mathcal{P}}(0)^*\mathcal{L} \dotplus \widehat{\mathcal{Q}}(0)^*\mathcal{L}.$$

Due to (4.12), (4.13), (4.17), (4.18) one obtains for every $u \in \mathcal{L}$

$$\widehat{\mathcal{P}}(0)^* u = \left\{ \begin{bmatrix} -\widetilde{C}_2^* u \\ u \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \quad \widehat{\mathcal{Q}}(0)^* u = \left\{ \begin{bmatrix} C_1^* u \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u \end{bmatrix} \right\}.$$

It follows from (4.20) that

$$\Gamma \widehat{\mathcal{P}}(0)^* u = \begin{bmatrix} u \\ 0 \end{bmatrix}, \quad \Gamma \widehat{\mathcal{Q}}(0)^* u = \begin{bmatrix} 0 \\ -u \end{bmatrix}$$

and hence the mapping $\Gamma : A^* \to \mathcal{L} \oplus \mathcal{L}$ is surjective. So $\{\mathcal{L}, \Gamma_1, \Gamma_2\}$ is a boundary triplet for A^*. □

4.4. **$\mathcal{L}$-resolvent matrix.** We define the block operator-functions $\mathcal{V}(\lambda) = \begin{bmatrix} -\mathcal{Q}(\lambda) \\ \mathcal{P}(\lambda) \end{bmatrix} \in [\mathcal{H}, \mathcal{L} \oplus \mathcal{L}]$ and $\widehat{\mathcal{V}}(\lambda)^* = \begin{bmatrix} -\widehat{\mathcal{Q}}(\lambda)^* \\ \widehat{\mathcal{P}}(\lambda)^* \end{bmatrix}$.

Definition 4.8. ([23]) A matrix-function $W_{\mathcal{L}}(\lambda)$ is called an $\mathcal{L}$-resolvent matrix for the symmetric linear relation A if it satisfies the identity

$$W_{\mathcal{L}}(\lambda) J W_{\mathcal{L}}(\mu)^* = J + i(\lambda - \bar{\mu}) \mathcal{V}(\lambda) \mathcal{V}(\mu)^* \quad (\lambda, \mu \in \rho(A, \mathcal{L})). \tag{4.22}$$

We emphasize that identity (4.22) defines a family of $\mathcal{L}$-resolvent matrices for the linear relation A. At the same time, it was shown in [11, 12] that with any boundary triplet $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ for A^* and any scale subspace $\mathcal{L}$ it is naturally associated the (unique) matrix function $W_{\Pi\mathcal{L}}(\cdot)$ that satisfies (4.22). It is called the $\mathcal{L}$-resolvent matrix of A corresponding to the triplet Π or just the $\Pi\mathcal{L}$-resolvent matrix of A. It was shown in [11, 12] that $W_{\Pi\mathcal{L}}(\cdot)$ is expressed in terms of boundary mappings $\Gamma_j\}$ and the function $\mathcal{V}(\cdot)$.

Theorem 4.9. ([11], [12]) *Let $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ be a boundary triplet for A^* and $\lambda \in \rho_s(A, \mathcal{L})$. Then the corresponding $\Pi\mathcal{L}$-resolvent matrix $W_{\Pi\mathcal{L}}$ is given by*

$$W_{\Pi\mathcal{L}}(\lambda) = [w_{ij}(\lambda)]_{i,j=1}^2 = \begin{bmatrix} -\Gamma_2 \widehat{\mathcal{Q}}(\lambda)^* & \Gamma_2 \widehat{\mathcal{P}}(\lambda)^* \\ -\Gamma_1 \widehat{\mathcal{Q}}(\lambda)^* & \Gamma_1 \widehat{\mathcal{P}}(\lambda)^* \end{bmatrix}^* = (\Gamma \widehat{\mathcal{V}}(\lambda)^*)^*. \tag{4.23}$$

Proposition 4.10. *Let the data set (B_1, B_2, C_1, C_2, K) satisfy the assumptions (A1)–(A3) and let the boundary triplex $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ for A^* be given by (4.20). Then the corresponding $\Pi\mathcal{L}$-resolvent matrix of A is given by*

$$W_{\Pi\mathcal{L}}(\lambda) = \begin{bmatrix} I_{\mathcal{L}} & 0 \\ -\lambda & I_{\mathcal{L}} \end{bmatrix} \left(I + i\lambda \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I_{\mathcal{H}} - \lambda B_1)^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^* J \right). \tag{4.24}$$

Proof. One obtains from (4.17), (4.12) and (4.20) that

$$\Gamma_2 \widehat{\mathcal{P}}(\lambda)^* v = -\bar{\lambda} v - \bar{\lambda} C_2 F(\lambda)^* v, \tag{4.25}$$

$$\Gamma_1 \widehat{\mathcal{P}}(\lambda)^* v = v + \bar{\lambda} C_1 F(\lambda)^* v. \tag{4.26}$$

Similarly (4.18), (4.13) and (4.20) imply

$$-\Gamma_2 \widehat{\mathcal{Q}}(\lambda)^* v = v - \bar{\lambda} C_2 (I_{\mathcal{H}} - \lambda B_1^*)^{-1} C_1^* v, \tag{4.27}$$

$$-\Gamma_1 \widehat{\mathcal{Q}}(\lambda)^* v = \bar{\lambda} C_1 (I_{\mathcal{H}} - \lambda B_1^*)^{-1} C_1^* v. \tag{4.28}$$

It follows from (4.25)–(4.28) and (4.23) that

$$W_{\Pi\mathcal{L}}(\lambda)^* = \begin{bmatrix} I_{\mathcal{L}} & -\bar{\lambda} \\ 0 & I_{\mathcal{L}} \end{bmatrix} - \bar{\lambda} \begin{bmatrix} C_2 \\ -C_1 \end{bmatrix} (I_{\mathcal{H}} - \bar{\lambda} B_1^*)^{-1} \begin{bmatrix} C_1^* & C_2^* - \bar{\lambda} C_1^* \end{bmatrix}$$

and hence

$$W_{\Pi\mathcal{L}}(\lambda) = \begin{bmatrix} I & 0 \\ -\lambda & I \end{bmatrix} - \lambda \begin{bmatrix} C_1 \\ C_2 - \lambda C_1 \end{bmatrix} (I_{\mathcal{H}} - \lambda B_1)^{-1} \begin{bmatrix} C_2^* & -C_1^* \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ -\lambda & I \end{bmatrix} \left(I_{\mathcal{L}\oplus\mathcal{L}} - \lambda \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I_{\mathcal{H}} - \lambda B_1)^{-1} \begin{bmatrix} C_2^* & -C_1^* \end{bmatrix} \right),$$

which coincides with the formula (4.24). □

We recall the basic properties of $\mathcal{L}$-resolvent matrix $W_{\Pi\mathcal{L}}$.

Proposition 4.11. ([8], [11]) *Let A be a symmetric lineal relation in $\mathcal{H}\oplus\mathcal{L}$ and $\rho_s(A,\mathcal{L}) \neq 0$. Let $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ be a boundary triplet for A^* and $W(\lambda) = W_{\Pi\mathcal{L}}(\lambda)$ be the $\mathcal{L}$-resolvent matrix of A corresponding to Π. Then*

1) *$0 \in \rho(W(\lambda))$ for all $\lambda \in \rho_s(A,\mathcal{L})$.*
2) *If $0 \in \rho(A,\mathcal{L})$ then there is a choice of the boundary triplet $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ such that $W(0) = I$ and*

$$W(\lambda) = I + i\lambda \mathcal{V}(\lambda)\mathcal{V}(0)^* J \quad (\lambda \in \rho(A,\mathcal{L})). \tag{4.29}$$

3) *The $\mathcal{L}$-resolvent matrix $W(\lambda)$ belongs to the generalized Potapov class $\mathcal{P}_\kappa(J)$ that is the kernel*

$$K_\omega(\lambda) = \frac{W(\lambda)JW(\omega)^* - J}{i(\lambda - \bar{\omega})} \quad (\lambda, \omega \in \rho(A,\mathcal{L}))$$

has κ negative squares on $\rho(A,\mathcal{L})$.

4) *The class of $\mathcal{L}$-resolvent matrices of A corresponding to Π is invariant under the multiplication by a right J-unitary factor.*

We obtain a formula in calculating the $\mathcal{L}$-resolvent matrix of A for a more general setting statement.

Corollary 4.12. *Let the data set (B_1, B_2, C_1, C_2, K) satisfy $(A1), (A2)$ and assume that*

(A3′) the operator $D = B_2 - \mu B_1$ is an isomorphism in $\mathcal{X}$ for some $\mu \in \mathbb{R}$, and the operators $B_1 D^{-1} : \mathcal{X} \to \mathcal{X}$, $G(\mu) : \mathcal{X} \to \mathcal{L}^2$ are bounded, where the operator $G(\mu) : \mathcal{X} \to \mathcal{L}^2$ is defined by

$$G(\mu) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (B_2 - \mu B_1)^{-1} \quad (\mu \in \mathbb{R}). \tag{4.30}$$

Then one of the $\mathcal{L}$-resolvent matrices of A can be found from

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ -\lambda & I \end{bmatrix}^{-1} W^\mu(\lambda) \\ &= I + i(\lambda - \mu) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (B_2 - \lambda B_1)^{-1} (B_2^* - \mu B_1^*)^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^* J. \end{aligned} \tag{4.31}$$

Proof. The data set

$$(B_1(B_2 - \mu B_1)^{-1}, I_{\mathcal{X}}, C_1(B_2 - \mu B_1)^{-1}, (C_2 - \mu C_1)(B_2 - \mu B_1)^{-1}, K)$$

satisfies the assumptions (A1)-(A3). Indeed, the assumption (1) verified by substitution into (A1) the vectors

$$h = (B_2 - \mu B_1)h', \quad g = (B_2 - \mu B_1)g',$$

and assumptions (2) and (3) follow from (3′).

Note that the linear relation A defined by the formula (4.2) has the same form for the problems $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ and $AIP_\kappa(B_1(B_2-\mu B_1)^{-1}, I_{\mathcal{X}}, C_1(B_2-\mu B_1)^{-1}, (C_2-\mu C_1)(B_2-\mu B_1)^{-1}, K)$. Consider the linear relation $A-\mu$

$$A-\mu = \left\{ \left\{ \begin{bmatrix} B_1(B_2-\mu B_1)^{-1}h \\ C_1(B_2-\mu B_1)^{-1}h \end{bmatrix}, \begin{bmatrix} h \\ (C_2-\mu C_1)(B_2-\mu B_1)^{-1}h \end{bmatrix} \right\} : h \in \mathcal{H} \right\}.$$

Due to (4.24) its $\mathcal{L}$-resolvent matrix $W(\lambda)$ satisfies the equality

$$\begin{aligned} &\begin{bmatrix} -I & 0 \\ \lambda & -I \end{bmatrix}^{-1} W(\lambda) \\ &= I_{\mathcal{L}\oplus\mathcal{L}} + i\lambda \begin{bmatrix} C_1 \\ C_2-\mu C_1 \end{bmatrix} (B_2-(\lambda+\mu)B_1)^{-1}(B_2^*-\mu B_1^*)^{-1} \begin{bmatrix} C_1 \\ C_2-\mu C_1 \end{bmatrix}^* J. \end{aligned}$$

Then the matrix $W^\mu(\lambda) = W(\lambda-\mu)$ is the $\mathcal{L}$-resolvent matrix of A and hence

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ -\lambda & I \end{bmatrix}^{-1} W^\mu(\lambda) = \begin{bmatrix} I & 0 \\ -\lambda & I \end{bmatrix}^{-1} W(\lambda-\mu) \\ &= \left(I_{\mathcal{L}\oplus\mathcal{L}} + i(\lambda-\mu) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (B_2-\lambda B_1)^{-1}(B_2^*-\mu B_1^*)^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^* J \right) \begin{bmatrix} I & 0 \\ \mu & I \end{bmatrix}. \end{aligned}$$

This prove (4.31) since the class of $\mathcal{L}$-resolvent matrices is invariant under the multiplication by a right J-unitary factor. □

Theorem 4.13. *Let A be a symmetric lineal relation in $\mathcal{H}\oplus\mathcal{L}$ let $\Pi = \{\mathcal{L}, \Gamma_1, \Gamma_2\}$ be a boundary triplet for A^* and $W_{\Pi\mathcal{L}}(\lambda)$ be the corresponding $\mathcal{L}$-resolvent matrix of A. Then the set of $\mathcal{L}$–resolvents of A is parametrized by the formula*

$$P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}\upharpoonright_{\mathcal{L}} = (w_{11}(\lambda)q(\lambda)+w_{12}(\lambda)p(\lambda))(w_{21}(\lambda)q(\lambda)+w_{22}(\lambda)p(\lambda))^{-1}, \tag{4.32}$$

where $\{p,q\}$ ranges over the set $\widetilde{\mathbf{N}}(\mathcal{L})$ of all equivalence classes of Nevanlinna pairs such that

$$0 \in \rho(w_{21}(\lambda)q(\lambda)+w_{22}(\lambda)p(\lambda)) \quad \textit{for some} \quad \lambda \in \mathbb{C}.$$

Moreover, the $\mathcal{L}$-resolvent $P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}\upharpoonright_{\mathcal{L}}$ defined by (4.32) is $\mathcal{L}$-regular if and only if the Nevanlinna pair $\{p,q\}$ satisfies the condition

$$(w_{11}(\lambda)q(\lambda)+w_{12}(\lambda)p(\lambda))(w_{21}(\lambda)q(\lambda)+w_{22}(\lambda)p(\lambda))^{-1} \in N_\kappa(\mathcal{L}). \tag{4.33}$$

Proof. The first statement of this Theorem was proved in [8, Prop. 5.3].

Let the $\mathcal{L}$-resolvent $P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}\upharpoonright_{\mathcal{L}}$ is $\mathcal{L}$-regular. Then the subspace

$$\mathcal{H}_0 = \overline{\operatorname{span}}\{P_{\mathcal{H}}(\widetilde{A}-\lambda)^{-1}\mathcal{L} : \lambda \in \rho(\widetilde{A})\},$$

is a nondegenerate Pontryagin space with the negative index equal to κ (Π_κ-space). As follows from Theorem 2.11 the generalized Fourier transform $\mathcal{F}$ maps isometrically the subspace $\mathcal{H}_0$ onto $\mathcal{H}(\varphi,\psi)$ where the functions φ,ψ are defined by the formula (2.3). Hence the space $\mathcal{H}(\varphi,\psi)$ is also a Π_κ-space and, therefore, $\{\varphi,\psi\} \in \mathbf{N}_\kappa(\mathcal{L})$. This statement is equivalent to the condition (4.33).

Now, let $\{\varphi,\psi\} \in \mathbf{N}_\kappa(\mathcal{L})$. Then the space $\mathcal{H}(\varphi,\psi)$ is a Π_κ-space and, therefore, the space $\mathcal{H}_0$ is Π_κ-space. The latter statement means that the linear relation $\widetilde{A}$ is $\mathcal{L}$-regular. □

4.5. **Parametrization of the set of solutions of AIP_κ.** To describe solutions of the AIP_κ it remains to combine Theorem 3.3 and Theorem 4.13. Let the operator valued function $\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$ be defined by

$$\Theta(\lambda) = \begin{bmatrix} I & 0 \\ \lambda & I \end{bmatrix} W(\lambda) = \left(I - \lambda \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I_{\mathcal{H}} - \lambda B_1)^{-1} \begin{bmatrix} C_2^* & -C_1^* \end{bmatrix} \right). \tag{4.34}$$

Theorem 4.14. *Let the data set $AIP_\kappa(B_1,\ B_2,\ C_1,\ C_2,\ K)$ satisfies (A1)–(A3) and $sq_-K = \kappa$. Then the formula*

$$\begin{bmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{bmatrix} = \Theta(\lambda) \begin{bmatrix} q(\lambda) \\ p(\lambda) \end{bmatrix} (w_{21}(\lambda)q(\lambda) + w_{22}(\lambda)p(\lambda))^{-1} \tag{4.35}$$

establishes a one-to-one correspondence between the set of all normalized solutions $\{\varphi, \psi\}$ of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ and the set of all equivalence classes of Nevanlinna pairs $\{p, q\} \in \widetilde{\mathbf{N}}(\mathcal{L})$ such that (4.33) *is satisfied. The corresponding mapping $F : \mathcal{X} \to \mathcal{H}(\varphi, \psi)$ in (C1)–(C2), is defined by the solution $\{\varphi, \psi\}$*

$$(Fg)(\mu) = \begin{bmatrix} \varphi(\mu) & -\psi(\mu) \end{bmatrix} G(\mu)g \quad (\mu \in \mathcal{O},\ g \in \mathcal{X}), \tag{4.36}$$

where $\mathcal{O}$ is a neighborhood of 0 *and*

$$G(\mu) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I_{\mathcal{H}} - \mu B_1)^{-1} \quad (\mu \in \mathcal{O}).$$

Proof. The description (4.35) is implied by the Theorem 4.13 and the formulas (3.4), (4.34). Indeed,

$$\begin{aligned} \begin{bmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{bmatrix} &= \begin{bmatrix} I_{\mathcal{L}} & 0 \\ \lambda & I_{\mathcal{L}} \end{bmatrix} \begin{bmatrix} P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}\restriction_{\mathcal{L}} \\ I_{\mathcal{L}} \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{L}} & 0 \\ \lambda & I_{\mathcal{L}} \end{bmatrix} \begin{bmatrix} \big(w_{11}(\lambda)q(\lambda) + w_{12}(\lambda)p(\lambda)\big)\big(w_{21}(\lambda)q(\lambda) + w_{22}(\lambda)p(\lambda)\big)^{-1} \\ I_{\mathcal{L}} \end{bmatrix} \\ &= \Theta(\lambda)W(\lambda)^{-1} \begin{bmatrix} \big(w_{11}(\lambda)q(\lambda) + w_{12}(\lambda)p(\lambda)\big)\big(w_{21}(\lambda)q(\lambda) + w_{22}(\lambda)p(\lambda)\big)^{-1} \\ I_{\mathcal{L}} \end{bmatrix}. \end{aligned} \tag{4.37}$$

Using the relation

$$W \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} w_{11}q + w_{12}p \\ w_{21}q + w_{22}p \end{bmatrix}$$

we rewrite the right side of (4.37) as

$$\begin{aligned} \begin{bmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{bmatrix} &= \Theta(\lambda)W(\lambda)^{-1}W(\lambda) \begin{bmatrix} q(\lambda) \\ p(\lambda) \end{bmatrix} \big(w_{21}(\lambda)q(\lambda) + w_{22}(\lambda)p(\lambda)\big)^{-1} \\ &= \Theta \begin{bmatrix} q(\lambda) \\ p(\lambda) \end{bmatrix} \big(w_{21}(\lambda)q(\lambda) + w_{22}(\lambda)p(\lambda)\big)^{-1}. \end{aligned}$$

Therefore, the formula (4.35) is proved.

Let $\mathcal{O}$ is a neighborhood of 0 such that the operator $(I_{\mathcal{H}} - \mu B_1)$ is invertible in $\mathcal{H}$ for $\mu \in \mathcal{O}$, and let $g \in (I - \mu B_1)\mathcal{X}$ $(\mu \in \mathcal{O})$. Applying (C1) to the vector $h = h_\mu := (I - \mu B_1)^{-1}g$, one obtains

$$\begin{aligned} (Fg)(\lambda) &= (Fh_\mu)(\lambda) - \mu(FB_1h_\mu)(\lambda) \\ &= \begin{bmatrix} \varphi(\lambda) & -\psi(\lambda) \end{bmatrix} G(\mu)g + (\lambda - \mu)(FB_1h_\mu)(\lambda). \end{aligned} \tag{4.38}$$

Setting in (4.38) $\lambda = \mu$ one obtains

$$(Fg)(\mu) = \begin{bmatrix} \varphi(\mu) & -\psi(\mu) \end{bmatrix} G(\mu)g \quad (\mu \in \mathcal{O}, g \in (I - \mu B_1)\mathcal{X}). \tag{4.39}$$

Let $g \in \mathcal{X}$, $g_n \in (I - \mu B_1)\mathcal{X}$ and $g_n \to g$ as $n \to \infty$ (where the convergence is meant in the Pontryagin space sense). Then taking the limit in

$$(Fg_n)(\mu) = \begin{bmatrix}\varphi(\mu) & -\psi(\mu)\end{bmatrix} G(\mu)g_n$$

one obtains (4.36) for $g \in \mathcal{X}$. □

Theorem 4.15. *Let the data set (B_1, B_2, C_1, C_2, K) satisfy (A1), (A2), (A3') ($sq_- K = \kappa$) and let*

$$\begin{aligned}\Theta^\mu(\lambda) &:= \begin{bmatrix} I & 0 \\ \lambda & I \end{bmatrix} W^\mu(\lambda) \\ &= \left(I + i(\lambda - \mu) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (B_2 - \lambda B_1)^{-1} (B_2^* - \mu B_1^*)^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^* J \right) V.\end{aligned}$$

Then the formula

$$\begin{bmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{bmatrix} = \Theta^\mu(\lambda) \begin{bmatrix} q(\lambda) \\ p(\lambda) \end{bmatrix} (w_{21}^\mu(\lambda)q(\lambda) + w_{22}^\mu(\lambda)p(\lambda))^{-1} \tag{4.40}$$

establishes the one-to-one correspondence between the set of all normalized solutions $\{\varphi, \psi\}$ of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ and the set of all equivalence classes of Nevanlinna pairs $\{p, q\} \in \widetilde{\mathbf{N}}(\mathcal{L})$ such that (4.33) *is satisfied.*

Proof. The theorem follows from the just proved Theorem 4.14 and Corollary 4.12. □

4.6. **Uniqueness of the mapping** F**.** In general, the mapping $F : \mathcal{X} \to \mathcal{H}(\varphi, \psi)$ in (C1)–(C2) is not uniquely defined by the solution $\{\varphi, \psi\}$ of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$. We impose an additional assumption on the data set which ensures the uniqueness of F.

(U) Let the mapping $B_2 - \lambda B_1$ be an isomorphism in $\mathcal{X}$ for all λ in nonempty domains $\mathcal{O}_\pm \subset \mathbb{C}_\pm$.

Let us set

$$G(\lambda) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (B_2 - \lambda B_1)^{-1} \quad (\lambda \in \mathcal{O}_\pm). \tag{4.41}$$

Proposition 4.16. *Let the data set (B_1, B_2, C_1, C_2, K) satisfies the assumptions (A1)–(A2) and (U) ($\nu \le \kappa$). Then the mapping $F : \mathcal{X} \to \mathcal{H}(\varphi, \psi)$ in (C1)–(C3) is uniquely defined by the solution $\{\varphi, \psi\}$ of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ by the formula*

$$(Fh)(\lambda) = \begin{bmatrix}\varphi(\lambda) & -\psi(\lambda)\end{bmatrix} G(\lambda)h \quad (\lambda \in \mathcal{O}_\pm). \tag{4.42}$$

Proof. Applying (C1) to the vector

$$h = h_\mu := (B_2 - \mu B_1)^{-1} g \quad (\mu \in \mathcal{O}_\pm,\ g \in \mathcal{X}),$$

one obtains

$$\begin{aligned}(Fg)(\lambda) &= (FB_2 h_\mu)(\lambda) - \mu(FB_1 h_\mu)(\lambda) \\ &= (FB_2 h_\mu)(\lambda) - \lambda(FB_1 h_\mu)(\lambda) + (\lambda - \mu)(FB_1 h_\mu)(\lambda) \\ &= \begin{bmatrix}\varphi(\lambda) & -\psi(\lambda)\end{bmatrix} G(\mu)g + (\lambda - \mu)(FB_1 h_\mu)(\lambda).\end{aligned} \tag{4.43}$$

Setting in (4.43) $\lambda = \mu$ one obtains $(Fg)(\mu) = \begin{bmatrix}\varphi(\mu) & -\psi(\mu)\end{bmatrix} G(\mu)g$. □

Remark 4.17. Let the data set (B_1, B_2, C_1, C_2, K) satisfies the assumptions (A1)–(A3). Then this data set satisfies the assumption (U) automatically. Hence in the conditions of Theorem 4.14 the mapping $F : \mathcal{X} \to \mathcal{H}(\varphi, \psi)$ in (C1)–(C2) is uniquely defined by the solution $\{\varphi, \psi\}$ of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$. The mapping F is defined by the formula (4.36).

5. A solubility of AIP_κ in the set of N_κ-function

A function $m(\lambda) \in N_\kappa(\mathcal{L})$ is called a solution of AIP_κ if the pair $\{I_{\mathcal{L}}, m\}$ is equivalent to the some N_κ-pair $\{\varphi, \psi\}$ which is solution of AIP_κ. In general the AIP_κ mayn't have solutions in the class of operator-function. We impose an additional assumption on the data set

(A4) that for some choice of numbers $\lambda_j \in \mathbb{C}_+$ $(j = 1, \ldots, \kappa)$ the following condition holds:

$$\ker \left[C_2^* \quad (1-\lambda_1 B_1^*)^{-1} C_2^* \quad (1-\lambda_2 B_1^*)^{-1} C_2^* \quad \cdots \quad (1-\lambda_\kappa B_1^*)^{-1} C_2^*\right] = \{0\}.$$

Apparently, for the first time a condition of this type appeared in [27]. A similar condition was used in [2] for solubility of some interpolation problem in a set of functions.

We can proceed to prove the main theorems.

Proof of Theorem 1.1. According to Theorem 3.3 it remains to show that every solution of $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ is operator-functions. Let the pair $\{\varphi, \psi\}$ be a solution of AIP_κ which is not equivalent to the pair of the form $\{I_{\mathcal{L}}, m\}$. This means that there is a point $\lambda_{\kappa+1} \in \mathbb{C}_+ (\neq \lambda_j)$ and vectors $f_j \in \mathcal{L}$ $(j = 1, \ldots, \kappa+1)$ such that the following conditions hold:

$$\varphi(\lambda_j) f_j = 0, \quad \psi(\lambda_j) f_j \neq 0 \quad (j = 1, \ldots, \kappa+1). \tag{5.1}$$

Since the N_κ-pair $\{\varphi, \psi\}$ is normalized then $\varphi(\lambda) - \lambda\psi(\lambda) = I_{\mathcal{L}}$. Let a lineal relation $\widetilde{A}$ be selfadjoint $\mathcal{L}$-regular extensions of $\widehat{A}$ with the exit to a Pontryagin space $\widetilde{\mathcal{H}} \oplus \mathcal{L} \supset \mathcal{H} \oplus \mathcal{L}$ and the formula (3.4) holds. Let vectors $h_j \in \widetilde{\mathcal{H}} \oplus \mathcal{L}$ be given by

$$h_j := \lambda_j (\widetilde{A} - \lambda_j)^{-1} f_j + f_j \quad (j = 1, \ldots, \kappa+1). \tag{5.2}$$

Then

$$\{h_j - f_j,\, \lambda_j h_j\} \in \widetilde{A} \quad (j = 1, \ldots, \kappa+1). \tag{5.3}$$

The vectors h_j belong to the space $\widetilde{\mathcal{H}}$, because

$$P_{\mathcal{L}} h_j = P_{\mathcal{L}}(\lambda_j(\widetilde{A} - \lambda_j)^{-1} + I_{\mathcal{L}}) f_j = \varphi(\lambda_j) f_j = 0.$$

Since the linear relation $\widetilde{A}$ is selfadjoint and the spaces $\widetilde{\mathcal{H}}$ and $\mathcal{L}$ are orthogonal it follows that for any $j, k = 1, \ldots, \kappa+1$

$$0 = \langle h_j - f_j,\, \lambda_k h_k \rangle_{\widetilde{\mathcal{H}} \oplus \mathcal{L}} - \langle \lambda_j h_j,\, h_k - f_k \rangle_{\widetilde{\mathcal{H}} \oplus \mathcal{L}} = (\lambda_j - \overline{\lambda}_k) \langle h_j,\, h_k \rangle_{\widetilde{\mathcal{H}}}.$$

Hence the span of $\kappa+1$ vectors h_j is a neutral subspace of the space $\widetilde{\mathcal{H}}$. Since the dimension of the neutral subspaces of space $\widetilde{\mathcal{H}}$ does not exceed κ then of the vectors $\{h_j\}_1^{\kappa+1}$ are linearly dependent, i.e. there are $\alpha_j \in \mathbb{C}$ $(j = 1, \ldots, \kappa+1)$ such that

$$\sum_{j=1}^{\kappa+1} \alpha_j h_j = 0. \tag{5.4}$$

Assume that $\alpha_{\kappa+1} = -1$, i.e.

$$h_{\kappa+1} = \sum_{j=1}^{\kappa} \alpha_j h_j.$$

Then the inclusions (5.3) yield

$$\left\{ \frac{1}{\lambda_{\kappa+1}} (h_{\kappa+1} - f_{\kappa+1}) - \sum_{j=1}^{\kappa} \frac{\alpha_j}{\lambda_j} (h_j - f_j),\, 0 \right\} \in \widetilde{A},$$

therefore

$$\left\{\sum_{j=1}^{\kappa}\alpha_j\left(\frac{1}{\lambda_{\kappa+1}}-\frac{1}{\lambda_j}\right)h_j-\frac{1}{\lambda_{\kappa+1}}f_{\kappa+1}+\sum_{j=1}^{\kappa}\frac{\alpha_j}{\lambda_j}f_j,\,0\right\}\in\widetilde{A}. \tag{5.5}$$

We define the vectors $g_0\in\mathcal{L}$ and $\widetilde{g}\in\widetilde{\mathcal{H}}$ by the formulas

$$g_0:=\sum_{j=1}^{\kappa}\frac{\alpha_j}{\lambda_j}f_j-\frac{1}{\lambda_{\kappa+1}}f_{\kappa+1},\quad \widetilde{g}:=\sum_{j=1}^{\kappa}\alpha_j\left(\frac{1}{\lambda_{\kappa+1}}-\frac{1}{\lambda_j}h_j\right). \tag{5.6}$$

Then the relation (5.5) can be rewritten as $\{g_0+\widetilde{g},\,0\}\in\widetilde{A}$.

Since $\widetilde{A}$ is a selfadjoint extension of $\widehat{A}=\left\{\left\{\begin{bmatrix}B_1x\\C_1x\end{bmatrix},\begin{bmatrix}x\\C_2x\end{bmatrix}\right\}:x\in\mathcal{X}\right\}$ then for all $x\in\mathcal{X}$

$$0=\left\langle\widetilde{g}+g_0,\,\begin{bmatrix}x\\C_2x\end{bmatrix}\right\rangle_{\widetilde{\mathcal{H}}\oplus\mathcal{L}}=(g_0,C_2x)_{\mathcal{L}}+\langle\widetilde{g},x\rangle_{\widetilde{\mathcal{H}}}. \tag{5.7}$$

Substituting expressions (5.2) for h_j into (5.6) one obtains

$$\begin{aligned}\widetilde{g}&=\sum_{j=1}^{\kappa}\alpha_j\left(\frac{1}{\lambda_{\kappa+1}}-\frac{1}{\lambda_j}\right)h_j\\&=\sum_{j=1}^{\kappa}\alpha_j\left(\frac{1}{\lambda_{\kappa+1}}-\frac{1}{\lambda_j}\right)(\lambda_j(\widetilde{A}-\lambda_j)^{-1}+I_{\mathcal{L}})f_j\\&=\sum_{j=1}^{\kappa}(\lambda_j(\widetilde{A}-\lambda_j)^{-1}+I)g_j,\end{aligned}$$

where the vectors $g_j\in\mathcal{L}$ are defined by

$$g_j=\alpha_j\left(\frac{1}{\lambda_{\kappa+1}}-\frac{1}{\lambda_j}\right)f_j\in\mathcal{L}.$$

It follows from (4.7) that

$$\begin{aligned}K_{\lambda_j}:&=\lambda_j(\widehat{A}-\lambda_j)^{-1}+I\\&=\left\{\left\{\begin{bmatrix}(I-\lambda_jB_1)x\\(C_2-\lambda_jC_1)x\end{bmatrix},\begin{bmatrix}x\\C_2x\end{bmatrix}\right\}:\ x\in\mathcal{H}\right\}\\&=\left\{\left\{\begin{bmatrix}x\\(C_2-\lambda_jC_1)(I-\lambda_jB_1)^{-1}x\end{bmatrix},\begin{bmatrix}(I-\lambda_jB_1)^{-1}x\\C_2(I-\lambda_jB_1)^{-1}x\end{bmatrix}\right\}:x\in\mathcal{H}\right\}.\end{aligned}$$

Therefore $K_{\lambda_j}^*=K_{\overline{\lambda}_j}$ as $\widetilde{A}\supset\widehat{A}$ is a selfadjoint lineal relation. So for all $x\in\mathcal{X}$

$$\begin{aligned}\langle\widetilde{g},\,x\rangle_{\widetilde{\mathcal{H}}}&=\sum_{j=1}^{\kappa}\langle(\lambda_j(\widetilde{A}-\lambda_j)^{-1}+I)g_j,\,x\rangle_{\widetilde{\mathcal{H}}}=\sum_{j=1}^{\kappa}\left\langle K_{\lambda_j}\begin{bmatrix}0\\g_j\end{bmatrix},\begin{bmatrix}x\\0\end{bmatrix}\right\rangle_{\widetilde{\mathcal{H}}\oplus\mathcal{L}}\\&=\sum_{j=1}^{\kappa}\left\langle K_{\lambda_j}\begin{bmatrix}0\\g_j\end{bmatrix},\begin{bmatrix}x\\(C_2-\overline{\lambda}_jC_1)(I-\overline{\lambda}_jB_1)^{-1}x\end{bmatrix}\right\rangle_{\widetilde{\mathcal{H}}\oplus\mathcal{L}}\\&=\sum_{j=1}^{\kappa}\left\langle\begin{bmatrix}0\\g_j\end{bmatrix},K_{\overline{\lambda}_j}\begin{bmatrix}x\\(C_2-\overline{\lambda}_jC_1)(I-\overline{\lambda}_jB_1)^{-1}x\end{bmatrix}\right\rangle_{\widetilde{\mathcal{H}}\oplus\mathcal{L}}\\&=\sum_{j=1}^{\kappa}\left\langle\begin{bmatrix}0\\g_j\end{bmatrix},\begin{bmatrix}(I-\overline{\lambda}_jB_1)^{-1}x\\C_2(I-\overline{\lambda}_jB_1)^{-1}x\end{bmatrix}\right\rangle_{\widetilde{\mathcal{H}}\oplus\mathcal{L}}=\sum_{j=1}^{\kappa}(g_j,\,C_2(I-\overline{\lambda}_jB_1)^{-1}x)_{\mathcal{L}}.\end{aligned}$$

Combining this equality with (5.7) one obtain

$$
\begin{aligned}
0 &= (g_0,\, C_2 x)_{\mathcal{L}} + \sum_{j=1}^{\kappa} (g_j,\, C_2(I - \overline{\lambda}_j B_1)^{-1} x)_{\mathcal{L}} \\
&= (C_2^* g_0,\, x)_{\mathcal{H}} + \sum_{j=1}^{\kappa} ((I - \lambda_j B_1^*)^{-1} C_2^* g_j,\, x)_{\mathcal{H}}.
\end{aligned} \tag{5.8}
$$

This equality holds for all vectors x which belong to the dense subset $\mathcal{X}$ of the Pontryagin space $\mathcal{H}$. It follows from (A4) that

$$g_0 = g_1 = g_2 = \cdots = g_\kappa = 0,$$

and therefore

$$\alpha_1 f_1 = \alpha_2 f_2 = \cdots = \alpha_k f_k = 0, \quad f_{\kappa+1} = 0.$$

This is a contradiction to the condition $f_{\kappa+1} \neq 0$.

Note, if the coefficient $\alpha_{\kappa+1}$ in (5.4) is equal to 0 we also obtain a contradiction. Indeed, representing any vector h_k via the linear combination of the other $\kappa - 1$ vectors h_j $(1 \leq j \leq \kappa,\, j \neq k)$, we prove that $f_k = 0$.

Thus, there is no vector $\{f_j\}_1^{\kappa+1}$ which satisfy the conditions (5.1). Hence the operator function $\varphi(\lambda)$ in the N_κ-pair $\{\varphi, \psi\}$ where $\varphi(\lambda)$ is invertible for all $\lambda \in \mathbb{C}_+$ with the exception of at most κ points. Therefore the N_κ-function $m(\lambda) = \psi(\lambda)\varphi^{-1}(\lambda)$ is a solution of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$. □

Proof of Theorem 1.2. Let a pair $\{\varphi, \psi\}$ be a solution of the $AIP_\kappa(B_1, B_2, C_1, C_2,\, K)$. It follows from the Theorem 4.14 that

$$\begin{bmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{bmatrix} = \begin{bmatrix} \theta_{11}(\lambda) q(\lambda) + \theta_{12}(\lambda) p(\lambda) \\ \theta_{21}(\lambda) q(\lambda) + \theta_{12}(\lambda) p(\lambda) \end{bmatrix} (w_{21}(\lambda) q(\lambda) + w_{22}(\lambda) p(\lambda))^{-1}.$$

It follows from the Theorem 1.1 that the solution of the $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ is the N_κ-function $m(\lambda) = \psi(\lambda)\varphi(\lambda)^{-1}$. Hence the formula (1.5) holds. □

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Received 02/04/2012; Revised 15/06/2012