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REALIZATIONS OF STATIONARY STOCHASTIC PROCESSES: APPLICATIONS OF PASSIVE SYSTEM THEORY

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Abstract. In the paper, we investigate realizations of a p-dimensional regular weak stationary discrete time stochastic process $y(t)$ as the output data of a passive linear bi-stable discrete time dynamical system. The state $x(t)$ is assumed to tend to zero as t tends to $-\infty$, and the input data is the m-dimensional white noise. The results are based on author's development of the Darlington method for passive impedance systems with losses of the scattering channels. Here we establish that considering realization for a discrete time process is possible, if the spectral density $\rho(e^{i\mu})$ of the process is a nontangential boundary value of a matrix valued meromorphic function $\rho(z)$ of rank m with bounded Nevanlinna characteristic in the open unit disk. A parameterization of all such realizations is given and minimal, optimal minimal, and *-optimal minimal realizations are obtained. The last two coincide with those which are obtained by Kalman filters. This is a further development of the Lindquist-Picci realization theory.

1. Introduction

In our works [6]–[11] we developed the Darlington method for passive linear time-invariant impedance systems. The present work was initiated by the Lindquist-Picci theory of the realization of p-dimensional weak stationary stochastic processes by Kalman filters, that was developed in [28]–[30]. This theory is closely connected with the Lax-Phillips scattering scheme and scattering matrix s for such a scheme. For details about this scheme, its scattering matrix and its connection with the theory of the characteristic functions of contractive and dissipative operators (or unitary operator nodes and dissipative Livshic-Brodsky nodes, respectively) see e.g. [26], [1], [5]. Some results of [6]–[7] were interpreted by the authors as respective results of Lindquist-Picci realization theory. This interpretation was first presented by the first author at the international conference in MAA–2007, dedicated to 100-birthday of M. G. Krein (see [8]) and later by the second author at the conference [10] (see [35]). In the present work we discuss this interpretation in more details.

Following Kolmogorov ([19], [20]), Krein ([21], [22]) and Wiener ([38], [39]) the study of stationary (in weak sense) stochastic processes usually connects with the factorization of the spectral density $\rho = \varphi^*\varphi$. If ρ has rank m, then φ is an outer matrix function of the size $m \times p$ from the Smirnov class in Ξ, which is defined as the open unit disc $\mathbb{D}$ for discrete time processes or the upper half plane $\mathbb{C}_+$ for continuous time processes (see [32], [33]). The matrix function ρ is the spectral density of the matrix function $c(z)$ from

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the Caratheodory class $\ell^{p\times p}$ of analytical in Ξ functions of order p with $\Re c(z) \geq 0$. This matrix function $c(z)$ has an integral representation in terms of ρ, given by the so-called Riesz-Herglotz-Nevanlinna formula. The matrix functions s and φ are also connected with each other in the certain way (see [28]).

In Kalman filtering and realization theory, the p-dimensional discrete-time regular stochastic stationary process $y(t)$ of rank m is considered as output data of the linear discrete time dynamical system

$$(1) \qquad (\Sigma_f) \begin{cases} x_f(t+1) = Ax_f(t) + Kw_f(t), \\ y(t) = Cx_f(t) + Lw_f(t) \end{cases}$$

with coefficients A, K, C, L, that are linear bounded operators acting between corresponding spaces. The main operator $A : X_o \to X_o$, where X_o is a Hilbert space. If c (as well as ρ, φ and s) is rational matrix function with MacMillan degree $\deg c < \infty$, then $\dim X_o = \deg c$; otherwise, $\dim X_o = \infty$. The other operators $K : \mathbb{C}^m \to X_o$, $C : X_o \to \mathbb{C}^p$, and $L : \mathbb{C}^m \to \mathbb{C}^p$. The input data of this system is m-dimensional white noise w_f.These realizations are discussed in more details in the body of the text. In the case of rational density ρ realizations by Kalman filters are more preferable than the models used by Kolmogorov and followers, where $y(t) = U^t y(0)$ with unitary operator U in Hilbert space X with $\dim X = \infty$ (see [37], [25] and [34]). For discrete-time processes Kolmogorov-Wiener's model is connected with the spectral theory of unitary operators, while for continuous-time processes it is connected with the spectral theory of selfadjoint operators and continuous groups of unitary operators (see [32]–[34]). In the class of spectral densities considered in this paper, matrix functions $c(z)$ have meromorphic pseudocontinuation in the exterior $\mathbb{D}_e$ of $\mathbb{D}$. We study realizations (1) with contractive main operators $A : X_o \to X_o$ such that

$$(2) \qquad A^t \to 0 \quad \text{and} \quad (A^*)^t \to 0 \quad \text{when} \quad t \to +\infty \quad (A \in C_{00}).$$

Moreover, they are the main operators of unitary operator nodes with characteristic matrix functions $\alpha(z)$ that belong to the class $S_{in}^{p\times p}$ of inner in $\mathbb{D}$ matrix functions of an order p. See [15] for discussion of the unitary operator nodes and their characteristic functions. Such unitary node corresponds to the Lax-Phillips scattering model and $\alpha(z)$ is a scattering matrix in this model (see [1], [3]). Thus the results of harmonic analysis of contractive operators from class C_{00} in a Hilbert space are applicable for the study of such realizations and these results are available, for example, in [15], [36]. Similarly, for realizations of continuous time processes α is characteristic function of Livshits-Brodsky's dissipative node with the main operator that is generator of a semigroup of contractions from class C_{00} ([5], [16]). The application of results of harmonic analysis of non-unitary and non-selfadjoint operators to correlation theory of non-stationary processes, presented in [27], chapter VIII, is different from that which is presented here.

In the present work, the matrix function $c(z)$ is considered as a block of a matrix function $\theta(z)$ of the form

$$(3) \qquad \theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}$$

that is holomorphic and $J_{p,m}$-inner in $\mathbb{D}$, i.e. it is $J_{p,m}$-contractive in $\mathbb{D}$

$$\theta(z)^* J_{p,m} \theta(z) \leq J_{p,m}, \quad z \in \mathbb{D}$$

with $J_{p,m}$-unitary non-tangential boundary values a.e. on the boundary of $\mathbb{D}$

$$\theta(\zeta)^* J_{p,m} \theta(\zeta) = J_{p,m} \quad \text{a.e.} \quad \zeta \in \partial\mathbb{D},$$

where

$$J_{p,m} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix}.$$

The matrix function θ is the transfer function of a conservative transmission system without losses and it is also the characteristic matrix function of the $J_{p,m}$-unitary node with contractive main operator from class C_{00}, see [4] and [7]. In [6]–[9] all such θ and corresponding transmission systems are described and some special types such as optimal, *-optimal, minimal θ were considered.

Note that the results of the present paper are directly connected with the work [17] where problems related to acoustic wave filters are studied. In such systems the input data are incoming waves and voltages, and the output data are the reflected waves and currents. The transfer function of such a filter is the so-called "mixing matrix"

$$\begin{bmatrix} \alpha & \beta \\ \gamma & c \end{bmatrix}$$

that coincides with the informative part of the transmission matrix θ of the system $\tilde{\Sigma}$ in our considerations.

We will make use of basic concepts and results of spectral operator theory in Hilbert spaces and the theory of stationary in the weak-sense discrete time stochastic processes which can be found, for example, in monographs [37], [34]. We also utilize the theory of stochastic realizations and Kalman filters which can be found, for example, in [18].

Notations and assumptions

All Hilbert spaces considered in this paper are assumed to be separable, and subspaces are closed and linear in the space;

$\mathbb{C}$ is the set of complex numbers;

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the set of all integers;

$\mathbb{Z}^+ = \{t \in \mathbb{Z} : t \geq 0\}$;

$\mathbb{Z}^- = \{t \in \mathbb{Z} : t < 0\}$;

$\mathbb{C}^p = \{u = \mathrm{col}\{u_k\}_{k=1}^p : u_k \in \mathbb{C}\}$;

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk;

$\mathbb{D}_e = \{z \in \mathbb{C} : 1 < |z| \leq \infty\}$ is the exterior of the unit disk in the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$;

$\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ is the unit circle;

I_p is the identity matrix of the order p;

I_U is the identity operator in a space U;

$\mathbb{B}(X, Y)$ is the space of linear bounded operators from a Hilbert space X to a Hilbert space Y; $\mathbb{B}(X) := \mathbb{B}(X, X)$;

$\bigvee_{\alpha \in A} \mathfrak{D}_\alpha$ is the closed linear span of vectors from $\mathfrak{D}_\alpha$ when α changes in the index set A;

$P_{\mathfrak{D}}$ is the orthogonal projection on a subspace $\mathfrak{D}$;

$A|_{\mathfrak{D}}$ is the restriction of the operator A on a subspace $\mathfrak{D}$;

Λ_f is the set of points where function f is holomorphic;

$f^{\sim}(z) = f(\bar{z})^*$;

$f^{\sharp}(z) = f(1/\bar{z})^*$;

$L_r^{p \times q}(\mathbb{T})$ with $1 \leq r < \infty$ is the space of measurable on $\mathbb{T}$ matrix functions $f(\zeta)$ of the size $p \times q$ such that

$$\|f\|_r^r = \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{trace}\{f(\zeta)^* f(\zeta)\}^{r/2} \, |d\zeta| < \infty;$$

$H_2^{p\times q}$ is the space of holomorphic in $\mathbb{D}$ matrix functions $f(z)$ of size $p\times q$ such that

$$\|f\|_2^2 = \sup_{r<1}\int_{\mathbb{T}} \operatorname{trace}\{f(r\zeta)^* f(r\zeta)\}\,|d\zeta| < \infty;$$

$\ell^{p\times p}$ is Caratheodory class of holomorphic in $\mathbb{D}$ matrix functions $c(z)$ of the order p such that $\Re c(z) \geq 0$ in $\mathbb{D}$;

$S^{p\times q}$ is Schur class of holomorphic in $\mathbb{D}$ matrix functions $s(z)$ of the size $p\times q$ such that $s(z)^* s(z) \leq I_q$ for all $z\in\mathbb{D}$;

$S_{in}^{p\times p}$ is the class of bi-inner matrix functions $s(z)$ of the order p, i.e. such that $s\in S^{p\times p}$ and $s(\zeta)^* s(\zeta) = I_p$ a.e. when $\zeta\in\mathbb{T}$;

$N^{p\times q}$ is the class of meromorphic in $\mathbb{D}$ matrix functions $f(z)$ of the size $p\times q$ with bounded Nevanlinna characteristic, i.e. such that

$$f = h^{-1} g,$$

where g is a holomorphic in $\mathbb{D}$ bounded matrix function of the size $p\times q$, and h is a holomorphic in $\mathbb{D}$ bounded scalar function;

$\Pi^{p\times q}$ is subclass of functions $f\in N^{p\times q}$ which have meromorphic pseudocontinuation in $\mathbb{D}_e$, i.e. such that there exists a meromorphic in $\mathbb{D}_e$ function f_- that satisfies the conditions

$$f_-^{\sharp}\in N^{q\times p}, \quad f(\zeta) := \lim_{r\uparrow 1} f(r\zeta) = \lim_{r\downarrow 1} f_-(r\zeta) \quad \text{a.e.} \quad \zeta\in\mathbb{T};$$

$\mathfrak{X}^{p\times q}\Pi$ is the intersection of the classes $\mathfrak{X}^{p\times q}$ and $\Pi^{p\times q}$.

2. Stochastic stationary vector processes

In this section we present some concepts and results of stochastic stationary (in a weak sense) process theory which will be used in the paper.

Let Ω be a space of elements ω, elementary events, with σ-algebra $\mathfrak{F}$ of ω-sets where the probability measure $P(d\omega)$ is defined, and $\mathbf{B}(\mathbb{C})$ is the smallest σ-algebra of sets of complex numbers which contains the sets $\{z:\ z = x+iy,\ a_1 < x \leq b_1,\ a_2 < y \leq b_2\}$ $(a_j, b_j\in\mathbb{R})$. A complex valued function $\xi(\omega)$ defined on the space Ω is called a complex random variable if for any $B\in\mathbf{B}(\mathbb{C})$ the condition $\{\omega : \xi(\omega)\in B\}\subset\mathfrak{F}$ hold. Such variables ξ will be considered further.

The primary numerical characteristics of an arbitrary random variable ξ are its mean $E\xi$, defined by the formula

$$E\xi = \int_\Omega \xi(\omega)dP(\omega) = \int_\Omega \xi dP,$$

and its dispersion (or variance)

$$D\xi = E|\xi - E\xi|^2.$$

Here we consider only ξ for which $E\xi$ and $D\xi$ are finite. We will denote by $\mathbb{H}$ the Hilbert space of all complex random variables ξ, which are defined in probability space Ω and have finite $E\,|\xi|^2$ (and, consequently, a finite mean and dispersion).

The scalar product in $\mathbb{H}$ is defined by the formula

$$\langle\xi,\eta\rangle = E\xi\bar{\eta}, \quad \xi,\eta\in\mathbb{H}.$$

In $\mathbb{H}$ the random variables that coincide with probability 1 are identified. Further we will consider the random variables ξ from the space $\mathbb{H}$ with zero mean. This case is always achieved by centering of the random variable, i.e. considering $\xi - E\xi$ instead of ξ. Thus the orthogonal complement in $\mathbb{H}$ to the space of random variables $\xi(\omega) = \text{const}$ will be considered.

The ordered set of p random variables $y(\omega) = \mathrm{col}\{y_k(\omega)\}_{k=1}^p$ is said to be a random vector of order p; the family $y(t,\omega) = \mathrm{col}\{y_k(t,\omega)\}_{k=1}^p$ of random p-dimensional vectors that depend on parameter $t \in \mathbb{Z}$ (time) is called a p-dimensional stochastic process with discrete time. We will use the shorter notations for $y(\omega)$ and $y(t,\omega)$: $y = \mathrm{col}\{y_k\}_{k=1}^p = \{y_k\}_{k=1}^p$ and $y(t) = y(t,\omega)$, respectively.

If $y(t) = \{y_k(t)\}_{k=1}^p$ is a p-dimensional stochastic process then matrix function $R(t,s) = \{R_{kj}(t,s)\}_{k,j=1}^p$ with elements

$$R_{kj}(t,s) = Ey_k(t)\overline{y_j(s)} := \langle y_k(t), y_j(s)\rangle, \quad 1 \le k, j \le p,$$

is called correlation function of the process $y(t)$. A stochastic process $y(t) = \{y_k(t)\}_{k=1}^p$ with zero mean is said to be stationary in the weak sense if its correlation matrix function

$$R(t,s) = \{Ey_k(t)\overline{y_j(s)}\}_{k,j=1}^p$$

depends on the difference $t - s$ only: $R(t,s) = R(t-s)$. Stationary in the weak sense stochastic processes

$$y(t) = \{y_k(t)\}_{k=1}^p, \quad x(t) = \{x_j(t)\}_{j=1}^m$$

are said to be stationary connected if the following correlation matrix function

$$R_{yx}(t,s) = \{Ey_k(t)\overline{x_j(s)}\}_{k=\overline{1,p}}^{j=\overline{1,m}}$$

depends on the difference $t - s$ only: $R_{yx}(t,s) = R_{yx}(t-s)$.

Let $y(t) = \{y_k(t)\}_{k=1}^p$ be a p-dimensional stationary (in a weak sense) stochastic processes and

$$\mathbf{H}(y) = \bigvee_{t\in\mathbb{Z}, 1\le k\le p} \{y_k(t)\}$$

be Hilbert space of the values of $y(t)$ that is the subspace of the space $\mathbb{H}$. Consider the unitary shift operator $\mathbf{U} : \mathbf{H}(y) \to \mathbf{H}(y)$ such that

$$\mathbf{U}y_k(t) = y_k(t+1), \quad k = 1, \dots, p, \quad t \in \mathbb{Z}. \tag{4}$$

The operator $\mathbf{U}$ admits spectral representation

$$\mathbf{U} = \int_{-\pi}^{\pi} e^{-i\mu} E(d\mu),$$

where $E(d\mu)$ is the spectral family of orthogonal projectors in the space $\mathbf{H}(y)$ (unit decomposition). For $\mathbf{U}^t$, $t \in \mathbb{Z}$ the next representation is true

$$\mathbf{U}^t = \int_{-\pi}^{\pi} e^{-it\mu} E(d\mu).$$

Then as $y_k(t) = \mathbf{U}^t y_k(0)$, $k = 1, \dots, p$, it follows from the last statement that

$$y_k(t) = \int_{-\pi}^{\pi} e^{-it\mu} E(d\mu) y_k(0) = \int_{-\pi}^{\pi} e^{-it\mu} F_k(d\mu), \quad k = 1, \dots, p. \tag{5}$$

In (5) quantities $F_k(d\mu) = E(d\mu)y_k(0)$ are random σ-additive measures with the properties: 1) $E|F_k(\Delta)|^2 \le E|y_k(0)|^2$, $k = 1, \dots, p$, for any measurable set $\Delta \subset [-\pi, \pi]$; 2) $EF_k(\Delta)\overline{F_l(\Delta')} = 0$ for all $k, l = 1, \dots, p$ and $\Delta \cap \Delta' = \emptyset$.

Equation (5) is called spectral representation of the process $y(t)$ and $F(d\mu) = \{F_k(d\mu)\}_{k=1}^p$ is called the spectral random measure of the p-dimensional stochastic process $y(t)$. Therefore an arbitrary stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^p$ admits spectral representation (5) as an integral with respect to its spectral random measure $F(d\mu) = \{F_k(d\mu)\}_{k=1}^p$.

The next integral representation holds for the correlation matrix function $R(t) = \{R_{kj}(t)\}_{k,j=1}^p$ of the stationary stochastic vector process $y(t) = \{y_k(t)\}_{k=1}^p$

$$R_{kj}(t) = \int_{-\pi}^{\pi} e^{-it\mu} \Phi_{kj}(d\mu),$$

where

$$\Phi_{kj}(d\mu) = EF_k(d\mu)\overline{F_j(d\mu)}, \quad 1 \le k, j \le p.$$

The matrix $\Phi = \{\Phi_{kl}\}_{k,l=1}^p$ is said to be the spectral measure of the process $y(t)$.

Setting $\sigma(\mu) = \Phi([-\pi, \mu))$ for $\mu \in (-\pi, \pi]$ and $\sigma(-\pi) = 0$ we will get that $\sigma(\mu) = \sigma(\mu - 0)$ and $\sigma(\mu)$ is nondecreasing bounded nonnegative valued matrix function on $[-\pi, \pi]$ of size $p \times p$ and

$$\Phi(\Delta) = \int_\Delta d\sigma(\mu).$$

The matrix function $R(t)$ can be presented in the form

$$R(t) = \int_{-\pi}^{\pi} e^{-it\mu} d\sigma(\mu), \quad t \in \mathbb{Z},$$

and $\sigma(\mu)$ under respective normalization can be defined via $R(t)$ using the Stieltjes inverse formula. The matrix function

$$c(z) = \frac{1}{2}R(0) + \sum_{t=1}^{\infty} R(t)z^t = \frac{1}{2}\int_{-\pi}^{\pi} \frac{e^{i\mu} + z}{e^{i\mu} - z} d\sigma(\mu)$$

belongs to the Caratheodory class $\ell^{p\times p}$.

In the case when $\sigma(\mu)$ is absolutely continuous on $[-\pi, \pi]$, i.e.

$$\sigma(\mu) = \int_{-\pi}^{\mu} \rho(e^{iu}) du, \quad \text{where} \quad \rho \in L_1^{p\times p}(\mathbb{T}),$$

matrix function $\rho(e^{i\mu})$ is called the spectral density of stochastic stationary process $y(t)$. In this case

$$R(t) = \int_{-\pi}^{\pi} e^{-it\mu} \rho(e^{i\mu}) d\mu.$$

A stationary stochastic process $w(t) = \{w_k(t)\}_{k=1}^m$ with the spectral density

$$\rho_w(e^{i\mu}) = \frac{1}{2\pi} I_m$$

is said to be white noise. The correlation matrix function of the white noise $w(t)$ is such that

$$R_w(t) = \begin{cases} I_m, & \text{if} \quad t = 0, \\ 0, & \text{if} \quad t \ne 0. \end{cases}$$

The Hilbert space $H(w)$ of the values of white noise $w(t)$ has the next property

$$H(w) = \bigoplus_{t=-\infty}^{+\infty} H_t(w), \quad \text{where} \quad H_t(w) = \bigvee \{w_k(t), k = 1, \dots, m\}.$$

Let $H(y)$ be the Hilbert space of values of stationary stochastic process $y(t)$; $H^-(y)$ and $H^+(y)$ are "past" and "future" subspaces of the process $y(t)$, i.e.

$$H^-(y) = \bigvee \{y_j(t) : t \in \mathbb{Z}^-; 1 \le j \le p\},$$

$$H^+(y) = \bigvee \{y_j(t) : t \in \mathbb{Z}^+; 1 \le j \le p\}.$$

Then it is obvious that

$$H(y) = H^-(y) \bigvee H^+(y).$$

The stationary stochastic process $y(t)$ is called regular if

$$\bigcap_{t<0} U^t H^-(y) = \{0\}.$$

This last condition holds if and only if

1) the spectral function of the process $y(t)$ is absolutely continuous and corresponding spectral density $\rho(e^{i\mu})$ has constant rank m a.e. on $[-\pi,\pi]$; and

2) there exists a holomorphic in $\mathbb{D}$ matrix function $\psi(z)$ of the size $p\times m$ that is the solution of factorization equation

$$\rho(e^{i\mu}) = \psi(e^{i\mu})\psi(e^{i\mu})^* \quad \text{a.e. on} \quad [-\pi,\pi] \tag{6}$$

and belongs to Hardy class $H_2^{p\times m}$. In this case ψ is called a spectral factor of the matrix function ρ.

The number $m := \dim\big(H^-(y)\ominus U^{-1}H^-(y)\big)$ is said to be the rank of the process $y(t)$. Note that if $y(t)$ satisfies condition 1) from above then it has rank m, and the spectral density $\rho(e^{i\mu})$ with the function ψ in (6) satisfy the following condition:

$$\operatorname{rank}\rho(e^{i\mu}) = \operatorname{rank}\psi(e^{i\mu}) \quad \text{a.e. on} \quad [-\pi,\pi]\,.$$

If for the given stationary (in a weak sense) stochastic process $y(t)$ of order p there exists white noise $w(t)$ of order m stationary connected with $y(t)$ and such that

– $H^-(y)\subset H^-(w)$,

– $H(y) = H(w)$, and unitary shift operators of the processes y and w are coincide in this space,

then process $y(t)$ is regular of rank m and there exists a spectral factor ψ of rank m of the density $\rho(e^{i\mu})$ such that

$$F_y(d\mu) = \psi(e^{i\mu})F_w(d\mu), \tag{7}$$

where $F_y(d\mu)$ is the spectral random measure of the process $y(t)$ and $F_w(d\mu)$ is the spectral random measure of white noise $w(t)$. It is possible to build white noise $w(t)$ of order m with above properties using an arbitrary factor ψ of rank m of the density ρ (see for example [33], [29]).

Matrix functions ψ_1 and ψ_2 of size $p\times m$ defined on $\mathbb{T}$ are said to be unitary equivalent if there exists a unitary matrix T of order m such that $\psi_1(e^{i\mu}) = \psi_2(e^{i\mu})T$ for almost all $\mu\in[\pi,\pi]$. Let us now identify all m-dimensional white noises w_1 and w_2 such that $w_1(t) = Tw_2(t)$ for all $t\in\mathbb{Z}$ where T is a unitary matrix of order m. Then it can be shown that there exists a one-to-one correspondence between white noises $w(t)$ with above properties and classes of unitary equivalent spectral factors ψ of the density ρ of the process $y(t)$.

3. Passive linear discrete time-invariant systems

3.1. **Linear time-invariant dynamical systems.** The evolution of the linear time-invariant dynamical system $\Sigma = (A,B,C,D;X,U,Y)$ with the discrete time $t\in\mathbb{Z}$ and Hilbert spaces of input data U and output data Y and state space X can be described by the equations

$$\begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$$

where $x(t)\in X$, $u(t)\in U$, $y(t)\in Y$, $t\in\mathbb{Z}$ and

$$A\in\mathbb{B}(X),\quad B\in\mathbb{B}(U,X),\quad C\in\mathbb{B}(X,Y),\quad D\in\mathbb{B}(U,Y).$$

Let

$$X_\Sigma^c = \bigvee_{k\geq 0} A^kBU,\quad X_\Sigma^o = \bigvee_{k\geq 0}(A^*)^kC^*Y.$$

System Σ is said to be
- *controllable* if $X = X_\Sigma^c$;
- *observable* if $X = X_\Sigma^o$;
- *simple* if $X = X_\Sigma^c \vee X_\Sigma^o$.

A system $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}; \hat{X}, U, Y)$ is called the dilation of the system

$$\Sigma = (A, B, C, D; X, U, Y)$$

if X is the subspace of $\hat{X}$ and for some subspaces $\mathfrak{D}_*$ and $\mathfrak{D}$ of the space $\hat{X}$ the following conditions hold:

$$\hat{X} = \mathfrak{D}_* \oplus X \oplus \mathfrak{D}, \quad \hat{A}^*\mathfrak{D}_* \subset \mathfrak{D}_*, \quad \hat{A}\mathfrak{D} \subset \mathfrak{D}, \quad \hat{B}^*\mathfrak{D}_* = \{0\}, \quad \hat{C}\mathfrak{D} = \{0\}, \tag{8}$$

and

$$A = P_X \hat{A}|_X, \quad B = P_X \hat{B}, \quad C = \hat{C}|_X, \quad D = \hat{D}. \tag{9}$$

In this case Σ is called *the restriction of the system* $\hat{\Sigma}$. System Σ is said to be *minimal* if it has no nontrivial restriction i.e. if it is not the dilation of any other system. It is known that system Σ is minimal if and only if it is controllable and observable i.e. $X = X_\Sigma^c = X_\Sigma^o$ (see for example [4]).

A $\mathbb{B}(U, Y)$-valued function θ_Σ defined by the formula

$$\theta_\Sigma(z) = D + zC(I - zA)^{-1}B, \quad z \in \Lambda_A, \tag{10}$$

is said to be the transfer function of the system $\Sigma = (A, B, C, D; X, U, Y)$. Here Λ_A is the subset of $\bar{\mathbb{C}} := \mathbb{C} \cup \infty$ of such $z \in \mathbb{C}$ for which $(I - zA)$ has bounded inverse, defined on whole space X, and $\infty \in \Lambda_A$ if A has inverse A^{-1} in $\mathbb{B}(X)$, and $\theta_\Sigma(\infty) = D + CA^{-1}B$.

If two holomorphic in the neighborhood of $z = 0$ functions $f_1(z)$ and $f_2(z)$ are such that $f_1(z) \equiv f_2(z)$ in some neighborhood of $z = 0$ we write $f_1 \simeq f_2$. If $\hat{\Sigma}$ is the dilation of the system Σ then $\theta_{\hat{\Sigma}} \simeq \theta_\Sigma$. If $\theta \simeq \theta_\Sigma$ then the system Σ is called the realization of the function $\theta(z)$; moreover, if Σ is a minimal system then it is called the minimal realization of the function $\theta(z)$.

Two systems $\Sigma_i = (A_i, B_i, C_i, D_i; X_i, U, Y)$, $i = 1, 2$, are called similar (unitary similar) if there exists an operator $R \in \mathbb{B}(X_1, X_2)$ with $R^{-1} \in \mathbb{B}(X_2, X_1)$ (unitary operator, respectively) such that

$$A_2 = RA_1R^{-1}, \quad B_2 = RB_1, \quad C_2 = C_1R^{-1}, \quad D_2 = D_1.$$

If the main operator A of the system Σ satisfies the condition

$$(a) \quad s - \lim_{n\to\infty} A^n = 0 \quad \text{or} \quad (b) \quad s - \lim_{n\to\infty} (A^*)^n = 0,$$

or both conditions (a) and (b) simultaneously, then system Σ is said to be stable, *-stable or bi-stable, respectively. If, in addition, A is contraction operator then we write: (a) $A \in C_{0\cdot}$, (b) $A \in C_{\cdot 0}$ or$A \in C_{00}$, respectively.

A system $\Sigma = (A, B, C, D; X, U, Y)$ is called a Φ-*forward passive* if, for any initial state $x(0)$ and for any input data $\{u(t)\}$, the following condition holds:

$$\|x(t+1)\|^2 - \|x(t)\|^2 \le \left(\Phi \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right)_{U\oplus Y}, \quad t \in \mathbb{Z}^+, \tag{11}$$

where $\Phi \in \mathbb{B}(U \oplus Y)$, $\Phi = \Phi^*$.

3.2. **Passive scattering systems.** A system $\Sigma = (A, B, C, D; X, U, Y)$ is called *passive scattering system* if, for any initial state $x(0)$ and for any input data $\{u(t)\}$, condition (11) holds with

$$\Phi = \begin{bmatrix} I_U & 0 \\ 0 & -I_Y \end{bmatrix}.$$

This condition means that operator

$$M_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Big(\in \mathbb{B}(X \oplus U, X \oplus Y)\Big)$$

is contractive. A system $\Sigma = (A, B, C, D; X, U, Y)$ is a passive scattering system if and only if the adjoint system $\Sigma^* = (A^*, C^*, B^*, D^*; X, Y, U)$ is a passive scattering system. The last statement is true because $M_{\Sigma^*} = M_\Sigma^*$ and the operator adjoint of a contraction is a contraction as well.

A system Σ is called a *conservative scattering system* if its block operator M_Σ is unitary, i.e. following equalities hold:

$$M_\Sigma^* M_\Sigma = I_{X\oplus U}, \quad M_\Sigma M_\Sigma^* = I_{X\oplus Y}.$$

An arbitrary passive scattering system is a restriction of some conservative scattering system and it is the dilation of a minimal passive scattering system, see [4].

The transfer function $\theta_\Sigma(z)$ of a passive scattering system Σ is called a scattering matrix. The main operator A of a passive scattering system is necessarily a contraction and that is why $\mathbb{D} \subset \Lambda_A$. It is known that a restriction of the scattering matrix of an arbitrary passive scattering system on the unit disk $\mathbb{D}$ belongs to the Schur class $S(U, Y)$ of holomorphic in $\mathbb{D}$ functions $s(z)$ with values from $\mathbb{B}(U, Y)$ such that $\|s(z)\| \leq 1$ in D. Conversely, an arbitrary function $\theta(z)$ from class $S(U, Y)$ is a restriction on $\mathbb{D}$ of the scattering matrix of some simple conservative scattering system that can be defined by θ up to unitary similarity, see [3], [4].

For any function $s(z)$ from class $S(U, Y)$ there exists

$$s - \lim_{r\uparrow 1} s(r\zeta) = s(\zeta) \quad \text{a.e.} \quad \zeta \in \mathbb{T}.$$

We denote $S_{in}(U, Y)$ the subclass of functions $b(z)$ from Schur class $S(U, Y)$ that are bi-inner, i.e. such $b \in S(U, Y)$ which has unitary boundary values $b(\zeta)$ a.e. on $\mathbb{T}$

$$b(\zeta)^* b(\zeta) = I_U, \quad b(\zeta) b(\zeta)^* = I_Y \quad \text{a.e.} \quad \zeta \in \mathbb{T}. \tag{12}$$

A function $b \in S(U, Y)$ is said to be inner (*-inner) if it satisfies the first (second) equality in (12) a.e. on the unit circle $\mathbb{T}$. A simple conservative scattering system is stable, *-stable or bi-stable if and only if the restriction on $\mathbb{D}$ of its scattering matrix is inner, *-inner or bi-inner (belongs to the class $S_{in}(U, Y)$), respectively, see [3].

In operator theory the colligation $\Sigma = (A, B, C, D; X, U, Y)$ where corresponding operator M_Σ is unitary, is called as an unitary node, and respective function θ_Σ is called characteristic function of this node. Thus the results of the theory of conservative scattering systems and their scattering matrices on the operator theory language are the results on the unitary nodes and their characteristic functions.

3.3. **Passive impedance systems.** A system $\Sigma = (A, B, C, D; X, U, Y)$ is called a *passive impedance system* if $Y = U$ and for any initial state $x(0)$ and any input data $\{u(t)\}$ condition (11) holds with

$$\Phi = \begin{bmatrix} 0 & I_U \\ I_U & 0 \end{bmatrix}.$$

This condition is equivalent to the following inequality for the coefficients of the system

$$\begin{bmatrix} I - A^*A & C^* - A^*B \\ C - B^*A & 2\Re D - B^*B \end{bmatrix} \geq 0. \tag{13}$$

For simplicity we denote a passive impedance system $\Sigma = (A, B, C, D; X, U)$ because $Y = U$. A system $\Sigma = (A, B, C, D; X, U)$ is a passive impedance system if and only if the adjoint system $\Sigma^* = (A^*, C^*, B^*, D^*; X, U)$ is passive impedance, i.e. the following condition holds:

$$\begin{bmatrix} I - AA^* & B - AC^* \\ B^* - CA^* & 2\Re D - CC^* \end{bmatrix} \geq 0 \tag{14}$$

(see [9], for example).

A passive impedance system Σ is said to be conservative impedance system if there are equalities in (13) and (14), i.e. if A is unitary, $C = B^*A$ and $2\Re D = B^*B$. For such a system

$$\theta_\Sigma(z) = i\Im D + \frac{1}{2}B^*(I + zA)(I - zA)^{-1}B.$$

A restriction $\Sigma = (A, B, C, D; X, U)$ of the passive impedance system $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}; \hat{X}, U)$ is also a passive impedance system. For any passive impedance system there exists a conservative impedance system which is its dilation. Passive impedance system Σ is minimal if it has no nontrivial restriction. Any passive impedance system has a restriction that is a minimal passive impedance system.

Transfer functions $\theta_\Sigma(z)$ of passive impedance systems are called *impedance matrices.* The main operator A of an arbitrary passive impedance system Σ is a contraction, that is why $\mathbb{D} \subset \Lambda_A$ and its impedance matrix is holomorphic in $\mathbb{D}$. Restrictions on the open unit disk $\mathbb{D}$ of impedance matrices of passive impedance systems form the class $\ell(U)$ of holomorphic in $\mathbb{D}$ functions $c(z)$ with values from $\mathbb{B}(U)$ that have $\Re c(z) \geq 0$ in $\mathbb{D}$. An arbitrary function $c \in \ell(U)$ is a restriction on $\mathbb{D}$ of an impedance matrix of some simple conservative impedance system which can be defined by $c(z)$ up to unitary similarity. Note that the impedance matrix θ_{Σ^*} of adjoint system Σ^* to the passive impedance system Σ is such that $\theta_{\Sigma^*}(z) = \theta_\Sigma^\sim(z),\ z \in \mathbb{D}$.

A passive impedance system $\Sigma_o = (A_o, B_o, C_o, D_o; X_o, U)$ with impedance matrix $\theta_{\Sigma_o}(z)$ is said to be *optimal* if for any other passive impedance system $\Sigma = (A, B, C, D; X, U)$ with impedance matrix $\theta_\Sigma(z) \equiv \theta_{\Sigma_o}(z)$ in $\mathbb{D}$ and for any $u(k) \in U$ and $n \geq 0$, the following condition holds:

$$\Big\| \sum_{k=0}^{n} A_o^k B_o u(k) \Big\| \leq \Big\| \sum_{k=0}^{n} A^k B u(k) \Big\|.$$

If Σ_o is an optimal passive impedance system then always $X^c_{\Sigma_o} \subset X^o_{\Sigma_o}$. That is why controllable optimal passive impedance system is always observable, and thus minimal.

An observable passive impedance system $\Sigma_1 = (A_1, B_1, C_1, D_1; X_1, U)$ is called **-optimal* if for any other observable passive impedance system $\Sigma = (A, B, C, D; X, U)$ with the same impedance matrix in $\mathbb{D}$ we have

$$\Big\| \sum_{k=0}^{n} A^k B u(k) \Big\| \leq \Big\| \sum_{k=0}^{n} A_1^k B_1 u(k) \Big\| \quad \forall u(k), \quad n \geq 0.$$

From an arbitrary conservative passive impedance system $\hat{\Sigma}$ with impedance matrix $\theta_{\hat{\Sigma}} = c(\in \ell(U))$ in $\mathbb{D}$ it is possible to get minimal passive impedance systems $\Sigma_\circ$ and $\Sigma_\bullet$ using the restriction $\hat{\Sigma}$ on the subspaces $X = X_\circ$ and $X = X_\bullet$, respectively, where

$$X_\circ = \overline{P_{X^o_{\hat{\Sigma}}} X^c_{\hat{\Sigma}}}, \quad X_\bullet = \overline{P_{X^c_{\hat{\Sigma}}} X^o_{\hat{\Sigma}}}. \tag{15}$$

Moreover, these minimal passive impedance systems $\Sigma_\circ$ and $\Sigma_\bullet$ are optimal and *-optimal, respectively. Minimal optimal and minimal *-optimal realizations of the function $c \in \ell(U)$ can be defined by c up to unitary similarity. A system

$$\Sigma = (A, B, C, D; X, U)$$

is a minimal *-optimal passive impedance system if and only if the adjoint system $\Sigma^* = (A^*, C^*, B^*, D^*; X, U)$ is minimal optimal passive impedance system.

A passive impedance system Σ is said to be a system with losses of scattering channels if the restriction on $\mathbb{D}$ of its impedance matrix $c(z)$ is such that at least one of following factorization inequalities

$$\varphi(z)^*\varphi(z) \le 2\Re c(z), \quad \psi(z)\psi(z)^* \le 2\Re c(z), \quad z \in \mathbb{D}, \tag{16}$$

has nontrivial solutions in classes of holomorphic in $\mathbb{D}$ functions with values from $\mathbb{B}(U, Y_\varphi)$ and $\mathbb{B}(U_\psi, U)$, respectively. Otherwise, Σ is said to be a passive impedance system without loses of scattering channels.

The case when factorization equations

$$(1)\ \ \varphi(\zeta)^*\varphi(\zeta) = 2\Re c(\zeta), \quad (2)\ \ \psi(\zeta)\psi(\zeta)^* = 2\Re c(\zeta) \quad \text{a.e.} \quad \zeta \in \mathbb{T}, \tag{17}$$

have nonzero solutions $\varphi \in H^2(U, Y_\varphi)$ and $\psi^\sim \in H^2(U, U_\psi)$ is most important for us. We treat equations (17) as follows: for any $u \in U$ and for almost all $\zeta \in \mathbb{T}$

$$\lim_{r\uparrow 1} \|\varphi(r\zeta)u\|^2 = \lim_{r\uparrow 1} 2\Re(c(r\zeta)u, u), \quad \lim_{r\uparrow 1} \|\psi(r\zeta)^*u\|^2 = \lim_{r\uparrow 1} 2\Re(c(r\zeta)u, u). \tag{18}$$

If factorization problems (17) are solvable then sets of their solutions, respectively, can be described by formulas

$$\varphi(z) = b_1(z)\varphi_e(z), \quad \psi(z) = \psi_e(z)b_2(z),$$

where φ_e is outer solution of the problem (1) in (17) with values from $\mathbb{B}(U, Y_{\varphi_e})$, i.e.

$$\bigvee_{n\ge 0} z^n\varphi_e(z)U = H^2(Y_{\varphi_e}),$$

ψ_e is *-outer solution of the problem (2) in (17) with values from $\mathbb{B}(U_{\psi_e}, U)$, i.e. $\psi_e^\sim$ is outer function; b_1 and b_2 are arbitrary inner and *-inner functions with values from $\mathbb{B}(Y_{\varphi_e}, Y_\varphi)$ and $\mathbb{B}(U_\psi, U_{\psi_e})$, respectively; $\dim Y_{\varphi_e} \le \dim Y_\varphi$, $\dim U_{\psi_e} \le \dim U_\psi$.

Under the normalization $\varphi_e(0)|_{Y_{\varphi_e}} > 0$, $\psi_e(0)^*|_{U_{\psi_e}} > 0$, $Y_{\varphi_e} \subset U$, $U_{\psi_e} \subset U$ functions φ_e and ψ_e are determined uniquely by c. If $\dim U < \infty$ then dimension of the spaces Y_{φ_e} and U_{ψ_e} are determined by the following equality: $\dim Y_{\varphi_e} = \operatorname{rank}\Re c(\zeta) = \dim U_{\psi_e}$ a.e.$\zeta \in \mathbb{T}$. Thus if factorization problems (17) are solvable then

$$m_c = \operatorname{rank}\Re c(\zeta) = \dim\left[\Re c(\zeta)U\right]$$

is constant a.e. on the unit circle $\mathbb{T}$.

3.4. **Conservative transmission SI-systems.** Let $\tilde{U}$ and $\tilde{Y}$ be Hilbert spaces. $J_1 \in \mathbb{B}(\tilde{U})$ and $J_2 \in \mathbb{B}(\tilde{Y})$ are signature operators, i.e.

$$J_i^* = J_i, \quad i = 1, 2; \quad J_1^2 = I_{\tilde{U}}, \quad J_2^2 = I_{\tilde{Y}}.$$

These operators determine the indefinite metrics $< .,. >$ in $\tilde{U}$ and $\tilde{Y}$ such that

$$\langle \tilde{u}, \tilde{u}'\rangle = (J_1\tilde{u}, \tilde{u}')\,, \quad \langle \tilde{y}, \tilde{y}'\rangle = (J_2\tilde{y}, \tilde{y}') \quad \tilde{u}, \tilde{u}' \in \tilde{U}, \quad \tilde{y}, \tilde{y}' \in \tilde{Y}.$$

A system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{X}, \tilde{U}, \tilde{Y})$ is called conservative transmission system if for any initial state $\tilde{x}(0) \in \tilde{X}$ and for any input data $\{\tilde{u}(t)\}$ the following condition holds

$$\|\tilde{x}(t+1)\|^2 - \|\tilde{x}(t)\|^2 = \left(\Phi_{J_1,J_2}\begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix}, \begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix}\right)_{\tilde{U}\oplus\tilde{Y}},$$

where

$$\Phi_{J_1,J_2} = \begin{bmatrix} J_1 & 0 \\ 0 & -J_2 \end{bmatrix}$$

for all $t \in \mathbb{Z}^+$, and the dual equality holds for the adjoint system $\tilde{\Sigma}^* = (\tilde{A}^*, \tilde{C}^*, \tilde{B}^*, \tilde{D}^*; \tilde{X}, \tilde{Y}^*, \tilde{U}^*)$ with operator

$$\Phi_{J_2,J_1} = \begin{bmatrix} J_2 & 0 \\ 0 & -J_1 \end{bmatrix}.$$

The fact that $\tilde{\Sigma}$ is conservative transmission system means that the operator

$$M_{\tilde{\Sigma}} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \in \mathbb{B}\left(\tilde{X} \oplus \tilde{U}, \tilde{X} \oplus \tilde{Y}\right)$$

is $(\tilde{J}_1, \tilde{J}_2)$-unitary, i.e.

$$M_{\tilde{\Sigma}}^* \tilde{J}_2 M_{\tilde{\Sigma}} = \tilde{J}_1, \quad M_{\tilde{\Sigma}} \tilde{J}_1 M_{\tilde{\Sigma}}^* = \tilde{J}_2, \quad \text{where} \quad \tilde{J}_i = \begin{bmatrix} I_X & 0 \\ 0 & J_i \end{bmatrix}, \quad i = 1, 2. \tag{19}$$

It was shown in [7] that an arbitrary passive impedance system $\Sigma = (A, B, C, D; X, U)$ with losses of scattering channels is the part of a conservative transmission system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; X, \tilde{U}, \tilde{Y})$ in the following sense. The outer spaces $\tilde{U}$, $\tilde{Y}$ and corresponding signature operators J_1 and J_2 of a conservative transmission system $\tilde{\Sigma}$ are such that

$$\tilde{U} = U_1 \oplus U \oplus U, \quad \tilde{Y} = Y_1 \oplus U \oplus U,$$

$$J_1 = \begin{bmatrix} I_{U_1} & 0 & 0 \\ 0 & 0 & -I_U \\ 0 & -I_U & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} I_{Y_1} & 0 & 0 \\ 0 & 0 & -I_U \\ 0 & -I_U & 0 \end{bmatrix} \tag{20}$$

and the operators $\tilde{\Sigma}$ have a special block structure

$$\tilde{A} = A, \quad \tilde{B} = \begin{bmatrix} K & B & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} M \\ C \\ 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} S & N & 0 \\ L & D & I_U \\ 0 & I_U & 0 \end{bmatrix}. \tag{21}$$

It follows from (19)–(21) that the operators $M \in \mathbb{B}(X, Y_1)$, $K \in \mathbb{B}(U_1, X)$, $S \in \mathbb{B}(U_1, Y_1)$, $N \in \mathbb{B}(U, Y_1)$ and $L \in \mathbb{B}(U_1, Y)$ are such that the following equalities hold:

$$\begin{bmatrix} I - A^*A & C^* - A^*B \\ C - B^*A & 2\Re D - B^*B \end{bmatrix} = \begin{bmatrix} M^*M & M^*N \\ N^*M & N^*N \end{bmatrix} = \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix},$$

$$\begin{bmatrix} I - AA^* & B - AC^* \\ B^* - CA^* & 2\Re D - CC^* \end{bmatrix} = \begin{bmatrix} KK^* & KL^* \\ LK^* & LL^* \end{bmatrix} = \begin{bmatrix} K \\ L \end{bmatrix} \begin{bmatrix} K^* & L^* \end{bmatrix},$$

$$L = B^*K + N^*S, \quad N = MC^* + SL^*,$$

and the operator

$$V = \begin{bmatrix} A & K \\ M & S \end{bmatrix} \in \mathbb{B}(X \oplus U_1, X \oplus Y_1) \tag{22}$$

is unitary. These conditions are equivalent to (19).

The inverse statement is also true. If $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; X, \tilde{U}, \tilde{Y})$ is a conservative transmission system with special block representation (21) of operators $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, and operators J_1 and J_2 which are defined in (20), then the corresponding system $\Sigma = (A, B, C, D; X, U)$ is a passive impedance system with losses of scattering channels. Systems $\tilde{\Sigma}$ of this type with special operators J_1 and J_2 of the form (20) and with corresponding block representations of coefficients of the form (21), are called conservative transmission SI-systems (scattering-impedance).

A restriction on $\mathbb{D}$ of the transfer function $\tilde{\theta}_{J_1,J_2}(z)$ of a conservative transmission SI-system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; X, \tilde{U}, \tilde{Y})$ is a holomorphic bi-(J_1, J_2)-contractive in $\mathbb{D}$ function, i.e. it is such that

$$\theta(z)^* J_2 \theta(z) \le J_1, \quad \theta(z) J_1 \theta(z)^* \le J_2, \quad z \in \mathbb{D},$$

with special block structure

$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & \delta(z) & I_U \\ 0 & I_U & 0 \end{bmatrix}, \quad \delta(z) = c(z), \quad z \in \mathbb{D}, \tag{23}$$

where the operators J_1 and J_2 are defined by the formula (20), and

$$\begin{aligned} \alpha(z) &= S + zM(I - zA)^{-1}K, \quad \beta(z) = N + zM(I - zA)^{-1}B, \quad z \in \mathbb{D}, \\ \gamma(z) &= L + zC(I - zA)^{-1}K, \quad \delta(z) = D + zC(I - zA)^{-1}B, \quad z \in \mathbb{D}. \end{aligned} \tag{24}$$

Moreover, an arbitrary function θ with block structure (23) that satisfies properties stated above is a restriction on $\mathbb{D}$ of the transmission matrix of some simple conservative transmission SI-system that can be determined by θ up to unitary similarity.

It was shown in [7] that a function $c(z)$ from $\mathbb{D}$ to $\mathbb{B}(U)$ is a restriction on $\mathbb{D}$ of the impedance matrix of some passive bi-stable impedance system $\Sigma = (A, B, C, D; X, U)$ if and only if there exists a bi-(J_1, J_2)-inner function θ with values from $\mathbb{B}(U_1 \oplus U \oplus U, Y_1 \oplus U \oplus U)$ with special block structure (23), where U_1 and Y_1 are some Hilbert spaces, and J_1 and J_2 are operators of the form (20). Here the function $\theta(z)$ is bi-(J_1, J_2)-inner in the sense that it is holomorphic in $\mathbb{D}$, takes bi-(J_1, J_2)-contractive values in $\mathbb{D}$ and for any $\tilde{u} \in U_1 \oplus U \oplus U$ and $\tilde{y} \in Y_1 \oplus U \oplus U$

$$\lim_{r \uparrow 1} (\theta(r\zeta)^* J_2 \theta(r\zeta)\tilde{u}, \tilde{u}) = (J_1 \tilde{u}, \tilde{u}), \quad \lim_{r \uparrow 1} (\theta(r\zeta) J_1 \theta(r\zeta)^* \tilde{y}, \tilde{y}) = (J_2 \tilde{y}, \tilde{y}) \tag{25}$$

a.e. on $\mathbb{T}$.

Blocks of the function $\theta(z)$ have the following properties:

1) $\beta \in H^2(U, Y_1)$ and $\gamma^{\sim} \in H^2(U, U_1)$ are the solutions of factorization problems (1) and (2) in (17);

2) $\delta (= c) \in \ell(U)$;

3) $\alpha(z)$ is a bi-inner scattering matrix of the conservative scattering system $\Sigma_{\text{scat}} = (A, K, M, S; X, U_1, Y_1)$, where operators $K \in \mathbb{B}(U_1, X)$, $M \in \mathbb{B}(X, Y_1)$ and $S \in \mathbb{B}(U_1, Y_1)$ appear as blocks of the unitary operator V in (22);

4) α, β and γ are connected with each other by the following relation:

$$\alpha(\zeta)^* \beta(\zeta) = \gamma(\zeta)^* \quad \text{a.e.} \quad \zeta \in \mathbb{T}.$$

A function $\theta(z)$ with given block $\delta(z) = c(z)$ in $\mathbb{D}$ that satisfies above properties is said to be a bi-(J_1, J_2)-inner SI-dilation of the function $c(z)$.

4. Realizations of stochastic stationary processes

In this section the authors results on the functional models of the forward and backward realizations of p-dimensional regular discrete time weak stationary stochastic processes of rank m with spectral density $\rho(e^{i\mu})$ where $\rho \in N^{p \times p}\Pi$ are presented. The results of the passive discrete time invariant impedance systems with losses theory that were discussed in the previous section are essentially used here. These results were obtained by authors under the influence of the works [28]–[31] on the forward and backward realizations of weak stationary stochastic processes via input/state/output linear time invariant systems. Below a reader will see how from the results of section 3 the new results for Lindquist-Picci realization theory follow.

4.1. **The problem of stochastic realization.** Let $y(t) = \{y_k(t)\}_{k=1}^p$ be the stationary (in a weak sense) regular stochastic process with spectral density $\rho(e^{i\mu})$ of rank m and taking values in the Hilbert space $H(y)$. The realizations of stochastic process $y(t)$ as an output data of the systems

$$(26) \quad (\Sigma_f)\begin{cases} x_f(t+1) = Ax_f(t) + Kw_f(t), \\ y(t) = Cx_f(t) + Lw_f(t), \end{cases} \qquad (\Sigma_b)\begin{cases} x_b(t-1) = \tilde{A}x_b(t) + \tilde{K}w_b(t), \\ y(t) = \tilde{C}x_b(t) + \tilde{L}w_b(t) \end{cases}$$

with $\mathbb{C}^m$, $\mathbb{C}^p$ and X as an input, output and state space, respectively, were considered in [28]–[29] (more specific information about the structure of the space X can be found in [28] or below). The first one of these systems, Σ_f, develops forward in time $t \in \mathbb{Z}$ (index f comes from $forward$), while the second one Σ_b develops backward in time $t \in \mathbb{Z}$ (index b comes from $backward$). There are vector white noises w_f and w_b of order m stationary connected with y in the systems (26) such that

$$(27) \qquad H(w_f) = H(w_b) = H(y), \quad H^-(y) \subset H^-(w_f), \quad H^+(y) \subset H^+(w_b)$$

and generate the same unitary shift operator on considering space; x_f and x_b are inner states such that

$$(28) \qquad H(x_f) \subset H(y), \quad \lim_{t\to-\infty} x_f(t) = 0, \quad x_b(t-1) = x_f(t);$$

$A, K, C, L, \tilde{A}, \tilde{K}, \tilde{C}, \tilde{L}$ are linear bounded operators between relevant subspaces such that $A \in C_{00}$ and

$$(29) \quad \tilde{A} = A^*, \quad I = AA^* + KK^* = A^*A + \tilde{K}\tilde{K}^*, \quad \tilde{C} = CA^* + LK^*, \quad C = \tilde{C}A + \tilde{L}\tilde{K}^*,$$

$$(30) \qquad E\{y(0)y(0)^*\} = CC^* + LL^* = \tilde{C}\tilde{C}^* + \tilde{L}\tilde{L}^*.$$

The following theorem is the criterion of solvability of the stochastic realization problem for stationary stochastic process $y(t)$ with the spectral density ρ.

Theorem 1. *The stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^p$ of rank m with spectral density $\rho(e^{i\mu})$ can be represented as an output data of stochastic systems (26) which have the main operator $A \in C_{00}$ and satisfy the properties (27)–(30) if and only if its spectral density is the nontangential boundary value of a function from class $N^{p\times p}$.*

The sufficient condition of this theorem will be proved in the next subsection. A detailed proof of the necessity will be presented by the second author in a separate paper.

4.2. **Realizations of stationary processes using the model of a passive impedance system with losses of scattering channels.** Let matrix function $\rho(e^{i\mu})$ be the spectral density of some stochastic stationary (in a weak sense) process $y(t) = \{y_k(t)\}_{k=1}^p$ of rank m. Suppose that $\rho(e^{i\mu})$ satisfies the conditions of Theorem 1. Our aim now is to construct stochastic systems of the form (26), the output data of which are the values of the stationary process $y(t)$ with given spectral density $\rho(e^{i\mu})$. To do this we are going to use our model of a passive impedance system with losses of scattering channels included in the conservative transmission SI-system, as described in Section 3.4. Also, we are going to give the description of a class of stationary stochastic processes with given spectral density, which can be received this way.

We assume the density $\rho(e^{i\mu})$ is the nontangential boundary value of some matrix function ρ from the class $N^{p\times p}$. Then $\rho \in N^{p\times p}\Pi$ because $\rho(e^{i\mu}) \geq 0$ a.e. on $[-\pi, \pi]$, and meromorphic pseudocontinuation of the function $\rho(z)$ can be determined by the symmetry principle

$$\rho(z) = \rho(1/\overline{z})^*, \quad z \in \mathbb{D}_e.$$

Let us now consider corresponding matrix function $c_\rho(z)$ from Caratheodory class $\ell^{p\times p}$ with spectral density $\rho(\zeta)$ determined by the formula

$$c_\rho(z) = \frac{1}{2}R(0) + \sum_{t=1}^{\infty} R(t)z^t, \tag{31}$$

where $R(t)$ are the Fourier coefficients of $\rho(\zeta)$. In this case c_ρ also has meromorphic pseudocontinuation in the exterior $\mathbb{D}_e$ of the open unit disk $\mathbb{D}$, thus $c_\rho \in \ell^{p\times p}\Pi$ and

$$2\Re c_\rho(\zeta) = \rho(\zeta), \quad \zeta = e^{i\mu}.$$

In [6], Theorem 1, it was shown that for $c_\rho(z)$ there exists a representation as a 2×2-block of a $J_{p,m}$-inner matrix function $\theta(z)$ of the form

$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & c_\rho(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}. \tag{32}$$

The matrix function θ is $J_{p,m}$-inner in that sense that it is holomorphic in the disk $\mathbb{D}$, takes $J_{p,m}$-contractive values in $\mathbb{D}$

$$\theta(z)^* J_{p,m}\theta(z) \le J_{p,m}, \quad z \in \mathbb{D}$$

and has $J_{p,m}$-unitary nontangential values a.e. on the unit circle

$$\theta(\zeta)^* J_{p,m}\theta(\zeta) = J_{p,m} \quad \text{a.e.} \quad \zeta \in \mathbb{T},$$

where

$$J_{p,m} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix}.$$

The matrix function θ is called the $J_{p,m}$-inner dilation of the matrix function c_ρ.

Let a matrix function $\theta(z)$ be such a dilation of c_ρ. Then blocks of $\theta(z)$ have the following properties:

$$\alpha \in S_{in}^{m\times m}, \quad \beta \in H_2^{m\times p}\Pi, \quad \gamma \in H_2^{p\times m}\Pi,$$
$$\beta(z)^*\beta(z) \le 2\Re c_\rho(z), \quad \gamma(z)\gamma(z)^* \le 2\Re c_\rho(z), \quad z \in \mathbb{D},$$
$$\beta(\zeta)^*\beta(\zeta) = \rho(\zeta)(= 2\Re c_\rho(\zeta)), \quad \gamma(\zeta)\gamma(\zeta)^* = \rho(\zeta)(= 2\Re c_\rho(\zeta)) \quad \text{a.e.} \quad \zeta \in \mathbb{T},$$
$$\alpha(\zeta)^*\beta(\zeta) = \gamma(\zeta)^* \quad \text{a.e.} \quad \zeta \in \mathbb{T}.$$

Using matrix function $\theta(z)$ and results of [7] (section 5, pp. 644–647) we construct the following functional model of a simple conservative transmission SI-system $\dot{\tilde{\Sigma}} = (\dot{\tilde{A}}, \dot{\tilde{B}}, \dot{\tilde{C}}, \dot{\tilde{D}}; \dot{X}, \tilde{U}, \tilde{Y})$ with the transfer function $\tilde{\theta}_{J_1,J_2} \equiv \theta$ in $\mathbb{D}$:

$$\tilde{U} = \tilde{Y} = \mathbb{C}^m \oplus \mathbb{C}^p \oplus \mathbb{C}^p, \quad \dot{X} = H_2^m \ominus \alpha H_2^m, \qquad w \in \mathbb{C}^m, \quad u \in \mathbb{C}^p, \quad x \in \dot{X};$$

$$\begin{bmatrix} \dot{\tilde{A}} & \dot{\tilde{B}} \\ \dot{\tilde{C}} & \dot{\tilde{D}} \end{bmatrix} = \begin{bmatrix} \dot{A} & \dot{K} & \dot{B} & 0 \\ \dot{M} & \dot{S} & \dot{N} & 0 \\ \dot{C} & \dot{L} & \dot{D} & I_p \\ 0 & 0 & I_p & 0 \end{bmatrix} : \ \dot{X} \oplus \tilde{U} \longrightarrow \dot{X} \oplus \tilde{Y};$$

$$\dot{\tilde{A}} = \dot{A} : \ \dot{X} \longrightarrow \dot{X}, \quad \dot{A}x = \zeta^{-1}\left[x(\zeta) - x(0)\right];$$

$$\dot{\tilde{B}} = \begin{bmatrix} \dot{K} & \dot{B} & 0 \end{bmatrix} : \ \tilde{U} \longrightarrow \dot{X};$$

$$\dot{K} : \ \mathbb{C}^m \longrightarrow \dot{X}, \quad \dot{K}w = \zeta^{-1}\left[\alpha(\zeta) - \alpha(0)\right]w;$$

$$\dot{B} : \ \mathbb{C}^p \longrightarrow \dot{X}, \quad \dot{B}u = \zeta^{-1}\left[\beta(\zeta) - \beta(0)\right]u;$$

$$\dot{\tilde{C}} = \begin{bmatrix} \dot{M} \\ \dot{C} \\ 0 \end{bmatrix} : \ \dot{X} \longrightarrow \tilde{Y};$$

$$\dot{M}: \ \dot{X} \longrightarrow \mathbb{C}^m, \quad \dot{M}x = x(0);$$
$$\dot{C}: \ \dot{X} \longrightarrow \mathbb{C}^p, \quad \dot{C}x = \int_{\mathbb{T}} \beta(\zeta)^* x(\zeta)\, |d\zeta|\,;$$
$$\dot{\tilde{D}} = \begin{bmatrix} \dot{S} & \dot{N} & 0 \\ \dot{L} & \dot{D} & I_p \\ 0 & I_p & 0 \end{bmatrix} : \ \tilde{U} \longrightarrow \tilde{Y};$$
$$\dot{S}: \ \mathbb{C}^m \longrightarrow \mathbb{C}^m, \quad \dot{S}w = \alpha(0)w;$$
$$\dot{N}: \ \mathbb{C}^p \longrightarrow \mathbb{C}^m, \quad \dot{N}u = \beta(0)u;$$
$$\dot{D}: \ \mathbb{C}^p \longrightarrow \mathbb{C}^p, \quad \dot{D}u = c_\rho(0)u;$$
$$\dot{L}: \ \mathbb{C}^m \longrightarrow \mathbb{C}^p, \quad (\dot{L}w, u)_{\mathbb{C}^p} = (\dot{K}w, \dot{B}u)_{L_2^m(\mathbb{T})} + (\dot{S}w, \dot{N}u)_{\mathbb{C}^m}.$$

By the construction system $\Sigma_{\text{scat}} = (\dot{A}, \dot{K}, \dot{M}, \dot{S}; \dot{X}, \mathbb{C}^m, \mathbb{C}^m)$ is a simple conservative scattering system with bi-inner scattering matrix $\alpha(z)$

$$\alpha(z) = \dot{S} + z\dot{M}(I_m - z\dot{A})^{-1}\dot{K}, \quad z \in \Lambda_{\dot{A}}. \tag{33}$$

It follows from the simplicity of Σ_{scat} that conservative transmission system $\dot{\tilde{\Sigma}}$ is simple. Minimality of $\dot{\tilde{\Sigma}}$ follows from the minimality of Σ_{scat} because its main operator belongs to the class C_{00}.

It follows from Theorem 3.1 in [7] that the system $\Sigma_{\text{imp}} = (\dot{A}, \dot{B}, \dot{C}, \dot{D}; \dot{X}, \mathbb{C}^p)$ is a passive bi-stable impedance system with the impedance matrix

$$c_\rho(z) = \dot{D} + z\dot{C}(I_m - z\dot{A})^{-1}\dot{B}, \quad z \in \mathbb{D}. \tag{34}$$

The operators of conservative transmission SI-system $\dot{\tilde{\Sigma}}$ are connected with each other via the following relations:

$$I_m - \dot{A}^*\dot{A} = \dot{M}^*\dot{M}, \quad \dot{C}^* - \dot{A}^*\dot{B} = \dot{M}^*\dot{N}, \quad 2\dot{D} - \dot{B}^*\dot{B} = \dot{N}^*\dot{N}, \tag{35}$$

$$I_m - \dot{A}\dot{A}^* = \dot{K}\dot{K}^*, \quad \dot{B} - \dot{A}\dot{C}^* = \dot{K}\dot{L}^*, \quad 2\dot{D} - \dot{C}\dot{C}^* = \dot{L}\dot{L}^*, \tag{36}$$

$$\dot{L} = \dot{B}^*\dot{K} + \dot{N}^*\dot{S}, \quad \dot{N} = \dot{M}\dot{C}^* + \dot{S}\dot{L}^*, \tag{37}$$

$$\dot{A}^*\dot{K} = -\dot{M}^*\dot{S}, \quad I_m - \dot{S}^*\dot{S} = \dot{K}^*\dot{K}, \quad \dot{A}\dot{M}^* = -\dot{K}\dot{S}^*, \quad I_m - \dot{S}\dot{S}^* = \dot{M}\dot{M}^*. \tag{38}$$

Let us consider simple conservative system

$$(\Sigma_{\text{scat}}) \quad \begin{cases} x_f(t+1) = \dot{A}x_f(t) + \dot{K}w_f(t), \\ w_b(t) = \dot{M}x_f(t) + \dot{S}w_f(t) \end{cases} \tag{39}$$

with bi-inner scattering matrix α, the input data of which is m-dimensional white noise $w_f(t)$, and states $x_f(t)$ satisfy the condition

$$s - \lim_{t\to-\infty} x_f(t) = 0. \tag{40}$$

It follows from [12], [13] that in this case the output data of the system Σ_{scat} is m-dimensional white noise $w_b(t)$. Moreover, from (39) and (40) it follows that

$$\hat{w}_b(\zeta) = \alpha(\zeta)\hat{w}_f(\zeta) \quad \text{a.e.} \quad \zeta \in \mathbb{T},$$

where $\hat{w}_f(\zeta) \in L_2^m(\mathbb{T})$ and $\hat{w}_b(\zeta) \in L_2^m(\mathbb{T})$ are the Fourier transforms of $w_f(t)$ and $w_b(t)$, respectively. White noises w_f and w_b are stationary connected and such that

$$H(w_b) = H(w_f), \quad H^-(w_b) \subset H^-(w_f)$$

with the same unitary shift operator. The spectral random measures $F_{w_f}(d\mu)$ and $F_{w_b}(d\mu)$ of these processes are connected via the equality

$$F_{w_b}(d\mu) = \alpha(e^{i\mu})F_{w_f}(d\mu) \tag{41}$$

and

$$w_b(t) = \int_{-\pi}^{\pi} e^{-it\mu}\alpha(e^{i\mu})F_{w_f}(d\mu).$$

Set

$$y(t) = \dot{C}x_f(t) + \dot{L}w_f(t), \quad t \in \mathbb{Z},$$

and consider the system

$$(\Sigma_f) \quad \begin{cases} x_f(t+1) = \dot{A}x_f(t) + \dot{K}w_f(t), \\ y(t) = \dot{C}x_f(t) + \dot{L}w_f(t). \end{cases} \tag{42}$$

By induction from (42) using (40) it follows that

$$y(t) = \sum_{k=-\infty}^{t-1} \dot{C}\dot{A}^{t-k-1}\dot{K}w_f(k) + \dot{L}w_f(t), \quad t \in \mathbb{Z}, \tag{43}$$

in particular, the considering above series converges. Moreover, the Fourier transform $\hat{y}(\zeta)$ of $y(t)$ is in $L_1^p(\mathbb{T})$ if $\hat{w}_f(\zeta) \in L_2^m(\mathbb{T})$ and

$$\hat{y}(\zeta) = \gamma(\zeta)\hat{w}_f(\zeta) \quad \text{a.e.} \quad \zeta \in \mathbb{T}. \tag{44}$$

Lemma 1. *$y(t)$ is stationary (in a weak sense) p-dimensional stochastic process with the sequence of correlations*

$$R(\tau) = \begin{cases} \dot{C}\dot{A}^{\tau-1}\dot{B}, & \tau > 0, \\ 2\dot{D}, & \tau = 0, \\ \dot{B}^*(\dot{A}^*)^{-\tau-1}\dot{C}^*, & \tau < 0. \end{cases} \tag{45}$$

Proof. It follows from (43) that

$$\begin{aligned} y_j(t) &= \sum_{k=-\infty}^{t-1} (\dot{C}\dot{A}^{t-k-1}\dot{K}w_f(k), e_j)_{\mathbb{C}^p} + (\dot{L}w_f(t), e_j)_{\mathbb{C}^p} \\ &= \sum_{k=-\infty}^{t-1} (w_f(k), \dot{K}^*(\dot{A}^*)^{t-k-1}\dot{C}^*e_j)_{\mathbb{C}^p} + (w_f(t), \dot{L}^*e_j)_{\mathbb{C}^p}, \end{aligned}$$

where $e_j, 1 \le j \le p$, is standard orthonormal basis in $\mathbb{C}^p$. Since in $\mathbb{H}$ all components of the last sum are orthogonal,

$$\|y_j(t)\|_{\mathbb{H}}^2 = \sum_{k=0}^{\infty} \|(w_f(t-k-1), \dot{K}^*(\dot{A}^*)^k\dot{C}^*e_j)_{\mathbb{C}^p}\|_{\mathbb{H}}^2 + \|(w_f(t), \dot{L}^*e_j)_{\mathbb{C}^p}\|_{\mathbb{H}}^2.$$

Moreover, if $h_s, 1 \le s \le m$, is standard orthonormal basis in $\mathbb{C}^m$, then

$$y_j(t) = \sum_{k=0}^{\infty}\sum_{s=1}^{m} (h_s, \dot{K}^*(\dot{A}^*)^{t-k-1}\dot{C}^*e_j)_{\mathbb{C}^m} w_{fs}(k) + \sum_{s=1}^{m} (h_s, L^*e_j)_{\mathbb{C}^m} w_{fs}(t). \tag{46}$$

If $\tau > 0$ from (46) with account that $w_{fs}(t), 1 \le s \le m$, is an orthonormal basis in $H(w_f)$, it follows that

$$\begin{aligned} (y_i(t+\tau), y_j(t))_{\mathbb{H}} = &\sum_{k=-\infty}^{t-1}\sum_{s=1}^{m} (h_s, \dot{K}^*(\dot{A}^*)^{t+\tau-k-1}\dot{C}^*e_i)_{\mathbb{C}^m} (\dot{K}^*(\dot{A}^*)^{t-k-1}\dot{C}^*e_j, h_s)_{\mathbb{C}^m} \\ &+ \sum_{s=1}^{m} (h_s, \dot{K}^*(\dot{A}^*)^{\tau-1}\dot{C}^*e_i)_{\mathbb{C}^m} (L^*e_j, h_s)_{\mathbb{C}^m} \end{aligned}$$

Last equality means that

$$R_{ij}(\tau, t) = \sum_{k=0}^{\infty}\sum_{s=1}^{m} (\dot{C}\dot{A}^{k+\tau}\dot{K})_{is}(\dot{K}^*(\dot{A}^*)^k\dot{C}^*)_{sj} + \sum_{s=1}^{m} (\dot{C}\dot{A}^{\tau-1}\dot{K})_{is}(\dot{L}^*)_{sj}.$$

This expression gives

$$R(\tau,t)=\sum_{k=0}^{\infty}\dot C\dot A^{k+\tau}\dot K\dot K^*(\dot A^*)^k\dot C^*+\dot C\dot A^{\tau-1}\dot K\dot L^*.$$

Using (36) we obtain

$$R(\tau,t)=R(\tau)=\dot C\dot A^{\tau-1}\dot B,\quad \tau>0.$$

Corresponding expressions for $R(\tau,t)$ in the cases $\tau=0$ and $\tau<0$ can be obtained analogically. Therefore, $y(t)$ is weak stationary p-dimensional stochastic process. □

Lemma 2. *In the settings of lemma 1, $y(t)$ is a regular stationary process with spectral density $\rho(e^{i\mu})$.*

Proof. Let π_+ be the orthogonal projection from $L_2^m(\mathbb{T})$ onto H_2^m. Then the formulas for $\dot A$ and $\dot B$ may be rewritten as

$$\dot Ax=\pi_+Ux,\quad x\in\dot X;\qquad \dot Bu=\pi_+U\beta u,\quad u\in\mathbb{C}^p,$$

where $\mathbf{U}$ is the operator of the multiplication on ζ^{-1} of vector functions $x(\zeta)$ from $L_2^m(\mathbb{T})$. Consequently,

$$\dot A^{k-1}x=\pi_+\mathbf{U}^{k-1}x,\quad \dot A^{k-1}\dot Bu=\pi_+\mathbf{U}^k\beta u\quad\Rightarrow$$

$$\sum_{k=1}^{\infty}\dot A^{k-1}\dot Buz^k=z\pi_+\mathbf{U}(I-z\mathbf{U})^{-1}\beta u=z\frac{\beta(\zeta)-\beta(z)}{\zeta-z}u$$

and

$$\sum_{k=1}^{\infty}R(k)z^k=\int_{\mathbb{T}}\beta(\zeta)^*z\frac{\beta(\zeta)-\beta(z)}{\zeta-z}\zeta^{-1}u\,|d\zeta|=z\int_{\mathbb{T}}\frac{\rho(\zeta)}{\zeta-z}u\,|d\zeta|\,.$$

Then, with the account that

$$\dot D=\frac{1}{2}R(0)=\frac{1}{2}\int_{\mathbb{T}}\beta(\zeta)^*\beta(\zeta)\,|d\zeta|=\frac{1}{2}\int_{\mathbb{T}}\rho(\zeta)\,|d\zeta|\,,$$

we obtain that

$$c_\rho=\frac{1}{2}R(0)+\sum_{k=1}^{\infty}R(k)z^k=\frac{1}{2}\int_{\mathbb{T}}\frac{\zeta-z}{\zeta+z}\rho(\zeta)\,|d\zeta|\,.$$

This establish the assertion of lemma. □

Thus, $y(t)$ is a regular stationary process with the spectral density

(47) $$\rho(e^{i\mu})=\gamma(e^{i\mu})\gamma(e^{i\mu})^*\quad\text{a.e.}\quad \mu\in[-\pi,\pi].$$

Hence, system $\Sigma_f:=(\dot A,\dot K,\dot C,\dot L;\dot X,\mathbb{C}^m,\mathbb{C}^p)$ in (42) is developing forward in time realization of the process $y(t)$ with the spectral density ρ as an output data. Block γ of the dilation θ is a restriction on the open unit disk $\mathbb{D}$ of the transfer function of system $\dot\Sigma_f$

(48) $$\gamma(z)=\dot L+z\dot C(I_m-z\dot A)^{-1}\dot K,\quad z\in\mathbb{D},$$

and it is a spectral factor of the rank m of density ρ, i.e. γ belongs to $H_2^{p\times m}\Pi$ and satisfies (47). The matrix function γ corresponds to white noise w_f in the sense that

$$H(w_f)=H(y),\quad H^-(y)\subset H^-(w_f),$$

the same unitary shift operator corresponds to y and w_f and the spectral random measures $F_y(d\mu)$ and $F_{w_f}(d\mu)$ of processes y and w_f, respectively, are connected via the relation

(49) $$F_y(d\mu)=\gamma(e^{i\mu})F_{w_f}(d\mu).$$

It follows from (49) that the values of the process $y(t)$ can be represented in the following integral form:

$$y(t) = \int_{-\pi}^{\pi} e^{-it\mu}\gamma(e^{it\mu})F_{w_f}(d\mu). \tag{50}$$

Using equalities $\dot{A}^*\dot{A} + \dot{M}^*\dot{M} = I$ and $\dot{A}^*\dot{K} + \dot{M}^*\dot{S} = 0$ and the first and the second equations in (42) and (39), respectively, we have

$$x_f(t) = (\dot{A}^*\dot{A} + \dot{M}^*\dot{M})x_f(t) + (\dot{A}^*\dot{K} + \dot{M}^*\dot{S})w_f(t) \qquad \Longrightarrow$$
$$x_f(t) = \dot{A}^*(\dot{A}x_f(t) + \dot{K}w_f(t)) + \dot{M}^*(\dot{M}x_f(t) + \dot{S}w_f(t)) \qquad \Longrightarrow$$

$$x_f(t) = \dot{A}^*x_f(t+1) + \dot{M}^*w_b(t). \tag{51}$$

Using equalities $\dot{K}^*\dot{A} + \dot{S}^*\dot{M} = 0$ and $\dot{K}^*\dot{K} + \dot{S}^*\dot{S} = I$ and also the first and the second equations in (42) and (39), respectively, we have

$$w_f(t) = (\dot{K}^*\dot{A} + \dot{S}^*\dot{M})x_f(t) + (\dot{K}^*\dot{K} + \dot{S}^*\dot{S})w_f(t) \qquad \Longrightarrow$$
$$w_f(t) = \dot{K}^*(\dot{A}x_f(t) + \dot{K}w_f(t)) + \dot{S}^*(\dot{M}x_f(t) + \dot{S}w_f(t)) \qquad \Longrightarrow$$

$$w_f(t) = \dot{K}^*x_f(t+1) + \dot{S}^*w_b(t). \tag{52}$$

It follows from the relations (51), (52) and results [12], [13], that simple conservative scattering system $\dot{\Sigma}^*_{\mathrm{scat}} = (\dot{A}^*, \dot{M}^*, \dot{K}^*, \dot{S}^*; \dot{X}, \mathbb{C}^m, \mathbb{C}^m)$ with scattering matrix $\alpha(\bar{z})^*$ in $\mathbb{D}$ turned backward in time with a shift of one step connects white noises w_f and w_b with each other in the following way:

$$\begin{cases} x_b(t-1) = \dot{A}^*x_b(t) + \dot{M}^*w_b(t), \\ w_f(t) = \dot{K}^*x_b(t) + \dot{S}^*w_b(t), \end{cases} \tag{53}$$

where

$$x_b(t) := x_f(t+1). \tag{54}$$

In this case

$$F_{w_f}(d\mu) = \alpha(e^{i\mu})^*F_{w_b}(d\mu),$$

and

$$w_f(t) = \int_{-\pi}^{\pi} e^{-it\mu}\alpha(e^{i\mu})^*F_{w_b}(d\mu).$$

Furthermore, using $\dot{C} = \dot{B}^*\dot{A} + \dot{N}^*\dot{M}$ and $\dot{L} = \dot{B}^*\dot{K} + \dot{N}^*\dot{S}$ from the second equation in (42) we have

$$y(t) = (\dot{B}^*\dot{A} + \dot{N}^*\dot{M})x_f(t) + (\dot{B}^*\dot{K} + \dot{N}^*\dot{S})w_f(t) \qquad \Longrightarrow$$
$$y(t) = \dot{B}^*(\dot{A}x_f(t) + \dot{K}w_f(t)) + \dot{N}^*(\dot{M}x_f(t) + \dot{S}w_f(t)) \qquad \Longrightarrow$$

$$y(t) = \dot{B}^*x_f(t+1) + \dot{N}^*w_b(t). \tag{55}$$

Consider the adjoint system $\Sigma_b := (\dot{A}^*, \dot{M}^*, \dot{B}^*, \dot{N}^*; \dot{X}, \mathbb{C}^m, \mathbb{C}^p)$. A restriction on $\mathbb{D}$ of the transfer function of this system is coincides with $\beta^\sim(z)$

$$\beta^\sim(z) = \dot{N}^* + z\dot{B}^*(I_m - z\dot{A}^*)^{-1}\dot{M}^*, \quad z \in \mathbb{D}. \tag{56}$$

It follows from (51), (55) and (54) that the system Σ_b turned backward in time with shift of one step is the realization of the process $y(t)$ such that if the values of the noise $w_b(t)$ are the input data, then the values of $y(t)$ are the output data of Σ_b

$$\begin{cases} x_b(t-1) = \dot{A}^*x_b(t) + \dot{M}^*w_b(t), \\ y(t) = \dot{B}^*x_b(t) + \dot{N}^*w_b(t). \end{cases} \tag{57}$$

Note that matrix function $\beta^{\sim}$ is a spectral factor of rank m of the density $\rho^{\sim}(e^{i\mu}) = \rho(e^{-i\mu})$ of the process $\tilde{y}(t) := y(-t)$, i.e. $\beta \in H_2^{m\times p}\Pi$ and

$$\beta^{\sim}(e^{i\mu})(\beta^{\sim}(e^{i\mu}))^* = \rho^{\sim}(e^{i\mu}) \quad \text{a.e.} \quad \mu \in [-\pi, \pi].$$

Since blocks α, β and γ of the dilation θ are connected via the relation

$$\gamma(e^{i\mu}) = \beta(e^{i\mu})^*\alpha(e^{i\mu}) \quad \text{a.e.} \quad \mu \in [-\pi, \pi],$$

and because of (49) and (41) the following equality holds for the spectral random measures of the processes y and w_b

$$F_y(d\mu) = \gamma(e^{i\mu})F_{w_f}(d\mu) = \beta(e^{i\mu})^*\alpha(e^{i\mu})F_{w_f}(d\mu) = \beta(e^{i\mu})^*F_{w_b}(d\mu). \tag{58}$$

Consequently, block β of $J_{p,m}$-inner dilation θ corresponds to the white noise $w_b(t)$ in the sense that

$$H(w_b) = H(y), \quad H^+(y) \subset H^+(w_b),$$

the same unitary shift operator corresponds to y and w_b and spectral random measures $F_y(d\mu)$ and $F_{w_b}(d\mu)$ of the processes y and w_b are connected via the relation (58). In this case the values of the process $y(t)$ can be represented in the integral form

$$y(t) = \int_{-\pi}^{\pi} e^{-it\mu}\beta(e^{it\mu})^*F_{w_b}(d\mu). \tag{59}$$

Thus, we can present now the following theorem.

Theorem 2. *Let $\rho(\zeta)$ be a rank m matrix function from $L_1^{p\times p}(\mathbb{T})$ which is nonnegative a.e. on $\mathbb{T}$ and it is the nontangential boundary value of a function $\rho(z)$ from the class $N^{p\times p}$; a matrix function $c_\rho \in \ell^{p\times p}\Pi$ is determined by ρ via formula (31).*

Let a realization model

$$\{\Sigma_f, \Sigma_b, \Sigma_{\mathrm{scat}}, \Sigma_{\mathrm{imp}}\}$$

is defined by ρ as it was done above in this subsection using an $J_{p,m}$-inner dilation θ of c_ρ. Then

1) *the system Σ_f is a forward in time realization of the stationary p-dimensional stochastic process $y(t)$ of rank m with given spectral density ρ, with the values of the white noise $w_f(t)$ as an input data*; *the system Σ_b is a backward in time realization of the process $y(t)$ with the values of the white noise $w_b(t)$ as an input data*;

2) *inner states of the systems Σ_f and Σ_b are connected via the relation*

$$x_b(t) = x_f(t+1);$$

3) *white noises w_f and w_b are connected with each other via the simple conservative bi-stable scattering system Σ_{scat}.*

Remark. Consider a matrix function

$$\theta^{\sim}(z) = \begin{bmatrix} \alpha^{\sim}(z) & \beta^{\sim}(z) & 0 \\ \gamma^{\sim}(z) & c_\rho^{\sim}(z) & I_p \\ 0 & I_p & 0 \end{bmatrix} = \begin{bmatrix} \alpha(\bar{z})^* & \gamma(\bar{z})^* & 0 \\ \beta(\bar{z})^* & c_\rho(\bar{z})^* & I_p \\ 0 & I_p & 0 \end{bmatrix}.$$

This function is a $J_{p,m}$-inner dilation of the matrix function $c_\rho^{\sim}(z)$. It is easy to see that $\theta^{\sim}(z)$ corresponds to stationary stochastic process $\tilde{y}(t) = y(-t)$ with spectral density $\rho^{\sim}(e^{i\mu}) = \rho(e^{-i\mu})$ and realization model $\{\tilde{\Sigma}_f, \tilde{\Sigma}_b, \tilde{\Sigma}_{\mathrm{scat}}, \tilde{\Sigma}_{\mathrm{imp}}\}$, where

$$\tilde{\Sigma}_{\mathrm{scat}} = \Sigma_{\mathrm{scat}}^*, \quad \tilde{\Sigma}_{\mathrm{imp}} = \Sigma_{\mathrm{imp}}^*,$$

$$\tilde{\Sigma}_f = (\dot{A}^*, \dot{M}^*, \dot{B}^*, \dot{N}^*; \dot{X}, \mathbb{C}^m, \mathbb{C}^p),$$

and realization $\tilde{\Sigma}_b$ of the process $\tilde{y}(t)$ is the system $\Sigma_f = (\dot{A}, \dot{K}, \dot{C}, \dot{L}; \dot{X}, \mathbb{C}^m, \mathbb{C}^p)$ turned backward in time with the one step shift.

For the description of the various realizations Σ_f and Σ_b of a stationary regular stochastic process with given spectral density $\rho \in \Pi^{p\times p}$ we need a parameterization of the set of all $J_{p,m}$-inner dilations of matrix function $c(z)$ from the class $\ell^{p\times p}\Pi$, that was presented in [6]. Namely, consider matrix function $c \in \ell^{p\times p}\Pi$ with

$$m = \operatorname{rank} 2\Re c(\zeta) \quad \text{a.e.} \quad \zeta \in \mathbb{T}.$$

Let functions $\varphi_e \in H_2^{m\times p}$ and $\psi_e \in H_2^{p\times m}$ be normalized outer and *-outer solutions of the factorization problems

$$2\Re c(\zeta) = \varphi(\zeta)^*\varphi(\zeta), \quad 2\Re c(\zeta) = \psi(\zeta)\psi(\zeta)^* \quad \text{a.e.} \quad \zeta \in \mathbb{T},$$

and $s_c \in N^{m\times m}$ be the scattering suboperator determined by the relation

$$s_c(z)\psi_e^\sharp(z) = \varphi_e(z), \quad z \in \Lambda_{\varphi_e} \cap \Lambda_{\psi_e^\sharp} \cap \Lambda_{s_c}$$

that takes unitary values a.e. on the unit circle. Then an arbitrary $J_{p,m}$-inner SI-dilation θ of matrix function $c(z)$ can be uniquely presented in the form

$$\begin{aligned}\theta(z) &= \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix} = \begin{bmatrix} b_1(z)s_c(z)b_2(z) & b_1(z)\varphi_e(z) & 0 \\ \psi_e(z)b_2(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix} \\ &= \begin{bmatrix} b_1(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \begin{bmatrix} s_c(z) & \varphi_e(z) & 0 \\ \psi_e(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix} \begin{bmatrix} b_2(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix},\end{aligned} \tag{60}$$

where $\{b_1, b_2\}$ is a denominator of matrix function s_c, i.e. ordered pair of functions from the class $S_{in}^{m\times m}$, such that the product $b_1 s_c b_2$ belongs to $S_{in}^{m\times m}$ (look at [6], [7] for the details).

In the following theorem the above realization models are parameterizing by denominators of s_c.

Theorem 3. *Let $\rho(\zeta)$ be a matrix function of rank m from $L_1^{p\times p}(\mathbb{T})$ that is nonnegative a.e. on $\mathbb{T}$ and it is the nontangential boundary value of a function $\rho(z)$ from $N^{p\times p}$; let corresponding matrix functions $c_\rho \in \ell^{p\times p}\Pi$, $\varphi_e \in H_2^{m\times p}$, $\psi_e \in H^{p\times m}$ and $s_c \in N^{m\times m}$ are determined by ρ as above.*

Let the $J_{p,m}$-inner dilation θ of the matrix function c_ρ be defined by the formula (60); let w_f and w_b be m-dimensional white noises with the spectral measures $F_{w_f}(d\mu)$ and $F_{w_b}(d\mu)$, respectively, such that

$$F_{w_b}(d\mu) = \alpha(e^{i\mu})F_{w_f}(d\mu) = b_1(e^{i\mu})s_c(e^{i\mu})b_2(e^{i\mu})F_{w_f}(d\mu), \tag{61}$$

where $\{b_1, b_2\}$ is a denominator of $s_c \in N^{m\times m}$ in representation (60). Determine $y(t)$ by the formulas

$$\begin{aligned} y(t) &= \int_{-\pi}^{\pi} e^{-it\mu}\gamma(e^{i\mu})F_{w_f}(d\mu) = \int_{-\pi}^{\pi} e^{-it\mu}\psi_e(e^{i\mu})b_2(e^{i\mu})F_{w_f}(d\mu) \\ &= \int_{-\pi}^{\pi} e^{-it\mu}\beta(e^{i\mu})^*F_{w_b}(d\mu) = \int_{-\pi}^{\pi} e^{-it\mu}\varphi_e(e^{i\mu})^*b_1(e^{i\mu})^*F_{w_b}(d\mu). \end{aligned} \tag{62}$$

Then $y(t)$ is stationary stochastic process of rank m with spectral density $\rho(e^{i\mu})$.

Proof. Let θ be a $J_{p,m}$-inner dilation of matrix function $c_\rho \in \ell^{p\times p}\Pi$. Its blocks can be uniquely presented in the form

$$\alpha = b_1 s_c b_2, \quad \beta = b_1\varphi_e, \quad \gamma = \psi_e b_2, \tag{63}$$

where $\{b_1, b_2\}$ is a denominator of matrix function $s_c \in N^{m\times m}$. Using these matrix functions it is possible to obtain a pair of the white noises $\{w_f, w_b\}$ and to construct realization systems $\{\Sigma_f, \Sigma_b, \Sigma_{\rm scat}, \Sigma_{\rm imp}\}$ as it was shown above. System $\Sigma_{\rm scat}$ connects processes w_f and w_b with each other, and their random spectral measures satisfy the condition (41). Using (63) we have (61).

The output data of the systems Σ_f and Σ_b are the values of some stationary process $y(t)$ of rank m with the spectral density $\rho(e^{i\mu})$. The values of this process can be presented in the form (50) and (59). Using this fact and equalities (63) we get (62). □

In end of this section we will notice that the following result holds.

Lemma 3. *Systems* $\Sigma_f = (\dot A, \dot K, \dot C, \dot L; \dot X, \mathbb{C}^m, \mathbb{C}^p)$ *and* $\tilde\Sigma_f = (\dot A^*, \dot M^*, \dot B^*, \dot N^*; \dot X, \mathbb{C}^m, \mathbb{C}^p)$ *are* Φ*-forward-passive with*

$$\Phi = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. For an arbitrary linear stationary dynamical system $\Sigma = (A, B, C, D; X, U, Y)$ the condition of Φ-forward-passivity (11) is equivalent to the following inequality for the operators of the system:

$$\begin{bmatrix} A^*A - I_X & A^*B \\ B^*A & B^*B \end{bmatrix} - \begin{bmatrix} 0 & C^* \\ I_X & D^* \end{bmatrix} \Phi \begin{bmatrix} 0 & I_X \\ C & D \end{bmatrix} \le 0. \tag{64}$$

For the realization system $\Sigma_f = (\dot A, \dot K, \dot C, \dot L; \dot X, \mathbb{C}^m, \mathbb{C}^p)$ we can write the left part of (64) using the relations (35) and (38) in the form

$$\begin{aligned} &\begin{bmatrix} \dot A^*\dot A - I_m & \dot A^*\dot K \\ \dot K^*\dot A & \dot K^*\dot K \end{bmatrix} - \begin{bmatrix} 0 & \dot C^* \\ I_m & \dot L^* \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ \dot C & \dot L \end{bmatrix} \\ &\quad = \begin{bmatrix} \dot A^*\dot A - I_m & \dot A^*\dot K \\ \dot K^*\dot A & \dot K^*\dot K - I_m \end{bmatrix} = \begin{bmatrix} -\dot M^*\dot M & -\dot M^*\dot S \\ -\dot S^*\dot M & -\dot S^*\dot S \end{bmatrix} \\ &\quad = -\begin{bmatrix} \dot M^* \\ \dot S^* \end{bmatrix} \begin{bmatrix} \dot M & \dot S \end{bmatrix} \le 0. \end{aligned}$$

For the realization system $\tilde\Sigma_f = (\dot A^*, \dot M^*, \dot B^*, \dot N^*; \dot X, \mathbb{C}^m, \mathbb{C}^p)$ the condition (11) of Φ-forward-passivity can be obtained in similarly way. □

4.3. **Minimal realizations of stationary stochastic processes.** We call a forward realization Σ_f of the stationary stochastic process $y(t)$ minimal if the system $\Sigma_f = (\dot A, \dot K, \dot C, \dot L; \dot X, \mathbb{C}^m, \mathbb{C}^p)$ is minimal. Similarly, a backward realization Σ_b of the process $y(t)$ is said to be minimal if the system $\tilde\Sigma_f = (\dot A^*, \dot M^*, \dot B^*, \dot N^*; \dot X, \mathbb{C}^m, \mathbb{C}^p)$ is minimal. It will be shown in this section how to construct minimal realizations of the process with given spectral density using corresponding $J_{p,m}$-inner dilations $\theta(z)$. A dilation θ of the matrix function $c(z)$ is called minimal if it can not be presented in the form

$$\theta(z) = \begin{bmatrix} u(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \tilde\theta(z) \begin{bmatrix} v(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\tilde\theta$ is a $J_{p,m}$-inner dilation of c, matrix functions u and v belong to $S_{in}^{m\times m}$ and at least one of them is not constant. A $J_{p,m}$-inner dilation θ of matrix function $c(z)$ is minimal if and only if the corresponding denominator $\{b_1, b_2\}$ of the scattering suboperator s_c in representation (60) is minimal, i.e. there exists no such a denominator $\{\tilde b_1, \tilde b_2\}$ of s_c that is a nontrivial divisor of denominator $\{b_1, b_2\}$. It means that, if

$$u_1 = \tilde b_1^{-1} b_1 \in S_{in}^{m\times m}, \quad u_2 = b_2 \tilde b_2^{-1} \in S_{in}^{m\times m} \quad \text{and} \quad \tilde b_1 s_c \tilde b_2 \in S_{in}^{m\times m},$$

then both u_1 and u_2 are constant unitary matrices.

The minimality condition for a $J_{p,m}$-inner dilation θ of the matrix function $c(z)$ can be reformulated in the form of the following two conditions:

$$(i)\ (\alpha,\gamma)_R = I, \quad (ii)\ (\alpha,\beta)_L = I. \tag{65}$$

Relation (i) (respectively, (ii)) in (65) means that matrix functions α and γ (respectively, α and β) have no nontrivial common bi-inner right (respectively, left) divider.

Assume now that stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^p$ of rank m with spectral density $\rho(e^{i\mu})$ and matrix function $c_\rho \in \ell^{p\times p}\Pi$, which is determined by ρ via (31), is realized as an output data of the systems Σ_f and Σ_b as it was show in subsection 4.2. Assume that a matrix function $\theta(z)$ is a $J_{p,m}$-inner dilation of c_ρ, and Σ_{imp} is the bi-stable passive impedance system with impedance matrix c_ρ constructed as part of the simple conservative transmission SI-system $\tilde{\Sigma}$ with transfer function $\theta(z)$ in $\mathbb{D}$ using the method described in the previous section. According to [7] the system Σ_{imp} with impedance matrix c_ρ is controllable (observable) if and only if the condition $(i)((ii))$ in (65) holds.

Controllability of the realization system Σ_f immediately follows from controllability of minimal conservative scattering system Σ_{scat} with scattering matrix $\alpha(z)$, described by the equations (39). It is easy to see that the system Σ_f is observable if and only if the corresponding passive impedance system Σ_{imp} is observable. Similarly, controllability of the backward realization system Σ_b follows from observability of the corresponding minimal conservative scattering system Σ_{scat}. And the realization Σ_b is observable if and only if the corresponding passive impedance system Σ_{imp} is controllable. Thus, as described above the following theorem is proved.

Theorem 4. *In the settings of theorems 2 and 3, the following assertions hold:*

1) *the forward realization Σ_f of the process y is minimal if and only if blocks of the dilation θ satisfy the condition (ii) in (65);*

2) *the backward realization Σ_b of the process y is minimal if and only if blocks of the dilation θ satisfy the condition (i) in (65);*

3) *the forward and backward realizations Σ_f, Σ_b of the process y are both minimal if and only if the corresponding $J_{p,m}$-inner dilation θ is minimal.*

Moreover, last condition holds if and only if in the representation (60) of θ the denominator $\{b_1, b_2\}$ of s_c is minimal, and this holds if and only if both conditions (i) and (ii) in (65) are satisfied.

4.4. **Minimal and optimal, minimal and *-optimal realizations and stationary Kalman filters.** We will now consider some special realizations of stochastic processes: minimal and optimal, minimal and *-optimal. Constructions of these realizations will lead us to the forward and backward stationary Kalman filters.

Let $\rho(e^{i\mu})$ be the spectral density of a stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^p$ of rank m, which is the nontangential boundary value of a matrix function ρ from the class $N^{p\times p}$; a matrix function $c_\rho \in \ell^{p\times p}\Pi$ is determined by ρ in (31); a matrix function θ of the form (60) is a $J_{p,m}$-inner dilation of c_ρ. The corresponding realization $\{\Sigma_f, \Sigma_b, \Sigma_{\mathrm{scat}}, \Sigma_{\mathrm{imp}}\}$ of the process y, constructed via blocks of θ using the method described in the subsection 4.2, is called optimal (*-optimal) if its passive impedance system Σ_{imp} is optimal (*-optimal). In [6] the definition of optimal (*-optimal) dilation was given. A $J_{p,m}$-inner dilation θ of the matrix function $c \in \ell^{p\times p}\Pi$ is said to be optimal if $\beta = \varphi_e$ in its representation (60), and *-optimal if $\gamma = \psi_e$ in (60).

All optimal $J_{p,m}$-inner dilations of the matrix function $c \in \ell^{p\times p}\Pi$ can be described by the formula

$$\theta_\circ(z) = \begin{bmatrix} s_c(z)b_\circ(z) & \varphi_e(z) & 0 \\ \psi_e(z)b_\circ(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}, \tag{66}$$

where $\{I_m, b_\circ\}$ is a (right) denominator of s_c. In this case an optimal dilation θ is minimal if and only if the corresponding right denominator $\{I_m, b_\circ\}$ of the matrix function s_c is minimal. Such a denominator exists and is essentially unique.

It follows from Theorem 3.3 in [7] that the passive impedance system Σ_{imp} with impedance matrix $c \in \ell^{p\times p}\Pi$, that is obtained in the considering above way, is optimal if and only if the transfer function θ of the corresponding conservative transmission SI-system $\tilde{\Sigma}$ (i.e. $J_{p,m}$-inner dilation of the function c) is optimal.

All *-optimal $J_{p,m}$-inner dilations of the matrix function $c \in \ell^{p\times p}\Pi$ can be described by the formula

$$\theta_\bullet(z) = \begin{bmatrix} b_\bullet(z)s_c(z) & b_\bullet(z)\varphi_e(z) & 0 \\ \psi_e(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}, \tag{67}$$

where $\{b_\bullet, I_m\}$ is a (left) denominator of s_c. In this case a *-optimal $J_{p,m}$-inner dilation θ is minimal if and only if the corresponding left denominator $\{b_\bullet, I_m\}$ of the matrix function s_c is minimal. Such a denominator exists and is essentially unique.

Consider a spectral density $\rho(e^{i\mu})$ of rank m which is the nontangential boundary value of a matrix function ρ from the class $N^{p\times p}$ and corresponding matrix function $c_\rho \in \ell^{p\times p}\Pi$, determined via ρ by the formula (31). Let $\theta_\circ$ and $\theta_\bullet$ be minimal optimal and minimal *-optimal $J_{p,m}$-inner dilations of the matrix function c_ρ, respectively. Using matrix functions $\theta_\circ$ and $\theta_\bullet$ construct corresponding conservative transmission SI-systems $\tilde{\Sigma}_\circ$ and $\tilde{\Sigma}_\bullet$ like it was shown in the subsection 4.2. By means of these systems we will have the realizations $\{\Sigma_{f\circ}, \Sigma_{b\circ}, \Sigma_{\mathrm{scat}\,\circ}, \Sigma_{\mathrm{imp}\,\circ}\}$ and $\{\Sigma_{f\bullet}, \Sigma_{b\bullet}, \Sigma_{\mathrm{scat}\,\bullet}, \Sigma_{\mathrm{imp}\,\bullet}\}$, respectively.

Let us first consider the realization $\{\Sigma_{f\circ}, \Sigma_{b\circ}, \Sigma_{\mathrm{scat}\,\circ}, \Sigma_{\mathrm{imp}\,\circ}\}$ that corresponds to the dilation $\theta_\circ$. As the matrix-function $\theta_\circ$ is a minimal $J_{p,m}$ - inner SI-dilation of the matrix function c_ρ, it follows from the Theorem 3 that the considered realization is minimal. Furthermore, it follows from the optimality of the dilation $\theta_\circ$ that its block $\beta = \varphi_e$. Then, using Theorem 3.3 in [7] we have that the passive impedance system $\Sigma_{\mathrm{imp}\,\circ} = (A_\circ, B_\circ, C_\circ, D_\circ; X_\circ, \mathbb{C}^p)$ is optimal. Consequently, the realization $\{\Sigma_{f\circ}, \Sigma_{b\circ}, \Sigma_{\mathrm{scat}\,\circ}, \Sigma_{\mathrm{imp}\,\circ}\}$ is optimal.

The outer matrix function φ_e is such that $\varphi_e^\sim(z)$ $(z \in \mathbb{D})$ is the transfer function of the backward realization $\Sigma_{b\circ} = (A_\circ^*, M_\circ^*, B_\circ^*, N_\circ^*; X_\circ, \mathbb{C}^m, \mathbb{C}^p)$ of stationary stochastic process $y_\circ$ with the spectral density ρ. The evolution of this system is described by the following equations:

$$(\Sigma_{b\circ}) \quad \begin{cases} x_{b\circ}(t-1) = A_\circ^* x_{b\circ}(t) + M_\circ^* w_{b\circ}(t), \\ y_\circ(t) = B_\circ^* x_{b\circ}(t) + N_\circ^* w_{b\circ}(t). \end{cases} \tag{68}$$

Let us consider now the realization $\{\Sigma_{f\bullet}, \Sigma_{b\bullet}, \Sigma_{\mathrm{scat}\,\bullet}, \Sigma_{\mathrm{imp}\,\bullet}\}$, that corresponds to a minimal and *-optimal $J_{p,m}$-inner dilation $\theta_\bullet$. The minimality of the systems $\Sigma_{f\bullet}$, $\Sigma_{b\bullet}$, $\Sigma_{\mathrm{imp}\,\bullet}$ immediately follow from the minimality of the dilation $\theta_\bullet$ according to the Theorem 3. As $\theta_\bullet$ is a *-optimal $J_{p,m}$-inner dilation of the matrix function c_ρ, its block $\gamma = \psi_e$, and the matrix function $\psi_e(\bar{z})^*$ is an outer matrix function of the size $m \times p$. The adjoint system $\Sigma_{\mathrm{imp}\,\bullet}^*$ is part of an adjoint transmission system $\tilde{\Sigma}_\bullet^*$ with transmission matrix $\theta_\bullet(\bar{z})^*$, which is an optimal $J_{p,m}$-inner dilation of $c_\rho(\bar{z})^*$ because of the properties of the matrix function ψ_e. Consequently, the minimal passive impedance system $\Sigma_{\mathrm{imp}\,\bullet}^*$ is optimal and, hence, the minimal passive impedance system $\Sigma_{\mathrm{imp}\,\bullet}$ is *-optimal. Therefore the realization $\{\Sigma_{f\bullet}, \Sigma_{b\bullet}, \Sigma_{\mathrm{scat}\,\bullet}, \Sigma_{\mathrm{imp}\,\bullet}\}$, constructed via $\theta_\bullet$, is *-optimal.

Furthermore, the *-outer matrix function ψ_e is the transfer function of the forward realization $\Sigma_{f\bullet} = (A_\bullet, K_\bullet, C_\bullet, L_\bullet; X_\bullet, \mathbb{C}^m, \mathbb{C}^p)$ of the process $y_\bullet$ with spectral density ρ. The evolution of this system is described by the equations:

$$(\Sigma_{f\bullet}) \quad \begin{cases} x_{f\bullet}(t+1) = A_\bullet x_{f\bullet}(t) + K_\bullet w_{f\bullet}(t), \\ y_\bullet(t) = C_\bullet x_{f\bullet}(t) + L_\bullet w_{f\bullet}(t). \end{cases} \tag{69}$$

We are ready now to present the following theorem.

Theorem 5. *The minimal backward realization $\Sigma_{b\circ}$ is such that its state space $X_\circ$ coincides with backward prediction space, i.e.*

$$X_\circ = \overline{P_{H^+(y_\circ)} H^-(y_\circ)},$$

where $H^-(y_\circ)$ and $H^+(y_\circ)$ are "past" and "future" subspaces of the stochastic process $y_\circ$, defined in subsection 2.1.

The state space $X_\bullet$ of the minimal forward realization $\Sigma_{f\bullet}$ coincides with forward prediction space, i.e.

$$X_\bullet = \overline{P_{H^-(y_\bullet)} H^+(y_\bullet)},$$

where $H^-(y_\bullet)$ and $H^+(y_\bullet)$ are "past" and "future" subspaces of the stochastic process $y_\bullet$.

The equations (68) and (69), which describe the evolution of minimal systems $\Sigma_{b\circ}$ and $\Sigma_{f\bullet}$, can be interpreted as a stationary backward and forward Kalman filters respectively.

Proof. Consider a simple conservative realization $\hat{\Sigma}_{\mathrm{imp}} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}; \hat{X}, \mathbb{C}^p)$ of the matrix function c_ρ

$$\hat{X} = L_2^m(\mathbb{T}), \quad x \in \hat{X}, \quad u \in \mathbb{C}^p;$$

$$\hat{A}x = \zeta^{-1} x(\zeta), \qquad \hat{B}u = \zeta^{-1}\beta_\circ(\zeta)u,$$
$$\hat{C} = \hat{B}^* \hat{A}, \qquad \hat{D} = \frac{1}{2}\hat{B}^*\hat{B}.$$

Then the minimal and optimal passive impedance system $\Sigma_{\mathrm{imp}\,\circ}$ is a restriction of the conservative impedance system $\hat{\Sigma}_{\mathrm{imp}}$. Statement of the theorem follows from the fact that

$$X^c_{\hat{\Sigma}_{\mathrm{imp}}} = \bigvee_{t\geq 0} \hat{A}^t \hat{B}\mathbb{C}^p = H^-(y_\circ), \quad X^o_{\hat{\Sigma}_{\mathrm{imp}}} = \bigvee_{t\geq 0} (\hat{A}^*)^t \hat{C}^* \mathbb{C}^p = H^+(y_\circ).$$

Proof of the statement about minimal forward realization $\Sigma_{f\bullet}$ is similar. ☐

It was shown in [28], [31] that the equations (68) of the system $\Sigma_{b\circ}$ and the equations (69) of the system $\Sigma_{f\bullet}$ can be written down in the form of stationary backward and forward Kalman filter respectively.

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